

**Equation-of-state-independent relations in neutron stars**Andrea Maselli,<sup>1,2</sup> Vitor Cardoso,<sup>3,4,5</sup> Valeria Ferrari,<sup>2</sup> Leonardo Gualtieri,<sup>2</sup> and Paolo Pani<sup>3,6</sup><sup>1</sup>*Institute of Cosmology & Gravitation, University of Portsmouth, Dennis Sciama Building, Portsmouth PO1 3FX, United Kingdom*<sup>2</sup>*Dipartimento di Fisica, Università di Roma “La Sapienza” & Sezione, INFN Roma1, P.A. Moro 5, 00185 Roma, Italy*<sup>3</sup>*CENTRA, Departamento de Física, Instituto Superior Técnico, Universidade Técnica de Lisboa - UTL, Avenida Rovisco Pais 1, 1049 Lisboa, Portugal*<sup>4</sup>*Perimeter Institute for Theoretical Physics Waterloo, Ontario N2J 2W9, Canada*<sup>5</sup>*Department of Physics and Astronomy, The University of Mississippi, University, Mississippi 38677, USA*<sup>6</sup>*Institute for Theory & Computation, Harvard-Smithsonian CfA, 60 Garden Street, Cambridge, Massachusetts 02138, USA*

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Neutron stars are extremely relativistic objects which abound in our universe and yet are poorly understood, due to the high uncertainty on how matter behaves in the extreme conditions which prevail in the stellar core. It has recently been pointed out that the moment of inertia  $I$ , the Love number  $\lambda$ , and the spin-induced quadrupole moment  $Q$  of an isolated neutron star, are related through functions which are practically independent of the equation of state. These surprising universal  $I - \lambda - Q$  relations pave the way for a better understanding of neutron stars, most notably via gravitational-wave emission. Gravitational-wave observations will probe highly dynamical binaries and it is important to understand whether the universality of the  $I - \lambda - Q$  relations survives strong-field and finite-size effects. We apply a post-Newtonian-affine approach to model tidal deformations in compact binaries and show that the  $I - \lambda$  relation depends on the inspiral frequency, but is insensitive to the equation of state. We provide a fit for the universal relation, which is valid up to a gravitational wave frequency of  $\sim 900$  Hz and accurate to within a few percent. Our results strengthen the universality of  $I - \lambda - Q$  relations, and are relevant for gravitational-wave observations with advanced ground-based interferometers. We also discuss the possibility of using the Love-compactness relation to measure the neutron-star radius with an uncertainty  $\leq 10\%$  from gravitational-wave observations.

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**I. INTRODUCTION**

Neutron stars (NSs) are extremely compact objects formed as the end-state of the collapse of massive stars, which populate the Universe either in isolation or in binaries. NS-NS binaries are one of the most promising sources for second-generation, ground-based detectors of gravitational waves, such as advanced Virgo [1], advanced LIGO [2] and KAGRA [3] (see also the proposed third-generation detector ET [4]); in addition, NSs are copious radio and x-ray emitters, and they can potentially be used as laboratories for high-energy and fundamental physics, to probe the behavior of matter in extreme conditions [5]. A persistent obstacle against exploring the full potential of NSs physics lies precisely in our ignorance on their inner structure, and, in particular, in the uncertainties of the equation of state (EOS) of matter at ultranuclear densities. The star radius  $R$ , mass  $M$ , moment of inertia  $I$ , and deformability, as measured by the tidal Love number  $\lambda$  and spin-induced quadrupole moment  $Q$ , all depend sensitively on the EOS. The lack of knowledge on the EOS therefore affects our understanding of the NS properties and prevents model-independent tests of gravity with these objects.

Despite the multitude of modern EOS proposed in the literature, leading to different NS configurations, Yagi and Yunes (hereafter, YY) recently discovered [6,7] some

universal relations between the moment of inertia, the tidal Love number, and the spin-induced quadrupole moment, which are essentially insensitive to the NS EOS. These tantalizing  $I - \lambda - Q$  relations open the interesting possibility of breaking degeneracies between these parameters and effectively make NSs viable laboratories for fundamental physics and astrophysics.

The tidal Love number used to derive the  $I - \lambda - Q$  relations was computed assuming a static, spherically symmetric star placed in a time-independent external quadrupolar tidal field [8]. Effectively, this means that these relations were derived for stars in isolation. However, gravitational-wave detectors in the next years will observe coalescing compact binaries, i.e. they will detect gravitational waves emitted by NSs at small orbital separations,  $d/R \leq 10$ , and high frequencies  $f_{\text{GW}} \geq 40$  Hz; thus gravitational-wave observations will probe highly dynamical NSs which strongly interact with their compact companion (either a black hole or another NS). The question then arises as to whether the  $I - \lambda - Q$  relations are actually useful or even correct in situations of physical interest.

The purpose of this article is to derive EOS-independent  $I - \lambda$  relations which hold true throughout almost the entire inspiralling phase of the binary coalescence, and to

provide accurate fits describing this relation at each orbital frequency.

## II. EVALUATION OF THE TIDAL LOVE NUMBER

Tidal deformability properties of NSs can be described in terms of a set of parameters, the *Love numbers* [8–12], which relate the mass multipole moments of the star to the (external) tidal field multipole moments. In particular, the dominant contribution to the stellar deformation is encoded in the electric,  $l = 2$  Love number, which we simply call tidal Love number  $\lambda$ , and is defined by the relation

$$Q_{ij} = -\lambda C_{ij}, \quad (1)$$

where  $Q_{ij}$  is the traceless quadrupole moment of the star, and  $C_{ij} = e_{(0)}^\alpha e_{(i)}^\beta e_{(0)}^\gamma e_{(j)}^\delta R_{\alpha\beta\gamma\delta}$  is the tidal tensor which induces the deformation;  $e_{(\mu)}^\alpha$  is a parallelly transported tetrad attached to the deformed star, and  $R_{\alpha\beta\gamma\delta}$  is the Riemann tensor.

Two approaches are currently used to evaluate the tidal Love number: a stationary and a dynamical approach. In the stationary approach used by YY [8–14], the compact bodies forming the binary system are assumed to be very far apart. Using spacetime perturbation theory [15] to study the  $l = 2$  stationary perturbations of a NS induced by a test tidal field, the quadrupole and tidal tensors are evaluated; the Love number is then computed from Eq. (1). As discussed in [9,16], this approach assumes that the time scale of the stellar deformation is much smaller than time scales associated with the orbital motion, an assumption which becomes less accurate in the last stages of coalescence.

In the dynamical approach [16,17], the evolution of the tidal deformation of NSs in compact binaries is modeled combining the post-Newtonian (PN) description of the two-body metric and of the orbital evolution, with an *affine description* of the NS as a deformable ellipsoid, subject to its self-gravity, to internal pressure forces and to the PN tidal field of the companion. The deformed NS is described in terms of five dynamical variables: the principal axes of the ellipsoid, and two angles describing the orientation of the principal frame; these quantities are determined by solving a set of ordinary differential equations in time, coupled with the PN equations of motion. This approach, called post-Newtonian affine (PNA), allows one to compute  $Q_{ij}(t)$  and  $C_{ij}(t)$  in terms of the dynamical variables, so that the tidal Love number can be evaluated during the inspiral. To parametrize the dynamical evolution of the system, it is convenient to use the orbital frequency  $f$ , instead of time or radial distance. The ratio between quadrupole and tidal tensors is then a function (the tidal Love function)  $\lambda(f)$  [16,17], and the Love number obtained in the stationary approach corresponds to the zero-frequency (i.e. infinite orbital separation) limit of this function.

The PNA approach also allows one to compute the moments of inertia  $I_i = I \times (a_i/R)^2$ , where  $i = 1, 2, 3$  indicate the star principal axes ( $i = 1$  corresponds to the axis pointing toward the companion),  $I$  and  $R$  are the moment of inertia and the radius of the spherical star, and  $a_i$  is the  $i$ th axis of the deformed, ellipsoidal star. During the inspiral,  $I_1$  increases, while  $I_2$  and  $I_3$  decrease.

## III. RESULTS

Using the PNA approach, we have computed the normalized Love function  $\bar{\lambda} = \lambda/M^5$  and the normalized moment of inertia corresponding to the axis pointing toward the companion,  $\bar{I} = I_1/M^3$ , as functions of the orbital frequency  $f$ ;  $M$  is the NS mass.

We have performed simulations of NS-NS binaries for three different EOS which are expected to cover a wide range of NS deformability, APR4, MS1, and H4, and masses in the range  $[1.2-2]M_\odot$ . In Table I we show the maximum mass, and the radius and compactness  $C = M/R$  of a  $1.4M_\odot$  star, for the EOS APR4, MS1, and H4. Comparing these values with those shown in the extensive survey of [13], we see that (excluding manifestly unphysical models), all EOS fall in the range of compactness considered in this paper within 10%. The EOS APR4 describes soft NS matter and yields models with high compactness and small deformability, whereas MS1 describes stiff matter and large deformability; H4 provides intermediate configurations. All EOS are modeled by parametrized piecewise polytropes as proposed by Read *et al.* [13].

Our results are summarized in Fig. 1. On the three left panels, we plot  $\bar{I}$  versus  $\bar{\lambda}$  for three different values of the gravitational wave frequency,  $f_{\text{GW}} \equiv 2f = 170, 500, 875$  Hz, for equal-mass NS-NS binaries with different EOS. The data have been fitted with the following function:

$$\ln \bar{I} = b_0 + b_1 \ln \bar{\lambda} + b_2 (\ln \bar{\lambda})^2 + b_3 (\ln \bar{\lambda})^3 + b_4 (\ln \bar{\lambda})^4, \quad (2)$$

where the fitting parameters  $b_i$  are functions of  $f_{\text{GW}}$  and are listed in Table II. The dashed lines in the left panels of Fig. 1 are the fits corresponding to the selected frequencies.

In the upper, right panel of Fig. 1, the relative error  $(\bar{I} - \bar{I}_{\text{fit}})/\bar{I}_{\text{fit}}$  is plotted versus  $\bar{\lambda}$ , for the selected frequencies. This error is always  $\lesssim 2\%$ . In the lower panel the ratio

TABLE I. Maximum mass, radius, and compactness of a  $1.4M_\odot$  neutron star, for the EOS APR4, H4, and MS1.

EOS	$M_{\text{max}}/M_\odot$	$R_{1.4}(km)$	$C_{1.4}$
APR4	2.20	11.12	0.186
H4	2.03	13.59	0.152
MS1	2.78	14.47	0.143

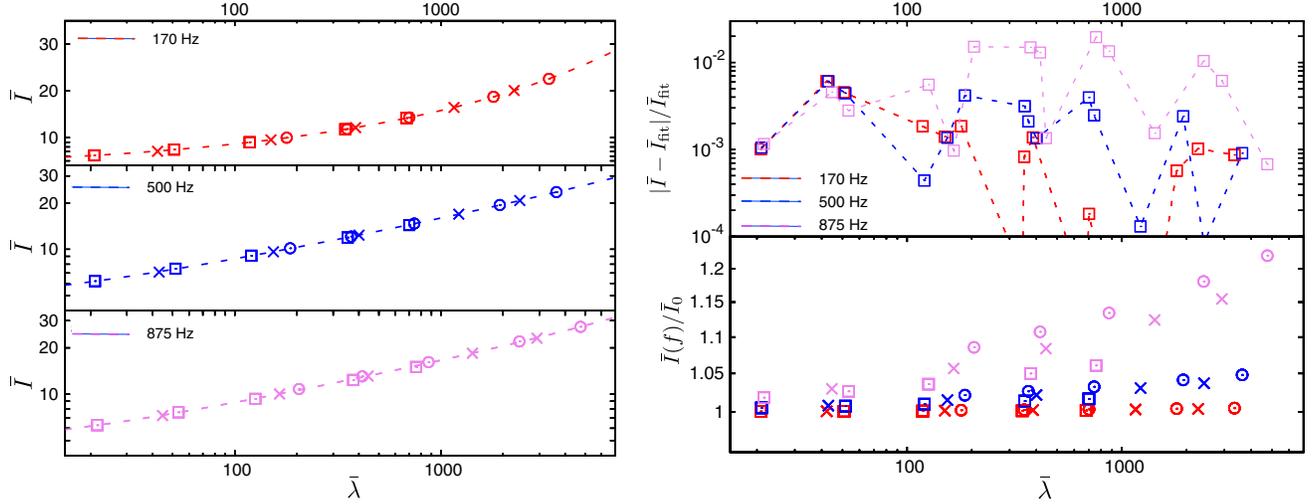


FIG. 1 (color online). Left: the  $\bar{I} - \bar{\lambda}$  relation is plotted for equal mass NS-NS binaries, three values of the gravitational wave frequency  $f_{\text{GW}} \equiv 2f = (170, 500, 875)$  Hz, and for the EOS APR4 (cross), MS1 (open circle), H4 (open square). Markers refer to numerical data, while dashed lines are the fits (2). Right top: relative fractional errors between fits and numerical results. Right bottom:  $\bar{I} - \bar{\lambda}$  relation with moment of inertia normalized by its value at infinity.

$\bar{I}(f)/\bar{I}_0$  is plotted versus  $\bar{\lambda}$ , where  $\bar{I}_0$  is the asymptotic value of  $\bar{I}$  when the stars are in isolation. This figure shows that, as the stars approach the merger, their moments of inertia change with respect to the asymptotic value, and grow as much as 10%–30%, depending on the EOS: for stiffer EOS, the variation with respect to the values at infinity is larger.

Nonetheless, the relative errors  $(\bar{I} - \bar{I}_{\text{fit}})/\bar{I}_{\text{fit}}$  are small and only mildly dependent on the EOS, suggesting that a simple frequency-independent relation can be found between  $\bar{I}$  and  $\bar{\lambda}$ . We find that

$$\ln \bar{I} = 1.95 - 0.373 \ln \bar{\lambda} + 0.155 (\ln \bar{\lambda})^2 - 0.0175 (\ln \bar{\lambda})^3 + 0.000775 (\ln \bar{\lambda})^4, \quad (3)$$

describes very well our numerical results. In the upper panel of Fig. 2, we compare the fit (3) with our numerical results, for the full set of binaries and frequencies we have considered. In the lower panel we plot the relative errors between the numerical results and the universal fit (3),

showing also how well YY's fit performs in the dynamical case. Our fit [Eq. (3)] reproduces the  $\bar{I} - \bar{\lambda}$  relation to within 5% at any frequency  $\lesssim 875$  Hz and for all EOS and masses we have considered, while the YY fit becomes less accurate as the frequency increases, with fractional errors which become of the order of 10%.

The general fit (3) also holds for unequal mass NS-NS binaries. For instance, we have checked that for  $M_1 = 1.2M_\odot$  and  $M_2 = 1.6M_\odot$ , the fit reproduces the  $\bar{I} - \bar{\lambda}$  relation at any frequency  $\lesssim 875$  Hz to within 5% for the  $1.2M_\odot$  star, and 3% for the  $1.6M_\odot$  star.

#### IV. DISCUSSION

NS-NS binaries are the prototypical sources for upcoming second-generation gravitational-wave detectors. Strong-field and finite-size effects are important to model the waveform during the latest stages of the inspiral. Our results show that the  $I - \lambda$  relations discovered by YY in the low frequency regime can be extended to describe

TABLE II. Fitting parameters of the  $\bar{I} - \bar{\lambda}$  relation given by Eq. (2), for several values of the gravitational wave frequency. These fits reproduce our data to within 2%, cf. Fig. 1. The last row corresponds to the fit (3) that reproduces data at *any* frequency to within 5% [cf. Fig. 2].

$f_{\text{GW}}$	$b_0$	$b_1$	$b_2$	$b_3$	$b_4$
170	1.54	$-3.72 \times 10^{-2}$	$5.49 \times 10^{-2}$	$-4.78 \times 10^{-3}$	$1.87 \times 10^{-4}$
300	1.58	$-6.53 \times 10^{-2}$	$6.26 \times 10^{-2}$	$-5.68 \times 10^{-3}$	$2.26 \times 10^{-4}$
500	1.60	$-8.34 \times 10^{-2}$	$6.83 \times 10^{-2}$	$-6.39 \times 10^{-3}$	$2.59 \times 10^{-4}$
700	1.64	$-1.18 \times 10^{-1}$	$7.89 \times 10^{-2}$	$-7.69 \times 10^{-3}$	$3.18 \times 10^{-4}$
800	1.68	$-1.46 \times 10^{-1}$	$8.76 \times 10^{-2}$	$-8.77 \times 10^{-3}$	$3.68 \times 10^{-4}$
875	1.71	$-1.72 \times 10^{-1}$	$9.54 \times 10^{-2}$	$-9.72 \times 10^{-3}$	$4.12 \times 10^{-4}$
any	1.95	$-3.73 \times 10^{-1}$	$1.55 \times 10^{-2}$	$-1.75 \times 10^{-3}$	$7.75 \times 10^{-4}$

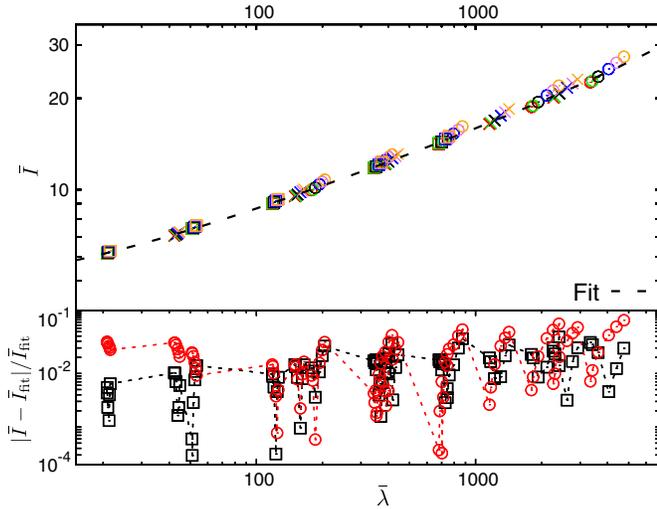


FIG. 2 (color online). Top: fitting curve (3) (dashed line) and numerical results of the  $\bar{I} - \bar{\lambda}$  relation, for data set including points up to  $f_{\text{GW}} = 875$  Hz and the EOS APR4 (cross), MS1 (open circle), H4 (open square). Bottom: relative fractional errors between fits and numerical results. Black squares and red circles refer to the fit of Eq. (3), and to the analytical relation found by YY, respectively.

the dynamical evolution of NSs during the late stages of the inspiral. These results open the possibility for strong-field, model-independent tests of NS properties. If, for instance, advanced gravitational-wave detectors LIGO/Virgo measure the tidal Love number in a compact binary coalescence to within (5–10)%, as estimated in [9,18], this would allow for an indirect estimate of the moment of inertia with roughly the same precision. This measurement would be independent of (and competitive to) the estimates coming from pulsar-timing observations [19]. In addition, as shown in YY, these estimates would allow one to set constraints on modified theories of gravity.

Although in this paper we have presented only NS-NS binaries, our approach also describes the dynamical evolution of mixed black hole-NS systems as well. We have computed the  $I - \lambda$  relation for mixed binaries (with mass ratio up to 5), finding similar universal relations during the entire inspiral.

In this work we have studied the  $I - \lambda$  relation, to understand the effects of the tidal interaction when the stars are at short orbital distance. Spin effects have been neglected. They have been considered by YY in the slow rotation, low frequency limit. It would be interesting to establish whether a simple EOS-independent, universal relation exists, between the tidal Love number and the spin-induced quadrupole moment  $Q$  in the fast rotation, high frequency regime. This matter will be investigated in a following work.

We conclude this discussion with some considerations on the relation between the tidal Love number and the

NS compactness. YY showed that this relation is more EOS-dependent than the  $I - \lambda - Q$  relations. However, they included in their study hot and young NSs, which are unlikely to be members of a coalescing binary system. If we consider only old and cold NSs, we find that the  $C - \lambda$  relation acquires a remarkable universality.<sup>1</sup> By computing  $\bar{\lambda}$  in the low frequency limit, for the EOS APR4, MS1, H4, and masses in the range  $[1.2-2]M_{\odot}$ , we find that  $C$  is well described by the fit

$$C = 3.71 \times 10^{-1} - 3.91 \times 10^{-2} \ln \bar{\lambda} + 1.056 \times 10^{-3} (\ln \bar{\lambda})^2. \quad (4)$$

This fit gives the compactness with a relative error  $\lesssim 2\%$ .

The  $C - \lambda$  relation can be extremely useful to extract information on the NS EOS from a detected gravitational wave signal emitted in a binary coalescence. If the tidal Love number is extracted by Advanced LIGO/Virgo with an error  $\sigma_{\ln \lambda} = \sigma_{\lambda} / \lambda \sim 60\%$  [7], we can determine the compactness with an error  $\sigma_C \sim \sqrt{\sigma_{\text{fit}}^2 + (\partial C / \partial \ln \lambda)^2 \sigma_{\ln \lambda}^2} \lesssim 10\% C$  [where we have assumed  $\sigma_{\text{fit}} \lesssim \max(C - C_{\text{fit}})$ ]. A much more optimistic estimate of the error on the measure of the tidal Love number,  $\sigma_{\ln \lambda} \sim 5\%$  [18], would imply a relative error on the compactness of the order of  $\sim 2\%$  (this remarkable decrease of the relative error can be traced back to the  $\sim R^5$  dependence of  $\bar{\lambda}$ ). Since the same detection would allow for an accurate estimate of the NS mass, we would then know the NS radius with an uncertainty of  $\sim 10\%$  or smaller. It should be noted that current estimates of NS radius based on astrophysical observations (see [21] and references therein), with a claimed error of  $\sim 10\%$ , are highly debated in the literature since they may depend on the way the NS surface emission is modeled [22]. A measurement of the NS radius, based on gravitational wave observations and on the  $C - \lambda$  relation, would have the same, or better, accuracy and it would be model independent. Such a measurement would be extremely useful to put constraints on the NS EOS [23].

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<sup>1</sup>This is consistent with the results of Ref. [20], who studied the  $Q - C$  relation using a set of EOS describing old, cold NSs, finding hints of universality. If we combine the  $Q - C$  and the  $Q - \lambda$  relation discovered by YY, the universal behavior of the  $C - \lambda$  relation naturally emerges.

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