Einstein gravity with Gauss-Bonnet entropic corrections

Guido Cognola,[1,*] Ratbay Myrzakulov,[2,†] Lorenzo Sebastiani,[2,‡] and Sergio Zerbini[1,§]

[1]*Dipartimento di Fisica, Università di Trento and Istituto Nazionale di Fisica Nucleare Gruppo Collegato di Trento,*
38123 Trento, Italy
[2]*Eurasian International Center for Theoretical Physics and Department of General and Theoretical Physics,*
Eurasian National University, Astana 010008, Kazhakhstan

A toy model of Einstein gravity with a Gauss-Bonnet classically "entropic" term mimicking a quantum correction is considered. The static black hole solution due to Tomozawa is found and generalized with the inclusion of nontrivial horizon topology, and its entropy is evaluated, deriving the first law by equations of motion. As a result the Bekenstein-Hawking area law turns out to be corrected by a logarithmic area term. A Misner-Sharp expression for the mass of a black hole is found. Within a Friedmann-Lemaître-Robertson-Walker cosmological setting, the model is used in order to derive modified Friedmann equations. Such new equations are shown to reproduce the first law with the same formal entropy and quasilocal energy of the static case, but here within a Friedmann-Lemaître-Robertson-Walker space-time interpreted as a dynamical cosmological black hole. A detailed analysis of cosmological solutions is presented, and it is shown that the presence of the correction term provides regular solutions and interesting phases of acceleration and deceleration, as well as, with negligible matter, exact de Sitter solutions.

I. INTRODUCTION

In a recent paper [1], Tomozawa put forward an argument to deal with finite one-loop quantum corrections to Einstein gravity, by reconsidering one of his old results. The techniques referred to within his proposal are well known and can be found in the celebrated textbook [2]. Here, we present a simplified approach motivated by alternative arguments, and in some sense, also inspired by the entropic approach proposed in Refs. [3,4], where Lagrangian surface terms are taken into account.

The paper is organized as follows. In Sec. II, we will give a short review of the traditional semiclassical approach to quantum corrections of Einstein's gravity. In Sec. III, the "entropic" approach is introduced and the spherically symmetric static case is investigated. In Sec. IV, the model is extended to the spherically symmetric dynamical Friedmann-Lemaître-Robertson-Walker (FLRW) space-time and in Sec. V some explicit solutions are presented. Conclusions are given in Sec. VI.

II. BRIEF REVIEW OF QUANTUM CORRECTIONS TO GRAVITY

Before discussing the model which mimics the Tomosawa proposal, we review the more traditional treatment as proposed by Starobinsky in the seminal paper [5]. The argument is based on the so-called semiclassical gravity approach, where the backreactions of quantum fields are taken into account, in order to correct the classical Einstein equation, namely,

$$ G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R = \chi \langle T_{ij} \rangle, \qquad \chi = 8\pi G_N, \quad (1) $$

g_{ij}, R, and R_{ij} being, respectively, the metric, the scalar curvature, and Ricci tensor; G_N the Newton constant; and $\langle T_{ij} \rangle$ the vacuum expectation value of the quantum energy stress tensor, which in general is not explicitly known. As usual, units of measure are chosen in such a way that the speed of light is equal to 1.

By taking the trace of the equation above one has

$$ R = -\chi \langle T_i^i \rangle, \quad (2) $$

in which the stress tensor trace appears. When one is dealing with a conformally coupled quantum field, a quantum conformal anomaly is present (see the review paper [6]). In four dimensions it reads

$$ \langle T_i^i \rangle = -(\alpha W + \beta G + \delta \Delta R). \quad (3) $$

Here W is the "square" of Weyl tensor C_{ijrs} and G the Gauss-Bonnet topological invariant. They read

$$ W = C^{ijrs} C_{ijrs}, \qquad G = R^{ijkl} R_{ijkl} - 4R^{ij} R_{ij} + R^2. \quad (4) $$

The coefficients α, β, and δ depend on the number of conformal fields present in the theory. In some conformal field theories [7], one has $\delta = 0$ and $\alpha = \beta = \frac{N^2}{64\pi^2}$, N being a very large parameter.

One might try to solve the anomaly driven trace equation (2) within the static spherically symmetric ansatz where the metric has the form

*cognola@science.unitn.it
†rmyrzakulov@gmail.com
‡l.sebastiani@science.unitn.it
§zerbini@science.unitn.it

$$ds^2 = -a(r)dt^2 + \frac{dr^2}{a(r)} + r^2 d\sigma_2^2. \qquad (5)$$

The account of conformal anomaly contribution to spherically symmetric space-time solutions (Schwarzschild de Sitter black hole or wormholes) has been done in Ref. [8] (see also Ref. [9] for the case of extended theories of gravity). In such a case one has

$$R = -\frac{1}{r^2} \frac{d^2}{dr^2} [r^2(a-1)],$$

$$G = \frac{2}{r^2} \frac{d^2}{dr^2} [a-1], \qquad (6)$$

$$W = \frac{r^2}{3} \left[\frac{d^2}{dr^2} \left(\frac{a-1}{r} \right) \right]^2.$$

As a result, the quantum corrected Einstein trace equation reduces to

$$\frac{d^2}{dr^2} [r^2(a-1) + 2\chi\beta(a-1)^2] + \frac{\chi\alpha}{3} \left[r^2 \frac{d^2}{dr^2} \left(\frac{a-1}{r} \right) \right]^2 = 0. \qquad (7)$$

This is a second order nonlinear equation in the unknown variable $a(r)$. Its exact solution appears to be difficult to find. However, one can directly verify that there exists the de Sitter solution

$$a(r) = 1 - \frac{r^2}{2\chi\beta}, \qquad (8)$$

which corresponds to the famous solution found by Starobinsky in FLRW coordinates.

On the other hand, putting $\xi = \chi\alpha = \chi\beta$, one can find perturbative solutions assuming ξ to be a small quantity, that is,

$$a(r) = a_0(r) + \xi a_1(r) + \xi^2 a_2(r) + \cdots. \qquad (9)$$

Starting from $\xi = 0$ one finds the unperturbed solution

$$\frac{d^2}{dr^2} [r^2(a-1)] = 0 \Rightarrow a(r) \sim a_0(r) = 1 - \frac{c_1}{r} - \frac{c_2}{r^2}, \qquad (10)$$

c_1, c_2 being constants of integration. If we choose $c_1 = 2M$ and $c_2 = 0$, then such a solution may be interpreted as the Schwarzschild solution generated by a body of mass equal to M.

Now, by taking into account a first order perturbation we get

$$a(r) \sim a_0(r) + \xi a_1(r) = 1 - \frac{c_1 + \xi c_3}{r} - \frac{c_2 + \xi c_4}{r^2}$$
$$- \frac{4\xi c_1^2}{r^4} - \frac{8\xi c_1 c_2}{r^5} - \frac{22\xi c_2^2}{5r^6}, \qquad (11)$$

c_3, c_4 being further constants of integration. In principle, one may investigate this quantum corrected approximate static solution, but this will not be done here.

In Ref. [10] (see also [11]), an attempt has been made to solve the anomaly driven trace equation, but in the special case of negligible α. In fact, for $\alpha = 0$ and $\xi = k\beta$ one obtains

$$a(r) = 1 - \frac{r^2}{4\xi} \left[1 \pm \sqrt{1 + \frac{16\xi^2}{r^4} \left(1 + \frac{c_1}{\xi} + c_2 r \right)} \right], \qquad (12)$$

where c_1, c_2 are arbitrary constants. From the mathematical point of view this is a quite interesting result, because it might provide quantum corrections to the Schwarzschild solution, but unfortunately it is not physically relevant since α and β are quantities of the same order. For this reason such an approach does not seem completely adequate to describe physical situations.

III. ENTROPIC CORRECTED STATIC SPHERICALLY SYMMETRIC METRIC

Let us come back to Tomosawa proposal, reformulating it within a classical Lagrangian approach and working first in n dimensions, and then making an entropic dimensional reduction to $n = 4$, also in the spirit of other different approaches (see, for example, [3,4,12]).

It is well known that in a four-dimensional manifold the Gauss-Bonnet term does not contribute to the classical equations of motion. At the quantum level, the situation changes due to the regularization procedure, and this is the key observation made in Ref. [1]. In order to mimic quantum corrections due to a Gauss-Bonnet invariant or alternatively in order to activate such a term in the variational principle, we will make use of the fact that the functional variation of the classical action in arbitrary n dimensions does not commute with the limit $n \to 4$.

Thus, let us consider the following classical action:

$$I = \frac{1}{2\chi} \int d^n x \sqrt{-g} \left(R - 2\Lambda + \frac{\xi G}{(n-4)} \right), \qquad (13)$$

where Λ is the cosmological constant, G the Gauss-Bonnet invariant in n dimensions and ξ an arbitrary parameter, which will be assumed to be proportional to $\chi\beta$. We shall derive the field equations and at the end of the calculation we shall perform the $n \to 4$ limit.

We shall look for static spherically symmetric solutions with arbitrary horizon topologies. The generic metric reads

$$ds^2 = -a(r)b^2(r)dt^2 + \frac{dr^2}{a(r)} + r^2 d\Sigma_k^2, \qquad (14)$$

where a, b are arbitrary functions to be determined and $d\Sigma_k^2$ is the metric of a maximally symmetric $n - 2$ dimensional manifold, which will reduce to the metric $d\sigma_k^2$ of a two-dimensional maximally symmetric manifold in the limit $n \to 4$, that is,

$$d\Sigma_k^2 \to d\sigma_k^2 = \frac{d\rho^2}{1 - k\rho^2} + \rho^2 d\phi^2, \quad \text{for } n \to 4.$$

Here, $k = 1, 0, -1$, respectively, for spherical, toroidal, and hyperbolic topology horizons.

Within this static ansatz, the Lagrangian becomes a function of $a[r]$, $b[r]$, and their derivatives, that is,

$$I = \Omega \int dr r^{n-1} L(a, a', a'', b, b', b''), \tag{15}$$

where Ω is a constant factor due to the integration on all variables except r.

Now, by means of the Weyl's method discussed, for example, in Ref. [13], one obtains the field equations, which in the limit $n \to 4$ are regular and read

$$\frac{b'}{rb}\left(1 + 2\xi \frac{k - a}{r}\right) = 0, \tag{16}$$

$$\frac{1}{r^2 b} \frac{d}{dr}\left[(k - a)r - \frac{\Lambda r^3}{3} + \frac{\xi(k - a)^2}{r}\right] = 0. \tag{17}$$

These equations can be obtained by an effective classical Lagrangian, which mimics quantum corrections, obtained as the limit $n \to 4$ of the one in (15) (apart from total derivatives).

Equation (16) has the trivial solution $b = \text{Const}$, and we can choose $b = 1$ without losing generality, while Eq. (17) gives

$$a(r) = k + \frac{r^2}{2\xi}\left(1 \pm \sqrt{1 + 4\xi\left(\frac{C}{r^3} + \frac{\Lambda}{3}\right)}\right), \tag{18}$$

where C is a constant of integration. Equation (18) represents the topological generalization of the Tomozawa black hole solution discussed in [1] in the presence of a non-vanishing cosmological constant [14]. It is the analogue of the black hole solution of Lovelock gravity in five dimensions [15,16]. Furthermore, it is formally identical to the black hole solution found in [17], as a particular limit of Horava-Lifshits gravity considered in [18].

What about the meaning of integration constant C? A possible approach consists in discussing the limit $\xi \to 0$ of solution (18) as in [1]. The limit is finite only for the solution in (18) with the minus sign in front of the square root. In such a case the result is

$$a(r) = k - \frac{C}{r} - \frac{\Lambda r^2}{3}. \tag{19}$$

As a consequence one may conclude that $C = 2M$, M being the mass of a black hole.

Another approach consists in the investigation of the Clausius relation $dM = T_H dS_H$, which relates mass, temperature, and entropy associated with the given black hole solution, as a direct consequence of field equations. As we

shall see, this approach has the merit to give information about the entropy S_H of the black hole solution.

First let us study under which conditions the solution (18) represents a black hole. As is well known, one should have a real positive solution of $a(r_H) = 0$, namely,

$$\xi k^2 - Cr_H + kr_H^2 - \frac{\Lambda}{3}r_H^4 = 0. \tag{20}$$

The general solution of this quartic algebraic equation is quite complicated and we will not write down its explicit form. For our purposes it is sufficient to know that, depending on parameters, there exist real positive roots. For example this is certainly true in an asymptotically flat manifold, that is, for $\Lambda = 0$ and $k = 1$. In such a case one gets

$$r_H = \frac{C}{2}\left(1 + \sqrt{1 - \frac{4\xi}{C^2}}\right), \tag{21}$$

which is positive for $\xi < C^2/4$ and so it defines an event horizon.

Associated with any event horizon there exists a Hawking temperature given by

$$T_H = \frac{\kappa_H}{2\pi}, \qquad \kappa_H = \frac{a_H'}{2} \Rightarrow a_H' = 4\pi T_H, \tag{22}$$

κ_H being the surface gravity related to the horizon at r_H. This result is robust and can be derived by several alternative methods [19–21].

Following [22] we evaluate the equation of motion (17) at the horizon, where $a_H = 0$ and $a_H' = 4\pi T_H$. We obtain

$$4\pi T_H\left(r_H + \frac{2k\xi}{r_H}\right) = k - \Lambda r_H^2 - \frac{\xi k^2}{r_H^2} = \frac{dC}{dr_H}, \tag{23}$$

where the latter expression is a direct consequence of (20), obtained by assuming Λ to be a given parameter and the integration constant $C = C(r_H)$ to depend only on the horizon radius.

Now, introducing the horizon area $A_H = V_k r_H^2$, V_k being the measure of the unit surface (for example $V_1 = 4\pi$) we can write the latter equation in the "thermodynamical" form

$$d\left(\frac{C}{2}\right) = T_H d\left(\frac{\pi A_H}{V_k} + 2\pi k\xi \ln \frac{A_H}{V_k}\right). \tag{24}$$

It is quite natural to interpret this identity as the Clausius relation $T_H dS_H = dE_H$ for the black hole solution with entropy S_H and quasilocal energy E_H evaluated on the horizon given, respectively, by

$$S_H = \frac{\pi A_H}{V_k G_N} + 2\pi k\xi \ln \frac{A_H}{V_k G_N}, \tag{25}$$

and by

$$E_H = \frac{C}{2G_N} = \frac{1}{2G_N}\left(kr_H - \frac{\Lambda}{3}r_H^3 + \frac{\xi k^2}{r_H}\right). \quad (26)$$

This result gives the energy of the black hole as a classical Misner-Sharp mass plus a correction which depends on the parameter ξ. Furthermore, with regard to the corrected black hole entropy, the area law ($S_H = \pi A_H/V_k G_N$) does not hold, because a quantum logarithmic correction is present. This is a well-known general result, and it has been derived many times. It has been proposed in [10], and it appears in the quantum field theory treatment of black hole entropy with heat kernel techniques [23,24], loop gravity [25], and other approaches (see the recent paper [26] and the references therein).

It should be noted that for a toroidal black hole, present when $\Lambda < 0$, the correction is absent, while for spherical and hyperbolic black holes the corrections have opposite sign.

The expressions (26) and (25) are depending on quantities like r_H, A_H, and χ_H which are scalars for a generic spherically symmetric (dynamical) space-times. It also follows that the Clausius relation should have this covariance form. We will verify this fact in a dynamical spherical symmetric space-time as the FLRW one.

IV. ENTROPIC CORRECTED FLRW SPACE-TIME

Anomaly driven FLRW models have already been considered in the past [5,27–30]. Here we shall consider the toy model described by the action (13), but in n-dimensional, spatially flat FLRW space-time defined by means of the metric

$$ds^2 = -e^{2\eta(t)}dt^2 + a(t)^2(dx_1^2 + \cdots dx_{n-1}^2), \quad (27)$$

where $\eta(t)$, $a(t)$ are arbitrary functions, which will play the role of Lagrangian coordinates. The quantity $\eta(t)$ will be set equal to zero at the end of calculations. With this choice the parameter t will become the standard cosmological time.

In the total action we must also include classical matter described by a perfect fluid with density $\rho(t)$ and a pressure $p = w\rho(t)$, $0 \le w \le \frac{1}{3}$ being a constant. Thus the total classical action will read

$$I = \frac{1}{2\chi}\int d^n x\sqrt{-g}\left(R - 2\Lambda - \frac{\xi G}{(n-4)}\right) + I_m, \quad (28)$$

I_m being the action of matter which assumes the form

$$I_m = -\frac{1}{2}\int d^n x\sqrt{-g}\,g^{ij}[(\rho + p)u_i u_j + (\rho - p)g_{ij}]. \quad (29)$$

Making the variations with respect to $\eta(t)$ and $a(t)$ in arbitrary dimension n, then taking the limit $n \to 4$ and finally putting $\eta(t) = 0$ we obtain the two equations

$$H^2 = \frac{\chi\rho}{3} + \frac{\Lambda}{3} + \xi H^4, \quad (30)$$

$$\dot{H} + \frac{3}{2}H^2 = -\frac{\chi p}{2} + \frac{\Lambda}{2} + \xi H^2\left(2\dot{H} + \frac{3}{2}H^2\right). \quad (31)$$

As usual $H = \dot{a}/a$ is the Hubble parameter. An equation similar to (30) has been obtained in a covariant renormalizable model for gravity [31].

As a crucial consistent check, it is easy to show that the latter equations give rise to a matter continuity law in agreement with diffeomorphism invariance, that is,

$$\dot{\rho} = -3H(1 + w)\rho, \quad (32)$$

and also to the generalized Raychaudhuri equation

$$\ddot{a} = \dot{H} + H^2 = -\frac{\chi(1 + 3w)\rho}{6} + \frac{\Lambda}{3} + \xi H^2(2\dot{H} + H^2). \quad (33)$$

In order to analyze the physical consequences, one may indifferently use two among the three equations (30)–(32). An equation formally identical to (30) has already been obtained within AdS/CFT holographic correspondence in [32,33], in brane cosmology [34], or assuming a Clausius relation and a logarithmic correction to the area law in [35].

The FLRW admits a dynamical trapping horizon (Hubble horizon) in the sense of Hayward [36] located at $R_H = \frac{1}{H}$, with associated surface gravity and "dynamical temperature" [37]

$$k_H = -\left(H + \frac{\dot{H}}{2H}\right), \qquad T_H = \frac{\kappa_H}{2\pi} = -\frac{1}{2\pi}\left(H + \frac{\dot{H}}{2H}\right). \quad (34)$$

The identification of temperature with the surface gravity, as it happens in the static case, is supported by a tunneling computation via the Hamilton-Jacobi method (see, for example, [37,38] and references therein).

Now we introduce the generalized Misner-Sharp energy evaluated on the dynamical Hubble horizon, by means of Eq. (26) obtained for the static case; then we get

$$E_H = \frac{1}{2G_N}\left(R_H - \frac{\Lambda}{3}R_H^3 - \frac{\xi}{R_H}\right), \quad (35)$$

and similarly for the entropy,

$$S_H = \frac{A_H}{4G_N} - 2\pi\xi\ln\frac{A_H}{4G_N}, \qquad A_H = 4\pi R_H^2 = \frac{4\pi}{H^2}. \quad (36)$$

Such quantities satisfy the Clausius relation as a consequence of field equations (30)–(33). In fact, one has

$$dE_H = T_H dS_H \frac{A_H}{4G_N} - \frac{\chi}{16\pi}T_{(2)}dV_H, \quad (37)$$

where $V_H = 4\pi R_H^3/3$ is the volume of the Hubble sphere, while $T_{(2)} = p - \rho = (w - 1)\rho$ is the reduced stress

tensor trace, a scalar quantity in a dynamical symmetric space-time [36]. As for the static case, the cosmological constant is considered a given quantity.

V. EXPLICIT SOLUTIONS

Here we assume $\rho \geq 0$, $\Lambda \geq 0$ and analyze the behavior of the solutions for all possible values of ξ. For simplicity we also use units for which $\chi = 1$.

First of all, in the absence of matter, namely, when $\rho(t) = 0$, there exist de Sitter solutions for which $\dot{H}_{dS} = 0$. In fact, Eqs. (30) and (31) have the trivial solutions

$$H_{dS}^2 = \frac{1}{2\xi}\left(1 \pm \sqrt{1 - \frac{4\xi\Lambda}{3}}\right), \quad \Lambda > 0, \quad \xi > 0, \quad \frac{4}{3}\xi\Lambda \leq 1, \tag{38}$$

$$H_{dS}^2 = \frac{1}{2|\xi|}\left(-1 + \sqrt{1 + \frac{4|\xi|\Lambda}{3}}\right), \quad \Lambda > 0, \quad \xi < 0. \tag{39}$$

In the limit $|\xi\Lambda| \ll 1$ one has (for expanding universe, $H > 0$)

$$\begin{cases} H_1 \simeq \sqrt{\frac{\Lambda}{3}}, & \Lambda > 0, & \xi \geq 0, \\ H_2 \simeq \sqrt{\frac{1}{\xi}}, & \Lambda \geq 0, & \xi > 0, \\ H_3 \simeq \sqrt{\frac{\Lambda}{3}}, & \Lambda > 0, & \xi \leq 0. \end{cases} \tag{40}$$

Both the solutions with $H(t) = H_1 = H_3$ could describe the current acceleration era independently on ξ, while the solution with $H(t) = H_2$ could describe the inflationary era independently on Λ. It has to be stressed that in the case $\xi > 0$, the model effectively describes only one of the two possible phases, because they correspond to distinct solutions.

Now we go back to the general model with arbitrary matter. We first observe that in some cases from the generalized Friedmann equation one gets constraints on the possible values of H and ρ and, as a consequence, the corresponding solution is not singular for any finite value of t. In fact, solving (30) for H^2, we obtain

$$H^2 = \frac{1}{2\xi}\left(1 \pm \sqrt{1 - \frac{4\xi(\rho + \Lambda)}{3}}\right), \quad \xi > 0, \tag{41}$$

$$H^2 = \frac{1}{2|\xi|}\left(-1 + \sqrt{1 + \frac{4|\xi|(\rho + \Lambda)}{3}}\right), \quad \xi < 0. \tag{42}$$

If $\xi > 0$, from (41) we get the constraints

$$\frac{3}{4\xi} \geq \rho + \Lambda \geq 0 \Rightarrow \begin{cases} \frac{1}{2} \geq \xi H^2 \geq 0, \\ \frac{1}{2} \leq \xi H^2 \leq 1, \end{cases} \quad \xi > 0. \tag{43}$$

If $\xi < 0$ Eq. (42) does not give any constraint on H and ρ.

Since we would like to describe possible changes of phase during the expansion, for example, the exit from inflation or the beginning of actual acceleration, we search for the stationary points of the quantity $\dot{a}(t)$. By solving the system of equations (30) and (33) for $\dot{H}$ we get

$$\begin{aligned} \ddot{a} &= H^2 - \frac{(1 + w)}{2(1 - 2\xi H^2)}\rho \\ &= -\frac{1}{2(1 - 2\xi H^2)}[(1 - 3w)\xi H^4 \\ &\quad + (1 + 3w)H^2 - (1 + w)\Lambda], \end{aligned} \tag{44}$$

and so $\ddot{a}(t)$ vanishes when $H^2(t)$ reaches the (positive) values

$$H_0^2 = \begin{cases} \frac{2}{3}\Lambda, & \xi \neq 0, \quad w = \frac{1}{3}, \\ \frac{1+3w}{2\xi(1-3w)}\left[-1 + \sqrt{1 + \frac{4\xi\Lambda(1+w)(1-3w)}{(1+3w)^2}}\right], & \xi > 0, \quad w \neq 1/3, \\ \frac{1+3w}{2|\xi|(1-3w)}\left[1 \pm \sqrt{1 - \frac{4|\xi|\Lambda(1+w)(1-3w)}{(1+3w)^2}}\right], & \xi < 0, \quad w \neq 1/3. \end{cases} \tag{45}$$

This means that when $H^2(t_0) = H_0^2 > 0$ the system goes from a decelerated to an accelerated expansion (or vice versa), and this happens only if H_0 is in the permitted range of H. If $H_0 = 0$, then there is a singularity [$a(t_0) = \infty$] or a bounce [$\dot{a}(t_0) = 0$], when $t_0 < \infty$.

In the case $\xi > 0$ we must take into account (43) and so, in order to have a possible change of phase, the free parameters have to satisfy the following constraints (we consider that matter or radiation contributes):

$$\left(\Lambda > 0, \xi > 0, 0, w = \frac{1}{3}\right), \quad H_0^2 = \frac{2}{3}\Lambda \Rightarrow \begin{cases} 0 < \frac{4}{3}\xi\Lambda \leq 1, \\ 1 \leq \frac{4}{3}\xi\Lambda \leq 2, \end{cases}$$

$$(\Lambda > 0, \xi > 0, w = 0), \quad H_0^2 = \frac{1}{2\xi}(-1 + \sqrt{1 + 4\xi\Lambda}) \Rightarrow \begin{cases} 0 < \frac{4}{3}\xi\Lambda \leq 1, \\ 1 \leq \frac{4}{3}\xi\Lambda \leq \frac{8}{3}. \end{cases} \tag{46}$$

When $\Lambda = 0$, the only stationary point of $\dot{a}(t)$ corresponds to $H_0 = 0$ and so the model with $(\Lambda = 0, \xi > 0)$ can describe only one phase of the expansion.

In the case $\xi < 0$, the algebraic equation (42) does not give any restriction on the values of H^2, but nevertheless, as we shall see in the following, the parameters ξ and Λ are not completely arbitrary. Moreover, from the third equation in (45) we also see that in this case, even for $\Lambda = 0$ there is a nonvanishing stationary point at $H_0^2 = (1 + 3w)/|\xi|(1 - 3w)$ with $w \neq 1/3$.

For the special case $\xi = 0$ one can find exact solutions for $a(t)$. In particular, the solutions of (30) and (31) with $\rho > 0$ read

$$a(t) = a_0 \left[\sinh^2 \left(\frac{1 + w}{2} \sqrt{3\Lambda} t \right) \right]^{\frac{1}{3(1+w)}}, \qquad (47)$$

$$H(t) = \coth \left(\frac{1 + w}{2} \sqrt{3\Lambda} t \right), \qquad (48)$$

$$\rho(t) = \Lambda \sinh^{-2} \left(\frac{1 + w}{2} \sqrt{3\Lambda} t \right), \qquad (49)$$

a_0 being an arbitrary constant. As one can see, $a(t)$ vanishes for a finite value of t ($t = 0$ with the chosen initial conditions) and as a consequence both $H(t)$ and $\rho(t)$ are divergent at that time.

As we have seen above, in the absence of matter $H(t) = \Lambda/3$ and $a(t) = a_0 \exp(\sqrt{\Lambda/3} t)$ (de Sitter solution).

If $\xi \neq 0$ we are not able to get explicit solutions for $a(t)$, but nevertheless we can get implicit solutions for $H(t)$, which permits us to understand the behavior of the system. We start with the two cases $(\Lambda = 0, \xi > 0)$ and $(\Lambda = 0, \xi < 0)$. Apart from arbitrary integration constants, we have

$$\frac{3(1 + w)}{2} t = \frac{1}{H} + \frac{\sqrt{\xi}}{2} \log \left| \frac{1 + \sqrt{\xi} H}{1 - \sqrt{\xi} H} \right|, \qquad \xi > 0, \qquad (50)$$

$$\frac{3(1 + w)}{2} t = \frac{1}{H} - \sqrt{|\xi|} \arctan(\sqrt{|\xi|} H), \qquad \xi < 0. \qquad (51)$$

It has to be noted that in order to have $\rho > 0$, in (50) $H(t)$ has to be restricted to the values $\xi H^2 \leq 1$ in agreement with (43). This means that $a(t)$ does not vanish and the density does not diverge. Equation (50) effectively corresponds to two different solutions related to the distinct algebraic equations in (41). The solution with $1/2 \geq \xi H^2 \geq 0$ has an asymptotic behavior of the kind $H(t) \sim 1/t \rightarrow 0$, while the other with $1/2 \leq \xi H^2 \leq 1$ goes as $H(t) \sim 1/\sqrt{\xi} = \text{Const}$, giving rise to a de Sitter asymptotic behavior for $a(t)$, that is,

$$a(t) \sim a_0 e^{t/\sqrt{\xi}}, \qquad \Lambda = 0, \qquad \xi > 0, \qquad \frac{1}{2} \leq \xi H^2 \leq 1. \qquad (52)$$

As we already said above, the model with such values of the free parameters describes only one phase of the expansion. It could be used in order to describe the inflationary era as in Ref. [5], but it does not provide a natural exit from that phase.

In the second case, Eq. (51), $H(t)$ has no restrictions and so there is a singularity for $t \rightarrow 0$ [$a(t) \rightarrow 0$ and $\rho(t) \rightarrow \infty$]. This model is not able to describe inflation, because it does not have a rapidly expanding phase, but it could describe current acceleration. In fact, putting for simplicity $w = 0$ in (45), we obtain $H_0^2 = 1/|\xi|$ and using (44) we see that $\ddot{a}(t)$ is negative or positive according to whether $H(t)$ is smaller or greater than H_0.

Finally we consider the two general cases with $(\Lambda > 0, \xi > 0)$ and $(\Lambda > 0, \xi < 0)$. Apart from arbitrary integration constants, the implicit solutions for $H(t)$ are given by

$$\frac{3(1 + w)}{2} t = \frac{1}{2\sqrt{\alpha_+}} \log \left| \frac{1 + H/\sqrt{\alpha_+}}{1 - H/\sqrt{\alpha_+}} \right|$$
$$+ \frac{1}{2\sqrt{\alpha_-}} \log \left| \frac{1 + H/\sqrt{\alpha_-}}{1 - H/\sqrt{\alpha_-}} \right| \qquad \Lambda > 0, \quad \xi > 0, \qquad (53)$$

where

$$\alpha_\pm = \frac{1}{2\xi} \left(1 \pm \sqrt{1 - \frac{4\xi\Lambda}{3}} \right), \qquad \xi\Lambda \leq \frac{3}{4}, \qquad (54)$$

and

$$\frac{3(1 + w)}{2} t = \frac{1}{2\sqrt{\beta_+}} \log \left| \frac{1 + H/\sqrt{\beta_+}}{1 - H/\sqrt{\beta_+}} \right|$$
$$- \frac{1}{\sqrt{\beta_-}} \arctan \left(\frac{H}{\sqrt{\beta_-}} \right), \qquad \Lambda > 0, \quad \xi < 0, \qquad (55)$$

where

$$\beta_\pm = \frac{1}{2|\xi|} \left(\sqrt{1 + \frac{4|\xi|\Lambda}{3}} \pm 1 \right). \qquad (56)$$

Also in this case in order to have $\rho \geq 0$, $H(t)$ in (53) has to be restricted to the values $\alpha_- \leq H^2 \leq \alpha_+$ and so $a(t)$ does not vanish for finite values of time. The equation describes two distinct solutions, both of which give rise to a de Sitter asymptotic behavior for $a(t)$ of the kind

$$a(t) \sim a_0 e^{\sqrt{\alpha_+} t}, \qquad \lim_{t \rightarrow \infty} H(t) = \sqrt{\alpha_+}, \qquad (57)$$

$$a(t) \sim a_0 e^{\sqrt{\alpha_-} t}, \qquad \lim_{t \rightarrow \infty} H(t) = \sqrt{\alpha_-}. \qquad (58)$$

Looking at (45) we see that in both the cases $w = 1/3$ and $w = 0$ the stationary point is out of the permitted range and so also this generalized solution can describe only one phase of the expansion.

In the last case, Eq. (55), ρ is non-negative for $H^2 \geq \beta_-$ and so, in contrast with the analog case with $\Lambda = 0$, there is a solution for which $a(t)$ is always finite for finite values of time. From (55), in fact, we have

$$\beta_- \leq H^2(t) \leq \beta_+, \qquad t_0 \leq t \leq \infty, \qquad (59)$$

$$\beta_+ \leq H^2(t) \leq \infty, \qquad \infty \geq t \geq t_0. \qquad (60)$$

Both the latter solutions give rise to a de Sitter asymptotic behavior for $a(t)$ similar to the one in (57), but with α_+ replaced by β_+.

Depending on the parameters, the previous solutions can describe decelerated and accelerated expansion phases. First choosing $w = 1/3$, from (45) we get

$$H_0^2 = \frac{2}{3}\Lambda \Rightarrow \begin{cases} \beta_- < H_0^2 < \beta + & \text{if } |\xi|\Lambda < \frac{9}{4}, \\ H_0^2 > \beta + & \text{if } |\xi|\Lambda > \frac{9}{4}, \end{cases} \qquad (61)$$

while in the case $w = 0$ we obtain

$$H_0^2 = H_\pm^2 = \frac{1}{2|\xi|}(1 \pm \sqrt{1 - 4|\xi|\Lambda})$$

$$\Rightarrow \begin{cases} \beta_- < H_\pm^2 < \beta + & \text{if } |\xi|\Lambda < \frac{1}{4}, \\ H_\pm^2 > \beta + & \text{never.} \end{cases} \qquad (62)$$

Using the latter results, now we are able to study the changes of phase of the system. For example, if $w = 1/3$ from (44) it follows that $\ddot{a}$ is negative or positive according to whether $H^2(t)$ is higher or lower than $H_0^2 = 2\Lambda/3$. This means that the system passes from a decelerated to an accelerated phase if $|\xi|\Lambda < 9/4$, while it passes from an accelerated to a decelerated phase if $|\xi|\Lambda > 9/4$. Of course such last situation has to be rejected for physical reasons, because both ξ and Λ have to be small quantities.

In the case of $w = 0$, the acceleration is negative for $H_- < H(t) < H_+$ and positive otherwise and this means that the model describes three phases, that is,

(1) accelerated expansion for $\beta_- < H^2(t) < H_-^2$;
(2) decelerated expansion for $H_-^2 < H^2(t) < H_+^2$;
(3) accelerated expansion for $H_+^2 < H^2(t) < \beta_+$.

quantum correction has been investigated. The static black hole solution due to Tomozawa has been recovered and generalized with the inclusion of nontrivial horizon topology, and its entropy has been evaluated, deriving the first law from equations of motion. As a result the Bekenstein-Hawking area law has acquired a corrected logarithmic area term. A Misner-Sharp expression for the mass of a black hole has been found. The same model has been used in order to derive Friedmann equations with corrected terms, which reproduce the first law with the same formal entropy and energy of the static case, but now related to a general FLRW space-time, interpreted as a dynamical cosmological black hole, with a "temperature" associated with Hayward's dynamical surface gravity: this result is in agreement with one obtained by the tunneling method applied to FLRW space-time in Ref. [37]. A detailed analysis of all possible cosmological solutions, including de Sitter and nonsingular solutions, has been provided. These solutions describe several phases; the more interesting one is present for $\xi < 0$ and $\Lambda > 0$. In this case, when $\omega = 0$, there is no singularity at $t = 0$, but an initial powerlike acceleration, followed by a deceleration phase, and by a final exponential acceleration.

As a consequence, this model may be a candidate to describe inflation and the current dark energy epoch with a natural exit and entrance in the various cosmological phases. Of course, it should be emphasized that in this qualitative work, we have focused only on the possibility of the realization of a unified scenario: important issues of inflationary cosmology, such as the reheating process and the generation of the curvature perturbations with a power spectrum consistent with the anisotropies of the cosmic microwave background, and the consistence with the observational constraints coming from observations of our Universe [39,40], are crucial in the choosing of boundary conditions and numerical tests on the model, which is not the aim of this work.

A generalization of this work may be done by using s-wave approximation and reduction to effective second theory as in Ref. [41]. It may also be interesting to generalize the results of this work for $F(R)$ gravity (for a review, see Refs. [42,43]), where a number of black hole solutions exist, as in the recent review [44].

VI. CONCLUDING REMARKS

In this paper, a toy model of Einstein gravity with a Gauss-Bonnet classically entropic term mimicking a

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