Probing deconfinement with Polyakov loop susceptibilities

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The susceptibilities of the real and imaginary parts, as well as of the modulus, of the Polyakov loop, are computed in SU(3) lattice gauge theory. We show that the ratios of these susceptibilities are excellent probes of the deconfinement transition, independent of the renormalization of the Polyakov loop and only weakly dependent on the system size. The ratios are almost temperature independent above and below the transition and exhibit a discontinuity at the transition temperature. This characteristic behavior can be understood in terms of the global Z_3 symmetry of the Yang-Mills Lagrangian and the general properties of the Polyakov loop probability distribution.

Systems described by a pure SU(N_c) gauge theory in (d + 1) dimensions undergo a phase transition at finite temperature. Owing to the conjectured universality with d-dimensional Z_{N_c} spin systems [1], this transition is of general interest.

For $N_c = 3$, the transition is first order and is characterized by spontaneous breaking of the global Z_3 center symmetry of the Yang-Mills Lagrangian [2–6].

The Polyakov loop, which is linked to the free energy of a static quark immersed in a hot gluonic medium [7,8], can be used to define an order parameter of the deconfinement transition. At low temperatures its thermal expectation value vanishes, implying color confinement, while at high temperatures it is nonzero, resulting in a finite energy of a static quark and consequently the deconfinement of color. While the basic thermodynamic functions of SU(3) pure gauge theory are well established within the lattice approach [2–6,9–12], the situation is less satisfactory for the renormalized Polyakov loop and, in particular, for the corresponding susceptibilities.

In a pure SU(3) gauge theory, the temperature of the confinement-deconfinement transition is uniquely defined by the discontinuity of the order parameter, since the transition is first order. More generally, for systems where the transition is continuous, e.g. QCD, the transition temperature is identified by a maximum of the fluctuations, quantified e.g. by one of the Polyakov loop susceptibilities. For $N_c \geq 3$, the Polyakov loop operator is complex-valued. Correspondingly, one can define susceptibilities of the real and imaginary parts as well as of the modulus of the Polyakov loop.

In a Yang-Mills theory, formulated on the lattice, the ultraviolet divergence of the bare quark-antiquark free energy implies that, in the continuum limit, the bare Polyakov loop vanishes at any temperature. Thus, in order to obtain a physically meaningful continuum limit, the Polyakov loop must be renormalized [13,14]. The renormalization of gluon correlation functions in general, and the Polyakov loop susceptibility in particular, are still subject to uncertainties.

In this paper we bypass these ambiguities by considering the ratios of Polyakov loop susceptibilities. In particular, we focus on their properties near the deconfinement transition. To this end, we compute the temperature dependence of the Polyakov loop susceptibilities within SU(3) lattice gauge theory on different-sized lattices and examine the relevance of the susceptibility ratios as probes of the deconfinement transition.

We argue that these characteristics are naturally understood in terms of the global Z_3 symmetry and the general properties of the Polyakov loop probability distribution. Moreover, they are independent of the renormalization of the Polyakov loop and depend only weakly on the volume. This implies that the susceptibility ratios are excellent observables for identifying the confinement-deconfinement phase transition in SU(3) pure gauge theory.

The Polyakov loop susceptibilities on the lattice.—On an $N_\sigma^3 \times N_\tau$ lattice, the Polyakov loop is defined as the trace of the product over temporal gauge links,

$$L_{\vec{x}}^{\text{bare}} = \frac{1}{N_c} \text{Tr} \prod_{\tau=1}^{N_\tau} U_{(\vec{x},\tau),4}, \qquad (1)$$

$$L^{\text{bare}} = \frac{1}{N_\sigma^3} \sum_{\vec{x}} L_{\vec{x}}^{\text{bare}}. \qquad (2)$$

Due to the Z_3 symmetry of the pure gauge action, this quantity vanishes, when averaged over all gauge field configurations. Furthermore the Polyakov loop is strongly N_τ dependent and must be renormalized.
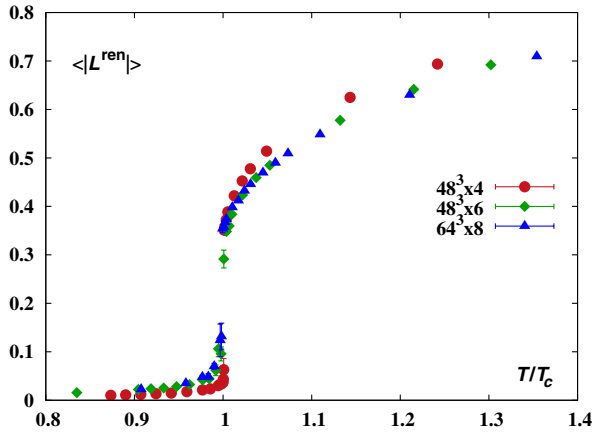
FIG. 1 (color online). The modulus of the renormalized Polyakov loop $\langle |L^{\mathrm{ren}}| \rangle$ obtained in SU(3) lattice gauge theory.

These problems are avoided by considering the renormalized Polyakov loop [13],

$$L^{\mathrm{ren}} = (Z(g^2))^{N_\tau} L^{\mathrm{bare}}, \qquad (3)$$

and introducing the ensemble average of the modulus thereof, $\langle |L^{\mathrm{ren}}| \rangle$. The latter is well defined in the continuum and thermodynamic limits and is an order parameter for the spontaneous breaking of the Z_3 center symmetry. The lattice gauge theory result for $\langle |L^{\mathrm{ren}}| \rangle$, as a function of temperature, is shown in Fig. 1.

As noted above, the location of the phase transition is correlated with a maximum (or divergence) of the fluctuations of the order parameter. For the confinement-deconfinement transition, these fluctuations are reflected in the renormalized Polyakov loop susceptibility,[1]

$$T^3 \chi_A = \frac{N_\sigma^3}{N_\tau^3} (\langle |L|^2 \rangle - \langle |L| \rangle^2). \qquad (4)$$

In the SU(3) gauge theory, the Polyakov loop operator is complex. Consequently, in addition to χ_A, one can also explore independent fluctuations of the real and imaginary parts of the Polyakov loop. Taking the Z_3 symmetry into account, we define a longitudinal and a transverse susceptibility,[2]

$$T^3 \chi_L = \frac{N_\sigma^3}{N_\tau^3} [\langle (L_L)^2 \rangle - \langle L_L \rangle^2], \qquad (5)$$

$$T^3 \chi_T = \frac{N_\sigma^3}{N_\tau^3} [\langle (L_T)^2 \rangle - \langle L_T \rangle^2], \qquad (6)$$

where $L_L = \mathrm{Re}(\tilde{L})$ and $L_T = \mathrm{Im}(\tilde{L})$. Here we have introduced the Z_3 transformed Polyakov loop, $\tilde{L} = L e^{2\pi n i/3}$,

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[1] In the following we deal only with the renormalized Polyakov loop, and hence drop the superscript on L^{ren}.
[2] There is no mixing between longitudinal and transverse susceptibilities.

with $n = 0, \pm 1$. The phase of the transformation is chosen such that for $T > T_c$ the expectation value of the transformed Polyakov loop, $\langle \tilde{L} \rangle$, is real. For $T < T_c$ the expectation value of the Polyakov loop vanishes, and we take $n = 0$. Thus, in the latter case $L_L = \mathrm{Re}(L)$ and $L_T = \mathrm{Im}(L)$.

We have computed the Polyakov loop susceptibilities, Eqs. (4)–(6), within SU(3) lattice gauge theory, using the (1,2)-tree-level Symanzik improved gauge action on an $N_\sigma^3 \times N_\tau$ lattice. We consider lattices of temporal size $N_\tau = 4$, 6 and 8 and spatial extent N_σ varying from 16 to 64. However, in order to make the figures more transparent, we show results only for the largest volumes, i.e. for $N_\sigma = 48$ and 64. We set the temperature for the three temporal lattice extents by varying the bare coupling and use the temperature scale determined by the zero-temperature string tension, as well as the critical couplings of the deconfinement transition [10,15]. The gauge field configurations were generated using one heatbath and four overrelaxation updates per sweep with 15 000 sweeps in general and up to 100 000 sweeps close to the critical temperature, T_c.

The ratios of susceptibilities.—In Figs. 2 and 3 we show SU(3) lattice gauge theory results for the ratios $R_A = \chi_A / \chi_L$ and $R_T = \chi_T / \chi_L$, as functions of temperature. Since renormalization as well as volume and temperature factors cancel in these ratios, they provide robust probes of the deconfinement transition. Indeed, both ratios exhibit a strong discontinuity at the deconfinement phase transition and are almost temperature and volume independent. A straightforward interpretation of the properties of R_A and R_T is obtained by using general considerations and the Z_3 center symmetry.

Consider first the ratio R_T for $T < T_c$. In the Z_3-symmetric phase, the expectation value of any symmetry-breaking operator, e.g. $\tilde{L}$ or $\tilde{L}^2$, must vanish. Hence,
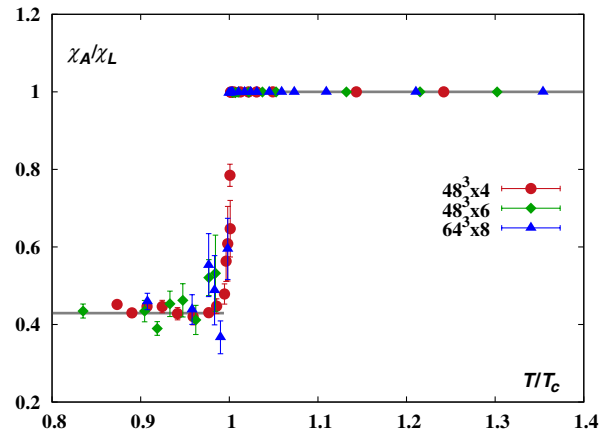

FIG. 2 (color online). The ratio of the modulus χ_A and longitudinal χ_L Polyakov loop susceptibilities obtained in SU(3) lattice gauge theory. The lines shows $\chi_A / \chi_L = 1$ for $T > T_c$ and $\chi_A / \chi_L = 2 - \pi/2$ for $T < T_c$ (see text).
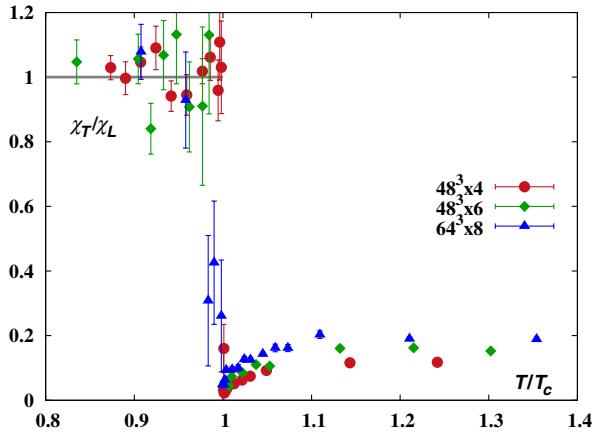
FIG. 3 (color online). The ratio of the transverse χ_T and longitudinal χ_L Polyakov loop susceptibilities. The line shows $\chi_T/\chi_L = 1$ for $T < T_c$ (see text).

$$V(\langle \tilde{L}^2 \rangle - \langle \tilde{L} \rangle^2) = \chi_L - \chi_T = 0, \qquad (7)$$

which implies that $\chi_L = \chi_T$. Since χ_L and χ_T are both nonzero, it follows that $R_T = 1$, as shown in Fig. 3.

The fact that $R_A \simeq 1$ in the deconfined phase, as shown in Fig. 2, follows from the following argument. In the broken-symmetry phase, we introduce shifted operators δL_L and δL_T,

$$L_L = L_0 + \delta L_L, \qquad (8)$$

$$L_T = \delta L_T, \qquad (9)$$

where $L_0 = \langle \tilde{L} \rangle = \langle L_L \rangle$ is the (real) expectation value of the transformed Polyakov loop and the shifted operators δL_i, with $i = (L, T)$, describe the fluctuations about the mean. The thermal average of the shifted operator squared yields the corresponding susceptibility, $V\langle (\delta L_i)^2 \rangle = \chi_i$, $i = L, T$. We then expand the modulus of the Polyakov loop $|L| = |\tilde{L}|$ in the shifted operators,

$$|L| = \sqrt{L_L^2 + L_T^2} \approx L_0 \left(1 + \frac{\delta L_L}{L_0} + \frac{1}{2} \frac{(\delta L_T)^2}{L_0^2} \right). \qquad (10)$$

Using the fact that by definition $\langle \delta L_i \rangle = 0$, we find

$$\langle |L| \rangle \simeq L_0 \left(1 + \frac{1}{2} \frac{\langle (\delta L_T)^2 \rangle}{L_0^2} \right), \qquad (11)$$

while

$$\langle |L|^2 \rangle = L_0^2 \left(1 + \frac{\langle (\delta L_T)^2 \rangle}{L_0^2} + \frac{\langle (\delta L_L)^2 \rangle}{L_0^2} \right). \qquad (12)$$

This implies that to leading order in the expansion of $|L|$, $\chi_A \simeq \chi_L$ and hence $R_A \simeq 1$ for $T > T_c$, as shown in Fig. 2.

The properties of R_T in the deconfined and R_A in the confined phase cannot be directly linked to the center symmetry. Using Eq. (11), we find that

$$\chi_T \simeq V(\langle |L| \rangle^2 - \langle L_L \rangle^2). \qquad (13)$$

Thus, in general, χ_T can be nonvanishing in the Z_3-broken phase. However, its value in the high-temperature phase is not constrained by symmetries or general principles.

In Fig. 3 we show that, above the phase transition, χ_T is in fact much smaller than χ_L. In the temperature range considered, we find that, for $T > T_c$, the ratio R_T is weakly dependent on the temperature and does not exceed $\simeq 0.2$. It has been argued [16,17] that in the broken Z_3 symmetry phase of the SU(3) gauge theory, R_T can be as large as 0.4. We note that in our results, a dependence of R_T on N_τ remains. Hence, we cannot at present draw firm conclusions on the continuum extrapolation of this quantity.

Finally, we turn to the value of R_A in the confined phase. In Fig. 2 we show that, for $T < T_c$, R_A is approximately temperature independent, with the lattice results clustering around a value slightly larger than 0.4. This property of R_A can be understood by assuming that in the symmetric phase, the probability distribution for the Polyakov loop is, to a good approximation, Gaussian, with the partition function[3]

$$Z = \int dL_L dL_T e^{-VT^3[\alpha(T)(L_L^2 + L_T^2)]}, \qquad (14)$$

where the integrations extend from $-\infty$ to ∞. The susceptibilities are then obtained by performing elementary integrals

$$\chi_L = \frac{1}{2\alpha T^3}, \quad \chi_T = \frac{1}{2\alpha T^3}, \quad \chi_A = \frac{1}{2\alpha T^3} \left(2 - \frac{\pi}{2} \right). \qquad (15)$$

Consequently, $R_A = (2 - \pi/2) \simeq 0.429$, in good agreement with the lattice results, shown in Fig. 2. We note that the Gaussian approximation is not expected to be valid close to T_c, where the coefficient $\alpha(T)$ in Eq. (14) is small and hence higher-order terms cannot be neglected.

In SU(2) gauge theory the Polyakov loop is real, so the corresponding integrals are one-dimensional, which implies a slightly different ratio, $R_A^{SU(2)} = (1 - 2/\pi) \simeq 0.363$. This value is indeed in agreement with lattice results for the SU(2) Polyakov loop susceptibilities below T_c, outside of the critical region [18]. These results indicate that in the symmetric phase, the effective Polyakov loop potential is well approximated by a Gaussian both in SU(2) and SU(3) lattice gauge theories.

Conclusions.—We have shown that the ratios of Polyakov loop susceptibilities provide an excellent signal for the deconfinement phase transition in SU(3) gauge theory. The ratios are discontinuous at the phase transition and only weakly temperature dependent on either side of T_c. Moreover, they are independent of the Polyakov loop renormalization and only weakly dependent on the system size.

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[3]More precisely, the quadratic terms of the effective action are responsible for the dominant contribution to the Polyakov loop susceptibility. However, higher-order, non-Gaussian terms are decisive for the determination of higher-order cumulants.

We have also shown that, with one exception, the ratios obtained outside of the transition region can be understood in terms of general symmetry arguments and the observation that, in the confined phase, the Polyakov loop probability distribution is well approximated by a Gaussian. There is, however, no restriction by symmetry on the ratio of the transverse to longitudinal susceptibility, χ_T/χ_L, in the deconfined phase. We find that, above T_c this ratio is fairly small and varies weakly with temperature.

In QCD, the global $\mathcal{Z}_3$ symmetry is explicitly broken by finite quark masses. Hence, the properties of the susceptibility ratios in QCD can differ from those in pure gauge theory. In particular, the discontinuity will most likely be smoothened, since in QCD the transition is continuous. Nevertheless, outside of the transition region, the ratios may approximately reflect the constraints from center symmetry and thus also provide a useful probe of the confinement-deconfinement transition in full QCD.

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