Phase transition, critical behavior, and critical exponents of Myers-Perry black holes

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The critical behavior of Myers-Perry black holes with equal angular momenta in even dimensions are studied. We include the corrections beyond the semiclassical approximation on Hawking temperature in the grand canonical ensemble. Having done so, we find that the critical behavior and critical exponents of Myers-Perry black holes correspond to those of a Van der Waals liquid-gas where this analogy holds in any dimension. Also, using Ehrenfest's equations, we calculate the order of the phase transition in the semiclassical approximation for the canonical ensemble and beyond the semiclassical approximation for the grand canonical ensemble near the critical point. Finally, the Ruppeiner curvature formula is used to investigate the thermodynamic geometry of Myers-Perry black holes.

I. INTRODUCTION

The Hawking temperature of a black hole is proportional to its surface gravity. Also, a black hole's entropy is proportional to its horizon area. This is known as the celebrated Bekenstein-Hawking area law, $S_{BH} = \frac{A}{4}$. There are several different methods for calculating the corrections to the semiclassical Bekenstein-Hawking entropy. These are based on statistical mechanical arguments, field theory methods, quantum geometry, the Cardy formula, the generalized uncertainty principle, etc. (The corresponding literature is rather extensive; for a partial selection, see Refs. [1–8].)

One of the black hole solutions in higher dimensions which has attracted a lot of attention is the Myers-Perry black hole [9–13], whose uncharged rotating version is a direct generalization of the Kerr black hole solution in general relativity. The classification of black hole species in higher dimensions was studied by Rodriguez [14]. Also, the corrections beyond the semiclassical approximation can lead to a corrected Hawking temperature and entropy for Myers-Perry (MP) black holes.

The lack of a statistical description for black hole systems has encouraged researchers to consider thermodynamic geometry [15–20], Ehrenfest's equations [21–23], and, recently, the analogy of a black hole with a Van der Waals system [24–26]. An important part in the theory of phase transitions is the exploration of the thermodynamic behavior of a system near its critical point using critical exponents. These critical exponents are supposed to be universal and are independent of the details of the interaction. In other words, different physical systems may share

the same critical exponents [27–30]. In this way, we may find the possibility of searching for the nature of phase transitions and critical exponents in the grand canonical ensemble (for which the angular velocity, Ω, is taken to be fixed) and in the canonical ensemble (for which the angular momentum, J, is also taken to be fixed). We consider the corrected temperature of MP black holes beyond the semiclassical approximation in the grand canonical ensemble and study the analogy of this system with a liquid-gas system. We also find that the critical exponents correspond to a Van der Waals system in the grand canonical ensemble. For both the canonical and grand canonical ensembles we search for the satisfication of Ehrenfest's equations at the critical point [31–33].

The outline of this paper is as follows. In Sec. II, we consider the semiclassical thermodynamic quantities and critical behavior of MP black holes in even dimensions with n nonzero equal spins J in the grand canonical and canonical ensembles. In Sec. III, corrections to the semiclassical Hawking temperature and entropy are investigated in the grand canonical ensemble and the analogous behavior of this system to a Van der Waals gas is studied. In Sec. IV, Ehrenfest's equations for these black hole systems are developed in both ensembles and the order of the phase transition near the critical points is analyzed. In Sec. V, the critical exponents near the critical point are calculated in the grand canonical ensemble. Section VI is devoted to the study of the thermodynamic geometry of MP black holes and it is found that the scalar curvature diverges exactly at the point where the heat capacity is divergent.

II. SEMICLASSICAL THERMODYNAMICS OF MYERS-PERRY BLACK HOLES

A summary of the thermodynamic quantities of MP black holes in even dimensions with $n = \frac{d-2}{2}$ nonzero

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equal spins J is presented below. More details and a complete solution can be found in Ref. [11]. The multiple-spin Kerr black hole's metric in Boyer-Lindquist coordinates for an even d is given by [12,13]

$$ds^2 = -dt^2 + r^2 d\alpha^2 + (r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2)$$
$$+ \frac{mr}{\Pi F}(dt - a_i \mu_i^2 d\phi_i)^2 + \frac{\Pi F}{\Pi - mr} dr^2, \qquad (1)$$

where $\mu_i^2 + \alpha^2 = 1$ and $m = \frac{16\pi GM}{(d-2)\Omega_{(d-2)}}$. The functions Π and F are defined as

$$\Pi = \prod_{i=1}^{(d-1)/2} (r^2 + a_i^2), \qquad (2)$$

$$F = 1 - \frac{a_i^2 \mu_i^2}{r^2 + a_i^2}. \qquad (3)$$

The metric is slightly modified for an odd d. The event horizon in the Boyer-Lindquist coordinates for an even d is defined by

$$\Pi(r_+) - mr_+ = 0. \qquad (4)$$

Moreover, the area of the event horizon for an even d can be expressed as follows:

$$A = \Omega_{(d-2)} \prod_i (r_+^2 + a_i^2). \qquad (5)$$

The Bekenstein-Hawking entropy for the MP black holes with an even d is given by

$$S = \prod_i (r_+^2 + a_i^2), \qquad (6)$$

where $k_B = \frac{1}{\pi}$ and $G = \frac{\Omega_{(d-2)}}{4\pi}$. Briefly put, the mass and Hawking temperature of the black hole are

$$M = \frac{d-2}{4} S^{(\frac{d-3}{d-2})} \sqrt{1 + \frac{4J^2}{S^2}} \qquad (7)$$

and

$$T = \frac{(d-3)}{4S^{\frac{1}{d-2}}} \frac{1 - \frac{4J^2}{(d-3)S^2}}{\sqrt{1 + \frac{4J^2}{S^2}}}. \qquad (8)$$

The angular velocity is defined as

$$\Omega = \left(\frac{\partial M}{\partial J}\right)_S = \frac{(d-2)S^{\frac{d-3}{d-2}}J}{\sqrt{1 + \frac{4J^2}{S^2}}S^2}. \qquad (9)$$

Thus, the heat capacity for MP black holes in d dimensions with $n = \frac{d-2}{2}$ nonzero equal spins in the canonical (constant J) ensemble is defined by
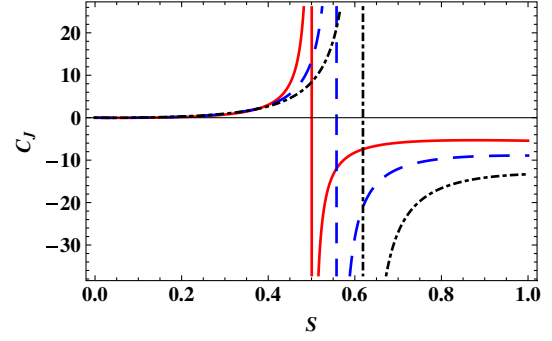

FIG. 1 (color online). Specific heat C_J with respect to S for $d = 6$ [red (solid) line], $d = 8$ [blue (dashed) line], $d = 10$ [black (dashed-dotted) line], and $J = 0.1$.

$$C_J(J, S) = T\left(\frac{dS}{dT}\right)_J$$
$$= \frac{(4J^2 + S^2)((d-3)S^2 - 4J^2)(d-2)S}{16(d-1)J^4 + 4(6 + (d-4)d)J^2 S^2 - (d-3)S^4}. \qquad (10)$$

Now, we can see that the heat capacity for the canonical ensemble in Eq. (10) is divergent at the critical entropy, which is given by

$$S_c = \frac{J}{\sqrt{d-3}}\Big(2(-4d + 6 + d^2$$
$$+ \sqrt{32d^2 - 64d - 8d^3 + 48 + d^4})\Big)^{\frac{1}{2}}. \qquad (11)$$

The discontinuity in the plot of the heat capacity indicates that a kind of phase transition occurs at this point (Fig. 1). Also, the Hawking temperature has positive values at this critical point, $S = S_c$ (Fig. 2). In order to find the nature of this phase transition, it is necessary to consider Ehrenfest's equations near the critical point. This phase transition will be dealt with in greater detail below.

The heat capacity in the grand canonical ensemble (constant Ω) is defined as
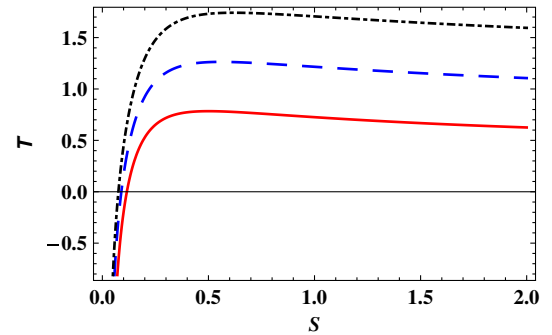

FIG. 2 (color online). Semiclassical Hawking temperature, T, with respect to S for $d = 6$ [red (solid) line], $d = 8$ [blue (dashed) line], $d = 10$ [black (dashed-dotted) line] and $J = 0.1$.

$$C_\Omega(\Omega, S) = T\left(\frac{dS}{dT}\right)_\Omega$$

$$= \frac{-S}{-d^3 + 7d^2 - 16d + 12 + 4d\Omega^2 S^{\frac{2}{d-2}} - 16\Omega^2 S^{\frac{2}{d-2}}}$$

$$\times (8S^{\frac{2}{d-2}}d^2\Omega^2 - d^4 + 9d^3 - 30d^2 - 36S^{\frac{2}{d-2}}d\Omega^2 + 44d + 40S^{\frac{2}{d-2}}\Omega^2 - 24 - 16S^{\frac{4}{d-2}}\Omega^4). \qquad (12)$$

Based on Eqs. (8), (9), and (12), we can plot the temperature and heat capacity with respect to S in the grand canonical ensemble (Figs. 3 and 4). We find that T is equal to zero at a special point whose value depends on the dimensions d and angular velocity Ω (for $d = 8$, $S = 0.000421875$). Since the Hawking temperature, T, has positive values before these specific points and the heat capacity is negative over the entire region, the black hole system is unstable throughout. The heat capacity is continuous for all values of d and Ω; therefore, the existence of a phase transition is ruled out.

III. ANALYSIS OF PHASE TRANSITION BEYOND THE SEMICLASSICAL APPROXIMATION

In the previous section, we showed that the heat capacity in the grand canonical ensemble (fixed Ω) is not divergent.
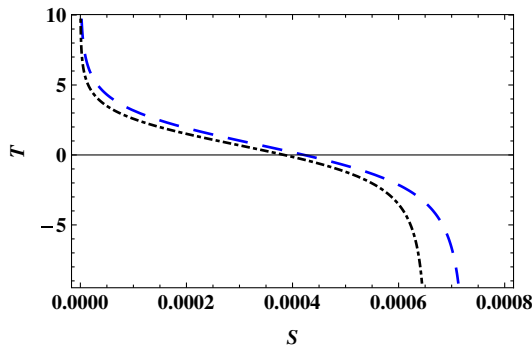

FIG. 3 (color online). Semiclassical Hawking temperature, T, with respect to S for $d = 8$ [blue (dashed) line], $d = 10$ [black (dashed-dotted) line], and $\Omega = 10$.
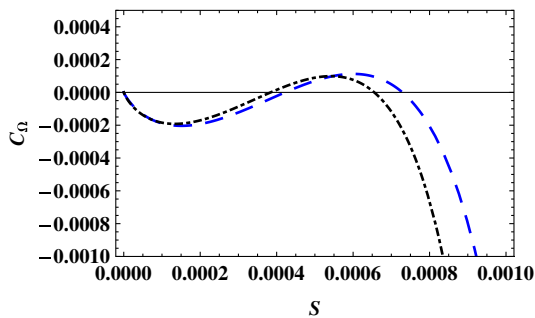

FIG. 4 (color online). Specific heat C_Ω with respect to S for $d = 8$ [blue (dashed) line], $d = 10$ [black (dashed-dotted) line] and $\Omega = 10$.

Recently, a phase transition has been investigated for the corrected thermodynamics of a Kerr black hole ($d = 4$) beyond the semiclassical approximation in Ref. [23]. In this section, we would like to explore these calculations for arbitrary dimensions, in this case for MP black holes. We also study the analogy between this system and a liquid-gas system.

We consider the corrections beyond the semiclassical approximation to the Bekenstein-Hawking entropy of black holes and the Hawking temperature, which can be obtained by using a variety of approaches based on statistical mechanical arguments, field theory methods, quantum geometry, the Cardy formula, the generalized uncertainty principle, etc. A review of these methods may be found in Refs. [1–8]. The first-order correction to the entropy can be expressed by

$$\tilde{S} = S + \beta_1 \frac{4}{(d - 2)} \log S. \qquad (13)$$

The first term is the semiclassical entropy $S = \frac{4Mr_+}{\hbar(d-2)}$ for MP black holes and the second term is the first-order quantum correction. Also, β_1 is a dimensionless constant and smaller than 1. To identify the coefficients of the leading corrections such as β_1, we can use the trace anomaly or other standard methods.

Hawking in Ref. [5] calculated one-loop corrections to the radiation process, associated with a trace anomaly, and showed that the backreaction has an interesting effect on the Hawking radiation and temperature. Considering the results of the renormalization group approach [6] and the generalized uncertainty principle [7,8] suggests that in the simplest cases the first-order correction term to the Hawking temperature is proportional to the inverse of the area (see also Ref. [1]). The above-mentioned theories suggest the following first-order correction to the Hawking temperature, which is also consistent with the first law of thermodynamics:

$$\tilde{T} = T\left(1 - \beta_1 \frac{4}{(d - 2)S}\right). \qquad (14)$$

It should be noted that a similar first-order quantum correction was also calculated in Ref. [34]; however, their assumptions have no theoretical justification and were criticized in Ref. [35].

Equations (8), (9), (13), and (14) can now be used to calculate the corrected temperature and specific heat up to the first-order correction as a function of entropy and angular velocity for these types of black holes,
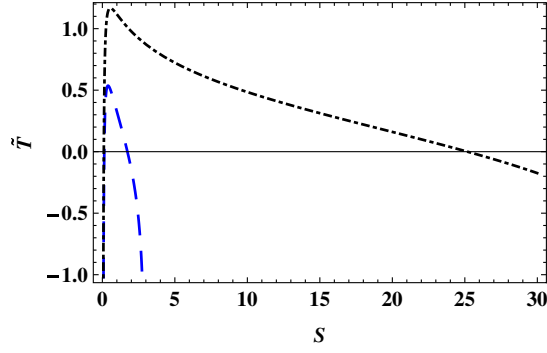
FIG. 5 (color online). Corrected Hawking temperature $\tilde{T}$ with respect to S for $\Omega < \Omega_c$, $\beta_1 = 0.2$, and $d = 8$ [blue (dashed) line], $d = 10$ [black (dashed-dotted) line].

$$\tilde{T}(S, \Omega) = \left(1 - \beta_1 \frac{4}{(d-2)S}\right)$$

$$\times \frac{(d-3)\left(1 - \frac{4S^2\Omega^2}{(d-3)(-4S^2\Omega^2 + S^{\frac{2(d-3)}{d-2}}(d-2)^2)}\right)}{4S^{\frac{1}{d-2}}\sqrt{1 + \frac{4S^2\Omega^2}{-4S^2\Omega^2 + S^{\frac{2(d-3)}{d-2}}(d-2)^2}}}, \quad (15)$$

and

$$\tilde{C}_\Omega(S, \Omega) = \tilde{T}\left(\frac{d\tilde{S}}{d\tilde{T}}\right)_\Omega$$

$$= \frac{((d-2)^2S^2 - 16\beta_1^2)}{\zeta}((d-3)(d-2)^3$$

$$- 4(d-2)(2d-5)S^{\frac{2}{d-2}}\Omega^2 + 16S^{\frac{4}{d-2}}\Omega^4), \quad (16)$$

where

$$\zeta = (d-2)((3-d)(d-2)^3S + 4(d-4)(d-2)S^{\frac{d}{d-2}}\Omega^2$$

$$+ 4((d-3)(d-2)^2(d-1) - 8(d-3)(d-1)S^{\frac{2}{d-2}}\Omega^2$$

$$+ 16S^{\frac{4}{d-2}}\Omega^4)\beta_1) - 2\Omega^2 + 16S^{\frac{4}{d-2}}\Omega^4. \quad (17)$$

Although the corrected specific heat $\tilde{C}_\Omega$ is divergent at two specific points $S_{(1,2)}$ for a given dimension, as d indicated
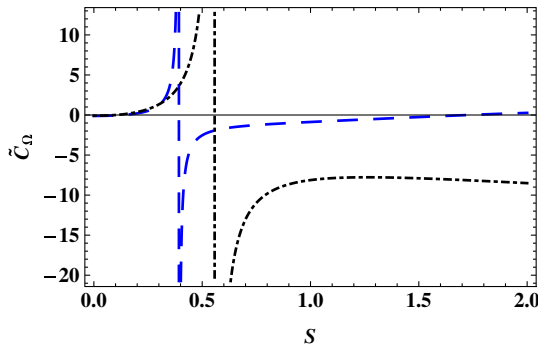

FIG. 6 (color online). Corrected specific heat $\tilde{C}_\Omega$ with respect to S for $\Omega < \Omega_c$, $\beta_1 = 0.2$ and $d = 8$ [blue (dashed) line], $d = 10$ [black (dashed-dotted) line].
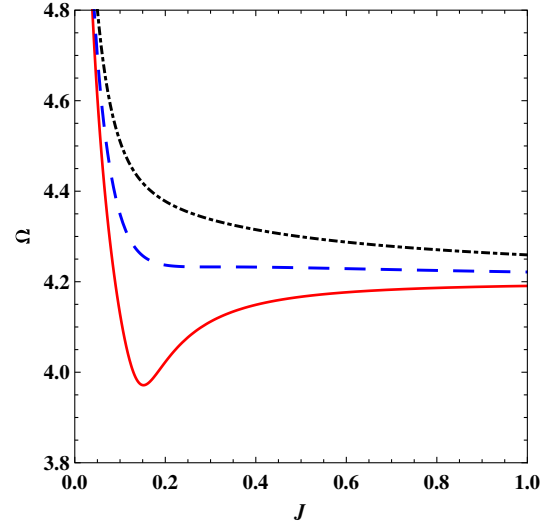

FIG. 7 (color online). The angular velocity Ω with respect to the angular momentum J at $\tilde{T} = \tilde{T}_c$ [blue (dashed) line], $\tilde{T} > \tilde{T}_c$ [black (dashed-dotted) line], and $\tilde{T} < \tilde{T}_c$ [red (solid) line] for $d = 6$ and $\beta_1 = 0.2$.

in Eq. (17), the corrected Hawking temperature in Eq. (15) does not have a real value at the larger point S_2. This means that the larger divergent point S_2 is nonphysical (Figs. 5 and 6). Based on Eqs. (9) and (14)), the angular velocity Ω can be depicted with respect to the angular momentum J as a "P-V diagram" at $\tilde{T} = \tilde{T}_c$, $\tilde{T} > \tilde{T}_c$, and $\tilde{T} < \tilde{T}_c$ for the given dimensions d (Fig. 7). An inflection point can be observed at $\tilde{T} < \tilde{T}_c$; this behavior is similar to that of a Van der Waals system. The critical point is obtained from $(\frac{\partial \Omega}{\partial J})_c = 0$ and $(\frac{\partial^2 \Omega}{\partial J^2})_c = 0$. Thus, by using the equation of state in higher dimensions d (Ω as a function of $\tilde{T}$, d, and J) and also $(\frac{\partial \Omega}{\partial J})_c = 0$ and $(\frac{\partial^2 \Omega}{\partial J^2})_c = 0$, we can obtain the values of Ω_c, S_c, and $\tilde{T}_c$ at the critical point when the discriminant of Eq. (17) vanishes, in other words, when two divergent points meet at one critical point (Fig. 8).

Stability is determined by the third derivative of Ω with respect to J. The inequality $(\frac{\partial^3 \Omega}{\partial J^3})_c < 0$ [27] shows the
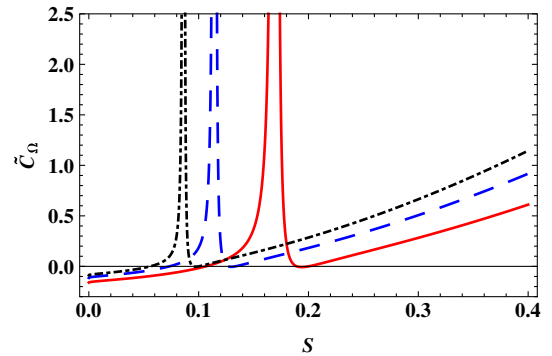

FIG. 8 (color online). Corrected specific heat $\tilde{C}_\Omega$ with respect to S for $\beta_1 = 0.2$, $\Omega = \Omega_c$ and $d = 6$ [red (solid) line], $d = 8$ [blue (dashed) line], and $d = 10$ [black (dashed-dotted) line].

TABLE I.   Summary of results for the values of Ω_c, S_c, $\tilde{T}_c$, and $(\frac{\partial^3\Omega}{\partial J^3})_c$ for $\beta_1 = 0.2$.

d	Ω_c	S_c	$\tilde{T}_c$	$(\frac{\partial^3\Omega}{\partial J^3})_c$
6	3.03198	0.1696	0.2371850	-25865.89
8	4.23272	0.11445	0.258412	-7383.72
10	5.36657	0.0863001	0.277826	-61407.8

stability at the critical point. The results are summarized in Table I.

The critical values of Ω_c, S_c, and $\tilde{T}_c$ depend on the coefficient β_1. This coefficient is not exactly known but is smaller than one. However, for $d = 4$, and for $\frac{P_c V_c}{T_c} = \frac{3}{8}$ which is universal and exactly the same as a Van der Waals fluid, the value for this coefficient will be $\beta_1^{d=4} = \frac{1}{24\sqrt{3}}$. Alternatively, its value for a given d may be determined by considering the relation $\frac{P_c V_c}{T_c} = \frac{2d-5}{4d-8}$ [25]; thus, $\beta_1^{d=6} = \frac{7(5-5^{\frac{1}{3}})}{1200} = 0.01919$ and $\beta_1^{d=8} = 0.01820$. It is shown below that we may obtain the ratification of both of Ehrenfest's equations and the critical exponents at the critical point in the canonical and grand canonical ensemble.

IV. EHRENFEST'S EQUATIONS AT THE CRITICAL POINT

In this section, we study the satisfaction of Ehrenfest's equations at the critical point in the grand canonical and the canonical ensemble for the black hole system [23,31,32]. Ehrenfest's equations for MP black holes in the grand canonical ensemble can be expressed as

$$-\left(\frac{\partial\Omega}{\partial\tilde{T}}\right)_{\tilde{S}} = \frac{\tilde{C}_{\Omega_2} - \tilde{C}_{\Omega_1}}{\tilde{T}J(\tilde{\alpha}_2 - \tilde{\alpha}_1)}, \quad (18)$$

$$-\left(\frac{\partial\Omega}{\partial\tilde{T}}\right)_J = \frac{\tilde{\alpha}_2 - \tilde{\alpha}_1}{\kappa_{\tilde{T}_2} - \kappa_{\tilde{T}_1}}, \quad (19)$$

where

$$\tilde{\alpha} = \frac{1}{J}\left(\frac{\partial J}{\partial\tilde{T}}\right)_\Omega, \quad (20)$$

$$\kappa_{\tilde{T}} = \frac{1}{J}\left(\frac{\partial J}{\partial\Omega}\right)_{\tilde{T}}. \quad (21)$$

A discontinuity in the plots of $\tilde{\alpha}$ and $\kappa_{\tilde{T}}$ is also necessary for a phase transition to take place. By using the chain rule of partial differentiation and the definitions of Ω and $\tilde{T}$ in Eqs. (9) and (15), respectively, we can plot $\tilde{\alpha}$ and $\kappa_{\tilde{T}}$ with respect to S. The results show that $\tilde{\alpha}$ and $\kappa_{\tilde{T}}$ are divergent exactly at the critical point.

In what follows we present the calculations to identify the validity of Ehrenfest's equations and the order of the phase transition for MP black holes in the grand canonical ensemble. Consider $\tilde{C}_\Omega = \frac{f(S)}{g(S)}$, $J\tilde{\alpha} = \frac{h(S)}{g(S)}$, and $J\kappa_{\tilde{T}} = \frac{k(S)}{g(S)}$

such that for the function $g(S_c) = 0$ near the critical point, we have

$$\tilde{C}_{\Omega_2} - \tilde{C}_{\Omega_1} = \frac{f(S_2)}{g(S_2)} - \frac{f(S_1)}{g(S_1)} = f(S_c)\left(\frac{1}{g(S_{c_2})} - \frac{1}{g(S_{c_1})}\right), \quad (22)$$

where $\tilde{C}_\Omega|_{S_i} = \tilde{C}_{\Omega_i}$, and $S_2 = S_c + \varepsilon$ and $S_1 = S_c - \varepsilon$. Thus, the rhs of both Ehrenfest's equations are given by

$$\frac{\tilde{C}_{\Omega_2} - \tilde{C}_{\Omega_1}}{\tilde{T}J(\tilde{\alpha}_2 - \tilde{\alpha}_1)} = \frac{f(S_c)}{\tilde{T}_c h(S_c)}, \quad (23)$$

$$\frac{J(\tilde{\alpha}_2 - \tilde{\alpha}_1)}{J(\kappa_{\tilde{T}_2} - \kappa_{\tilde{T}_1})} = \frac{h(S_c)}{k(S_c)}. \quad (24)$$

Both sides of the first Ehrenfest's equation are calculated for any values of d and β_1 at the critical point in the grand canonical ensemble,

$$\text{rhs} = \text{lhs} = \left(1 + \frac{4\beta_1}{(d-2)S_c}\right)S_c^{\frac{1}{d-2}}((d-2)^2 - 4\Omega_c^2 S_c^{\frac{2}{d-2}})^{\frac{3}{2}}. \quad (25)$$

It is also observed that both sides of the second Ehrenfest's equations are equal. Ehrenfest's equations for the MP black holes in the canonical ensemble are given by

$$-\left(\frac{\partial J}{\partial T}\right)_S = \frac{C_{J_2} - C_{J_1}}{T\Omega(\alpha_2 - \alpha_1)}, \quad (26)$$

$$-\left(\frac{\partial J}{\partial T}\right)_\Omega = -\frac{\alpha_2 - \alpha_1}{\kappa_{T_2} - \kappa_{T_1}}, \quad (27)$$

where

$$\alpha = -\frac{1}{\Omega}\left(\frac{\partial\Omega}{\partial T}\right)_J, \quad (28)$$

$$\kappa_T = \frac{1}{\Omega}\left(\frac{\partial\Omega}{\partial J}\right)_T. \quad (29)$$

In this ensemble we can see that α and κ_T are divergent exactly at the point where the specific heat is divergent (at S_c) and also that the first and second Ehrenfest's equations are satisfied at the critical point. So, a second-order phase transition is taking place in the grand canonical and canonical ensembles.

V. GIBBS FREE ENERGY

In this section, we consider the behavior of the corrected Gibbs free energy and the corrected specific heat as a function of temperature for different values of Ω in the grand canonical ensemble. The corrected Gibbs free energy for MP black holes can be described by Eqs. (9), (13), and (14), and the following relation:

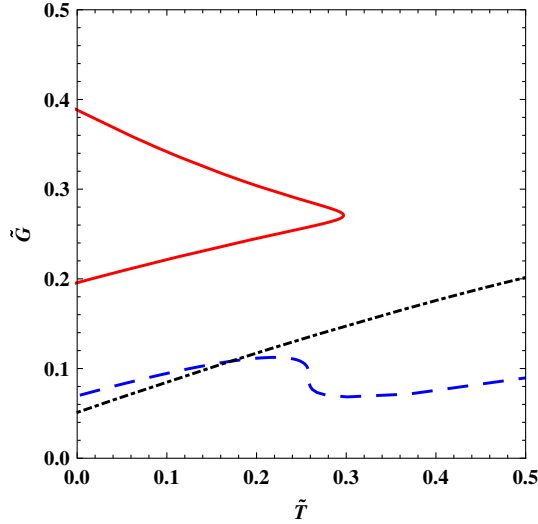$$\tilde{G}(d, \Omega, \tilde{S}) = M - \tilde{T}\tilde{S} - \Omega J. \quad (30)$$

FIG. 9 (color online). The corrected Gibbs free energy with respect to the corrected temperature for $\beta_1 = 0.2$ and $\Omega < \Omega_c$ [red (solid) line], $\Omega = \Omega_c$ [blue (dashed) line], and $\Omega > \Omega_c$ [black (dashed-dotted) line].

The corrected specific heat in the grand canonical ensemble, however, is determined by Eq. (16). Now we can plot the corrected Gibbs free energy and the corrected specific heat with respect to $\tilde{T}$ at $\Omega = \Omega_c$, $\Omega > \Omega_c$, and $\Omega < \Omega_c$ (Figs. 9 and 10). Consider the $\tilde{G} - \tilde{T}$ plot; there are two wings at $\Omega < \Omega_c$, which are joined at the special point $\tilde{T}_1$ (Fig. 9). The corrected heat capacity is divergent and changes from negative values (unstable phase) to positive ones (stable phase) at $\tilde{T}_1$; the black hole system also exists just at $\tilde{T} < \tilde{T}_1$. Since the corrected Gibbs free energy for the upper wing does not have its minimum value, the upper wing behaves like a metastable state. It
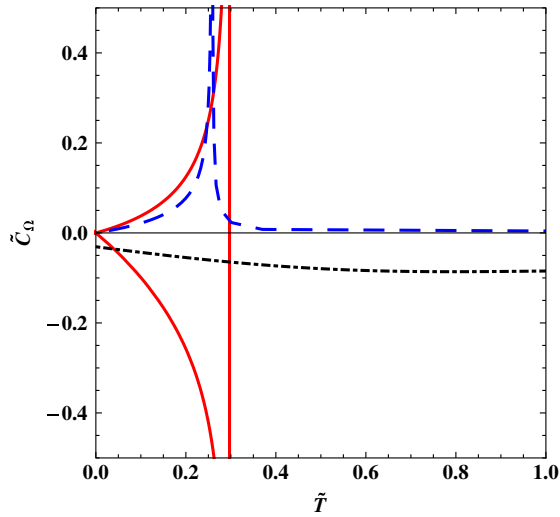

FIG. 10 (color online). The corrected heat capacity with respect to the corrected temperature $\beta_1 = 0.2$ and $\Omega < \Omega_c$ [red (solid) line], $\Omega = \Omega_c$ [blue (dashed) line], and $\Omega > \Omega_c$ [black (dashed-dotted) line].

is also observed that the black holes experience a second-order phase transition at $\tilde{T}_1$ which is not in a metastable state, as shown in Fig. 9. Clearly, the critical exponents are completely different from a Van der Waals system at this point. In other words, at this critical point we get $\frac{\partial^2 \Omega}{\partial J^2} \neq 0$ and this behavior leads to different values for the critical exponents. Thus, what have been left out are the critical exponents of this second-order phase transition.

For $\Omega = \Omega_c$, the corrected Gibbs free energy is positive for any value of $\tilde{T}$ and the corrected specific heat is divergent at $\tilde{T} = \tilde{T}_c$. For $\Omega > \Omega_c$, the corrected specific heat is negative for all values of $\tilde{T}$ and is not divergent at any special point (Fig. 10). These calculations can be expanded for $d \geq 4$ dimensions, the results being thus independent of dimensions.

VI. CRITICAL EXPONENTS

In this section, we calculate the critical exponents of MP black holes near the critical point in the grand canonical ensemble. It was shown in the previous section that not only do the black holes in the grand canonical ensemble behave similarly to a van der Waals system at the critical point, but also that the corrected Gibbs free energy has positive values throughout. If we consider the entropy near the critical point S_c, where $\tilde{C}_\Omega$ is divergent, we get [28,29]

$$S = S_c(1 + \Delta), \quad (31)$$

where $|\Delta| \ll 1$. Since the corrected temperature is a function of the semiclassical entropy, we have

$$\check{T} = \tilde{T}_c(1 + \epsilon). \quad (32)$$

Here $|\epsilon| \ll 1$. The critical exponent α is associated with the singular behavior of the corrected specific heat $\tilde{C}_J$. Since $\tilde{C}_J$ does not diverge at this critical point S_c (where $\tilde{C}_\Omega$ is divergent), we find that the critical exponent $\alpha = 0$. By expanding $\Omega(S, \tilde{T})$ and $J(S, \Omega)$ near the critical point, we get the following two equations:

$$
\begin{aligned}
\Omega(S, \tilde{T}) = \Omega_c &+ \left[\left(\frac{\partial \Omega}{\partial S}\right)\right]_{\tilde{T}=\tilde{T}_c, S=S_c} (S - S_c) \\
&+ \frac{1}{2}\left[\left(\frac{\partial^2 \Omega}{\partial S^2}\right)\right]_{\tilde{T}=\tilde{T}_c, S=S_c} (S - S_c)^2 \\
&+ \frac{1}{6}\left[\left(\frac{\partial^3 \Omega}{\partial S^3}\right)\right]_{\tilde{T}=\tilde{T}_c, S=S_c} (S - S_c)^3 \\
&+ \left[\left(\frac{\partial \Omega}{\partial \tilde{T}}\right)\right]_{\tilde{T}=\tilde{T}_c, S=S_c} (\tilde{T} - \tilde{T}_c) \\
&+ \left[\left(\frac{\partial^2 \Omega}{\partial S \partial \tilde{T}}\right)\right]_{\tilde{T}=\tilde{T}_c, S=S_c} (\tilde{T} - \tilde{T}_c)(S - S_c) \quad (33)
\end{aligned}
$$

and

$$J(S, \Omega) = J_c + \left[\left(\frac{\partial J}{\partial S}\right)\right]_{\Omega=\Omega_c, S=S_c} (S - S_c)$$
$$+ \text{ higher order terms.} \quad (34)$$

Since $(\frac{\partial \Omega}{\partial J})_c = 0$ and $(\frac{\partial^2 \Omega}{\partial J^2})_c = 0$ at the critical point, we can use the chain rule of partial differentiation at this point and rewrite Eq. (33) in the following form:

$$\Omega(S, \tilde{T}) = \Omega_c + \frac{1}{6}\left[\left(\frac{\partial^3 \Omega}{\partial S^3}\right)\right]_{\tilde{T}=\tilde{T}_c, S=S_c} (S - S_c)^3$$
$$+ \left[\left(\frac{\partial \Omega}{\partial \tilde{T}}\right)\right]_{\tilde{T}=\tilde{T}_c, S=S_c} (\tilde{T} - \tilde{T}_c)$$
$$+ \left[\left(\frac{\partial^2 \Omega}{\partial S \partial \tilde{T}}\right)\right]_{\tilde{T}=\tilde{T}_c, S=S_c} (\tilde{T} - \tilde{T}_c)(S - S_c). \quad (35)$$

Differentiating $\Omega(S, \tilde{T})$ for a fixed $\tilde{T}$ and using both Eq. (34) and Maxwell's equal-area law [24] yields the critical exponent $\beta = \frac{1}{2}$. Let us obtain the critical exponent γ associated with $\kappa_{\tilde{T}}^{-1} \propto (\frac{\partial \Omega}{\partial J})_{\tilde{T}}$. $J(S, \tilde{T})$ is first expanded near the the critical point,

$$J(S, \tilde{T}) = J_c + \left[\left(\frac{\partial J}{\partial S}\right)\right]_{\tilde{T}=\tilde{T}_c, S=S_c} (S - S_c)$$
$$+ \left[\left(\frac{\partial \Omega}{\partial \tilde{T}}\right)\right]_{\tilde{T}=\tilde{T}_c, S=S_c} (\tilde{T} - \tilde{T}_c)$$
$$+ \text{ higher order terms.} \quad (36)$$

Equations (35) and (36) are differentiated with respect to entropy to obtain

$$\left(\frac{\partial J}{\partial \Omega}\right)_{\tilde{T}} = \left(\frac{\frac{\partial J}{\partial S}}{\frac{\partial \Omega}{\partial S}}\right)_{\tilde{T}} = \frac{\left[\left(\frac{\partial^2 \Omega}{\partial S \partial \tilde{T}}\right)\right]_{\tilde{T}=\tilde{T}_c, S=S_c}(\tilde{T} - \tilde{T}_c)}{\left[\left(\frac{\partial J}{\partial S}\right)\right]_{\tilde{T}=\tilde{T}_c, S=S_c}}. \quad (37)$$

Hence, the critical exponent $\gamma = 1$. Also, by using Eqs. (34) and (35) we obtain the critical exponent δ defined at $\tilde{T} = \tilde{T}_c$ in the following form:

$$\Omega(S, \tilde{T}) - \Omega_c \propto (J - J_c)^3 \Rightarrow \delta = 3. \quad (38)$$

The results show that the calculated values of the critical exponents of MP black holes with quantum corrections at the critical point correspond to a Van der Waals system and that they do not depend on the dimensionality of the system.

VII. THERMODYNAMIC GEOMETRY OF AN MP BLACK HOLE

The phase transitions of black holes may also be viewed from the viewpoint of thermodynamic state space (Ruppeiner) geometry [15]. Thus, the Ruppeiner metric can be expressed as [16]

$$dS^2 = g_{ij}^R dX^i dX^j, \quad (39)$$

where $g_{ij} = -\frac{\partial^2 S(X^k)}{\partial X^i \partial X^j}$. Also, the Weinhold metric can be defined as [17]

$$dS_W^2 = g_{ij}^W dX^i dX^j. \quad (40)$$

Here, $g_{ij}^W = \frac{\partial^2 M(X^k)}{\partial X^i \partial X^j}$, and $i, j = 1, 2$, $X^1 = J$ and $X^2 = S$. The relationship between the Ruppeiner metric and the Weinhold metric is expressed by

$$dS_R^2 = \frac{1}{T} dS_W^2. \quad (41)$$

Here, T is the temperature of the black hole system. The Ruppeiner curvature of MP black holes with quantum corrections is not flat at the critical point in the grand canonical ensemble and is divergent at $S = S_c$ (Fig. 11). So, the second-order phase transition of MP black holes at the critical point can be investigated by using the Ruppeiner curvature formula in the grand canonical ensemble.

If we consider the entropy of black holes as a function of the internal energy $u = M - \Omega J$ and Ω (angular velocity), then, for a fixed value of J, we may obtain the Ruppeiner metric and also the scalar curvature of geometry [18]. In this way, the first law of thermodynamics can be written as $du = TdS - Jd\Omega$. Since the thermodynamic metric introduced in Ruppeiner's theory is defined by the second derivatives of mass divided by temperature, we may replace the mass inducing the metric by the function regarded as the internal energy and the extensive variables in ordinary thermodynamic systems as follows:

$$g_{ij}^R = \frac{1}{T} \frac{\partial^2 M(X^k)}{\partial X^i \partial X^j}. \quad (42)$$

Here, $i, j = 1, 2$, $X^1 = \Omega$, and $X^2 = u$. Using these, we find that the Ruppeiner curvature scalar (R) is not flat but diverges at the critical point where the heat capacity is divergent in the canonical ensemble, indicating a second-order phase transition.
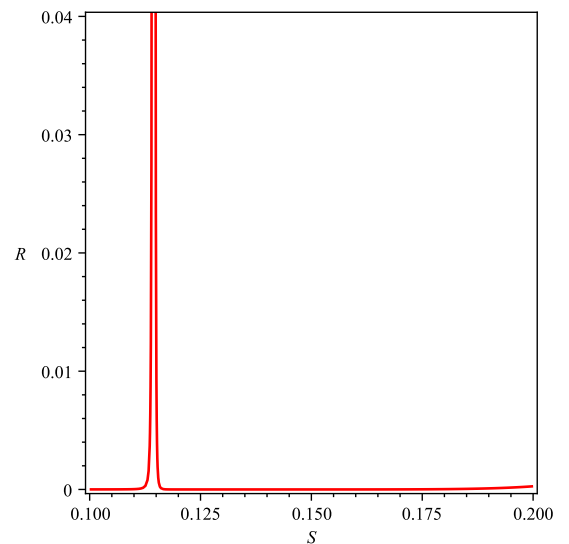

FIG. 11 (color online). Ruppeiner curvature scalar (R) for $\Omega = \Omega_c$, $\beta_1 = 0.2$, and $d = 8$.

VIII. CONCLUSION

In this paper, we considered the thermodynamics of MP black holes in even dimensions with $n = \frac{d-2}{2}$ nonzero equal spins J. We showed that while a second-order phase transition in the canonical ensemble is taking place in the semiclassical approximation, it is not possible to obtain a critical behavior similar to that of a Van der Waals fluid. It was also found that taking the corrections beyond the semiclassical approximation in the grand canonical ensemble gives rise to a critical behavior similar to a Van der Waals fluid. In this way, we obtain the value of the leading coefficient correction to be $\beta_1^{d=4} = \frac{1}{24\sqrt{3}}$ in order to find the universal number $\frac{PV}{T} = \frac{3}{8}$ in $d = 4$. Also, the coefficient β_1 can be calculated for a given d from the relation $\frac{P_c V_c}{T_c} = \frac{2d-5}{4d-8}$ [25]. It is straightforward to obtain $\beta_1^{d=6} = \frac{7(5-5^{\frac{1}{3}})}{1200} = 0.01919$ and $\beta_1^{d=8} = 0.01820$. These properties motivated us to check the validity of Ehrenfest's equations and the critical exponents. Our calculations showed that both Ehrenfest's equations are satisfied in the canonical and the grand canonical ensemble, indicating that a second-order phase transition happens. We also found that the corrected Gibbs free energy had positive values for any value of $\tilde{T}$ for $\Omega = \Omega_c$. We extended these calculations to the higher-order correction terms. The results showed that the critical behavior and the critical exponents of black holes behave similar to a Van der Waals system. In another part of this paper, we calculated the Ruppeiner curvature scalar (R) and investigated its behavior at the critical point. It was shown that R diverges exactly at the critical point where the specific heat and corrected specific heat are divergent in both the canonical and the grand canonical ensembles, respectively. This indicated that the Ruppeiner curvature formula could be exploited to investigate the second-order phase transition for MP black holes at the critical point.

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