

Gaussian curvature and global effects: Gravitational Aharonov-Bohm effect revisitedM. Nouri-Zonoz^{*} and A. Parvizi[†]*Department of Physics, University of Tehran, North Karegar Avenue, Tehran 14395-547, Iran*

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By using the Gauss-Bonnet formula, the integral of the Gaussian curvature over a 2-surface enclosed by a curve in the asymptotically flat region of a static spacetime was found to be a measure of a gravitational analogue of the Aharonov-Bohm effect by Ford and Vilenkin in the linearized regime. Employing the $1 + 3$ formulation of spacetime decomposition, we study the same effect in the context of the full Einstein field equations for stationary spacetimes. Applying our approach to static tubelike and cylindrical distributions of dust, not only do we recover their result but we also obtain an extra term which is interpreted to be representing the classical version of the Colella-Overhauser-Werner effect (the Colella-Overhauser-Werner experiment).

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I. INTRODUCTION

Aharonov and Bohm in their celebrated paper [1] have shown that the interference pattern of an electrically charged ensemble of particles traveling on a closed path in a region where the electromagnetic fields are absent can be affected by the electric and magnetic fields in the region from which they are excluded. This result is usually interpreted as a manifestation of nonlocality of electromagnetism or as a direct physical effect attributed to the electromagnetic four-potential. Looking for gravitational analogues, physical effects analogous to the electromagnetic Aharonov-Bohm have been considered by many authors through different approaches and perspectives [2–9]. Matter wave interferometry experiments have also been proposed to measure this effect in the weak field limit of a uniform gravitational field [10] and also more recently in a gravitational potential produced by a pair of spherical masses [11].

Basically, two different versions of the gravitational Aharonov-Bohm effect are discussed in the literature which could be attributed to the spacetime under consideration being static or stationary.

In the first version, in closer analogy to the electromagnetic case, the effect arises in stationary spacetimes which possess a gravitomagnetic field, and so it should be called the *gravitomagnetic Aharonov-Bohm effect*. In this case taking advantage of the analogy between (weak) gravitational fields and electromagnetism, it is shown, by solving the Klein-Gordon equation, that the energy spectrum of a scalar particle in the region where the gravitomagnetic field vanishes depends on the gravitomagnetic flux in the region from which it is excluded [12,13]. Other studies following essentially the same approach but in different contexts could be found in Refs. [14,15]. Also, loop space

variables in the gravitational field of a slowly rotating long cylindrical shell as well as a static cosmic string are being employed to investigate gravitational analogues of the Aharonov-Bohm effect [16,17]. It is shown that when one parallel transports a vector around a rotating source it acquires a gravitationally induced phase shift which is proportional to the angular momentum of the source as a local quantity, and this happens despite the fact that the Riemann tensor does not depend on the angular momentum.

Closely related to the above version of the effect, Stachel considers globally stationary but locally static spacetimes and solves the eikonal equation in such backgrounds [6]. He shows that the eikonal function is proportional to the period of a one-form which in the $1 + 3$ (or threading) formulation of spacetime decomposition is called the *threading vector*. It can be shown that the period is nothing but the gravitomagnetic flux defined in the same formulation of spacetime decomposition. The important point about Stachel's approach is that the dependence of the phase shift on the rotation of the source is a classical result and could be obtained without an appeal to the weak field approximation.

In the present paper, we turn our attention to the second version of the effect investigated by Ford and Vilenkin [5]. Since the role of potentials and fields in the electromagnetism are played in general relativity by the metric and Riemann curvature tensors, respectively, one can study physical effects that originated from the regions of nonzero curvature on particles moving in a region where curvature vanishes. In other words, in this case there is no need for the notion of a gravitomagnetic field for the effect to be realized, and it could even arise in a static spacetime, which is why it should be called the *gravitoelectric Aharonov-Bohm effect*. For this purpose Ford and Vilenkin consider a tubelike distribution of matter producing a spacetime with two Killing vectors $\xi = \xi^a \partial_a$ and $\eta = \eta^a \partial_a$ [18] (which are timelike and spacelike, respectively) and asymptotically flat in the direction

^{*}Corresponding author.

nouri@khayam.ut.ac.ir

[†]a.parvizi@ut.ac.ir

perpendicular to the tube. Then they make use of a form of Gauss-Bonnet theorem stating that if a vector is parallel transported along a closed curve C in a 2-surface S , it will not go back to itself but undergo a rotation by an angle α given by the following area integral:

$$\alpha = \int_{S_0} K da, \quad (1)$$

where K is the Gaussian curvature of the 2-surface and S_0 is the subsurface of S enclosed by the curve. Now by assuming that the curve C lies in the asymptotically flat region, then the particle moving on this curvatureless region still feels the effect from the nonzero-curvature region, hence a version of gravitational Aharonov-Bohm effect. To give an explicit example, they have applied their formulation to the case of a tubelike dust source using the weak field approximation to calculate the Gaussian curvature in terms of the components of the energy-momentum tensor of the source.

In this paper, we employ the same approach but in the context of the 1 + 3 decomposition of spacetimes which leads to the quasi-Maxwell form of the Einstein field equations and the so-called *gravitoelectromagnetism*. This decomposition introduces a differentiable 3-manifold with a metric element prescribing spatial distances in a given stationary spacetime. This in turn allows one to express the Gaussian curvature of a 2-surface in terms of the components of the energy-momentum tensor of the source and the gravitoelectric field of the underlying spacetime without using the weak field approximation. The outline of the paper is as follows: First we introduce briefly the 1 + 3 decomposition of a stationary spacetime in Sec. II, and then, using the projection tensor defined on that basis, in Sec. III we calculate an expression for the Gaussian curvature in terms of the three-dimensional Ricci scalar and Ricci tensor. In Sec. IV, we apply this expression to different solutions including static dust solutions and spacetime of a cosmic string. In the last section, the results are summarized.

II. 1 + 3 (THREADING) FORMULATION OF SPACETIME DECOMPOSITION

Considering the light propagation in a stationary spacetime between two nearby spatial points x^α and $x^\alpha + dx^\alpha$, the spacetime metric could be written in the following general form [19]:

$$ds^2 = d\tau_{\text{syn}}^2 - dl^2 = h(dx^0 - A_\alpha dx^\alpha)^2 - \gamma_{\alpha\beta} dx^\alpha dx^\beta, \quad (2)$$

where $h \equiv g_{00}$, $A_\alpha = -\frac{g_{0\alpha}}{g_{00}}$, and

$$\gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}$$

is the spatial metric. In this so-called 1 + 3 formulation of spacetime decomposition, $d\tau_{\text{syn}} = \sqrt{h}(dx^0 - A_\alpha dx^\alpha)$ is

the infinitesimal interval of *synchronized proper time*, and dl is the infinitesimal spatial distance between the two events. The spacetime being stationary [$g_{ij} \neq g_{ij}(x^0)$] will allow one to define distance along a curve in a finite region of the spacetime by the integral $\int dl$. In a more rigorous mathematical language, a stationary spacetime (\mathcal{M}, g_{ab}) is decomposed into spatial and temporal sections by a congruence of timelike curves generated by the timelike Killing vector field ξ^a of the spacetime. This is achieved through the introduction of the following projection tensor:

$$h_{ab} = -g_{ab} + u_a u_b,$$

where u^a is the normalized tangent vector to the timelike curves

$$u^a = \frac{\xi^a}{|\xi|}, \quad |\xi| = (\xi_a \xi^a)^{1/2}.$$

By using a coordinate system (denoted by the sign \doteq) in which $\xi^a \doteq (1, 0, 0, 0)$ [and $\xi_a \doteq (g_{00}, g_{\alpha 0})$], i.e., adapted to the timelike Killing vector field, then

$$g_{00} \doteq |\xi|^2 = \xi_0, \quad A_\alpha \doteq -\frac{\xi_\alpha}{\xi_0}, \quad h_{00} \doteq 0,$$

and

$$\gamma_{\alpha\beta} \doteq h_{\alpha\beta} \doteq -g_{\alpha\beta} + \frac{1}{\xi_0} \xi_\alpha \xi_\beta,$$

so that the spacetime metric will take the following form:

$$ds^2 \doteq \xi_0 \left(dx^0 + \frac{\xi_\alpha}{\xi_0} dx^\alpha \right)^2 - h_{\alpha\beta} dx^\alpha dx^\beta.$$

It should be noted that Σ_3 is a differentiable 3-manifold but not a hypersurface in \mathcal{M} . Indeed, it is called the quotient space $\frac{\mathcal{M}}{G_1}$, where G_1 is the one-dimensional group of motions generated by the timelike Killing vector field of the spacetime \mathcal{M} [20].

Working in the general coordinate system of (2), in the 1 + 3 decomposition, gravitoelectric and gravitomagnetic fields are defined in terms of the derivatives of the metric components as follows [21]:

$$\mathbf{E}_g = -\frac{\nabla h}{2h} \doteq -\nabla \ln |\xi|, \quad \mathbf{B}_g = \nabla \times \mathbf{A}.$$

In terms of the above fields, Einstein field equations for a perfect fluid source could be rewritten in the following *quasi-Maxwell* form [21]:

$$\nabla \times \mathbf{E}_g = 0, \quad \nabla \cdot \mathbf{B}_g = 0, \quad (3)$$

$$\nabla \cdot \mathbf{E}_g = 1/2hB_g^2 + E_g^2 - 8\pi \left[\frac{\rho + p}{1 - v^2} - \frac{\rho - p}{2} \right], \quad (4)$$

$$\nabla \times (\sqrt{h}\mathbf{B}_g) = 2\mathbf{E}_g \times (\sqrt{h}\mathbf{B}_g) - 16\pi \left[\frac{\rho + p}{1 - v^2} \mathbf{v} \right], \quad (5)$$

$${}^{(3)}R_{\mu\nu} = -E_g^{\mu;\nu} + h(B_g^\mu B_g^\nu - B_g^2 \gamma^{\mu\nu}) + E_g^\mu E_g^\nu + 8\pi \left[\frac{p + \rho}{1 - v^2} v^\mu v^\nu + \frac{\rho - p}{2} \gamma^{\mu\nu} \right]. \quad (6)$$

The following points need to be mentioned with respect to the above equations:

- (1) ${}^{(3)}R^{\mu\nu}$ is the three-dimensional Ricci tensor of the 3-space Σ_3 constructed from the three-dimensional metric $\gamma_{\alpha\beta}$ in the same way that the usual four-dimensional Ricci tensor R^{ab} is made out of g_{ab} .
- (2) In the above equations all the differential operations are defined in the 3-space Σ_3 with metric $\gamma_{\alpha\beta}$ [19,21].
- (3) The 3-velocities v^μ are defined with respect to the synchronized proper time, i.e., $v^\mu = \frac{dx^\mu}{d\tau_{\text{syn}}}$.

Another feature in the above equations, which will be employed later, is the simple fact that by Eq. (5) a non-vacuum, static ($B_g \doteq 0$) solution produced by a perfect fluid, in general, has to be in the comoving frame ($v^\mu \doteq 0$) with respect to the dust particles. In other words, the same frame in which the spacetime has no cross terms is also comoving [22].

III. GAUSSIAN CURVATURE IN 1 + 3 FORMALISM AND GRAVITATIONAL AHARONOV-BOHM EFFECT

After introducing their formulation of the gravitational Aharonov-Bohm effect, based on (1), Ford and Vilenkin faced the problem of expressing Gaussian curvature in terms of the four-dimensional Ricci tensor so as, through Einstein field equations, to be able to calculate the phase shift (1) in terms of local quantities such as mass and angular momentum [5]. To do so, they have appealed to the weak field limit to express K in terms of the energy density of a dust source. In what follows, using the 1 + 3 formulation and quasi-Maxwell form of the Einstein field equations and without employing the weak field limit, we show that the Gaussian curvature can be expressed in terms of the gravitoelectromagnetic fields of the source and consequently in terms of the components of its energy-momentum tensor. To follow Ford and Vilenkin we consider a stationary spacetime corresponding to a tubelike distribution of matter with two Killing vectors $\xi = \xi^a \partial_a$ and $\eta = \eta^a \partial_a$, which are timelike and spacelike, respectively. Let S be a 2-surface in Σ_3 orthogonal to the two Killing vectors and C a closed curve in the 2-surface which may or may not encircle the tube containing the source. Now according to the relation (1) if a vector is parallel transported around C , it will acquire a rotation angle α due to the nonzero curvature region in the 2-surface S . To relate the Gaussian curvature of the 2-surface to the gravitoelectromagnetic fields of the underlying spacetime, we introduce, using the spacelike Killing vector η , a projection tensor from Σ_3 to S as follows:

$$\tilde{h}_{\alpha\beta} = \gamma_{\alpha\beta} - n_\alpha n_\beta, \quad \alpha, \beta = 1, 2, 3, \quad (7)$$

where now $n_\alpha = \frac{\eta_\alpha}{|\eta|}$ is the unit vector normal to S .

By choosing a preferred coordinate system in which $\eta \doteq \partial_z$, i.e., η^α takes the following form:

$$\eta^\alpha \doteq (0, 1, 0), \quad x^1 = r, \quad x^2 = z, \quad x^3 = \phi,$$

for 2-surface S with metric [23]

$$\tilde{g}_{ij} = \gamma_{ij}, \quad i, j = 1, 3, \quad (8)$$

the Gaussian curvature K is given by

$$K = \frac{1}{2} \tilde{g}^{ik} \tilde{g}^{jl(2)} R_{ijkl} = \frac{1}{2} {}^{(2)}R, \quad i, j, k, l = 1, 3. \quad (9)$$

The upper left indices indicate the dimension of the space for which the geometric entity is computed, so in the above equation the upper left index (2) shows that the Riemann tensor is calculated for the 2-surface S .

Since η is a Killing vector of the Σ_3 space and orthogonal to the 2-surface S , one can show that the extrinsic curvature of the 2-surface is zero, and hence by the Gauss-Codazzi equation we have ${}^{(2)}R_{kilj} = {}^{(3)}R_{kilj}$ ($i, j, k, l = 1, 3$). This could be used in turn to calculate the two-dimensional Ricci tensor as follows:

$${}^{(2)}R_{ij} = \tilde{g}^{kl(2)} R_{kilj} = \tilde{h}^{\alpha\beta(3)} R_{\alpha i \beta j}, \quad i, j = 1, 3,$$

where in the last equality by replacing for the projection tensor (7) we end up with

$${}^{(2)}R_{ij} = {}^{(3)}R_{ij} - \frac{\eta^\alpha \eta^\beta}{|\eta|^2} {}^{(3)}R_{\alpha i \beta j}, \quad i, j = 1, 3. \quad (10)$$

Therefore the two-dimensional Ricci scalar of the 2-surface S with coordinates $\{x^1, x^3\}$ and metric \tilde{g}^{ij} is given by

$${}^{(2)}R = \tilde{g}^{ij(2)} R_{ij} = \gamma^{ij} \left({}^{(3)}R_{ij} - \frac{\eta^\alpha \eta^\beta}{|\eta|^2} {}^{(3)}R_{\alpha i \beta j} \right), \quad i, j = 1, 3,$$

where use is made of the fact that $\tilde{g}^{ij} = \gamma^{ij}$. Now applying the relation ${}^{(3)}R_{\alpha\beta} = \gamma^{\mu\nu(3)} R_{\alpha\mu\beta\nu}$ to the above equation, we obtain the following relation for the Gaussian curvature of the 2-surface in terms of the three-dimensional Ricci scalar and Ricci tensor of the 3-space Σ_3 :

$${}^{(2)}R = 2K = {}^{(3)}R - 2 \frac{\eta^\alpha \eta^\beta}{|\eta|^2} {}^{(3)}R_{\alpha\beta}. \quad (11)$$

Up to now we have considered stationary tubelike spacetimes which possess two Killing vectors, but it is obvious that the above formulation could also be applied to cylindrically symmetric spacetimes which have an extra Killing vector $\zeta \doteq \partial_\phi$. In the next two subsections, first we apply the above relation to a static dust solution and then to cylindrically symmetric spacetimes to study a gravitational analogue of the Aharonov-Bohm effect.

A. Static tubelike distribution of dust: Aharonov-Bohm effect vs the Colella-Overhauser-Werner effect

As is clear from its derivation, relation (11) can be applied to any tubelike *stationary spacetime* with a given energy-momentum tensor, but to compare our results with those of Ford and Vilenkin [5] first we apply it to a static tubelike distribution of dust given by the following general metric:

$$ds^2 = g_{tt}dt^2 - g_{rr}dr^2 - g_{zz}dz^2 - g_{\phi\phi}d\phi^2, \quad (12)$$

where all the metric components are functions of r and ϕ . For such a spacetime from contraction of Eq. (6) (with $B_g = 0$, $p = 0$, and $v = 0$), we arrive at

$${}^{(3)}R = -\nabla \cdot \mathbf{E}_g + E_g^2 + 12\pi\rho = 16\pi\rho, \quad (13)$$

where we have used (4) in the last step. For the second term in the right-hand side of (11), we obtain

$$2\frac{\eta^\alpha\eta^\beta}{|\eta|^2}{}^{(3)}R_{\alpha\beta} = 8\pi\rho - \mathbf{E}_g \cdot \nabla \ln|\eta|^2, \quad (14)$$

and after substituting Eqs. (13) and (14) in (11), the Gaussian curvature for static dust yields

$$K = 4\pi\rho + \mathbf{E}_g \cdot \nabla \ln|\eta|. \quad (15)$$

The first term in (15) is the exact same term that was obtained by Ford and Vilenkin [5] for a tubelike distribution of dust in the weak field regime, but the second term which depends on the gravitoelectric field of the spacetime and is generally nonzero is absent in their approach. Now taking the spacetime (12) to be asymptotically flat (where $E_g \rightarrow 0$ as $r \rightarrow \infty$), by Eq. (1), a particle parallel transported along a closed path C in this zero-curvature region will nevertheless be affected by the nonzero curvature in the interior region.

Since the second term is proportional to the gravitoelectric field, when integrated over a closed path, it reminds one of the Colella-Overhauser-Werner (COW) experiment and effect [24]. This term, arising from a purely geometrical consideration, shows that the gravitationally induced phase shift has a classical origin [25]. The presence of this term shows that even when the closed path of a particle does not encircle the source, there still will be an effect in the interference pattern resulting from the presence of the gravitoelectric field. To investigate in more detail the effect of this extra term, in the next subsection we consider cylindrically symmetric dust spacetimes and sources and their matching to exterior (vacuum) solutions.

B. Static cylindrically symmetric counterrotating dust

Consider matching of a cylindrically symmetric static dust solution, at a given radius R , to an exterior (vacuum) asymptotically flat static solution, both of the general type (12) but now all metric components are functions only of r .

As an explicit example, one can think of the interior solution introduced by Teixeira and Som [27], representing counterrotating dust particles with net zero angular momentum and the following energy-momentum tensor [28]:

$$T_b^a = \frac{1}{2}\rho(u_a u^b + v_a v^b), \quad (16)$$

where $u^a \doteq (u^0, 0, 0, \omega)$ and $v^a \doteq (u^0, 0, 0, -\omega)$ are the four velocities of the counterrotating particles with $u^a u_a = v^a v_a = 1$. This interior spacetime is matched to the well-known exterior Levi-Civita metric [29]

$$ds^2 = r^{4\sigma} dt^2 - r^{-4\sigma} [r^{8\sigma^2} (dr^2 + B^2 dz^2) + C^2 r^2 d\phi^2], \quad (17)$$

in which B and C are scaling parameters and σ , for small values, could be interpreted as the effective gravitational mass per unit proper length [30]. Now two cases could be considered:

- (I) The closed path of the parallel transported particle encircles the nonvacuum cylindrical region (Fig. 1) so that the rotation angle (1) is given by

$$\begin{aligned} \alpha &= \int_{S_0 = S_0^{\text{ext}} \cup S_0^{\text{int}}} K da \\ &= \int_{S_0^{\text{ext}}} (K da)^{\text{ext}} + \int_{S_0^{\text{int}}} (K da)^{\text{int}}, \end{aligned} \quad (18)$$

where the upper indices show the region of spacetime in which the quantities are calculated. Repeating the calculation leading to (15), now with the energy-momentum tensor of the perfect fluid given by (16), we end up with

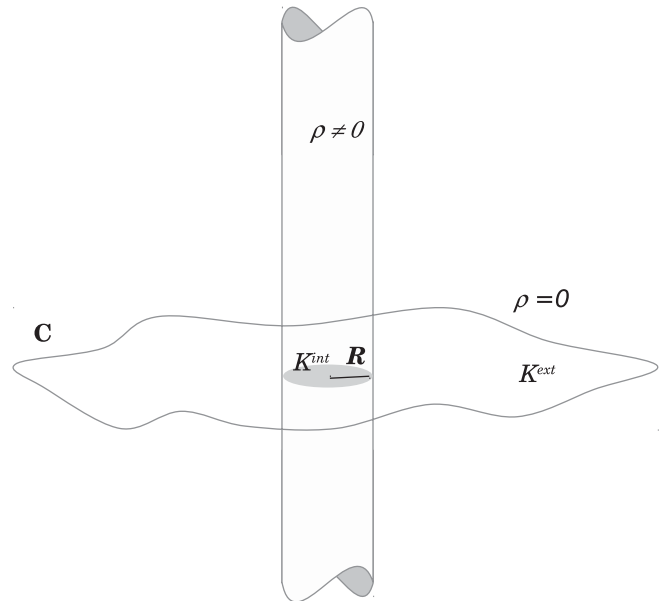


FIG. 1. Closed path C of a particle encircling the nonvacuum cylindrical region at the asymptotically flat region.

$$K^{\text{int}} = \frac{4\pi\rho}{1-v^2} + (\mathbf{E}_g \cdot \nabla \ln|\eta|)^{\text{int}} \quad (19)$$

for the Gaussian curvature of the interior region, where v^2 is the squared norm of the three-velocity of the dust particles defined by $v^\mu = \frac{dx^\mu}{d\tau}$ (τ being the proper time in the corresponding static spacetime). Indeed, using the relation between the components of this three-velocity and the corresponding four-velocity $u^\mu = \frac{dx^\mu}{ds}$ [19], one can obtain the following relation:

$$v^2 = \frac{g_{\phi\phi}\omega^2}{g_{\phi\phi}\omega^2 - 1}. \quad (20)$$

Equation (19) is an interesting result showing that not only the density of the particles but also the norm of their velocity, both as local quantities, have global effects. The presence of v^2 is a manifestation of the fact that the kinetic energy of the particles gravitates even though their net angular momentum is zero. Obviously, for the exterior (vacuum) static solution ($\rho = 0$), we end up with the following relation for the Gaussian curvature:

$$K^{\text{ext}} = (\mathbf{E}_g \cdot \nabla \ln|\eta|)^{\text{ext}}. \quad (21)$$

The overall result in the arrangement of Fig. 1, as in the case of a simple static dust solution, for a path C in the asymptotically (locally) flat region, is a rotation angle (18) which is a manifestation of a gravitational Aharonov-Bohm effect.

- (II) In this case the closed path of the parallel transported particle does not encircle the nonvacuum cylindrical region (Fig. 2), so that the rotation angle (1) is given by

$$\alpha = \int_{S_0^{\text{ext}}} (K da)^{\text{ext}} = \int (\mathbf{E}_g \cdot \nabla \ln|\eta| da)^{\text{ext}}, \quad (22)$$

which is obviously of the COW-type effect leading to a gravitationally induced phase shift on the transported particle. In other words, the Gaussian curvature having a nonzero contribution from the second term in (15) represents the COW-type effect for a cylindrical gravitational field represented by the exterior metric (17). Indeed, in this approach, both the gravitational Aharonov-Bohm effect and the COW effect for cylindrically symmetric spacetimes, were treated as different phase shifts under the same formulation.

C. Deficit angle of a cosmic string

As another somewhat trivial example of the application of the formalism introduced in Sec. III, we turn our attention to the spacetime of a cosmic string that seems to be naturally adapted to the investigations on the gravitational Aharonov-Bohm effect. The spacetime metric of a cosmic

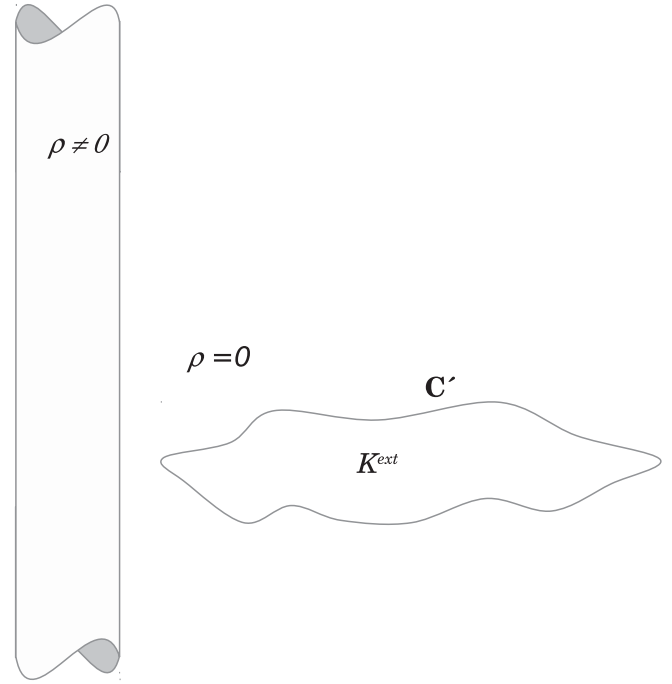


FIG. 2. Closed path C' of a particle which does not encircle the nonvacuum cylindrical region.

string is given by setting $\sigma = \frac{1}{2}$ in (17) (with the rescaling parameter B removed) [31]:

$$ds^2 = dt^2 - dr^2 - dz^2 - C^2 r^2 d\phi^2. \quad (23)$$

The corresponding energy-momentum tensor is given by [32]

$$T_t^t = T_z^z = \mu \frac{\delta(\mathbf{r})}{\sqrt{\tilde{g}}}, \quad T_r^r = T_\phi^\phi = 0, \quad (24)$$

where μ is the linear mass density and \tilde{g} is the determinant of the metric on the $t = \text{const}$ and $z = \text{const}$ 2-surface S_0 . To compute the Gaussian curvature, we rewrite the last of the quasi-Maxwell equations, Eq. (6), for the above space-time metric and energy-momentum tensor as follows (with $E_g = B_g = 0$):

$${}^{(3)}R^{\mu\nu} = 8\pi \left(T^{\mu\nu} - \frac{1}{2} g^{\mu\nu} T \right). \quad (25)$$

Contraction with $\gamma_{\mu\nu}$ gives the three-dimensional Ricci scalar as follows:

$${}^{(3)}R = 16\pi \left(\mu \frac{\delta(\mathbf{r})}{\sqrt{\tilde{g}}} \right). \quad (26)$$

On the other hand, it is an easy task to see that in the adapted coordinate system [in which $\eta^\mu \doteq (0, 1, 0)$] the second term in the right-hand side of relation (11) vanishes, so that the Gaussian curvature in this case is given by

$$K = \frac{1}{2} {}^{(3)}R = 8\pi \left(\mu \frac{\delta(\mathbf{r})}{\sqrt{\tilde{g}}} \right). \quad (27)$$

After performing the area integral of K over 2-surface S_0 , we have the rotation angle $\alpha = 8\pi\mu$. Now employing the Gauss-Bonnet theorem

$$\int_S K da = 2\pi\chi(S) - \int_{\partial S} k_g dl \quad (28)$$

to the above case, we can find the deficit angle $2\pi\delta$ corresponding to a cosmic string. To do so, we take $\chi(S) = 1$ and $k_g = \frac{1}{r}$ for the Euler characteristic and geodesic curvature of the corresponding 2-surface S_0 and its encircling path ∂S_0 , respectively, i.e.,

$$\int_{S_0} 8\pi\mu\delta(\mathbf{r})d^2\mathbf{r} = 2\pi - \int_{\partial S_0} \frac{1}{r} C r d\phi, \quad (29)$$

leading to the scaling factor $C \equiv 1 - \delta = 1 - 4\mu$ of a cosmic string in terms of its linear mass density.

IV. CONCLUSION

With the help of a projection tensor defined in the context of $1 + 3$ formulation of spacetime decomposition, we introduced an expression for the Gaussian curvature of a 2-surface in terms of the Ricci scalar and Ricci tensor of the corresponding 3-space Σ_3 . On the other hand, using the quasi-Maxwell form of the Einstein field equations we have related the components of the energy-momentum tensor of a source and the gravitoelectromagnetic fields of the underlying spacetime to the three-dimensional Ricci tensor of Σ_3 . This enabled us to relate the Gaussian curvature of a 2-surface in Σ_3 to local quantities such as the energy density of the source and velocity norm of its constituent particles. The expression is then applied to the case of a tubelike perfect fluid source to investigate a version of the gravitational Aharonov-Bohm effect introduced by Ford and Vilenkin. While they have applied their formalism to a static dust solution in the weak field limit, here we have shown that our formulation allows one to apply it to stationary spacetimes and without appeal to the weak field limit. In doing so, not only have we recovered, for a static dust solution, the same result as that of Ford and Vilenkin, but we also obtained, interestingly enough, another phase shift related to the presence

of the gravitoelectric field of the underlying spacetime. Since the particles traversing the closed path in the background spacetime are treated classically (i.e., without any reference to their quantum mechanical characteristics reflected in the quantum mechanical equation they might satisfy), this phase shift could be interpreted as the *classical* version of the well-known COW (experiment) effect. The presence of this kind of phase shift was discussed explicitly in the context of a static cylindrically symmetric vacuum solution matched to an interior static dust solution. Indeed, applying our formulation to the case of a dust solution produced by counterrotating particles, we have arrived at a Gaussian curvature which is given in terms of the energy density of the source as well as the norm of the particle velocities, despite the fact that their net angular momentum is zero. As another trivial example of the application of our formalism, also showing its consistency, the deficit angle of a cosmic string was obtained by using the Gauss-Bonnet theorem. On the experimental side, it should be noted that the two experimental proposals to measure versions of this effect [10,11] both use quantum mechanical particles (atoms) to gain a gravitationally induced phase as a function of the underlying gravitational potential. In our case, particles traversing the closed paths are classical but possess a vector-valued characteristic, and the induced phase shift α on that vector, given by (1), is a purely geometrical entity. So, in principle, to have any experimental realization of the above version of the effect, one needs a static cylindrically symmetric gravitational field and a classical particle possessing a vector quantity (for example, a spinning top) traversing a closed path either around the source or outside it, which is obviously not an easy task to do, unless one could come up with an indirect experiment somewhat similar to that in Ref. [11], but with cylindrical masses instead of spheres.

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