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Monodromy transform and the integral equation method for solving the string gravity and supergravity equations in four and higher dimensions

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The monodromy transform and corresponding integral equation method described here give rise to a general systematic approach for solving integrable reductions of field equations for gravity coupled bosonic dynamics in string gravity and supergravity as well as for pure vacuum gravity in four and higher dimensions. For physically different types of fields in space-times of $D \geq 4$ dimensions with $d = D - 2$ commuting isometries—stationary fields with spatial symmetries, interacting waves or evolution of partially inhomogeneous cosmological models—the string gravity equations govern the dynamics of interacting gravitational, dilaton, antisymmetric tensor, and any number $n \geq 0$ of Abelian vector gauge fields (all depending only on two coordinates). The equivalent spectral problem constructed earlier allows one to parametrize the entire infinite-dimensional space of (normalized) local solutions of these equations by two pairs of arbitrary coordinate-independent holomorphic $d \times d$- and $d \times n$- matrix functions $\{\mathbf{u}_\pm(w), \mathbf{v}_\pm(w)\}$ of a spectral parameter w which constitute a complete set of monodromy data for a normalized fundamental solution of this spectral problem. The "direct" and "inverse" problems of such monodromy transform—calculating the monodromy data for any local solution and constructing the field configurations for any chosen monodromy data—always admit unique solutions. We construct the linear singular integral equations which solve this inverse problem. For any *rational* and *analytically matched* [i.e. $\mathbf{u}_+(w) \equiv \mathbf{u}_-(w)$ and $\mathbf{v}_+(w) \equiv \mathbf{v}_-(w)$] monodromy data the solution of these integral equations and corresponding solution for string gravity equations can be found explicitly. Simple reductions of the space of monodromy data leads to the similar constructions for solving of other integrable symmetry reduced gravity models, e.g. 5D minimal supergravity or vacuum gravity in $D \geq 4$ dimensions.

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I. INTRODUCTION

Motivated by various physical reasons, numerous modern studies of strong gravitational fields in different gravity models—pure gravity as well as the bosonic sectors of string gravity and supergravity in four and higher dimensions—brought us many interesting discoveries concerning the existence and properties of a large variety of higher-dimensional space-time structures—black holes, black rings, black lens, etc. (see, for example, [1]), the existence of which distinguishes these models in higher dimensions from those in 4D space-times. It can be expected that the same may concern also different types of fields, e.g. interacting waves and cosmological solutions. In all cases, an explicit construction of appropriate solutions can play an even more crucial role than it was in four dimensions, where our physical intuition is better adapted.

The construction of solutions for different gravity models needs a generalization of methods developed earlier for the 4D case to make these available for solving field equations in higher dimensions and to include various nongravitational fields. For these purposes, in particular, it is very important to find the cases in which the symmetry reduced dynamical equations occur to be completely integrable. Such integrable reductions may arise if space-times of D dimensions admit $D - 2$ commuting isometries and therefore, all field components and potentials depend only on two space-time coordinates. Now we have a large experience in development and application of various solution generating methods for Einstein's field equations in 4D space-times with two commuting isometries (see, e.g., the survey [2] and the references therein). In the 4D case, the developed methods allow, in particular, one to construct explicitly the hierarchies of solutions with an arbitrary large number of free parameters. These are the N-soliton solutions generated on (or, in other words, interacting with) arbitrarily chosen backgrounds such as Belinski and Zakharov vacuum solitons [3] and Einstein-Maxwell solitons [4], as well as some classes of nonsoliton solutions [5–7]. Besides that, for these equations one can solve the (effectively, two-dimensional) boundary value problems for stationary fields (see [8] and the references there) and characteristic initial value problems [9–11]. However, a generalization of these methods for higher dimensional gravity models is not trivial and it may need to overcome some specific difficulties.

In the last two decades, many authors used the Belinski and Zakharov soliton generating transformations for constructing stationary axisymmetric solutions for pure vacuum 5D gravity. This "dressing" method for generating solitons on arbitrary (vacuum) backgrounds can be

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generalized directly from 4D to higher dimensions without any essential changes of the procedure. Some attempts were made also to apply the Belinski and Zakharov construction of the spectral problem to other gravity models in 4D and to the bosonic sector of 5D minimal supergravity. However, the constructed spectral problems had not been equivalent to the dynamical equations. As a result, generating soliton solutions in general do not satisfy all necessary conditions (see the discussion in [12]).

The spectral problem of another kind constructed in [13] is *equivalent* to the dynamical equations of the bosonic sector of heterotic string theory in D dimensions with $d = D - 2$ commuting isometries. Certain constraints for the potentials reduce this system to the dynamical equations and equivalent spectral problem for the bosonic sector of 5D minimal supergravity.

The spectral problem [13] possesses the monodromy preserving properties, which allow one to generalize for this case the monodromy transform approach suggested in [14] for symmetry reduced vacuum Einstein equations, electrovacuum Einstein-Maxwell and Einstein-Maxwell-Weyl equations in general relativity (see also [15,16]).

In this paper we present some sketch of the monodromy transform for heterotic string gravity model in D dimensions and 5D minimal supergravity which includes the following:

(i) the solution of a "direct" problem—a parametrization of the whole space of local solutions of symmetry reduced bosonic equations for heterotic string gravity by two pairs of coordinate-independent matrix valued holomorphic functions of spectral parameter $\{\mathbf{u}_\pm^{d\times d}(w), \mathbf{v}_\pm^{d\times n}(w)\}$ which arise as the monodromy data of the normalized fundamental solution of the associated spectral problem;

(ii) the solution of the "inverse" problem—a construction of the local solutions for *any* given monodromy data. For this we construct a system of linear singular integral equations the solution of which allows us to calculate all field components in quadratures;

(iii) description of a special class of *analytically matched* monodromy data determined by the conditions $\mathbf{u}_+(w) = \mathbf{u}_-(w) \equiv \mathbf{u}(w)$, $\mathbf{v}_+(w) = \mathbf{v}_-(w) \equiv \mathbf{v}(w)$. For any *rational* $\mathbf{u}(w)$ and $\mathbf{v}(w)$ the corresponding solutions can be found explicitly.

The monodromy data defined above play the role of "coordinates" in the infinite-dimensional space of local solutions of symmetry reduced dynamical equations. Solving the inverse problem of our monodromy transform for arbitrary rational analytically matched monodromy data, we can construct explicitly infinite hierarchies of solutions which include many physically interesting known solutions and which give us their multiparametric generalizations. For constructing solutions with more general monodromy data describing, e.g., colliding waves or cosmological models which are singular at $\alpha = 0$ see [7].

II. MASSLESS BOSONIC DYNAMICS IN STRING GRAVITY

The massless bosonic part of string effective action in space-times with $D \geq 4$ dimensions in string frame is

$$\mathcal{S} = \int e^{-\hat{\Phi}}\Big\{\hat{R}^{(D)} + \nabla_M\hat{\Phi}\nabla^M\hat{\Phi} - \frac{1}{12}H_{MNP}H^{MNP} - \frac{1}{2}\sum_{\mathfrak{p}=1}^{n} F_{MN}{}^{(\mathfrak{p})}F^{MN(\mathfrak{p})}\Big\}\sqrt{-\hat{G}}d^Dx, \tag{1}$$

where $M, N, \ldots = 1, 2, \ldots, D$ and $\mathfrak{p} = 1, \ldots n$; $\hat{G}_{MN}$ possesses the "most positive" Lorentz signature. Metric $\hat{G}_{MN}$ and dilaton field $\hat{\Phi}$ are related to the metric G_{MN} and dilaton Φ in the Einstein frame as

$$\hat{G}_{MN} = e^{2\Phi}G_{MN}, \qquad \hat{\Phi} = (D-2)\Phi. \tag{2}$$

The components of a three-form H and two-forms $F^{(\mathfrak{p})}$ are determined in terms of antisymmetric tensor field B_{MN} and Abelian gauge field vector potentials $A_M{}^{(\mathfrak{p})}$:

$$H_{MNP} = 3\Big(\partial_{[M}B_{NP]} - \sum_{\mathfrak{p}=1}^{n} A_{[M}{}^{(\mathfrak{p})}F_{NP]}{}^{(\mathfrak{p})}\Big),$$
$$F_{MN}{}^{(\mathfrak{p})} = 2\partial_{[M}A_{N]}{}^{(\mathfrak{p})}, \qquad B_{MN} = -B_{NM}.$$

III. SPACE-TIME SYMMETRY ANSATZ

We consider the space-times with $D \geq 4$ dimensions which admit $d = D - 2$ commuting Killing vector fields. All field components and potentials are assumed to be functions of only two coordinates x^1 and x^2, one of which can be timelike or both are spacelike coordinates. We assume also the following structure of metric components:

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & G_{ab}\end{pmatrix}, \qquad \begin{array}{c}\mu, \nu, \ldots = 1, 2, \\ a, b, \ldots = 3, 4, \ldots D,\end{array} \tag{3}$$

while the components of field potentials take the forms

$$B_{MN} = \begin{pmatrix} 0 & 0 \\ 0 & B_{ab}\end{pmatrix}, \qquad A_M{}^{(\mathfrak{p})} = \begin{pmatrix} 0 \\ A_a{}^{(\mathfrak{p})}\end{pmatrix}. \tag{4}$$

We choose x^1, x^2 so that $g_{\mu\nu}$ takes a conformally flat form

$$g_{\mu\nu} = f\eta_{\mu\nu}, \qquad \eta_{\mu\nu} = \begin{pmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2\end{pmatrix}, \qquad \begin{array}{c}\epsilon_1 = \pm 1, \\ \epsilon_2 = \pm 1,\end{array}$$

where $f(x^\mu) > 0$ and the sign symbols ϵ_1 and ϵ_2 allow us to consider various types of fields. The field equations imply that the function $\alpha(x^1, x^2) > 0$ is a "harmonic" one:

$$\det\|G_{ab}\| \equiv \epsilon\alpha^2, \qquad \eta^{\mu\nu}\partial_\mu\partial_\nu\alpha = 0, \qquad \epsilon = -\epsilon_1\epsilon_2,$$

where $\eta^{\mu\nu}$ is inverse to $\eta_{\mu\nu}$, and therefore, the function $\beta(x^\mu)$ can be defined as "harmonically" conjugated to α:

$$\partial_\mu\beta = \epsilon\varepsilon_\mu{}^\nu\partial_\nu\alpha, \qquad \varepsilon_\mu{}^\nu = \eta_{\mu\gamma}\varepsilon^{\gamma\nu}, \qquad \varepsilon^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}.$$

Using the functions (α, β), we construct a pair (ξ, η) of real null coordinates in the hyperbolic case or complex conjugated to each other coordinates in the elliptic case:

$$\begin{cases} \xi = \beta + j\alpha, \\ \eta = \beta - j\alpha, \end{cases} \qquad j = \begin{cases} 1, \epsilon = 1 & - \text{ hyperbolic case,} \\ i, \epsilon = -1 & - \text{ elliptic case.} \end{cases}$$

In particular, for stationary axisymmetric fields $\xi = z + i\rho$, $\eta = z - i\rho$, whereas for plane waves or for cosmological solutions $\xi = -x + t$, $\eta = -x - t$, or these may have more complicated expressions in terms of x^1, x^2.

IV. DYNAMICAL EQUATIONS

The symmetry reduced dynamical equations for the action (1) can be presented in the form of real matrix Ernst-like equations for the string frame matrix variables—a symmetric $d \times d$-matrix $\mathcal{G}$, antisymmetric $d \times d$-matrix $\mathcal{B}$, a rectangular $d \times n$-matrix $\mathcal{A}$, the scalars $\hat{\Phi}$ and α:

$$\mathcal{G} = e^{2\Phi} \| G_{ab} \|, \qquad \mathcal{B} = \| B_{ab} \|, \qquad \mathcal{A} = \| A_a{}^{(\mathrm{p})} \|,$$

which should satisfy the system of equations

$$\begin{cases} \eta^{\mu\nu} \partial_\mu (\alpha \partial_\nu \mathcal{E}) - \alpha \eta^{\mu\nu} (\partial_\mu \mathcal{E} - 2 \partial_\mu \mathcal{A} \mathcal{A}^T) \mathcal{G}^{-1} \partial_\nu \mathcal{E} = 0, \\ \eta^{\mu\nu} \partial_\mu (\alpha \partial_\nu \mathcal{A}) - \alpha \eta^{\mu\nu} (\partial_\mu \mathcal{E} - 2 \partial_\mu \mathcal{A} \mathcal{A}^T) \mathcal{G}^{-1} \partial_\nu \mathcal{A} = 0, \\ \eta^{\mu\nu} \partial_\mu \partial_\nu \alpha = 0, \end{cases} \tag{5}$$

where T means a matrix transposition and

$$\mathcal{E} = \mathcal{G} + \mathcal{B} + \mathcal{A} \mathcal{A}^T, \qquad \det \mathcal{G} = \epsilon \alpha^2 e^{2\hat{\Phi}}. \tag{6}$$

The equations in (5) imply the existence of antisymmetric $d \times d$ potential $\tilde{\mathcal{B}}$ and $d \times n$ potential $\tilde{\mathcal{A}}$ defined as

$$\begin{aligned} \partial_\mu \tilde{\mathcal{B}} &= -\epsilon \alpha \varepsilon_\mu{}^\nu \mathcal{G}^{-1} (\partial_\nu \mathcal{B} - \partial_\nu \mathcal{A} \mathcal{A}^T + \mathcal{A} \partial_\nu \mathcal{A}^T) \mathcal{G}^{-1}, \\ \partial_\mu \tilde{\mathcal{A}} &= -\epsilon \alpha \varepsilon_\mu{}^\nu \mathcal{G}^{-1} \partial_\nu \mathcal{A} + \tilde{\mathcal{B}} \partial_\mu \mathcal{A}. \end{aligned} \tag{7}$$

The remaining (nondynamical) part of field equations determines the conformal factor f in quadratures, provided the solution of dynamical equations is found [13].

V. EQUIVALENT SPECTRAL PROBLEM

As it was described in [13], the dynamical equations (5) admit an *equivalent* reformulation in terms of the spectral problem for the four $(2d + n) \times (2d + n)$ matrix functions depending on two real (in a hyperbolic case) or two complex conjugated (in the elliptic case) coordinates ξ and η and a free complex ("spectral") parameter $w \in \mathbb{C}$

$$\mathbf{\Psi}(\xi, \eta, w), \quad \mathbf{U}(\xi, \eta), \quad \mathbf{V}(\xi, \eta), \quad \mathbf{W}(\xi, \eta, w), \tag{8}$$

which should satisfy the following linear system for $\mathbf{\Psi}$ with algebraic constraints on its matrix coefficients

$$\begin{cases} 2(w - \xi) \partial_\xi \mathbf{\Psi} = \mathbf{U}(\xi, \eta) \mathbf{\Psi} \\ 2(w - \eta) \partial_\eta \mathbf{\Psi} = \mathbf{V}(\xi, \eta) \mathbf{\Psi} \end{cases} \Bigg\| \begin{array}{l} \mathbf{U} \cdot \mathbf{U} = \mathbf{U}, \ \mathrm{tr} \mathbf{U} = d \\ \mathbf{V} \cdot \mathbf{V} = \mathbf{V}, \ \mathrm{tr} \mathbf{V} = d \end{array} \tag{9}$$

The supplemental condition is that the system (9) should admit a symmetric matrix integral $\mathbf{W}_o(w)$ such that

$$\begin{cases} \mathbf{\Psi}^T \mathbf{W} \mathbf{\Psi} = \mathbf{W}_o(w) \\ \mathbf{W}_o^T(w) = \mathbf{W}_o(w) \end{cases} \Bigg\| \ \frac{\partial \mathbf{W}}{\partial w} = \mathbf{\Omega}, \quad \mathbf{\Omega} = \begin{pmatrix} 0 & I_d & 0 \\ I_d & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{10}$$

where I_d is a $d \times d$ unit matrix and $\mathbf{\Omega}$ is a $(2d+n) \times (2d+n)$ matrix. We should impose also the reality conditions

$$\overline{\mathbf{\Psi}(\xi, \eta, \bar{w})} = \mathbf{\Psi}(\xi, \eta, w), \quad \overline{\mathbf{W}_o(\bar{w})} = \mathbf{W}_o(w), \quad \mathbf{W}_{(3)(3)} = I_n, \tag{11}$$

where $\mathbf{W}_{(3)(3)}$ is the lower right $n \times n$ block of $\mathbf{W}$ which, in accordance with (8)–(10), is a constant matrix and therefore the last condition in (11) is pure gauge.

VI. FIELD VARIABLES AND POTENTIALS

As it can be shown by direct calculations (the detail will be published elsewhere), the conditions (8)–(10) imply, in particular, that $\mathbf{W}$ possesses a special structure

$$\mathbf{W} = (w - \beta) \mathbf{\Omega} + \mathbf{G}, \quad \text{where}$$

$$\mathbf{G} = \begin{pmatrix} \epsilon \alpha^2 \mathcal{G}^{-1} - \tilde{\mathcal{B}} \mathcal{G} \tilde{\mathcal{B}} + \tilde{\mathcal{A}} \tilde{\mathcal{A}}^T & \tilde{\mathcal{B}} \mathcal{G} + \tilde{\mathcal{A}} \mathcal{A}^T & \tilde{\mathcal{A}} \\ -\mathcal{G} \tilde{\mathcal{B}} + \mathcal{A} \tilde{\mathcal{A}}^T & \mathcal{G} + \mathcal{A} \mathcal{A}^T & \mathcal{A} \\ \tilde{\mathcal{A}}^T & \mathcal{A}^T & I_n \end{pmatrix} \tag{12}$$

and $\alpha = (\xi - \eta)/2j$, $\beta = (\xi + \eta)/2$; $d \times d$ matrix blocks $\mathcal{G}$ and $\tilde{\mathcal{B}}$ are symmetric and antisymmetric, respectively, and, together with $d \times n$ matrices $\mathcal{A}$ and $\tilde{\mathcal{A}}$, these satisfy (5)–(7). This allows us to calculate all field components and potentials for any solution (8) of our spectral problem.

VII. THE SPACE OF NORMALIZED LOCAL SOLUTIONS

We consider now the space of all (normalized) local solutions of (5) near a chosen regular "initial" point (ξ_o, η_o) and corresponding solutions of our spectral problem (8)–(11) considered also locally in ξ and η, but "for all" w,

$$(\xi, \eta) \in (\Omega_{\xi_o} \times \Omega_{\eta_o}), \qquad w \in \bar{\mathbb{C}},$$

where Ω_{ξ_o}, Ω_{η_o} are local regions near ξ_o, η_o, respectively, where $\mathbf{U}$ and $\mathbf{V}$ are holomorphic functions of (ξ, η).

Without any loss of generality we impose on the field components and auxiliary matrix functions a set of normalization conditions which provide unambiguous correspondence between local solutions of (5) and (8)–(11):

$$\Psi(\xi_o, \eta_o, w) = \mathbf{I}, \qquad \mathbf{W}_o(w) = (w - \beta_o)\boldsymbol{\Omega} + \mathbf{G}_o,$$
$$\mathbf{G}_o = \begin{pmatrix} \epsilon\alpha_o^2\mathcal{G}_o^{-1} & 0 & 0 \\ 0 & \mathcal{G}_o & 0 \\ 0 & 0 & I_n \end{pmatrix}, \quad \begin{array}{l} \mathcal{G}(\xi_o, \eta_o) = \mathcal{G}_o, \\ \mathcal{B}(\xi_o, \eta_o) = 0, \\ \mathcal{A}(\xi_o, \eta_o) = 0, \\ \tilde{\mathcal{A}}(\xi_o, \eta_o) = 0, \end{array} \tag{13}$$

where $\alpha_o = (\xi_o - \eta_o)/2j$, $\beta_o = (\xi_o + \eta_o)/2$, and $d \times d$ matrix $\mathcal{G}_o = \mathrm{diag}\{\varepsilon_1, \ldots, \varepsilon_d\}$ with $\mathcal{G}_o \cdot \mathcal{G}_o = I_d$, which can be achieved using the symmetries admitted by (8)–(11).

VIII. GENERAL ANALYTIC STRUCTURE OF Ψ ON w PLANE

A. Global structure of Ψ on the spectral plane

The conditions (9) and (13) imply that the normalized fundamental solution $\Psi(\xi, \eta, w)$ of (9) possesses in general only four singular (branching) points $w = \xi_o$, $w = \xi$, $w = \eta_o$, $w = \eta$, respectively of the orders $(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$ for $\Psi(\xi, \eta, w)$ and opposite for its inverse. To construct a holomorphic branch of $\Psi(\xi, \eta, w)$, we make two local cuts on the w plane: L_+ goes from $w = \xi_o$ to $w = \xi$ and L_-; from $w = \eta_o$ to $w = \eta$ each belong to the corresponding local region $\Omega_+ = \{w | w = \xi \in \Omega_{\xi_o}\}$ and $\Omega_- = \{w | w = \eta \in \Omega_{\eta_o}\}$ (see Fig. 1).

B. Local structure of Ψ near the cuts

It can be shown (see [16]) that the normalized matrix function Ψ and its inverse possess near the cuts $L_\pm$ the local structures

$$\Psi = \begin{cases} \lambda_+^{-1}\boldsymbol{\psi}_+(\xi, \eta, w) \cdot \mathbf{k}_+(w) + \mathbf{M}_+(\xi, \eta, w), & w \in \Omega_+ \\ \lambda_-^{-1}\boldsymbol{\psi}_-(\xi, \eta, w) \cdot \mathbf{k}_-(w) + \mathbf{M}_-(\xi, \eta, w), & w \in \Omega_- \end{cases},$$
$$\Psi^{-1} = \begin{cases} \lambda_+\mathbf{l}_+(w) \cdot \boldsymbol{\varphi}_+(\xi, \eta, w) + \mathbf{N}_+(\xi, \eta, w), & w \in \Omega_+ \\ \lambda_-\mathbf{l}_-(w) \cdot \boldsymbol{\varphi}_-(\xi, \eta, w) + \mathbf{N}_-(\xi, \eta, w), & w \in \Omega_- \end{cases}, \tag{14}$$

where "$\cdot$" means matrix multiplication and the functions $\lambda_\pm$ are defined by the expressions $\lambda_\pm(w = \infty) = 1$ and

$$\lambda_+ = \sqrt{(w - \xi)/(w - \xi_0)}, \qquad \lambda_- = \sqrt{(w - \eta)/(w - \eta_0)};$$

all other "fragments" of the local structures (14) —$d \times (2d + n)$ matrix functions $\mathbf{k}_\pm(w)$ and $\boldsymbol{\varphi}_\pm(\xi, \eta, w)$, $(2d + n) \times d$ matrix functions $\mathbf{l}_\pm(w)$ and $\boldsymbol{\psi}_\pm(\xi, \eta, w)$ as well as $(2d + n) \times (2d + n)$ matrix functions $\mathbf{M}_\pm(\xi, \eta, w)$ and $\mathbf{N}_\pm(\xi, \eta, w)$—are holomorphic in Ω_+ or Ω_-, respectively, and should satisfy there the algebraic relations

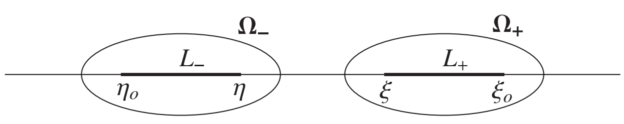

FIG. 1. Cuts on the w plane in the hyperbolic case ($\epsilon = 1$). In the elliptic case ($\epsilon = -1$), the cuts are located symmetrically to each other with respect to the real axis.

$$\begin{array}{lll} \mathbf{k} \cdot \mathbf{N} = 0, & \mathbf{N} \cdot \boldsymbol{\psi} = 0, & \mathbf{l}(w) = \Sigma_o^2 \mathbf{W}_o^{-1}(w) \cdot \mathbf{k}^T(w), \\ \mathbf{M} \cdot \mathbf{l} = 0, & \boldsymbol{\varphi} \cdot \mathbf{M} = 0, & \Sigma_o^2 \equiv (w - \xi_o)(w - \eta_o). \end{array} \tag{15}$$

Here and below we use instead of a pair of functions with indices "+" and "−" defined and holomorphic in Ω_+ and Ω_-, respectively, a one function defined in $\Omega_+ \cup \Omega_-$ and represented in Ω_+ and Ω_- by the corresponding functions with the indices "+" or "−".

IX. MONODROMY DATA

The coordinate-independent functions $\mathbf{k}_\pm(w)$ and $\mathbf{l}_\pm(w)$ play a very important role in our construction. These functions determine the monodromy of Ψ on the cuts $L_\pm$ because the analytical continuation of Ψ along a simple path which goes from one edge of a cut L_+ or L_- to its other edge leads [in accordance with (14)] to the linear transformations $\Psi \to \Psi \cdot \mathbf{T}_\pm(w)$ with

$$\mathbf{T}_\pm(w) = \mathbf{I} - 2\mathbf{l}_\pm(w) \cdot (\mathbf{k}_\pm(w) \cdot \mathbf{l}_\pm(w))^{-1} \cdot \mathbf{k}_\pm(w), \tag{16}$$

where the upper and lower signs correspond to L_+ and L_-, respectively. Note also that $\mathbf{T}_\pm^2(w) \equiv \mathbf{I}$.

We are going to characterize unambiguously the solutions of (5) by a complete set of independent functional parameters in (16) which we call the monodromy data of a solution. In view of (15), $\mathbf{l}_\pm(w)$ can be expressed in terms of $\mathbf{k}_\pm(w)$ and therefore, $\mathbf{T}_\pm(w)$ are completely determined by $\mathbf{k}_\pm(w)$. On the other hand, it is easy to see, that Ψ remains unchanged after the transformations

$$\mathbf{k}_\pm(w) \to \mathbf{c}_\pm(w) \cdot \mathbf{k}_\pm(w), \qquad \boldsymbol{\psi}_\pm(w) \to \boldsymbol{\psi}_\pm(w) \cdot \mathbf{c}_\pm^{-1}(w),$$

where $\mathbf{c}_\pm(w)$ are arbitrary nondegenerate $d \times d$ matrix functions holomorphic in Ω_+ and Ω_-, respectively. Thus, not all components of $\mathbf{k}_\pm(w)$ are important in (16), and to reduce this ambiguity we should consider $\mathbf{k}_\pm(w)$ taking the values in Grassmann manifold $\mathbb{G}_{d,2d+n}(\mathbb{C})$. One more restriction on $\mathbf{k}_\pm(w)$ arises from the reality condition (11). It takes the form $\overline{\mathbf{k}_\pm(\bar{w})} = \mathbf{k}_\pm(w)$ in the hyperbolic case $(\epsilon = 1)$ or $\overline{\mathbf{k}_\pm(\bar{w})} = \mathbf{k}_\mp(w)$ in the elliptic case $(\epsilon = -1)$. Just such functions $\mathbf{k}_\pm(w)$ parametrize unambiguously the whole space of the local solutions of (5).

For $\mathbf{k}_\pm(w)$, as the elements of $\mathbb{G}_{d,2d+n}(\mathbb{C})$, it is convenient to use an affine parametrization. Namely, in general, there should exist d linear independent columns of $\mathbf{k}_+(w)$ such that the corresponding d columns of $\mathbf{k}_-(w)$ are also linear independent. Using the global symmetries of our spectral problem we can locate these columns at the first d positions and choose $\mathbf{c}_\pm(w)$ so that the first $d \times d$ blocks of $\mathbf{k}$ reduce to the unit matrix (the exceptional cases for which this can not be done should be considered separately). Thus, in general we obtain

$$\mathbf{k}_\pm(w) = \{\mathbf{I}_d, \mathbf{u}_\pm(w), \mathbf{v}_\pm(w)\}, \tag{17}$$

where $d \times d$ matrix functions $\mathbf{u}_\pm(w)$ and $d \times n$ matrix functions $\mathbf{v}_\pm(w)$ holomorphic in $\Omega_\pm$, respectively, and satisfying the reality conditions, play the role of "coordinates" in the infinite-dimensional space of local solutions.

X. LINEAR SINGULAR INTEGRAL EQUATIONS

In accordance with the well-known theorems of complex analysis, the analytical structure of $\mathbf{\Psi}(\xi, \eta, w)$ allows one to present it and its inverse as the Cauchy-type integrals

$$\mathbf{\Psi} = \mathbf{I} + \frac{1}{i\pi}\int_L \frac{[\mathbf{\Psi}]_\zeta}{\zeta - w} d\zeta, \quad \mathbf{\Psi}^{-1} = \mathbf{I} + \frac{1}{i\pi}\int_L \frac{[\mathbf{\Psi}^{-1}]_\zeta}{\zeta - w} d\zeta, \tag{18}$$

where $[\mathbf{\Psi}]_\zeta$ and $[\mathbf{\Psi}^{-1}]_\zeta$ denote the jumps (i.e. half of the difference of left and right limits) of $\mathbf{\Psi}$ and $\mathbf{\Psi}^{-1}$ at the points $\zeta \in L$. Then the "continuous parts" $\{\mathbf{\Psi}\}_\zeta$ and $\{\mathbf{\Psi}^{-1}\}_\zeta$ (i.e. half of the sums of left and right limits) of $\mathbf{\Psi}$ and $\mathbf{\Psi}^{-1}$ on L are determined by the integrals of the form (18) in which, however, we put $w = \tau \in L$ and the singular integrals should be considered as the Cauchy principal value integrals (Sokhotski-Plemelj formula) [17]. From (14) we obtain at the point $\zeta \in L = L_+ \cup L_-$,

$$\begin{aligned} [\mathbf{\Psi}]_\zeta &= [\lambda^{-1}]_\zeta \boldsymbol{\psi}(\xi, \eta, \zeta) \cdot \mathbf{k}(\zeta), & \{\mathbf{\Psi}\}_\zeta &= \mathbf{M}(\xi, \eta, \zeta), \\ [\mathbf{\Psi}^{-1}]_\zeta &= [\lambda]_\zeta \mathbf{l}(\zeta) \cdot \boldsymbol{\varphi}(\xi, \eta, \zeta), & \{\mathbf{\Psi}^{-1}\}_\zeta &= \mathbf{N}(\xi, \eta, \zeta), \end{aligned} \tag{19}$$

where we use that $\{\lambda_\pm\} \equiv 0$ on $L_\pm$. Using these expressions for $\mathbf{M}$ and $\mathbf{N}$ as Cauchy principal value integrals in the relations $\mathbf{k} \cdot \mathbf{N} = 0$, $\mathbf{M} \cdot \mathbf{l} = 0$, we obtain the relations

$$-\frac{1}{i\pi} \fint_L \frac{[\lambda]_\zeta (\mathbf{k}(\tau) \cdot \mathbf{l}(\zeta))}{\zeta - \tau} \cdot \boldsymbol{\varphi}(\xi, \eta, \zeta) d\zeta = \mathbf{k}(\tau), \tag{20}$$

$$-\frac{1}{i\pi} \fint_L \boldsymbol{\psi}(\xi, \eta, \zeta) \cdot \frac{[\lambda^{-1}]_\zeta (\mathbf{k}(\zeta) \cdot \mathbf{l}(\tau))}{\zeta - \tau} d\zeta = \mathbf{l}(\tau), \tag{21}$$

which are the linear singular integral equations for matrices $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$, provided the monodromy data are given.

The matrix equations (20) and (21) are equivalent to each other, and for constructing solutions it is enough to solve only one of them. For pure technical reasons we choose (20) as the basic one. The theory of such equations is well developed [17], and it allows us to show that the solution of this integral equation [at least, for (ξ, η) close enough to (ξ_o, η_o)] always exists for any choice of the monodromy data $\{\mathbf{u}(w), \mathbf{v}(w)\}$.

XI. CALCULATION OF THE FIELD COMPONENTS

For any given monodromy data and for the corresponding solution $\boldsymbol{\varphi}(\xi, \eta, w)$ of the integral equation (20) the field components can be determined in quadratures. In accordance with (18) and (19), for $w \to \infty$ we have

$$\mathbf{\Psi}^{-1} = \mathbf{I} - w^{-1}\mathbf{R} + \cdots,$$
$$\mathbf{R} = \frac{1}{i\pi}\int_L [\lambda]_\zeta \mathbf{l}(\zeta) \cdot \boldsymbol{\varphi}(\xi, \eta, \zeta) d\zeta,$$

where the $(2d+n)\times(2d+n)$ matrix $\mathbf{R}(\xi, \eta)$ is determined by the integral over $L_+ \cup L_-$. Then we obtain from (8)–(11)

$$\mathbf{U} = 2\partial_\xi \mathbf{R}, \quad \mathbf{V} = 2\partial_\eta \mathbf{R}, \quad \mathbf{W} = \mathbf{W}_o(w) - \mathbf{\Omega} \cdot \mathbf{R} - \mathbf{R}^T \cdot \mathbf{\Omega}.$$

Due to (12), all field components can be calculated algebraically in terms of the components of the matrix $\mathbf{R}$.

XII. A CLASS OF ANALYTICALLY MATCHED MONODROMY DATA

If the initial point (ξ_o, η_o) is chosen on the boundary $\alpha = 0$ (e.g., on the "axis of symmetry") and the monodromy data are "analytically matched," i.e. such that

$$\mathbf{u}_+(w) = \mathbf{u}_-(w) \equiv \mathbf{u}(w), \qquad \mathbf{v}_+(w) = \mathbf{v}_-(w) \equiv \mathbf{v}(w), \tag{22}$$

the corresponding solutions are regular on this axis near the initial point and these monodromy data can be expressed in terms of the axis values of the Ernst potentials:

$$\mathbf{u}(\beta) = -\frac{\mathcal{E}(\beta) - \mathcal{E}(\beta_o)}{2(\beta - \beta_o)}, \quad \mathbf{v}(\beta) = -\frac{\mathcal{A}(\beta) - \mathcal{A}(\beta_o)}{(\beta - \beta_o)}, \tag{23}$$

provided $\mathcal{G}_o = \mathrm{diag}\{\varepsilon_1, \ldots, \varepsilon_d\}$ with $\mathcal{G}_o \cdot \mathcal{G}_o = I_d$. Besides that, for any choice of *rational* functions $\mathbf{u}(w)$ and $\mathbf{v}(w)$,

$$\mathbf{u}(w) = \frac{\mathbf{U}_{N_u}(w)}{Q_{N_q}(w)}, \qquad \mathbf{v}(w) = \frac{\mathbf{V}_{N_v}(w)}{Q_{N_q}(w)},$$

where $Q_{N_q}(w)$ is any scalar and $\mathbf{U}_{N_u}(w)$, $\mathbf{V}_{N_v}(w)$ are arbitrary matrix polynomials, the integral equations (20) reduce to an algebraic system, and the corresponding infinite hierarchies of solutions can be found explicitly.

XIII. 5D MINIMAL SUPERGRAVITY

The equations for the bosonic fields in 5D minimal supergravity determined by the action

$$S^{(5)} = \int \left\{ R^{(5)} * \mathbf{1} - 2 * F \wedge F + \frac{8}{3\sqrt{3}} F \wedge F \wedge A \right\}$$

coincide with $D = 5$ and $n = 1$ bosonic equations of heterotic string gravity (1) if we impose there the constraints $\Phi = 0$ and $H_{ABC} = *F_{ABC}$ with a subsequent rescaling $F_{AB} \to (2/\sqrt{3}) F_{AB}$. In our present context, these constraints are equivalent to the relations

$$\det \mathcal{G} = \epsilon \alpha^2, \quad \tilde{\mathcal{B}}^{ab} = -\epsilon^{abc} \mathcal{A}_c, \quad \mathcal{B}_{ab} = -\epsilon_{abc} \tilde{\mathcal{A}}^c, \tag{24}$$

where ϵ^{abc} is the Levi-Civita symbol. Thus, the space of solutions of 5D minimal supergravity is embedded into the space of solutions of heterotic string gravity and to obtain these solutions we have to impose the appropriate constraints on the choice of monodromy data. To find these constraints for analytically matched data one can use (23); however, a more general kind of monodromy data needs a more complicated analysis.

XIV. SOLUTIONS AND THEIR MONODROMY DATA

We consider here some very simple examples of 5D solutions. The first of them is the 5D Minkowski metric

$$ds^2 = -dt^2 + d\rho_1^2 + d\rho_2^2 + \rho_1^2 d\varphi^2 + \rho_2^2 d\psi^2 \qquad (25)$$

with $\alpha = \rho_1\rho_2$, $\beta = z = \frac{1}{2}(\rho_2^2 - \rho_1^2)$. If the point of normalization is on the "axis" $\rho_2 = 0$ at $\rho_1 = \sqrt{-2z_o}$, the (analytically matched) monodromy data for (25) are

$$\mathbf{u} = \mathrm{diag}\{0, -(2z_o)^{-1}, (2z_o)^{-1}\}, \qquad \mathbf{v} = 0.$$

The four-parametric solution for a charged rotating black hole in 5D minimal supergravity [18] also possesses analytically matched monodromy data with one pole: $\mathbf{u}(w) = \mathbf{u}_0 + \mathbf{u}_1/(w-h)$ and $\mathbf{v}(w) = \mathbf{v}_1/(w-h)$ where the matrices $\mathbf{u}_0$ and $\mathbf{u}_1$, the vector $\mathbf{v}_1$, and the scalar h are real and constant. For simplicity, we show their structure for a nonrotating case, the 5D Reissner-Nordström solution,

$$\mathbf{u}(w) = \begin{pmatrix} \frac{a_0}{w-h} & 0 & 0 \\ 0 & b_0 & c_0 \\ 0 & -c_0 & d_0 \end{pmatrix}, \qquad \mathbf{v}(w) = \begin{pmatrix} \frac{p_0}{w-h} \\ 0 \\ 0 \end{pmatrix},$$

where the parameters a_0, b_0, c_0, d_0, p_0, and h depend on the mass m and charge $q(= sm)$ of a black hole and on the position $z = z_0$ of the point of normalization:

$$a_0 = \frac{m^2 - 4z_0 h}{4z_0^2 - m^2}, \qquad d_0 = \frac{(2z_0 - m)^2 - 8ms^2 z_0}{2(z_0 - h)(4z_0^2 - m^2)},$$

$$b_0 = -\frac{1}{2(z_0 - h)}, \qquad c_0 = \frac{p_0}{z_0 - h},$$

$$p_0 = \frac{ms\sqrt{1+s^2}}{\sqrt{4z_0^2 - m^2}}, \qquad h = m(1+2s^2)/2.$$

XV. CONCLUDING REMARKS

The infinite hierarchies of multiparametric families of solutions which can be constructed explicitly using the monodromy transform approach described above for pure vacuum or heterotic string gravity in D dimensions as well as 5D minimal supergravity include many physically important solutions (such as, e.g., black holes in a 4D case or black holes, black rings, etc. in a 5D case) as well as various their generalizations.

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[1] R. Emparan and H. Reall, Living Rev. Relativity **11**, 6 (2008).
[2] G. A. Alekseev, *Proceedings of the Twelfth Marcel Grossmann Meeting on General Relativity*, edited by T. Damour, R. T. Jantzen, and R. Ruffini (World Scientific, Singapore, 2011) Part A, Plenary and Review Talks, pp. 645–666.
[3] V. A. Belinski and V. E. Zakharov, Sov. Phys. JETP **48**, 985 (1978); **50**, 1 (1979).
[4] G. A. Alekseev, JETP Lett. **32**, 277 (1980).
[5] G. A. Alekseev, Sov. Phys. Dokl. **28**, 17 (1983).
[6] G. A. Alekseev, in *Abstracts of Contributed Papers 13th International Conference on General Relativity and Gravitation*, edited by P. W. Lamberty and O. E. Ortiz (Huerta Grande, Cordoba, Argentina, 1992), p. 3.
[7] G. A. Alekseev and J. B. Griffiths, Phys. Rev. Lett. **84**, 5247 (2000).
[8] R. Meinel, M. Ansorg, A. Kleinwächter, G. Neugebauer, and D. Petroff, *Relativistic Figures of Equilibrium* (Cambridge University Press, Cambridge, England, 2008).
[9] G. A. Alekseev, Theor. Math. Phys. **129**, 1466 (2001).
[10] G. A. Alekseev and J. B. Griffiths, Phys. Rev. Lett. **87**, 221101 (2001).
[11] G. A. Alekseev and J. B. Griffiths, Classical Quantum Gravity **21**, 5623 (2004).
[12] P. Figueras, E. Jamsin, J. V. Rocha, and A. Virmani, Classical Quantum Gravity **27**, 135011 (2010).
[13] G. A. Alekseev, Phys. Rev. D **80**, 041901(R) (2009).
[14] G. A. Alekseev, Sov. Phys. Dokl. **30**, 565 (1985).
[15] G. A. Alekseev, Trudy Mat. Inst. Steklov. **176**, 211 (1987)
[Proc. Steklov Inst. Math. **3**, 215 (1988)].
[16] G. A. Alekseev, Theor. Math. Phys. **143**, 720 (2005).
[17] F. D. Gakhov, *Boundary Problems* (in Russian) (Nauka, Moscow, 1977), 3rd ed.; *Boundary Value Problems* (English translation) (Pergamon, Oxford, 1966), 2nd ed.
[18] M. Cvetič and D. Youm, Nucl. Phys. **B476**, 118 (1996).