Decay constants of heavy-light vector mesons from QCD sum rules

P. Gelhausen, A. Khodjamirian, A. A. Pivovarov, and D. Rosenthal

Theoretische Physik 1, Naturwissenschaftlich-Technische Fakultät, Universität Siegen, D-57068 Siegen, Germany

We revisit QCD sum rules for the decay constants of heavy-light mesons. In the sum rules for the vector mesons $B^*_{(s)}$ and $D^*_{(s)}$ we improve the accuracy of OPE, taking into account the $O(\alpha_s^2)$ terms in the perturbative part and calculating the $O(\alpha_s)$ corrections to the quark-condensate contribution. With this accuracy, we obtain the ratios of decay constants: $f_{B^*}/f_B = 1.02^{+0.07}_{-0.03}$, $f_{D^*}/f_D = 1.20^{+0.10}_{-0.07}$. The sum rule predictions for the decay constants of pseudoscalar mesons are updated with the results $f_B = (207^{+17}_{-9})$ MeV, $f_{B_s} = (242^{+17}_{-12})$ MeV, $f_D = (201^{+12}_{-13})$ MeV, $f_{D_s} = (238^{+13}_{-23})$ MeV. In order to assess the sensitivity of our calculation to the form of the sum rule, we consider alternative versions such as the power moments and Borel sum rules with different weights of the spectral density. We also investigated the heavy-quark limit of the sum rules for vector and pseudoscalar mesons, estimating the violations of the heavy-quark spin and flavor symmetry.

I. INTRODUCTION

The decay constants of heavy-light mesons are the simplest hadronic matrix elements relevant for heavy-flavor physics. The constants of pseudoscalar mesons, $f_{D_{(s)}}$ and f_B, received much attention because they determine the long-distance QCD dynamics in the leptonic weak decays $D_{(s)} \to \mu\nu_\mu$ and $B \to \tau\nu_\tau$, respectively (see, e.g., the review [1] and a more recent measurement [2]). The accuracy of $f_{B_{s(d)}}$ is vital for the analysis of the leptonic flavor-changing neutral current decay $B_{s(d)} \to \mu^+\mu^-$ [3]. Recent determinations of the decay constants of B and D mesons in lattice QCD with dynamical flavors quote an impressive precision [4–7].

The decay constants f_{B^*} and f_{D^*} of the heavy-light vector mesons B^* and D^* cannot be directly probed in weak decays. Nonetheless, they also play an important role in heavy flavor phenomenology. To bring only one example: an accurate knowledge of f_{B^*} and f_{D^*} is needed in the calculation of the strong couplings $B^*B\pi$ and $D^*D\pi$ from QCD light-cone sum rules [8]. Furthermore, the deviation of $f_{B^*(D^*)}$ from $f_{B(D)}$ calculated at finite masses allows one to assess the violation of the heavy-quark spin symmetry. In lattice QCD, we are aware of only one recent calculation of f_{D^*} and $f_{D^*_s}$ in [9].

In continuum QCD, the hadronic decay constants are determined from QCD (SVZ) sum rules [10] based on the operator-product expansion (OPE) for the two-point correlation functions of quark currents. The sum rules for heavy-light mesons have a long history, starting from the very early papers [11,12] (see also, e.g., [13,14]) in the framework of full QCD, as well as using the heavy-quark expansion [15], followed by sum rules in heavy-quark effective theory (HQET) [16–19]. The gluon radiative corrections to the correlation functions at $O(\alpha_s)$ (two-loop) level calculated in [20,21] play an important role. The sum rule for the heavy-light vector mesons with this accuracy can be found, e.g., in [22].

A substantial improvement of the OPE was achieved in [23] where the $O(\alpha_s^2)$ (three-loop) contribution to the perturbative part of the correlation function was calculated. The results are available in a seminumerical form. These NNLO corrections were implemented in the sum rule determination of the $B_{(s)}$ decay constants for finite heavy-quark masses in [24], and in the framework of HQET in [25]. Note that the $\overline{\text{MS}}$ scheme chosen for the heavy-quark mass in [24] leads to a reasonable suppression of gluon radiative corrections in the perturbative part of the sum rules.

In this paper we revisit QCD sum rules for heavy-light meson decay constants. Our main goal is to upgrade the accuracy of the sum rules for the vector-meson decay constants f_{B^*} and f_{D^*}, including the three-loop, $O(\alpha_s^2)$ terms in the OPE for the correlation function and calculating the missing $O(\alpha_s)$ corrections to the quark condensate contribution. In parallel, we update the decay constants of pseudoscalar heavy-light mesons. We also evaluate the ratios of decay constants, sensitive to the violation of the heavy-quark and $SU(3)_{\text{fl}}$ symmetries in bottom and charmed mesons. In addition to the standard sum rule calculation, we obtain upper bounds for decay constants following from the positivity of the spectral density in the dispersion relations and independent of the quark-hadron duality assumption. Furthermore, to assess the sensitivity of our results to the particular form of the sum rule with Borel exponent, we employ certain modifications of sum rules with different weights in the dispersion integrals, as well as their power moments.

In what follows, in Sec. II we present a brief outline of the method. Section III contains a discussion of the input parameters and the numerical results. Section IV is devoted to the alternative sum rules and contains their comparison with the conventional Borel sum rules. In Sec. V we consider the heavy-quark limit of the sum rules. Our concluding discussion is presented in Sec. VI.

II. OUTLINE OF THE SUM RULE DERIVATION

Here we remind the main steps leading to the standard QCD sum rule [10] for a decay constant. For the heavy-light vector mesons, which are the main objects of our study, one needs the following two-point correlation function:

$$\Pi_{\mu\nu}(q) = i \int d^4x e^{iqx} \langle 0|T\{j_\mu(x)j_\nu^\dagger(0)\}|0\rangle$$
$$= (-g_{\mu\nu}q^2 + q_\mu q_\nu)\tilde{\Pi}_T(q^2) + q_\mu q_\nu \Pi_L(q^2),$$
$$(1)$$

where $j_\mu = \bar{q}\gamma_\mu Q$ is the interpolating heavy-light quark current, $Q = c, b$ and $q = u, d, s$. In this paper finite quark masses in $\overline{\text{MS}}$ scheme are considered. In (1), the relevant invariant amplitude is $\tilde{\Pi}_T(q^2)$, multiplying the transverse kinematic structure. To avoid kinematical singularities, it is more convenient to rescale it, introducing

$$\Pi_T(q^2) \equiv q^2 \tilde{\Pi}_T(q^2), \qquad (2)$$

that is, to consider the coefficient at $-g_{\mu\nu}$ in $\Pi_{\mu\nu}(q)$. In what follows, we also need the standard definition of the correlation function with pseudoscalar heavy-light currents:

$$\Pi_5(q) = i \int d^4x e^{iqx} \langle 0|T\{j_5(x)j_5^\dagger(0)\}|0\rangle, \qquad (3)$$

where $j_5 = (m_Q + m_q)\bar{q}i\gamma_5 Q$.

The decay constants of the heavy-light vector meson $H^* = \{B^*, D^*\}$ and pseudoscalar meson $H = \{B, D\}$ are defined in a standard way,

$$\langle 0|j_\mu|H^*(q)\rangle = m_{H^*}\epsilon_\mu^{(H^*)}f_{H^*}, \qquad \langle 0|j_5|H(q)\rangle = m_H^2 f_H,$$
$$(4)$$

where $\epsilon_\mu^{(H^*)}$ is the polarization vector of H^*. These decay constants squared enter the ground-state pole terms of the hadronic dispersion relations for the correlation functions (1) and (3). The invariant amplitudes defined according to (2) and (3) obey double-subtracted dispersion relations in the variable q^2:

$$\Pi_{T(5)}(q^2) - \Pi_{T(5)}(0) - q^2\left(\frac{d\Pi_{T(5)}(q^2)}{dq^2}\right)\bigg|_{q^2=0}$$
$$= \frac{(q^2)^2}{\pi}\int ds \frac{\text{Im}\Pi_{T(5)}(s)}{s^2(s-q^2)}. \qquad (5)$$

The spectral densities,

$$\rho_T(s) \equiv \frac{1}{\pi}\text{Im}\Pi_T(s)$$
$$= m_{H^*}^2 f_{H^*}^2 \delta(s - m_{H^*}^2) + \rho_T^h(s)\theta(s - (m_H + m_P)^2),$$
$$(6)$$

$$\rho_5(s) \equiv \frac{1}{\pi}\text{Im}\Pi_5(q)$$
$$= m_H^4 f_H^2 \delta(s - m_H^2) + \rho_5^h(s)\theta(s - (m_{H^*} + m_P)^2),$$
$$(7)$$

are positive definite and, according to unitarity, are given by the sums over all hadronic contributions with the quantum numbers of H^* and H, respectively. In the above, the ground-state contribution is written separately and a generic notation $\rho_{T(5)}^h(s)$ is introduced for the spectral density of excited and continuum states; m_P is the mass of the lightest pseudoscalar meson, π or K, depending on the flavor content of $H^{(*)}$. Note that in the pseudoscalar channel there is a gap between the ground state H and the threshold $m_{H^*} + m_P$ of the lowest continuum state, whereas in the vector $D^*(B^*)$ channel the threshold $m_H + m_P$ lies below (above but very close to) the ground state H^*. Note also that the vector mesons with strangeness D_s^* and B_s^* are strongly coupled to DK and BK continuum states, respectively, both having larger thresholds than the vector meson masses, whereas the channels $D_s\pi$ and $B_s\pi$ are decoupled in the limit of isospin symmetry.

In QCD, the local OPE for the correlation function (1) is valid at $q^2 \ll m_Q^2$, far from hadronic thresholds. This expansion includes a perturbative part and contributions of the vacuum condensates. The latter, ordered according to their operator dimension d, are taken into account up to $d = 6$:

$$\Pi_{T(5)}^{\text{OPE}}(q^2) = \Pi_{T(5)}^{\text{(pert)}}(q^2) + \Pi_{T(5)}^{\langle\bar{q}q\rangle}(q^2) + \Pi_{T(5)}^{\langle GG\rangle}(q^2)$$
$$+ \Pi_{T(5)}^{\langle\bar{q}Gq\rangle}(q^2) + \Pi_{T(5)}^{\langle\bar{q}q\bar{q}q\rangle}(q^2), \qquad (8)$$

where the contributions of the quark, gluon, quark-gluon, and four-quark condensates with dimensions $d = 3, 4, 5, 6$ are indicated by the indices $\langle\bar{q}q\rangle$, $\langle GG\rangle$, $\langle\bar{q}Gq\rangle$ and $\langle\bar{q}q\bar{q}q\rangle$, respectively. The contributions of $d = 4, 5, 6$ condensates are very small in the region where we consider OPE; hence it is justified to neglect the terms of OPE stemming from the condensates of larger dimension. The perturbative part of the OPE is usually represented in a form of a dispersion integral:

$$\bar{\Pi}_{T(5)}^{\text{(pert)}}(q^2) = (q^2)^2 \int_{(m_Q+m_q)^2}^\infty ds \frac{\rho_{T(5)}^{\text{(pert)}}(s)}{s^2(s-q^2)}. \qquad (9)$$

Here, as in (5), the two subtractions are needed due to the $s \to \infty$ asymptotics of the leading order (LO) perturbative contributions:

$$\rho_T^{\text{(pert,LO)}}(s) = \frac{1}{8\pi^2}s(1-z)^2(2+z), \qquad (10)$$

$$\rho_5^{\text{(pert,LO)}}(s) = \frac{3(m_Q + m_q)^2}{8\pi^2}s(1-z)^2, \qquad (11)$$

given by the simple heavy-light loop diagrams, where $z = \frac{m_Q^2}{s}$. In the above, for simplicity, the light-quark masses are neglected, except in the factor $(m_Q + m_q)$ for the pseudoscalar-current density. The relevant mass corrections are presented in Appendix A 1.

The perturbative part of OPE includes NLO (two-loop) and NNLO (three-loop) terms:

$$\rho_{T(5)}^{(\text{pert})}(s) = \rho_{T(5)}^{(\text{pert,LO})}(s) + \left(\frac{\alpha_s}{\pi}\right)\rho_{T(5)}^{(\text{pert,NLO})}(s)$$
$$+ \left(\frac{\alpha_s}{\pi}\right)^2 \rho_{T(5)}^{(\text{pert,NNLO})}(s). \qquad (12)$$

The NLO corrections are known from early papers [20,21] and are determined by a sum of three two-loop diagrams originating from gluon exchanges in the heavy-light quark loop. We have once more recalculated these diagrams for vector- and pseudoscalar-current correlation functions in both pole- and $\overline{\text{MS}}$-mass (our default) scheme and confirmed the results of previous calculations [20,21], given for pseudoscalar currents, e.g., in [24] and for vector currents in [22]. The analytical formulas for the NLO gluon radiative corrections are presented in Appendix A 1.

The NNLO corrections were calculated in [23] and implemented in the sum rule determination of $f_{B_{(s)}}$ in [24] and $f_{B,D}$ in [25]. In this paper, we include these corrections also in the OPE of the vector-current correlation function. To this end, we follow the α_s expansion of the invariant amplitude $\Pi^v(q^2)$ introduced in [23], which in our notation corresponds to $\tilde{\Pi}_T(q^2) = \Pi_T(q^2)/q^2$. For the three-loop, $O(\alpha_s^2)$ term in this expansion we make use of the color-structure decomposition given in Eq. (8) of [23]. For the imaginary parts of separate contributions we employ the formulas obtained from the seminumerical Pade procedure and encoded in the program Rvs.m made available by the authors of Ref. [23].

In the OPE (8) the contribution of the $d = 3$ quark condensate includes LO and NLO [one-loop, $O(\alpha_s)$] contributions. The latter for the pseudoscalar-current correlation function were calculated in [16,24]. We repeated this calculation and confirm their result. The NLO correction to the quark-condensate term in the vector-current correlation function is new. Finally, for the $d = 4$, 5, 6 contributions of gluon, quark-gluon and four-quark condensates to OPE, we use the LO expressions known from the literature [10,12,22]. For completeness, these expressions are also collected in Appendix A 2.

In the case of nonstrange heavy-light mesons, the u, d quark masses are neglected everywhere in the correlation function, being negligibly small numerically in comparison with all other energy-momentum scales. In the sum rules for the strange heavy-light mesons the s quark mass is taken into account in the prefactor $(m_Q + m_s)^2$ of Π_5. Apart from that, we include the $O(m_s^2)$ terms in the LO [$O(m_s)$ in the NLO] perturbative part and the $O(m_s)$ terms

in the LO quark-condensate contribution in both correlation functions (see the expressions in Appendix A 1).

Substituting the OPE result (8) in the hadronic dispersion relation (5), we perform the standard Borel transformation, after which the subtraction terms vanish. One has

$$\Pi_T^{\text{OPE}}(M^2) = m_{H^*}^2 f_{H^*}^2 e^{-m_{H^*}^2/M^2}$$
$$+ \int_{(m_H+m_P)^2}^{\infty} ds \rho_T^h(s) e^{-s/M^2} \qquad (13)$$

and a similar relation for the pseudoscalar meson case. The truncated OPE on the lhs is reliable at $M^2 > M_{\text{min}}^2$, where the $d = 4$, 5, 6 condensate contributions remain sufficiently small numerically. On the other hand, below a certain upper boundary $M^2 < M_{\text{max}}^2$ the contributions of higher states accumulated in the integral on the rhs in (13) are suppressed with respect to the ground-state term due to the Borel exponent. As usual, we apply the quark-hadron duality approximation for the hadronic spectral density $\rho_T^h(s)$ and replace the integral on the rhs of (13) by the integral over the perturbative spectral density, introducing an effective threshold:

$$\rho_T^h(s)\theta(s - (m_H + m_P)^2) = \rho_T^{(\text{pert})}(s)\theta(s - s_0^{H^*}). \quad (14)$$

After subtracting the part of the dispersion integral over the perturbative spectral density from both sides of (13), the final form of QCD sum rule reads

$$f_{H^*}^2 = \frac{e^{m_{H^*}^2/M^2}}{m_{H^*}^2}\{\Pi_T^{(\text{pert})}(M^2, s_0^{H^*}) + \Pi_T^{\langle\bar{q}q\rangle}(M^2)$$
$$+ \Pi_T^{(d456)}(M^2)\}. \qquad (15)$$

The analogous sum rule for the pseudoscalar meson channel is

$$f_H^2 = \frac{e^{m_H^2/M^2}}{m_H^4}\{\Pi_5^{(\text{pert})}(M^2, s_0^H) + \Pi_5^{\langle\bar{q}q\rangle}(M^2)$$
$$+ \Pi_5^{(d456)}(M^2)\}. \qquad (16)$$

In above equations the notation

$$\Pi_{T(5)}^{(\text{pert})}(M^2, s_0) = \int_{(m_Q+m_q)^2}^{s_0} ds e^{-s/M^2} \rho_{T(5)}^{(\text{pert})}(s), \qquad (17)$$

$$\Pi_{T(5)}^{(d456)}(M^2) = \Pi_{T(5)}^{\langle GG\rangle}(M^2) + \Pi_{T(5)}^{\langle\bar{q}Gq\rangle}(M^2) + \Pi_{T(5)}^{\langle\bar{q}q\bar{q}q\rangle}(M^2), \qquad (18)$$

is used for brevity. In (17) the α_s expansion (12) of the perturbative spectral density is then substituted. In terms of (17), the sum over excited states and continuum contributions is simply equal to $\Pi_{T(5)}^{(\text{pert})}(M^2, \infty) - \Pi_{T(5)}^{(\text{pert})}(M^2, s_0)$.

A specific "Borel window" is usually adopted for each sum rule, defined as an interval $M_{\text{min}}^2 < M^2 < M_{\text{max}}^2$ where the higher-dimensional condensate terms in OPE and the

excited and continuum contributions are suppressed simultaneously. The actual size of this interval depends on the quantum numbers of quark currents in the correlation function. The effective threshold is usually fixed by fitting the mass of the ground-state meson calculated from the sum rule to its experimentally measured value. To obtain an equation for the meson mass squared $m_{H^*}^2$ (m_H^2) from the sum rule (15) [(16)], it is sufficient to differentiate the rhs of each sum rule over $(-1/M^2)$ and formally equate the result to zero.

Apart from the sum rule, the dispersion relation for the invariant amplitude $\Pi_{T(5)}^{QCD}$, after Borel transformation, yields an upper bound for the decay constant which simply follows from the positivity of the hadronic spectral density and is independent of the quark-hadron duality approximation (for earlier uses of the bounds see, e.g., [14,26]). Formally, the upper bounds for $f_{H^{(*)}}^2$ are obtained, putting $s_0^{H^{(*)}} \to \infty$ in (15) and (16).

III. NUMERICAL ANALYSIS OF BOREL SUM RULES

In Table I the adopted intervals of QCD parameters entering the sum rules (15) and (16) are collected. As is well known, these sum rules are very sensitive to the value of the heavy-quark mass. For a correlation function of highly virtual quarks in full QCD involved in our calculation, we use, as in [24], the quark masses in the $\overline{MS}$ scheme. Currently, there is a good agreement between various lattice- and continuum-QCD determinations of b and c quark masses; hence we simply take the intervals of their $\overline{MS}$ values from [27]. In particular, the heavy-quark masses extracted from the QCD quarkonium sum rules [30,31] are very close to the average values we are using. For the strange quark mass we double the theoretical uncertainty quoted in [27], having in mind that the latter uncertainty is dominated by the most recent lattice determinations. With our choice, the interval of continuum QCD determinations of m_s (e.g. from the QCD sum rules [32]) with typically

larger errors is also covered. The nonstrange quark condensate density $\langle 0|\bar{q}q|0\rangle \equiv \langle \bar{q}q\rangle$, $(q = u, d)$, is calculated from the chiral perturbation theory (ChPT) relations derived in [28], using the s quark mass as an input. The details can be found, e.g., in [33]. Note that the $SU(3)_{fl}$ symmetry violation in OPE originates not only from the quark mass difference $m_s - m_{u,d}$, but also from the difference between strange and nonstrange quark-condensate densities. For their ratio we adopt a rather broad interval from the review [29], where also the $d = 4$, 5, 6 condensate densities are taken from. The latter are parametrized in a standard way: $(\alpha_s/\pi)\langle 0|G_{\mu\nu}^a G^{a\mu\nu}|0\rangle \equiv \langle GG\rangle$, $\langle 0|g_s\bar{q}G_{\mu\nu}^a t^a \sigma^{\mu\nu}q|0\rangle \equiv \langle \bar{q}Gq\rangle = m_0^2\langle \bar{q}q\rangle$. Furthermore, following [10], the four-quark condensate density is factorized with an intermediate vacuum insertion into the square of quark condensates $r_{vac}\langle \bar{q}q\rangle^2$, with an additional coefficient r_{vac}, parameterizing the deviation from the factorization.

For the running of the QCD coupling and quark masses with an appropriate loop accuracy we employ the numerical program RunDec available from [34]. We adopt a uniform renormalization scale μ for the correlation function, strong coupling and quark masses. The running of the quark-condensate density is taken into account in the same approximation as for the quark masses; the gluon-condensate density is renorminvariant and the running of quark-gluon and four-quark condensates is negligible, i.e., their densities $m_0^2\langle \bar{q}q\rangle$ and $\alpha_s r_{vac}\langle \bar{q}q\rangle^2$, respectively, are taken at a low scale $\mu = 1$ GeV.

The choice of the three remaining parameters—Borel mass M, renormalization scale μ and effective threshold $s_0^{H^{(*)}}$—generally depends on the quantum numbers of quark currents in the correlation function. In our analysis, for all mesons containing one and the same heavy quark, b or c, a uniform range of M^2 and μ is chosen, whereas the effective threshold depends also on the light-quark flavor and spin-parity of the interpolating quark currents.

In the heavy-quark limit of the sum rules (see Sec. V below), one expects that an "optimal Borel window" discussed in the previous section is located around

TABLE I. Input parameters used in the sum rules.

Parameters	Values (comments)	
Quark masses	$\bar{m}_b(\bar{m}_b) = 4.18 \pm 0.03$ GeV $\bar{m}_c(\bar{m}_c) = 1.275 \pm 0.025$ GeV $\bar{m}_s(2 \text{ GeV}) = 95 \pm 10$ MeV (error doubled)	[27]
Strong coupling	$\alpha_s(M_Z) = 0.1184 \pm 0.0007$ $\alpha_s(3 \text{ GeV}) = 0.255 \pm 0.003$ $\alpha_s(1.5 \text{ GeV}) = 0.353 \pm 0.006$	[27]
Quark condensate	$\langle \bar{q}q\rangle(2 \text{ GeV}) = -(277^{+12}_{-10} \text{ MeV})^3$ (ChPT $\oplus$ m_s) $\langle \bar{s}s\rangle/\langle \bar{q}q\rangle = 0.8 \pm 0.3$	[27,28]
$d = 4$, 5, 6 condensates	$\langle GG\rangle = 0.012^{+0.006}_{-0.012}$ GeV^4 $m_0^2 = 0.8 \pm 0.2$ GeV^2 $\langle \bar{s}Gs\rangle/\langle \bar{q}Gq\rangle = \langle \bar{s}s\rangle/\langle \bar{q}q\rangle$ $r_{vac} = 0.1\text{–}1.0$	[29]

$M^2 \sim 2m_Q\tau$, where $\tau \sim 1$ GeV $\gg \Lambda_{\rm QCD}$ does not scale with the heavy mass. Specifically, for the correlation functions with b quark (c quark) we adopt the lower boundary of the Borel parameter $M^2_{\rm min} = 4.5$ GeV^2 ($M^2_{\rm min} = 1.5$ GeV^2), so that the magnitude of the sum over $d = 4$, 5, 6 condensate contributions to the OPE does not exceed $\pm 5\%$ of the perturbative part. The renormalization scale μ, being generally in the ballpark of M, is chosen so that one retains the hierarchy of NNLO and NLO terms in the perturbative part of OPE. We fix $\mu = 3.0$ GeV ($\mu = 1.5$ GeV) as "default" renormalization scale in the correlation functions with b (c) quarks. With this choice, at $M^2 \geq M^2_{\rm min}$, the NNLO terms in OPE with b quark (c quark) are less than 15% (30%) of NLO terms. The scale dependence will be investigated by varying μ within the intervals indicated in Table II.

Finally, in the sum rule for each decay constant $f_{H^{(*)}}$ we determine the effective threshold $s_0^{H^{(*)}}$, demanding that the measured mass of the $H^{(*)}$ meson is reproduced from the differentiated sum rule with an accuracy no less than 0.5%. Although we work in the isospin symmetry limit, in the numerical analysis, for definiteness, mesons with the flavor content $c\bar{d}$ and $b\bar{d}$ are taken. The Borel parameter is constrained from above, $M^2 < M^2_{\rm max}$, so that the relative contribution of excited and continuum states to the sum rule, which in our approximation is equal to $1 - \Pi_{T(5)}^{\rm (pert)}(M^2, s_0)/\Pi_{T(5)}^{\rm (pert)}(M^2, \infty)$, remains less than 50%.

The decay constants are numerically calculated from the sum rules at $M^2_{\rm min} < M^2 < M^2_{\rm max}$, taking for each M^2 the fitted value of $s_0^{H^{(*)}}$. The results obtained at central input are presented in Table II. To estimate the total uncertainty of the calculated decay constants, we vary each input parameter separately, assuming (conservatively) that they are uncorrelated. Note on the other hand that the values of the effective threshold $s_0^{H^{(*)}}$, after fixing the meson mass, are correlated with the Borel parameter. Hence, we do not attribute a separate uncertainty related to the choice of $s_0^{H^{(*)}}$. To take into account, albeit rather conservatively, the neglected $d > 6$ terms in the OPE, we attribute to each calculated value of the decay constant an additional theoretical error equal to the sum of $d = 4$, 5, 6 condensate contribution. All individual uncertainties—except very small ones, with a magnitude ≤ 0.1 MeV—are presented in Table II. Adding them in quadrature, we arrive at the final results for the decay constants of heavy-light vector and pseudoscalar mesons:

$$f_{B^*} = (210^{+10}_{-12})[261] \text{ MeV},$$
$$f_B = (207^{+17}_{-9})[258] \text{ MeV}, \tag{19}$$

$$f_{B_s^*} = (267^{+14}_{-20})[310] \text{ MeV},$$
$$f_{B_s} = (242^{+17}_{-12})[285] \text{ MeV}, \tag{20}$$

$$f_{D^*} = (242^{+20}_{-12})[297] \text{ MeV},$$
$$f_D = (201^{+12}_{-13})[237] \text{ MeV}, \tag{21}$$

$$f_{D_s^*} = (314^{+19}_{-14})[367] \text{ MeV},$$
$$f_{D_s} = (238^{+13}_{-23})[266] \text{ MeV}, \tag{22}$$

TABLE II. Details of the numerical analysis of QCD sum rules for decay constants. Here ΔM^2, $\Delta \mu$, etc. denote the individual uncertainty of the decay constant (in MeV) due to the variation of M^2, μ, etc. within the adopted intervals.

Corr. function	$\Pi_T^b(M^2)$		$\Pi_5^b(M^2)$		$\Pi_T^c(M^2)$		$\Pi_5^c(M^2)$	
Default M^2 [GeV^2]	5.5				2.0			
(range)	(4.5–6.5)				(1.5–2.5)			
Default μ [GeV]	3.0				1.5			
(range)	(3.0–5.0)				(1.3–3.0)			
Sum rule for	f_{B^*}	$f_{B_s^*}$	f_B	f_{B_s}	f_{D^*}	$f_{D_s^*}$	f_D	f_{D_s}
Meson mass [GeV] [27]	5.325	5.415	5.280	5.367	2.010	2.112	1.870	1.968
Eff. threshold [GeV^2]	34.1	36.7	33.9	35.6	6.2	7.6	5.6	6.3
Central value [MeV]	210.3	266.7	206.7	241.7	241.9	313.8	201.0	237.4
ΔM^2	$+0.1$ -1.8	$+5.7$ -5.1	$+6.1$ -4.5	$+8.1$ -5.8	$+3.6$ -5.0	$+13.4$ -8.9	$+10.7$ -12.1	$+8.6$ -19.4
$\Delta \mu$	$+0.0$ -5.3	$+0.0$ -14.7	$+13.0$ -0.0	$+10.3$ -0.0	$+17.3$ -3.9	$+8.3$ -0.6	$+1.3$ -3.5	$+3.5$ -9.3
Δm_Q	$+9.0$ -8.7	$+10.2$ -10.0	$+7.6$ -7.5	± 8.2	± 7.5	± 8.1	$+1.6$ -1.9	$+1.7$ -2.1
Δm_s	–	± 2.8	–	± 1.6	–	$+3.8$ -3.9	–	± 3.1
$\Delta \langle q\bar{q} \rangle$	± 3.2	± 2.2	$+2.8$ -2.9	$+2.1$ -2.2	± 4.0	± 2.7	± 3.0	± 2.2
$\Delta \frac{\langle s\bar{s} \rangle}{\langle q\bar{q} \rangle}$	–	$+6.7$ -6.9	–	$+6.5$ -6.7	–	$+8.1$ -8.3	–	$+6.6$ -6.9
$\Delta \langle GG \rangle$	$+0.4$ -0.2	$+0.3$ -0.2	$+0.1$ -0.3	$+0.1$ -0.3	$+1.9$ -0.9	$+1.6$ -0.8	$+0.4$ -0.8	$+0.4$ -0.8
Δm_0^2	± 0.9	± 0.7	± 0.3	± 0.2	± 0.7	± 0.5	± 0.5	± 0.4
$\Delta d456$	± 4.0	± 3.0	± 0.9	± 0.6	± 4.6	± 3.5	± 2.8	± 2.5

where the duality-independent upper bounds are presented in square brackets, with the uncertainties included in the same way as in [26]. Note that for charmed mesons the upper bounds are more restrictive than for bottom mesons.

As we can see from the above results, the QCD sum rule predictions for decay constants of vector and pseudoscalar mesons have similar uncertainties. We also calculated the ratios of vector and pseudoscalar meson decay constants dividing the corresponding sum rules by each other:

$$f_{B^*}/f_B = 1.02^{+0.07}_{-0.03}, \qquad f_{B^*_s}/f_{B_s} = 1.10^{+0.05}_{-0.06}, \qquad (23)$$

$$f_{D^*}/f_D = 1.20^{+0.10}_{-0.07}, \qquad f_{D^*_s}/f_{D_s} = 1.32^{+0.04}_{-0.10}. \qquad (24)$$

The individual uncertainties for the above ratios are treated in the same way as for separate sum rules; in this case the correlations result in somewhat smaller total uncertainties. Finally, we also obtain the $SU(3)_{\rm fl}$ violating ratios of decay constants:

$$f_{B_s}/f_B = 1.17^{+0.04}_{-0.03}, \qquad f_{B^*_s}/f_{B^*} = 1.27^{+0.05}_{-0.06}, \qquad (25)$$

$$f_{D_s}/f_D = 1.18^{+0.04}_{-0.05}, \qquad f_{D^*_s}/f_{D^*} = 1.30^{+0.08}_{-0.05}. \qquad (26)$$

IV. OTHER VERSIONS OF SUM RULES

Apart from the uncertainties caused by the input parameters, the overall accuracy of QCD sum rules is influenced by the quark-hadron duality approximation (14). One can argue that the "semilocal" duality used here can be trusted, having in mind the positivity of the spectral function and the fact that the mass of the ground-state hadron is reproduced from the differentiated sum rule with a high precision.

One possible strategy to assess the "systematic" uncertainty of the adopted calculational procedure is to employ other versions of QCD sum rules, based on the same correlation function and same OPE, but differing from the standard Borel sum rules by the weight function multiplying the spectral density in the dispersion integrals. In the standard version, after Borel transformation, the role of the weight function is played by the exponent $\exp{(-s/M^2)}$. Transforming the initial, q^2-dependent dispersion integrals differently, one modifies the weight

function resulting in a redistribution of the spectral density between the ground-state hadron and excited states. Below we consider a few different versions of QCD sum rules for decay constants and carry out their numerical analysis.

A. Power moments

First, we employ the very familiar power moments of the QCD sum rules [10], obtained by differentiating over q^2 the hadronic dispersion relations (5) for the invariant amplitude at some spacelike value $q_0^2 \ll m_Q^2$. Minimum two differentiations are needed to get rid of subtraction terms. For the lhs of (5) we use the OPE result (8) and for the hadronic spectral density of excited and continuum states the quark-hadron duality approximation (14). The decay constants squared are then determined as

$$f_{H^*}^2 = \frac{(m_{H^*}^2 - q_0^2)^{n+1}}{m_{H^*}^2} \Pi_T^{(n)}(s_0^{H^*}, q_0^2),$$

$$f_H^2 = \frac{(m_H^2 - q_0^2)^{n+1}}{m_H^4} \Pi_5^{(n)}(s_0^H, q_0^2), \qquad (27)$$

where the nth moment of the sum rule is

$$\Pi_{T(5)}^{(n)}(s_0, q_0^2) \equiv \int_{(m_Q+m_q)^2}^{s_0} \frac{ds}{(s-q_0^2)^{n+1}} \rho_{T(5)}^{(\rm pert)}(s)$$
$$+ \left(\frac{d}{dq^2}\right)^n [\Pi_{T(5)}^{\langle \bar{q}q \rangle}(q^2) + \Pi_{T(5)}^{(d456)}(q^2)]|_{q^2=q_0^2}, \qquad (28)$$

with a power weight function in the dispersion integral. In the numerical analysis we employ only the $n = 2, 3$ moments to avoid the growth of higher-dimensional condensate terms at larger n. The effective threshold is again estimated by forming the ratio of the second and third moments and fitting the ground-state meson mass. For bottom mesons the power moments of sum rules work well at $q_0^2 = 0$, with both NNLO and $d \geq 4$ corrections sufficiently small. To fulfill the same criteria for charmed mesons, it is necessary to take $q_0^2 < 0$. The results for the decay constants obtained from the power moments are collected in Table III.

TABLE III.   Decay constants calculated from different sum rules at central input. In the power moments $q_0^2 = 0$ ($q_0^2 = -4.0$ GeV^2) is taken for bottom (charmed) mesons.

Method	Decay constant [MeV]							
	f_{B^*}	$f_{B^*_s}$	f_B	f_{B_s}	f_{D^*}	$f_{D^*_s}$	f_D	f_{D_s}
Power moments	196	252	198	231	228	307	203	238
Borel SR with $1/s$ weight	$\cdots$	$\cdots$	211	248	$\cdots$	$\cdots$	220	260
Borel SR with s weight	208	261	201	233	232	289	175	207
Borel SR w/o radial excit.	208	267	208	242	243	315	204	239
Standard Borel SR	210	267	207	242	242	314	201	238

B. Modified Borel sum rule

Let us consider a combination of invariant amplitudes:

$$\Delta_{T(5)}(q^2) \equiv \frac{\Pi_{T(5)}(q^2) - \Pi_{T(5)}(0)}{q^2}. \qquad (29)$$

Note that this expression is finite at $q^2 = 0$. Approximating the lhs by OPE, the rhs by the dispersion relations and performing the Borel transformation, we obtain

$$\Delta_T^{(OPE)}(M^2) = f_{H^*}^2 e^{-m_{H^*}^2/M^2} + \int_{(m_H + m_P)^2}^{\infty} ds \frac{\rho^h(s)}{s} e^{-s/M^2}, \qquad (30)$$

and a similar relation for the pseudoscalar-meson channel. An additional $1/s$ factor appears in the weight function multiplying the spectral density. Applying the quark-hadron duality approximation (14), we calculate the decay constants $f_{H^{(*)}}$ from these sum rules. Dividing (30) by the conventional Borel sum rule we obtain a relation for the inverse mass squared of the ground-state meson, allowing us to adjust the threshold $s_0^{H^{(*)}}$.

It is even simpler to obtain another modified Borel sum rule with an extra power of s in the integral. One needs to multiply both parts of (15) and (16) by $e^{m_{H^*}^2/M^2}$ and $e^{m_H^2/M^2}$, respectively, and after that differentiate them over $-1/M^2$. In this case the effective threshold is estimated by dividing the modified sum rule by the standard Borel sum rule and adjusting the result to the mass squared of $H^{(*)}$.

The numerical results for the decay constants obtained from the modified Borel sum rules with $1/s$ and s weights are given in Table III. Here we compare the results of different sum rules obtained at one and the same central input (as specified in Tables I and II). However, it turns out that for the vector-meson decay constants calculated from $1/s$ sum rules, the central values of Borel parameter specified in Table II are not suitable; hence the corresponding results are missing.

C. Excluding the first radial excitation

Radial (i.e., same spin-parity) excitations of heavy-light mesons form the resonance part of the hadronic spectrum above the ground states in the spectral densities (6) and (7). Including these resonances in the hadronic spectral density explicitly, one improves the accuracy of the quark-hadron duality approximation. Following this strategy, we separate the first radial excitation $H^{*'}$ from the rest of the hadronic spectrum in the vector-meson channel, transforming the duality ansatz (14) to the following form:

$$\rho_T^h(s)\theta(s - (m_H + m_P)^2) = m_{H^{*'}}^2 f_{H^{*'}}^2 \delta(s - m_{H^{*'}}^2) + \rho_T^{(pert)}(s)\theta(s - s_0^{H^{*'}}), \quad (31)$$

where the total width of $H^{*'}$ is neglected for simplicity (it can be easily restored employing a Breit-Wigner ansatz) and $s_0^{H^{*'}}$ generally differs from the effective threshold in

(14). In the same way, the spectral density in the pseudo-scalar channel is modified introducing the excited state H'.

Currently, only limited experimental data on the radially excited charmed mesons are available. The resonances $D(2550)$ and $D(2600)$, observed in [35] (see also [27]), represent realistic candidates for the first radially excited D' and $D^{*'}$ states, respectively. The mass differences between these resonances and ground states D and D^* are in the same ballpark as for the light unflavored mesons, cf. the mass difference between the first radial excitation $\rho' = \rho(1450)$ and the ground-state ρ meson. Here we assume that the mass differences between the first excited and ground states for all heavy-light mesons are approximately the same:

$$m_{B'} - m_B \simeq m_{D'} - m_D \simeq m_{D_s'} - m_{D_s}, \qquad (32)$$
$$m_{B^{*'}} - m_{B^*} \simeq m_{D^{*'}} - m_{D^*} \simeq m_{D_s^{*'}} - m_{D_s^*}.$$

Without introducing extra parameters, such as the decay constants of the radially excited mesons, we suggest a modified QCD sum rule in which the spectral density is multiplied by an additional factor $(m_{H^{(*)'}}^2 - s)$ vanishing at the position of the first radially excited state $H^{(*)'}$. To derive this sum rule, one simply multiplies the initial, q^2-dependent dispersion relation for $\Pi_{5(T)}(q^2)$ by an overall factor $(m_{H^{(*)'}}^2 - q^2)$. After Borel transformation the following expression, e.g., for the decay constant of the vector heavy-light meson is obtained:

$$f_{H^*}^2 = \frac{e^{m_{H^*}^2/M^2}}{m_{H^*}^2(m_{H^{*'}}^2 - m_{H^*}^2)}$$
$$\times \left\{ \int_{(m_Q + m_q)^2}^{s_0^{H^{*'}}} ds(m_{H^{*'}}^2 - s)e^{-s/M^2}\rho_T^{(pert)}(s) \right.$$
$$\left. + \left(m_{H^{*'}}^2 - \frac{d}{d(-1/M^2)}\right)[\Pi_T^{\langle \bar{q}q \rangle}(M^2) + \Pi_T^{(d456)}(M^2)] \right\}. \qquad (33)$$

It is straightforward to derive the analogous sum rule for f_H.

We take as an input $m_{D'} = 2.55$ GeV, $m_{D^{*'}} = 2.60$ GeV and estimate the masses of other radially excited states from (32). In fact, in the above sum rule we do not necessarily need a precise value of the mass $m_{H^{(*)'}}$. Important is that, due to a partial cancellation between the two intervals, below and above $s = m_{H^{(*)'}}^2$, the region above the ground state and adjacent to the first radial excitation is suppressed in the integral over the weighted spectral density. As a result, the rhs of (33) becomes less sensitive to the duality approximation, allowing us to simplify the choice of the effective threshold. Here we simply adopt $s_0^{H^{(*)'}} = m_{H^{(*)'}}^2$, without adjusting the threshold parameter. Interestingly, as our numerical analysis shows, this choice reproduces the masses of the ground states from

differentiated sum rules (33) within 1% accuracy. The decay constants obtained from (33) and from the analogous sum rule for vector mesons are surprisingly close to the decay constants obtained from standard Borel sum rules with fitted effective thresholds (see Table III).

Assessing the mutual deviations between the decay constants calculated from various sum rules considered in this section, we have to include the variations of all entries in Table III due to the input parameters. After that, we find that the predicted intervals of all decay constants calculated from different sum rules overlap. The same is valid for the ratios of decay constants.

V. HEAVY-QUARK LIMIT OF THE SUM RULES

The heavy-quark mass expansion in QCD sum rules for decay constants was pioneered in [15]. Later, the sum rule

technique was applied [16–18] in the framework of the HQET, considering, instead of the quark currents with a finite m_Q, their HQET counterparts, with a possibility to systematically resum the logarithms $\ln(m_Q/\mu)$ emerging in the OPE. The "static" value of $f_{H^{(*)}}$ calculated from the HQET sum rule in the $m_Q \to \infty$ limit receives large inverse heavy-mass corrections which have to be estimated separately [17,19,25].

To obtain the heavy-quark limit of the sum rules (15) and (16), one has to rescale the m_Q-dependent parameters:

$$m_H = m_Q + \bar{\Lambda}, \qquad s_0^H = m_Q^2 + 2m_Q\omega_0, \qquad M^2 = 2m_Q\tau. \tag{34}$$

After this replacement, the sum rule for the heavy-light vector meson decay constant transforms to

$$f_{H^*}^2 m_{H^*}\left(\frac{m_{H^*}}{m_Q}\right)e^{-\frac{\bar{\Lambda}}{\tau}-\frac{\bar{\Lambda}^2}{2m_Q\tau}} = \frac{\tau^3}{\pi^2}\int_0^{\omega_0/\tau}dz\,e^{-z}\left(\frac{z^2}{1+\frac{2z\tau}{m_Q}}\right)\left(2+\frac{1}{1+\frac{2z\tau}{m_Q}}\right)\left\{1+\frac{2\alpha_s}{\pi}\left[\ln\left(\frac{m_Q}{2\tau}\right)+\frac{3}{2}+\frac{2\pi^2}{9}-\ln(z)+\frac{2}{3}\mathcal{K}_T\left(\frac{2z\tau}{m_Q}\right)\right]\right\}$$
$$-\langle q\bar{q}\rangle\left\{1+\frac{2\alpha_s}{3\pi}\left(3+\left(\frac{2\tau}{m_Q}\right)\int_0^\infty dz\frac{e^{-z}}{(1+\frac{2z\tau}{m_Q})^2}\right)\right\}-\frac{\langle GG\rangle}{12m_Q}+\frac{m_0^2\langle q\bar{q}\rangle}{16\tau^2}$$
$$+\frac{\pi\alpha_s r_{\text{vac}}\langle q\bar{q}\rangle^2}{162\tau^3}\left\{1-\frac{16\tau}{m_Q}-\frac{32\tau^2}{m_Q^2}\right\}, \tag{35}$$

where

$$\mathcal{K}_T(x) = 2\text{Li}_2(-x) + \ln(x)\ln(1+x) + \frac{x}{(3+2x)}\ln(x)$$
$$+\frac{(1+2x)(2+x)(1+x)}{(3+2x)x^2}\ln(1+x)$$
$$+\frac{6x^2+3x-8}{4(3+2x)x}-\frac{9}{4}, \tag{36}$$

so that

$$\lim_{x\to 0}\mathcal{K}_T(x) = \frac{4}{3}x\ln(x) - \frac{29}{18}x + \mathcal{O}(x^2). \tag{37}$$

Note that deriving (35) we use the pole-mass scheme for m_Q, as it is more convenient for the matching with HQET. The expression for the $O(\alpha_s)$ correction was accordingly modified.

The rescaled sum rule in the pseudoscalar channel [16] is

$$f_H^2 m_H\left(\frac{m_H}{m_Q}\right)^3 e^{-\frac{\bar{\Lambda}}{\tau}-\frac{\bar{\Lambda}^2}{2m_Q\tau}} = \frac{3\tau^3}{\pi^2}\int_0^{\omega_0/\tau}dz\,e^{-z}\left(\frac{z^2}{1+\frac{2z\tau}{m_Q}}\right)\left\{1+\frac{2\alpha_s}{\pi}\left[\ln\left(\frac{m_Q}{2\tau}\right)+\frac{13}{6}+\frac{2\pi^2}{9}-\ln(z)+\frac{2}{3}\mathcal{K}_5\left(\frac{2z\tau}{m_Q}\right)\right]\right\}$$
$$-\langle q\bar{q}\rangle\left\{1-\frac{2\alpha_s}{3\pi}\left(-1+3\frac{2\tau}{m_Q}\int_0^\infty dz\frac{e^{-z}}{1+\frac{2z\tau}{m_Q}}\right)\right\}+\frac{\langle GG\rangle}{12m_Q}+\frac{m_0^2\langle q\bar{q}\rangle}{16\tau^2}\left\{1-\frac{4\tau}{m_Q}\right\}$$
$$+\frac{\pi\alpha_s r_{\text{vac}}\langle q\bar{q}\rangle^2}{162\tau^3}\left\{1+\frac{6\tau}{m_Q}-\frac{48\tau^2}{m_Q^2}\right\}, \tag{38}$$

where

$$\mathcal{K}_5(x) = 2\text{Li}_2(-x) + \ln(x)\ln(1+x) - \frac{x}{1+x}\ln(x)$$
$$+\frac{1+x}{x}\ln(1+x) - 1, \tag{39}$$

and

$$\lim_{x\to 0}\mathcal{K}_5(x) = -\frac{3}{2}x + \mathcal{O}(x^2). \tag{40}$$

It is now possible to take the limit $m_Q \to \infty$ in (35) and (38), whereas the logarithms have to be treated separately. Neglecting the gluon radiative corrections, one reproduces the well-known heavy-quark limit of decay constants:

$$f_H = f_{H^*} = \frac{\hat{f}}{\sqrt{m_H}}, \tag{41}$$

where

$$\hat{f} = e^{\frac{\bar{\Lambda}}{2\tau}}\left(\frac{3\tau^3}{\pi^2}\int_0^{\frac{\omega_0}{\tau}} dz z^2 e^{-z} - \langle q\bar{q}\rangle \right.$$

$$\left. + \frac{m_0^2\langle q\bar{q}\rangle}{16\tau^2} + \frac{\pi\alpha_s r_{\text{vac}}\langle q\bar{q}\rangle^2}{162\tau^3}\right)^{1/2}. \tag{42}$$

Thus, the rescaled decay constant $\hat{f}$ receives contributions from the perturbative loop, quark condensate and the $d = 5, 6$ condensates. The $d = 4$ gluon condensate term enters the sum rules at the $1/m_Q$ level, together with other inverse mass corrections.

The radiative correction to the ratio of decay constants obtained from (35) and (38) in the heavy-quark limit,

$$\frac{f_{H^*}}{f_H} = 1 - \frac{2\alpha_s}{3\pi}, \tag{43}$$

is in accordance with the well-known $O(\alpha_s)$ correction to the heavy-quark spin symmetry relation which follows from the matching of HQET and full QCD heavy-light currents.

It is interesting to compare our predictions for the ratios of decay constants (23) and (24), obtained from QCD sum rules with finite quark masses, with the HQET relation [19]:

$$\frac{f_{H^*}}{f_H} = \left(1 - \frac{2\alpha_s(m_Q)}{3\pi}\right)[1 + \delta/m_Q], \tag{44}$$

where we introduce a short-hand notation for the combination of HQET parameters determining the inverse mass correction. For the values of pole masses we use $m_b^{\text{pole}} = 4.6$ GeV and $m_c^{\text{pole}} = 1.5$ GeV which correspond [with $O(\alpha_s)$ accuracy] to the $\overline{\text{MS}}$ quark masses in Table I. For bottom mesons the interval for the lhs of (44) taken from (23) corresponds to $\delta = 180 \div 650$ MeV. Assuming the same value of this parameter for charmed mesons and neglecting $O(1/m_Q^2)$ corrections, we obtain from (44) the ratio $f_{D^*}/f_D = 1.03 \div 1.33$, which agrees with our predicted interval in (24). Note that the HQET parameters contributing to δ were estimated from sum rules in HQET [19] yielding $f_{B^*}/f_B = 1.07 \pm 0.02$ and $f_{D^*}/f_D = 1.35 \pm 0.05$ which is in a satisfactory agreement with our results from full QCD sum rules. Numerically, the heavy-quark spin symmetry for decay constants is violated by an inverse mass correction of about 12%–14% (20%–40%) for bottom (charmed) mesons. Let us finally estimate the heavy-quark flavor symmetry violation. For example, the leading-order ratio in HQET including radiative corrections [19],

$$\frac{f_B}{f_D} = \sqrt{\frac{m_D}{m_B}}\left(\frac{\alpha_s(m_c)}{\alpha_s(m_b)}\right)^{6/25}\left(1 + 0.894\frac{\alpha_s(m_c) - \alpha_s(m_b)}{\pi}\right)$$

$$\simeq 0.69, \tag{45}$$

has to be compared with the interval $f_B/f_D \simeq 0.93 \div 1.19$ allowed by the intervals of sum rule predictions presented in (19) and (21), assuming no correlation between uncertainties.

VI. DISCUSSION

In this paper we calculated the decay constants of heavy-light vector and pseudoscalar mesons employing the well-established method of QCD sum rules. The sum rules for $B^*_{(s)}$ and $D^*_{(s)}$ mesons have acquired the same level of accuracy as the sum rules for $B_{(s)}$ and $D_{(s)}$ mesons: $O(\alpha_s^2)$ in the perturbative part and $O(\alpha_s)$ in the quark condensate term. We correspondingly updated the numerical values of all decay constants, together with their upper bounds and ratios. The uncertainties for f_B and f_D caused by the input variation became somewhat smaller than in the earlier sum rule determinations where $f_B = 210 \pm 19$ MeV, $f_{B_s} = 244 \pm 21$ MeV [24] and $f_B = 206 \pm 20$ MeV, $f_D = 195 \pm 20$ MeV [25] were obtained, with a typical error of about ± 20 MeV. This improvement is mainly due to smaller uncertainties of quark $\overline{\text{MS}}$ masses achieved in recent years. In this paper we investigated different versions of sum rules: power moments and Borel sum rules with a modified weight of the spectral density. Within uncertainties, their predictions agree with the ones obtained from the standard Borel sum rules.

Reducing the uncertainties of quark masses and condensate densities is one of the few remaining possibilities to further improve the accuracy of QCD sum rules for decay constants. It is also desirable to obtain a fully analytic form of the $O(\alpha_s^2)$ corrections, in order to achieve a better control over the renormalization scale dependence. Calculating the $O(\alpha_s)$ correction to the quark-gluon condensate term can also be useful, at least for the ratio of sum rules for vector and pseudoscalar mesons where this contribution is enhanced.

Turning to the comparison with recent sum rule determinations of pseudoscalar meson decay constants, let us note that, contrary to the analysis presented in [36], we do not attempt to fit the heavy-quark mass simultaneously with the decay constants. We also cannot confirm the total uncertainties and upper bounds quoted in [36], which are both systematically smaller than what is obtained here. In [37], while determining the pseudoscalar heavy-light meson decay constants with $O(\alpha_s^2)$ accuracy, an explicit polynomial dependence of the effective threshold on the Borel parameter is introduced. The claim that this dependence improves the sum rules and allows one to estimate a related systematic error remains obscure to us. A determination of $f_{B_{(s)}}$ and $f_{D_{(s)}}$ from finite-energy sum rules [38], which is a different method based on the same correlation function and OPE, yields somewhat smaller decay constants than the ones obtained here.

In Table IV we compare our predictions for decay constants and their ratios with the most recent lattice QCD

TABLE IV. Decay constants of heavy-light mesons, comparison with lattice QCD results.

Decay constant	Lattice QCD [Reference]	This work
f_B [MeV]	196.9 ± 9.1 [4] 186 ± 4 [6]	207^{+17}_{-9}
f_{B_s} [MeV]	242.0 ± 10.0 [4] 224 ± 5 [6]	242^{+17}_{-12}
f_{B_s}/f_B	1.229 ± 0.026 [4] 1.205 ± 0.007 [6]	$1.17^{+0.04}_{-0.03}$
f_D [MeV]	218.9 ± 11.3 [4] 213 ± 4 [5]	201^{+12}_{-13}
f_{D_s} [MeV]	260.1 ± 10.8 [4] 248.0 ± 2.5 [5]	238^{+13}_{-23}
f_{D_s}/f_D	1.188 ± 0.025 [4] 1.164 ± 0.018 [5]	$1.15^{+0.04}_{-0.05}$
f_{D^*} [MeV]	$278 \pm 13 \pm 10$ [9]	242^{+20}_{-12}
$f_{D_s^*}$ [MeV]	311 ± 9 [9]	314^{+19}_{-14}
$f_{D_s^*}/f_{D^*}$	$1.16 \pm 0.02 \pm 0.06$ [9]	$1.30^{+0.08}_{-0.05}$

determinations, revealing a good agreement within the uncertainties.

We can also compare the predictions for f_D and f_{D_s} with the averages [1] over various experiments measuring $D_{(s)} \to \ell \nu_\ell$ decay widths: $f_D^{(\text{exp.av.})} = 206.7 \pm 8.5 \pm 2.5$ MeV and $f_{D_s}^{(\text{exp.av.})} = 260.0 \pm 5.4$ MeV. We notice some tension of our prediction (and also of the lattice result [5]) for f_{D_s} with the above interval. Furthermore, the most recent measurement of $B \to \tau \nu_\tau$ [2] yields $f_B^{(\text{exp})} = (211 \pm 22 \pm 14)$ MeV$/(|V_{ub}|/0.0035)$ taking for $|V_{ub}|$ a typical value obtained [27,39] from exclusive semileptonic B decays. A future reduction of the experimental uncertainty in this measurement opens up a possibility to use f_B calculated in QCD for an independent $|V_{ub}|$ determination. On the other hand, the $|V_{ub}|$ independent ratio of $B \to \pi \ell \nu_e$ and $B \to \tau \nu_\tau$ widths can be used [39] to check QCD calculations of form factors and decay constants.

We conclude our discussion mentioning the role of radial excitations of heavy-light mesons in the sum rules. As we found, excluding the first radial excitation from the hadronic spectrum makes the sum rule less sensitive to the value of the effective threshold. One can turn this argument around, anticipating that the sum rules considered in this paper are also capable of yielding estimates of the decay constants for the first radial excitations of heavy-light mesons. We plan a separate study in this direction.

ACKNOWLEDGMENTS

APPENDIX A: OPE EXPRESSIONS

1. Perturbative spectral density

Here we collect the expressions for NLO, $O(\alpha_s)$ contributions to the spectral density $\rho_{T(5)}^{(\text{pert})}(s)$ in the $\overline{\text{MS}}$ scheme for the heavy-quark mass m_Q. For the vector heavy-light quark currents, according to our convention for the invariant amplitude $\Pi_T(q^2)$, we extract the coefficient at $-g_{\mu\nu}$. The corresponding spectral density reads

$$\rho_T^{(\text{pert,NLO})}(s) = \frac{3C_F}{16\pi^2} s \left[1 - \frac{5}{2}z + \frac{2}{3}z^2 + \frac{5}{6}z^3 \right.$$
$$+ \frac{1}{3}z(-5 - 4z + 5z^2)\ln(z)$$
$$- \frac{1}{3}(1-z)^2(4 + 5z)\ln(1-z)$$
$$+ \frac{2}{3}(1-z)^2(2+z)(2\text{Li}_2(z) + \ln(z)\ln(1-z))$$
$$\left. - z(1-z^2)\left(3\ln\left(\frac{\mu^2}{m_Q^2}\right) + 4\right) \right], \quad \text{(A1)}$$

where $z = m_Q^2/s$ and $\text{Li}_2(z) = -\int_0^z \frac{\ln(1-t)}{t} dt$. For the pseudoscalar heavy-light quark currents, one has

$$\rho_5^{(\text{pert,NLO})}(s) = \frac{3C_F}{16\pi^2}(m_Q + m_q)^2 s(1-z) \left[\frac{9}{2}(1-z) \right.$$
$$+ (3-z)(1-2z)\ln(z)$$
$$- (1-z)(5-2z)\ln(1-z) + 2(1-z)(2\text{Li}_2(z)$$
$$+ \ln(z)\ln(1-z)) + (1-3z)\left(3\ln\left(\frac{\mu^2}{m_Q^2}\right) + 4\right) \right]. \quad \text{(A2)}$$

For NNLO corrections we are using the results from [23] calculated in the pole mass scheme. Hence, to properly apply the $\overline{\text{MS}}$ scheme for m_Q to α_s^2 accuracy, we have to add to the NNLO part the corrections which arise from expanding the pole mass in the LO and NLO in terms of $\overline{\text{MS}}$ mass. For the vector-current correlation function they are

$$\Delta_1 \rho_T^{(\text{pert,NNLO})}(s) = -\frac{3}{8\pi^2} s[(3 - 7z^2)r_m^{(1)^2} - 2(1 - z^2)r_m^{(2)}], \quad \text{(A3)}$$

$$\Delta_2 \rho_T^{(\text{pert,NNLO})}(s) = -\frac{1}{16\pi^2} C_F r_m^{(1)} s[-z(1-z^2)(24\text{Li}_2(z)$$
$$+ 12\ln(z)\ln(1-z))$$
$$- 2z(9 + 6z - 17z^2)\ln(z)$$
$$+ 2(1-z)(4 + 9z + 17z^2)\ln(1-z)$$
$$- z(1-z)(17 + 15z)], \quad \text{(A4)}$$

respectively, and for the pseudoscalar-current correlation function,

$$\Delta_1\rho_{5,\text{LO}}^{(\text{pert,NNLO})}(s) = \frac{3(m_Q + m_q)^2}{8\pi^2}s[(3 - 20z + 21z^2)r_m^{(1)^2}$$
$$- 2(1 - z)(1 - 3z)r_m^{(2)}], \qquad (\text{A5})$$

$$\Delta_2\rho_5^{(\text{pert,NLO})}(s)$$
$$= -\frac{3(m_Q + m_q)^2}{8\pi^2}C_F r_m^{(1)}s\Big[(1 - z)(1 - 3z)(4\text{Li}_2(z)$$
$$+ 2\ln(z)\ln(1 - z)) + (3 - 22z + 29z^2 - 8z^3)\ln(z)$$
$$- (1 - z)(7 - 21z + 8z^2)\ln(1 - z)$$
$$+ \frac{1}{2}(1 - z)(15 - 31z)\Big], \qquad (\text{A6})$$

where $r_m^{(1,2)}$ are the well-known coefficients in the perturbative relation between the pole and $\overline{\text{MS}}$ quark masses [given, e.g., in Eqs. (B5–B9)] in [24].

The corrections due to the nonzero light-quark mass, after expanding the complete answer in powers of m_q read

$$\delta\rho_T^{(\text{pert,LO},m_q)}(s) = \frac{3m_q}{8\pi^2}[2m_Q(1 - z) - m_q(1 + z^2)], \quad (\text{A7})$$

$$\delta\rho_T^{(\text{pert,NLO},m_q)}(s)$$
$$= \frac{m_q m_Q}{8\pi^2}C_F\Big[-24(1 - z)(2\text{Li}_2(z) + \ln(z)\ln(1 - z))$$
$$+ 12(3 - 4z - z^2)\ln(z) - 12(1 - z)(5 + z)\ln(1 - z)$$
$$+ 6(17 - 26z + z^2) + 6(6 - 12z)\ln\Big(\frac{\mu^2}{m_Q^2}\Big)\Big]. \quad (\text{A8})$$

The analogous corrections to the perturbative part of the pseudoscalar-current correlation function are

$$\delta\rho_5^{(\text{pert,LO},m_q)}(s) = \frac{3(m_Q + m_q)^2}{8\pi^2}[2(1 - z)m_Q m_q - 2m_q^2], \quad (\text{A9})$$

$$\delta\rho_5^{(\text{pert,NLO},m_q)}(s) = \frac{3(m_Q + m_q)^2}{8\pi^2}C_F m_Q m_q\Big[(1 - z)(4\text{Li}_2(z)$$
$$+ 2\ln(z)\ln(1 - z) - 2(4 - z)\ln(1 - z))$$
$$+ 2(3 - 5z + z^2)\ln(z) + 2(7 - 9z)$$
$$+ 3(2 - 3z)\ln\Big(\frac{\mu^2}{m_Q^2}\Big)\Big]. \qquad (\text{A10})$$

We include the above corrections only for the s quark and up to the second (first) power in m_s in LO (NLO). We checked that the higher-power corrections in m_s are vanishingly small.

2. Condensate contributions

We present the condensate contributions in two forms: with an explicit q^2 dependence (needed, e.g., for the power moments) and after Borel transformation.

In the correlation function of vector currents, the total contribution of the quark condensate is

$$\Pi_{\mu\nu}^{\langle\bar{q}q\rangle}(q^2) = \langle\bar{q}q\rangle\frac{m_Q}{m_Q^2 - q^2}\Big[g_{\mu\nu}\Big(1 - \frac{m_q m_Q}{2(m_Q^2 - q^2)}$$
$$+ \frac{\alpha_s C_F}{2\pi}f_{V,1}(z)\Big) - \frac{q_\mu q_\nu}{q^2}\frac{\alpha_s C_F}{\pi}f_{V,2}(z)\Big], \quad (\text{B1})$$

with the NLO terms given by

$$f_{V,1}(z) = 2 - z + z(1 - z)L_z - \frac{z}{z - 1}\Big(3\ln\frac{\mu^2}{m^2} + 4\Big), \quad (\text{B2})$$

$$f_{V,2}(z) = 1 - 2z + 2z(1 - z)L_z, \qquad (\text{B3})$$

where the short-hand notations $z = \frac{m_Q^2}{q^2}$ and $L_z = \ln\big(\frac{z-1}{z}\big)$ are used. In the case $q = s$ the first-order $O(m_q)$ correction included in (B1) provides a sufficient accuracy. For our purpose, only the coefficient $\Pi_T^{\langle\bar{q}q\rangle}(q^2)$ of the structure $-g_{\mu\nu}$ is needed. The Borel-transformed expression of this amplitude is

$$\Pi_T^{\langle\bar{q}q\rangle}(M^2) = -m_Q\langle\bar{q}q\rangle e^{-\frac{m_Q^2}{M^2}}\Big(1 - \frac{m_q m_Q}{2M^2}$$
$$+ \frac{\alpha_s C_F}{2\pi}\Big[1 - 3\frac{m_Q^2}{M^2}\ln\frac{\mu^2}{m_Q^2} - 4\frac{m_Q^2}{M^2}$$
$$+ \frac{m_Q^2}{M^2}e^{\frac{m_Q^2}{M^2}}\Gamma\Big(-1, \frac{m_Q^2}{M^2}\Big)\Big]\Big), \qquad (\text{B4})$$

with the incomplete gamma function $\Gamma(a, z) = \int_z^\infty t^{a-1}e^{-t}dt$. The NLO part in (B1) originating from one-loop diagrams has an imaginary part at $q^2 \to s \geq m_Q^2$. The latter, in addition to the terms proportional to $\delta(s - m_Q^2)$ and its derivatives, contains also a part which does not vanish at $s > m_Q^2$, that is proportional to $\theta(s - m_Q^2)$. Since we include the latter in the OPE spectral density involved in the quark-hadron duality approximation, we present here also the spectral density of the condensate contribution:

$$\rho_T^{\langle\bar{q}q\rangle}(s) = -m_Q\langle\bar{q}q\rangle\Big(\delta(m_Q^2 - s) - \frac{1}{2}m_q m_Q\delta'(s - m_Q^2)$$
$$+ \frac{\alpha_s C_F}{2\pi}\Big[\delta(m_Q^2 - s) - m_Q^2\Big(3\ln\frac{\mu^2}{m_Q^2} + 4\Big)\delta'(m_Q^2 - s)$$
$$+ \frac{m_Q^2}{s^2}\theta(s - m_Q^2)\Big]\Big). \qquad (\text{B5})$$

In the pseudoscalar-meson channel, the quark condensate contribution to the correlation function in the same approximation reads

$$\Pi_5^{\langle \bar{q}q \rangle}(q^2) = -\langle \bar{q}q \rangle \frac{(m_Q + m_q)^2 m_Q}{m_Q^2 - q^2} \left(1 - \frac{m_q}{2m_Q} \right.$$

$$\left. - \frac{m_q m_Q}{2(m_Q^2 - q^2)} - \frac{\alpha_s C_F}{2\pi} f_5(z) \right), \qquad \text{(B6)}$$

with the coefficient (see also [24])

$$f_5(z) = 3 \frac{z}{z - 1} (L_z (2 - z) - 1)$$

$$+ \frac{1}{z - 1} \left(3 \ln \frac{\mu^2}{m^2} + 7 - 3L_z \right). \qquad \text{(B7)}$$

The Borel-transform of (B6) yields

$$\Pi_5^{\langle \bar{q}q \rangle}(M^2) = -(m_Q + m_q)^2 m_Q \langle \bar{q}q \rangle e^{-\frac{m_Q^2}{M^2}} \left(1 - \frac{m_q}{2m_Q} - \frac{m_q m_Q}{2M^2} \right.$$

$$- \frac{\alpha_s C_F}{2\pi} \left[\left(3 \ln \frac{\mu^2}{m_Q^2} + 4 \right) \frac{m_Q^2}{M^2} - 7 - 3 \ln \frac{\mu^2}{m_Q^2} \right.$$

$$\left. \left. + 3 \Gamma \left(0, \frac{m_Q^2}{M^2} \right) e^{\frac{m_Q^2}{M^2}} \right] \right). \qquad \text{(B8)}$$

The spectral density derived from (B6) reads

$$\rho_5^{\langle \bar{q}q \rangle}(s) = -(m_Q + m_q)^2 m_Q \langle \bar{q}q \rangle \left(\delta(m_Q^2 - s) \right.$$

$$+ \frac{\alpha_s C_F}{2\pi} \left[\left(7 + 3 \ln \frac{\mu^2}{m_Q^2} \right) \delta(m_Q^2 - s) \right.$$

$$\left. \left. - m_Q^2 \left(4 + 3 \ln \frac{\mu^2}{m_Q^2} \right) \delta'(m_Q^2 - s) - \frac{3}{s} \theta(s - m_Q^2) \right] \right). \qquad \text{(B9)}$$

The expressions for $d \geq 4$ condensate contributions for the vector-current correlation function read

$$\Pi_{\mu\nu}^{\langle GG \rangle}(q^2) = \frac{\langle GG \rangle}{12(m_Q^2 - q^2)} g_{\mu\nu},$$

$$\Pi_{\mu\nu}^{\langle \bar{q}Gq \rangle}(q^2) = -\frac{m_0^2 \langle \bar{q}q \rangle m_Q^3}{2(m_Q^2 - q^2)^3} g_{\mu\nu}, \qquad \text{(B10)}$$

$$\Pi_{\mu\nu}^{\langle \bar{q}q\bar{q}q \rangle}(q^2) = \frac{8\pi \alpha_s r_{vac} \langle \bar{q}q \rangle^2}{81(m_Q^2 - q^2)^4} [(9m_Q^4 - 16m_Q^2 q^2 + 4q^4) g_{\mu\nu}$$

$$+ (10m_Q^2 - 4q^2) q_\mu q_\nu]. \qquad \text{(B11)}$$

The Borel-transformed form is

$$\Pi_T^{\langle GG \rangle}(M^2) = -\frac{\langle GG \rangle}{12} e^{-\frac{m_Q^2}{M^2}},$$

$$\Pi_T^{\langle \bar{q}Gq \rangle}(M^2) = \frac{m_0^2 \langle \bar{q}q \rangle m_Q^3}{4M^4} e^{-\frac{m_Q^2}{M^2}}, \qquad \text{(B12)}$$

$$\Pi_T^{\langle \bar{q}q\bar{q}q \rangle}(M^2) = -\frac{32\pi \alpha_s r_{vac} \langle \bar{q}q \rangle^2}{81M^2} \left(1 + \frac{m_Q^2}{M^2} - \frac{m_Q^4}{8M^4} \right) e^{-\frac{m_Q^2}{M^2}}. \qquad \text{(B13)}$$

The corresponding condensate contributions to the correlation function with pseudoscalar currents are

$$\Pi_5^{\langle GG \rangle}(q^2) = \frac{\langle GG \rangle m_Q^2}{12(m_Q^2 - q^2)},$$

$$\Pi_5^{\langle \bar{q}Gq \rangle}(q^2) = -\frac{m_0^2 \langle \bar{q}q \rangle m_Q^3}{2(m_Q^2 - q^2)^2} \left(1 - \frac{m_Q^2}{m_Q^2 - q^2} \right), \qquad \text{(B14)}$$

$$\Pi_5^{\langle \bar{q}q\bar{q}q \rangle}(q^2) = -\frac{8\pi \alpha_s r_{vac} \langle \bar{q}q \rangle^2 m_Q^2 q^2}{27(m_Q^2 - q^2)^4} (2q^2 - 3m_Q^2), \qquad \text{(B15)}$$

yielding after Borel transformation

$$\Pi_5^{\langle GG \rangle}(M^2) = \frac{\langle GG \rangle m_Q^2}{12} e^{-\frac{m_Q^2}{M^2}},$$

$$\Pi_5^{\langle \bar{q}Gq \rangle}(M^2) = -\frac{m_0^2 \langle \bar{q}q \rangle m_Q^3}{2M^2} \left(1 - \frac{m_Q^2}{2M^2} \right) e^{-\frac{m_Q^2}{M^2}}, \qquad \text{(B16)}$$

$$\Pi_5^{\langle \bar{q}q\bar{q}q \rangle}(M^2) = -\frac{16\pi \alpha_s r_{vac} \langle \bar{q}q \rangle^2 m_Q^2}{27M^2}$$

$$\times \left(1 - \frac{m_Q^2}{4M^2} - \frac{m_Q^4}{12M^4} \right) e^{-\frac{m_Q^2}{M^2}}. \qquad \text{(B17)}$$

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