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Holographic charge density waves

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We show that strongly coupled holographic matter at finite charge density can exhibit charge density wave phases which spontaneously break translation invariance while preserving time-reversal and parity invariance. We show that such phases are possible within Einstein-Maxwell-dilaton theory in general spacetime dimensions. We also discuss related spatially modulated phases when there is an additional coupling to a second vector field, possibly with nonzero mass. We discuss how these constructions, and others, should be associated with novel spatially modulated ground states.

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I. INTRODUCTION

The charge density in a metal is usually highly uniform due to strong Coulombic interactions. However, a wide variety of metals exhibit stable charge density wave (CDW) phases in which the charge density spontaneously becomes spatially modulated. These phases were first predicted by Peierls in the context of weakly coupled one-dimensional systems [1]. They are also known to occur in strongly correlated systems and in some cases the CDWs are present with additional order. For example, in the high-temperature cuprate superconductors there are striped phases in which CDWs appear with spin density waves, with the latter breaking time-reversal invariance (for a review see Ref. [2]). In this paper we will discuss CDWs, that preserve time-reversal and parity invariance (T and P), within the holographic framework of the AdS/CFT correspondence.

The first constructions of holographic spatially modulated phases, but without CDWs, were made in the context of $D = 5$ Einstein-Maxwell theory with a Chern-Simons term [3,4] (see also Ref. [5]). At finite charge density and high temperatures, the system is described by the standard electrically charged AdS-Reissner-Nordström (AdS-RN) black brane corresponding to a spatially homogeneous and isotropic phase. For sufficiently large Chern-Simons coupling, the AdS-RN black brane becomes unstable at a critical temperature and a new branch of black hole solutions appears corresponding, in the dual field theory, to a phase with spatially modulated currents with helical order [3]. The fully backreacted black hole solutions were constructed in Ref. [4] where it was shown that a helical current phase is thermodynamically preferred and that the phase transition is second order. Furthermore, at zero temperature the black hole solutions of Ref. [4] revealed new ground states with a helical structure similar to those studied in Ref. [6]. These helical current phases break P and T.

In subsequent work, a $D = 4$ Einstein-Maxwell theory coupled to a pseudoscalar φ was shown to admit black hole solutions corresponding to phases with spatially modulated currents and in addition, CDWs [7]. In this class of models the key coupling driving the spatially modulated phase transition is the axion-like coupling $\varphi F \wedge F$, where F is the field strength of the Maxwell field. The fully back-reacted black hole solutions require solving partial differential equations and some results have recently appeared in Ref. [8]. While the models of Ref. [7] realize CDWs the spatially modulated currents break P and T. Holographic constructions of spatially modulated phases at finite charge density[1] have also been made in Refs. [16–23] and P and T are again not preserved.

It is natural to ask, therefore, if the breaking of P and/or T is necessary to realize a spatially modulated phase at finite charge density in the context of holography. Here we will show that it is not.[2] Specifically, we will discuss two general classes of models associated with CDW phases, without current density waves, and preserving P and T. Our general strategy will be to consider setups in which the zero-temperature limit of the electrically charged black holes that describe the unbroken high-temperature phase are domain-wall solutions interpolating between some UV fixed point and an electrically charged $AdS_2 \times \mathbb{R}^{D-2}$ solution in the far IR. We then construct CDW-type modes that violate the AdS_2 Breitenlohner-Freedman (BF) bound; the existence of these modes necessarily implies that the finite-temperature unbroken-phase black holes become unstable at some critical temperature. Furthermore, at this critical temperature the unbroken-phase black holes will admit a static normalizable mode corresponding to the existence of a new branch of electrically charged black holes dual to the CDW phase.

The first class of models, discussed in Sec. II, involve Einstein-Maxwell theory coupled to a single scalar field ϕ, often called the dilaton. These models have a single conserved U(1) symmetry in the dual field theory. We will see

[1] Spatially modulated phases preserving the U(1) symmetry have also been shown to exist in the presence of magnetic fields in Refs. [9–15], where the magnetic field itself breaks T.

[2] For a discussion of constructions exploiting magnetic gaugings, see Ref. [24].

that the relevant CDW mode involves the gauge field, the dilaton and the metric.

The second class of models, discussed in Sec. III, involves Einstein-Maxwell-dilaton models with an additional coupling to a second vector field which can either be massive or massless. This class of models is somewhat simpler than the first in that the standard electrically charged AdS-RN black brane solution solves the equations of motion. The AdS-RN black branes describe the high-temperature unbroken phase of a dual CFT when held at finite charge density with respect to the U(1) symmetry associated with the Maxwell field. When the second vector field is massless the dual theory has a second global U(1) symmetry. We show that there can be spatially modulated modes just involving the dilaton and the second vector field, which correspond, in the massless case, to CDWs for the second U(1) symmetry.

In Sec. IV we conclude with some discussion on how our constructions should be associated with novel spatially modulated ground states. One route is to follow the spatially modulated phases down to zero temperature. A different route, following Ref. [25], utilises AdS_2 solutions which contain spatially modulated modes that are dual to relevant operators in the renormalization group (RG) sense.

II. EINSTEIN-MAXWELL-DILATON MODELS

In this section we consider Einstein-Maxwell-dilaton models in D spacetime dimensions, which couple gravity to a gauge field A, with field strength $F = dA$, and a scalar field ϕ, the "dilaton." The Lagrangian density is given by

$$\mathcal{L} = R - V(\phi) - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}\tau(\phi)F^2, \tag{2.1}$$

where $F^2 = F_{\mu\nu}F^{\mu\nu}$ and V, τ are functions that we will partially restrict below. The equations of motion derived from the action (2.1) can be written as

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{D-2}Vg_{\mu\nu} + \frac{1}{2}\partial_\mu\phi\partial_\nu\phi \\ &\quad + \frac{1}{2}\tau\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{2(D-2)}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right), \\ \nabla_\mu(\tau F^{\mu\nu}) &= 0, \qquad \nabla^2\phi - V' - \frac{1}{4}\tau' F^2 = 0. \end{aligned} \tag{2.2}$$

This class of models has been studied in a holographic context in Ref. [26] (see also e.g., Refs. [27–37]).

We will consider particular classes of models that admit electrically charged $AdS_2 \times \mathbb{R}^{D-2}$ solutions of the form

$$\begin{aligned} ds^2 &= L^2(ds^2(AdS_2) + dx_1^2 + \cdots + dx_{D-2}^2), \\ F &= E\,\mathrm{Vol}(AdS_2), \qquad \phi = \phi_0, \end{aligned} \tag{2.3}$$

where E, ϕ_0, L are constants and we have written the metric and field strength using two-dimensional anti de-Sitter space with unit radius. Using the equations of motion (2.2) the existence of this class of solutions imposes the conditions

$$\begin{aligned} V(\varphi_0)\tau'(\varphi_0) &= -\tau(\varphi_0)V'(\varphi_0), \\ V(\varphi_0) < 0, &\qquad \tau(\varphi_0) > 0, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} L^2 &= -\frac{1}{V(\varphi_0)}, \\ E^2 &= \frac{2}{-V(\varphi_0)\tau(\varphi_0)}. \end{aligned} \tag{2.5}$$

We will take $E > 0$. We will need the first few terms in the expansion of the potential V and the function τ around the value φ_0. In general, it can be brought to the convenient form

$$\begin{aligned} V &= v_0\left(1 - \tau_1(\varphi - \varphi_0) - \frac{v_2}{2}(\varphi - \varphi_0)^2 + \cdots\right), \\ \tau &= \tau_0\left(1 + \tau_1(\varphi - \varphi_0) - \frac{\tau_2}{2}(\varphi - \varphi_0)^2 + \cdots\right), \\ &\qquad v_0 < 0, \qquad \tau_0 > 0, \end{aligned} \tag{2.6}$$

where we have incorporated the conditions in Eq. (2.4).

We now wish to study linearized perturbations around the background (2.3). For simplicity we will continue with $D = 4$ but we will also quote the final key result for $D = 5$. More specifically, using a coordinate system for AdS_2 such that

$$ds^2(AdS_2) = -r^2dt^2 + \frac{dr^2}{r^2}, \tag{2.7}$$

we are interested in the perturbation

$$\begin{aligned} \delta g_{tt} &= L^2r^2e^{-i\omega t}h_{tt}(r)\cos(kx_1), \\ \delta g_{x_ix_i} &= L^2e^{-i\omega t}h_{x_ix_i}(r)\cos(kx_1), \\ \delta g_{tx_1} &= L^2e^{-i\omega t}h_{tx_1}(r)\sin(kx_1), \\ \delta A_t &= -Ee^{-i\omega t}a_t(r)\cos(kx_1), \\ \delta A_{x_1} &= -Ee^{-i\omega t}a_{x_1}(r)\sin(kx_1), \\ \delta\varphi &= e^{-i\omega t}h(r)\cos(kx_1), \end{aligned} \tag{2.8}$$

with $i = 1, 2$, which involves seven functions of the radius $\{h_{tt}, h_{x_ix_i}, h_{tx_1}, a_t, a_{x_1}, h\}$ in a self-consistent manner, as we shall see.

The linearised perturbation is in a radial gauge but there is still some remaining gauge freedom. Specifically, the combined coordinate transformation $x^\mu \to x^\mu + \delta x^\mu$ and U(1) gauge transformation $A_\mu \to A_\mu + \partial_\mu\delta\Lambda$ with

$$\begin{aligned} \delta t &= -\frac{1}{2r^2}\partial_t f + g_1, \\ \delta r &= rf, \\ \delta x_1 &= -\ln r\,\partial_{x_1}f + g_2, \\ \delta x_2 &= 0, \\ \delta\Lambda &= -\frac{E}{r}\partial_t f + Eg_3, \end{aligned} \tag{2.9}$$

keeps the form of the perturbation (2.8) invariant when $f = f(t, x_1)$ and $g_i = g_i(t, x_1)$. Indeed for such gauge transformations we have

$$\begin{aligned} \delta g_{tt} &= -2L^2\left(r^2 f - \frac{1}{2}\partial_t^2 f + r^2\partial_t g_1\right),\\ \delta g_{x_1x_1} &= 2L^2(-\ln r\partial_{x_1}^2 f + \partial_{x_1} g_2), \qquad \delta g_{x_2x_2} = 0,\\ \delta g_{tx_1} &= L^2\left(\frac{1}{2}\partial_t\partial_{x_1} f - r^2\partial_{x_1} g_1 - \ln r\partial_t\partial_{x_1} f + \partial_t g_2\right),\\ \delta A_t &= E\left(-rf - \frac{1}{2}\frac{1}{r}\partial_t^2 f - r\partial_t g_1 + \partial_t g_3\right),\\ \delta A_{x_1} &= E\left(-\frac{1}{2r}\partial_t\partial_{x_1} f + \partial_{x_1} g_3 - r\partial_{x_1} g_1\right), \end{aligned} \tag{2.10}$$

and we see that two gauge-invariant combinations are given by

$$\frac{1}{L^2 r^2}\delta g_{tt} - \frac{2}{E}\partial_r \delta A_t, \qquad \delta g_{x_2x_2}. \tag{2.11}$$

This will be useful shortly.

We now return to the linearised perturbation (2.8) about the $AdS_2 \times \mathbb{R}^2$ solution (2.3). After substituting into the equations of motion (2.2) we obtain three ordinary differential equations (ODEs) from the gauge field equation of motion, seven from Einstein's equations and one from the scalar equation of motion,

$$\begin{aligned} -2r^2 a_t' + r^3(g_{x_1x_1}' + g_{x_2x_2}') + \omega^2(g_{x_1x_1} + g_{x_2x_2}) - 2ik\omega g_{tx_1} - r^4 g_{tt}'' - 3r^3 g_{tt}' + k^2r^2 g_{tt} - r^2 g_{tt} &= 0,\\ -i\omega(-r(g_{x_1x_1}' + g_{x_2x_2}') + g_{x_1x_1} + g_{x_2x_2}) + krg_{tx_1}' - 2kg_{tx_1} &= 0,\\ 2a_t' - r(r(g_{x_1x_1}'' + g_{x_2x_2}'' - g_{tt}'') + g_{x_1x_1}' + g_{x_2x_2}' - 3g_{tt}') + g_{tt} &= 0,\\ r^2(2a_t' + 2rg_{x_1x_1}' + 2\tau_1 h + r^2 g_{x_1x_1}'' + (1+k^2)g_{tt} - k^2 g_{x_2x_2}) + \omega^2 g_{x_1x_1} - 2ik\omega g_{tx_1} &= 0,\\ \omega^2 g_{x_2x_2} + r^2(2a_t' + 2rg_{x_2x_2}' + 2\tau_1 h - k^2 g_{x_2x_2} + r^2 g_{x_2x_2}'' + g_{tt}) &= 0,\\ r^2(2\tau_1 a_t' + r^2 h'' + 2rh' - k^2 h - (\tau_2 + v_2)h + \tau_1 g_{tt}) + \omega^2 h &= 0,\\ i\omega(2a_t' + g_{x_1x_1} + g_{x_2x_2} + g_{tt} + 2\tau_1 h) + 2kr^2 a_{x_1}' + 2kg_{tx_1} &= 0,\\ 2ik\omega a_{x_1} - 2k^2 a_t + r^2(2a_t'' + g_{x_1x_1}' + g_{x_2x_2}' + g_{tt}' + 2\tau_1 h') &= 0,\\ r^2(2a_{x_1}' + g_{tx_1}'') + ik\omega g_{x_2x_2} &= 0,\\ -k(2a_t + r(r(g_{tt}' - g_{x_2x_2}') + g_{tt})) + i\omega(2a_{x_1} + g_{tx_1}') &= 0,\\ r^2(r^2 a_{x_1}'' + 2ra_{x_1}' + g_{tx_1}') + \omega^2 a_{x_1} + ik\omega a_t &= 0. \end{aligned} \tag{2.12}$$

Some analysis now shows that four of these ODEs are implied by the remaining seven, as expected from the Bianchi identity for the Einstein tensor. Furthermore, we find that in this system of seven ODEs the three functions $\{h, a_t, h_{x_2x_2}\}$ enter with second-order derivatives in r while the four functions $\{h_{x_1x_1}, h_{tx_1}, h_{tt}, a_{x_1}\}$ enter with first-order derivatives. This indicates that there are three propagating degrees of freedom and four constraints.

To proceed we now exploit our previous discussion on gauge invariance and introduce the variables

$$\Phi_1 = h_{tt} + 2a_t', \qquad \Phi_2 = g_{x_2x_2}, \qquad \Phi_3 = h, \tag{2.13}$$

to find that the system of seven ODEs can be written in terms of $\{\Phi_1, \Phi_2, \Phi_3\}$ and $\{h_{x_1x_1}, h_{tx_1}, h_{tt}, a_{x_1}\}$ in the form

$$
\begin{aligned}
&\omega^2\Phi_3 + r^3(r\Phi_3'' + 2\Phi_3') + r^2(\tau_1\Phi_1 - (k^2 + v_2 + \tau_2)\Phi_3) = 0,\\
&\omega^2\Phi_2 + r^3(r\Phi_2'' + 2\Phi_2') + r^2(\Phi_1 - k^2\Phi_2 + 2\tau_1\Phi_3) = 0,\\
&2ik\omega(ra_{x_1} + g_{tx_1}) + 2k^2r^2a_t' - 2k^2ra_t - \omega^2 g_{x_1x_1} + 2r^3\tau_1\Phi_3' - r^2(2\tau_1\Phi_3 + \Phi_1)\\
&\quad - k^2r^2\Phi_1 + k^2r^2\Phi_2 + r^3\Phi_1' - \omega^2\Phi_2 = 0,\\
&-2k^2r^2a_t' + r^3g_{x_1x_1}' + \omega^2 g_{x_1x_1} - 2ik\omega g_{tx_1} + 2r^2\tau_1\Phi_3 + (k^2+1)r^2\Phi_1 + (\omega^2 - k^2r^2)\Phi_2 + r^3\Phi_2' = 0,\\
&i\omega(-2k^2r^2a_t' + 2r^2\tau_1\Phi_3 + (k^2+1)r^2\Phi_1 - k^2r^2\Phi_2 + r^2\Phi_2 + \omega^2\Phi_2 + (r^2+\omega^2)g_{x_1x_1})\\
&\quad - kr^3g_{tx_1}' + 2k(r^2+\omega^2)g_{tx_1} = 0,\\
&2ik(k^2+1)r^3\omega a_{x_1} + 2k^2r^5a_t'' + 2k^2r^4(k^2+1)a_t' + 2k^2r^2\omega^2a_t' - 2k^2(k^2+1)r^3a_t - \omega^2((k^2+1)r^2+\omega^2)g_{x_1x_1}\\
&\quad + 2ik\omega g_{tx_1}((k^2+1)r^2+\omega^2) + 2k^2r^5\tau_1\Phi_3' - 2r^2\tau_1(k^2r^2+\omega^2)\Phi_3 + (k^4r^4 - \omega^2(r^2+\omega^2))\Phi_2\\
&\quad + k^2r^5\Phi_2' - r^2(k^2((k^2+2)r^2+\omega^2)+\omega^2)\Phi_1 = 0,\\
&\quad i\omega(g_{x_1x_1} + 2\tau_1\Phi_3 + \Phi_1 + \Phi_2) + 2k(r^2a_{x_1}' + g_{tx_1}) = 0.
\end{aligned}
\tag{2.14}
$$

The first two equations comprise part of a second-order system for the three gauge-invariant fields $\{\Phi_1, \Phi_2, \Phi_3\}$. We can solve the third equation for Φ_1'. Differentiating this expression and then using the remaining equations, we find the remaining second-order equation. Introducing the three-vector $\mathbb{V}^T = (\Phi_1, \Phi_2, \Phi_3)$ we can write these in matrix form as the standard system of three mixed modes propagating on AdS_2,

$$\left(\frac{\omega^2}{r^2} + r^2\partial_r^2 + 2r\partial_r\right)\mathbb{V} - M^2\mathbb{V} = 0, \tag{2.15}$$

with the mass matrix

$$M^2 = \begin{pmatrix} 2+\tau_1^2+k^2 & -2k^2 & 2\tau_1(2-k^2-\tau_2-v_2) \\ -1 & k^2 & -2\tau_1 \\ -\tau_1 & 0 & k^2+v_2+\tau_2 \end{pmatrix}. \tag{2.16}$$

In addition, once we have specified these three fields the remaining four functions are determined by solving first-order equations arising from Eq. (2.14) and demanding regularity at the Poincaré horizon of the AdS_2 space at $r = 0$.

To summarize, for $D = 4$ we have shown that the linearized perturbation (2.8) corresponds to three propagating modes in AdS_2 with the mass matrix (2.16). We note that the mass matrix (2.16) only depends on two parameters appearing in Eq. (2.6), τ_1 and $\tau_2 + v_2$. As a check we note that when $\tau_1 = 0$ the scalar field decouples from the metric and gauge-field fluctuations and we obtain the mass-squared eigenvalues

$$m_1^2 = k^2 + v_2 + \tau_2, \qquad m_\pm^2 = 1 + k^2 \pm \sqrt{1+2k^2}. \tag{2.17}$$

In particular, the masses $m_\pm^2$ agree with previous studies of longitudinal perturbations of Einstein-Maxwell theory around its $AdS_2 \times \mathbb{R}^2$ solution [38].

The AdS_2 BF bound is violated if any of the three eigenvalues m_i^2 of the mass matrix (2.16) is such that $m_i^2 < -1/4$. The signal for spatially modulated phases are modes that violate the BF bound with the smallest mass-squared occurring at $k \neq 0$. This is easily achieved by suitable choices of the parameters τ_1 and $\tau_2 + v_2$, including cases where BF-violating modes only occur for $k \neq 0$. Indeed, the mass-squared eigenvalues when $k = 0$ are simply

$$m_1^2 = 0, \qquad m^2 = 2, \qquad m_3^2 = 2\tau_1^2 + \tau_2 + v_2, \tag{2.18}$$

and demanding that $2\tau_1^2 + \tau_2 + v_2 \geq -1/4$ there are still broad choices for the parameters which have modes with $k \neq 0$ that violate the BF bound. Note from Eq. (2.17) that while there can be instabilities when $\tau_1 = 0$ they will not be spatially modulated.

To illustrate we can consider the specific choice of V, τ given by

$$V = v_0 e^{-\gamma\phi}, \qquad \tau = e^{\gamma\phi}, \tag{2.19}$$

where γ is a constant, that was studied in Ref. [26]. The eigenvalues of the mass matrix are given by k^2, $1 + k^2 \pm \sqrt{1 + 2(1+\gamma^2)k^2}$. At $k = 0$ the eigenvalues are 0, 0, and 2 and hence do not violate the BF bound. If $\gamma > 1$ there are BF-violating modes for a range of $k \neq 0$, with the minimum mass-squared occurring at $k^2 = \gamma^2(2+\gamma^2)/(2(1+\gamma^2))$.

We emphasize that the spatially modulated BF-violating modes that we have identified are associated with CDWs preserving P and T. To see this[3] we first note that the mode that is relevant for this discussion is static with $\omega = 0$. From Eq. (2.14) we deduce that these modes have $h_{tx_1} = a_{x_1} = 0$ and hence the linearized perturbation in Eq. (2.8) has $\delta g_{tx_1} = \delta A_{x_1} = 0$.

[3]We will elaborate on this discussion in a simpler setting in the next section in the context of another class of models.

Thus, the gauge-field part of the linearized perturbation only involves a spatially modulated A_t component corresponding to a spatially modulated CDW in the dual theory. It is also clear from the perturbation in Eq. (2.8) that the CDW preserves P and T. Indeed it is worth noting that the mass-matrix in Eq. (2.8) only depends on k^2 in contrast to other cases that have been analyzed in the literature [e.g., Eq. (3.17) of Ref. [3] and Eq. (2.17) of Ref. [7]] where it depends linearly on k.

Finally, we have focussed on Einstein-Maxwell-dilaton theory in $D = 4$ spacetime dimensions, but very similar comments apply to other D. For example, when $D = 5$, the analogue of the mass matrix given in Eq. (2.16) is presented in the Appendix.

III. EINSTEIN-MAXWELL-DILATON-VECTOR MODELS

We now consider a theory in D bulk spacetime dimensions which couples gravity to a massless U(1) gauge field A, a scalar dilaton field ϕ and an additional vector field B of mass m_v. The Lagrangian density is given by

$$\mathcal{L} = R - \frac{1}{2}(\partial\phi)^2 - V(\phi) - \frac{1}{4}t(\phi)F^2 - \frac{1}{4}v(\phi)G^2 - \frac{1}{2}m_v^2 B^2 - \frac{1}{2}u(\phi)FG, \tag{3.1}$$

where $F = dA$ and $G \equiv dB$ and $FG = F_{\mu\nu}G^{\mu\nu}$. The corresponding equations of motion are given by

$$\begin{aligned}
R_{\mu\nu} &= \frac{1}{D-2}Vg_{\mu\nu} + \frac{1}{2}m_v^2 B_\mu B_\nu + \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}t\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{2(D-2)}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)\\
&\quad + \frac{1}{2}v\left(G_{\mu\rho}G_\nu{}^\rho - \frac{1}{2(D-2)}g_{\mu\nu}G_{\rho\sigma}G^{\rho\sigma}\right) + u\left(G_{(\mu}{}^\rho F_{\nu)\rho} - \frac{1}{2(D-2)}g_{\mu\nu}G_{\rho\sigma}F^{\rho\sigma}\right),\\
&\nabla_\mu(tF^{\mu\nu} + uG^{\mu\nu}) = 0, \qquad \nabla_\mu(vG^{\mu\nu} + uF^{\mu\nu}) - m_v^2 B^\nu = 0,\\
&\nabla^2\phi - V' - \frac{1}{4}t'F^2 - \frac{1}{4}v'G^2 - \frac{1}{2}u'FG = 0.
\end{aligned} \tag{3.2}$$

We will assume that the functions V, t, u and v have the following expansion:

$$\begin{aligned}
V(\phi) &= -\frac{1}{L^2} + \frac{1}{2}m_s^2\phi^2 + \cdots,\\
t(\phi) &= 1 - \frac{1}{2}nL^2\phi^2 + \cdots,\\
u(\phi) &= \frac{1}{\sqrt{2}}sL\phi + \cdots,\\
v(\phi) &= 1 + \cdots
\end{aligned} \tag{3.3}$$

Then AdS_D, with radius squared ℓ^2, where

$$\ell^2 = L^2(D-1)(D-2), \tag{3.4}$$

and $\phi = A = B = 0$, solves the equations of motion and is dual to a CFT in $D-1$ spacetime dimensions. The gauge field A is dual to a conserved current for a global U(1) symmetry, while ϕ and B are dual to neutral scalar and vector operators, respectively. In the special case that $m_v = 0$ the dual CFT has a second global U(1) symmetry and B is dual to the conserved current.

We are interested in the CFT held at fixed chemical potential μ with respect to the global U(1) symmetry associated with the gauge field A. The high-temperature phase is described by the standard electrically charged AdS-RN black brane solution with $\phi = B = 0$. At zero temperature, the solution interpolates between AdS_D in the UV and $\mathrm{AdS}_2 \times \mathbb{R}^{D-2}$ in the IR, with the latter solution given by

$$\begin{aligned}
ds_4^2 &= L^2(ds^2(\mathrm{AdS}_2) + dx_1^2 + \cdots + dx_{D-2}^2),\\
F &= E\,\mathrm{vol}(\mathrm{AdS}_2), \qquad \phi = G = 0,
\end{aligned} \tag{3.5}$$

where $E = \sqrt{2}L$.

In the following subsection we will first examine CDW-type instabilities of this $\mathrm{AdS}_2 \times \mathbb{R}^{D-2}$ solution. Such instabilities imply that electrically charged AdS-RN will have analogous instabilities at finite temperature, and these will be analyzed in the subsequent subsection. We also note that the spatially modulated instabilities that we saw in the last section are not present in the models that we consider in this section because in Eq. (3.3) we have assumed $t'(0) = 0$.

A. Instabilities of the $\mathrm{AdS}_2 \times \mathbb{R}^{D-2}$ solution

We consider the following linearized perturbation about Eq. (3.5), using the coordinates for AdS_2 given in Eq. (2.7),

$$\begin{aligned}
\delta\phi &= e^{-i\omega t + ikx_1}\Phi(r),\\
\delta B &= e^{-i\omega t + ikx_1}\left(r^2 b_t(r)dt + i\omega b_r(r)\frac{dr}{r^2}\right).
\end{aligned} \tag{3.6}$$

We now substitute this into the equations of motion (3.2). The x_1 component of the equation of motion for the gauge field B implies the constraint[4]

[4]Note that when $m_v^2 \neq 0$ the equations of motion (3.2) imply that $\nabla_\mu \delta B^\mu = 0$ and hence $\omega(b_t + b_r') = 0$. We also note that when $m_v^2 = 0$ we can work in a gauge with $\delta B_r = 0$ if desired.

$$k\omega(b_t + b_r') = 0. \tag{3.7}$$

We also obtain a coupled system of equations which we can write in matrix form as

$$\left(\frac{\omega^2}{r^2} + r^2\partial_r^2 + 2r\partial_r\right)\mathbb{V} - L^2M^2\mathbb{V} = 0, \tag{3.8}$$

where $\mathbb{V}^T = (\Phi, b_r)$ and the mass matrix is given by

$$M^2 = \begin{pmatrix} m_s^2 + n + s^2 + p^2 & -s(m_v^2 + p^2) \\ -s & m_v^2 + p^2 \end{pmatrix}, \tag{3.9}$$

with $p = k/L$. The matrix (3.9) yields the AdS_2 mass spectrum

$$m_\pm^2 = \frac{1}{2}(\tilde{m}_s^2 + m_v^2 + s^2) + p^2 \pm \frac{1}{2}\sqrt{(\tilde{m}_s^2 - m_v^2)^2 + 2(\tilde{m}_s^2 + m_v^2 + 2p^2)s^2 + s^4}, \tag{3.10}$$

where

$$\tilde{m}_s^2 = m_s^2 + n. \tag{3.11}$$

It is now straightforward to choose parameters such that there are modes violating the BF bound associated with spatially modulated phases. As a specific example, which we will return to in the next subsection, if we set $D = 4$, $L^2 = 1/24$, $m_s^2 = -8$, $m_v^2 = 0$, $n = -96$, $s = 16.2$ we find $L^2m_-^2 < -1/4$ in the range

$$3.52 \lesssim |p| \lesssim 8.92. \tag{3.12}$$

Note that when $\omega = 0$ the spatially modulated BF-violating modes involve the scalar field and only the time component of the gauge field, B_t. To be more explicit, consider the static mode

$$\delta\phi = e^{ikx_1}v_1 r^\lambda, \qquad \delta B = e^{ikx_1}v_2 r^{\lambda+1}dt, \tag{3.13}$$

where v_1, v_2 are constants and λ corresponds to the scaling dimension of an operator in the one-dimensional CFT dual to the $AdS_2 \times \mathbb{R}^{D-2}$ solution. We obtain a solution to the linearized equations of motion provided that

$$\left(\lambda(\lambda+1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - L^2\begin{pmatrix} m_s^2 + n + p^2 & \frac{s}{L^2}(\lambda+1) \\ \lambda s & m_v^2 + p^2 \end{pmatrix}\right)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0. \tag{3.14}$$

This can be solved for λ and it is straightforward to check that $\lambda(\lambda + 1) = m_\pm^2$ as expected. As usual, the BF bound is violated when λ becomes imaginary, the onset of which happens when $\lambda = -1/2$. These BF-violating modes are associated with spatially modulated phases that preserve P and T. Furthermore, in the special case that $m_v^2 = 0$, when there is a second global U(1) symmetry in the dual CFT, the unstable modes correspond to CDWs for the second U(1). We will make this more explicit in the next subsection.

B. AdS-RN black hole instabilities

In this subsection we will show, for an illustrative case, that the instabilities that we deduced for the $AdS_2 \times \mathbb{R}^{D-2}$ solutions are associated with instabilities of the finite-temperature AdS-RN black brane solution. In particular, we will calculate the value of the critical temperature at which the AdS-RN black brane becomes unstable as a function of wave number, finding the usual "bell curve"-type behavior.

We will work in $D = 4$. The electrically charged AdS-RN black brane is given by

$$ds^2 = -fdt^2 + \frac{dr^2}{f} + r^2(dx_1^2 + dx_2^2), \qquad A = \left(1 - \frac{r_+}{r}\right)dt, \qquad \phi = B = 0, \tag{3.15}$$

with

$$f = \frac{r^2}{\ell^2} - \left(\frac{r_+^2}{\ell^2} + \frac{1}{4}\right)\frac{r_+}{r} + \frac{r_+^2}{4r^2}. \tag{3.16}$$

This describes the high-temperature phase of the dual $d = 3$ CFT at finite temperature T and nonvanishing chemical potential μ with respect to the global symmetry corresponding to the gauge field A. Notice, for convenience, that we have scaled to set $\mu = 1$ and that $T = (12r_+^2 - \ell^2)/(16\pi r_+\ell^2)$. We will set $m_v^2 = 0$ so that the CFT has a second global symmetry, corresponding to the gauge field B. The instabilities that we discuss correspond to CDW phases associated with this second U(1) symmetry.

Specifically, following from the analysis above, we consider the following linearized perturbation:

$$\delta\phi = \phi(r)\cos(kx_1), \qquad \delta B = b_t(r)\cos(kx_1)dt. \tag{3.17}$$

For illustration we will take the scalar field to have mass given by $m_s^2 = -2/\ell^2$. This corresponds to having an operator in the dual $d = 3$ CFT with scaling dimension $\Delta = 1, 2$ and we will choose the boundary conditions so that $\Delta = 2$. Since we are interested in instabilities associated with phases that spontaneously break translation invariance, we consider the following UV expansion as $r \to \infty$, in which the sources are set to zero:

$$\phi(r) \approx \frac{\phi_2}{r^2} + \cdots, \qquad b_t(r) \approx \frac{q}{r} + \cdots \tag{3.18}$$

We will also demand regularity on the black hole horizon at $r = r_+$, by demanding that the functions admit the analytic expansion,

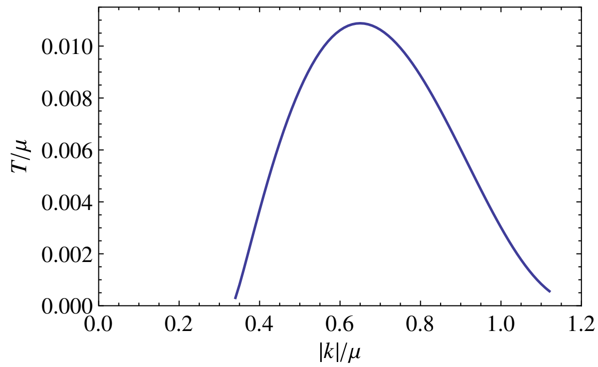

FIG. 1 (color online). A plot of the critical temperature versus wave number at which the electrically charged AdS-RN black brane becomes unstable to the formation of a CDW phase preserving P and T. The plot is for the model (3.1) and (3.3) with $n = -96$, $s = 16.2$, $m_s^2 = -8$, and $m_v^2 = 0$. For these values the highest critical temperature occurs at $T_c/\mu \approx 0.011$ with $|k_c|/\mu \approx 0.645$.

$$\phi(r) \approx \phi_{(0)} + \mathcal{O}(r - r_+),$$
$$b_t(r) \approx b_{(0)}(r - r_+) + \mathcal{O}(r - r_+)^2. \qquad (3.19)$$

The linearized equations of motion lead to two second-order differential equations for $\phi(r)$, $b_t(r)$ and so a solution is specified by four integration constants. For a given k we have five parameters, r_+, ϕ_2, q, $\phi_{(0)}$, $b_{(0)}$ entering the ODEs. Since the ODEs are linear, we can always scale one of the parameters to unity. This means that for a given k, we expect solutions to exist, if at all, for specific critical values of T. These are precisely the static modes appearing at the onset of the instability. In Fig. 1 we have plotted the critical temperature versus wave number for these modes for the specific choice of parameters given by $n = -96$, $s = 16.2$ and $\ell = 1/2$. Recall that we used the same parameters in the last subsection and we also note that in contrast to the last subsection, the equations now depend on m_s^2 and n individually and not just in the combination (3.11). To make a comparison with the wave numbers of the $AdS_2 \times \mathbb{R}^2$ analysis in the last subsection given in Eq. (3.12), we should rescale the spatial coordinates by a factor of $\sqrt{2}$ leading to dividing p by $4\sqrt{3}$ so that Eq. (3.12) corresponds to $0.508 \lesssim |k| \lesssim 1.29$, in good agreement with Fig. 1.

The static mode with the highest critical temperature, T_c, associated with the wave number k_c, corresponds to the onset of CDW phases in the dual CFT. Indeed, for temperatures just below T_c the charge density $j_t^{(B)}$ for the second U(1), corresponding to the gauge field B, becomes spatially modulated with

$$\langle j_t^{(B)} \rangle \sim q \cos(k_c x_1). \qquad (3.20)$$

At the same time the operator $\mathcal{O}_\phi$, dual to the scalar field ϕ, also spontaneously acquires an expectation value given by $\langle \mathcal{O}_\phi \rangle \sim \phi_2 \cos(k_c x_1)$. It is clear that this new CDW phase preserves P and T.

IV. DISCUSSION

We have shown that two classes of holographic models with electrically charged $AdS_2 \times \mathbb{R}^{D-2}$ solutions can have spatially modulated BF-violating modes that are associated with CDWs. Specifically, these modes demonstrate the existence of rich classes of spatially modulated black hole solutions that are dual to the appearance of CDW phases, preserving both P and T, in holographic matter at finite charge density. For the models discussed in Sec. III with two U(1) vector fields, we deduced the critical temperature at which these black hole solutions will appear. Similar calculations are also possible for the Einstein-Maxwell-dilaton theory considered in Sec. II, by choosing specific functions V, τ, constructing the unbroken-phase black holes and then analyzing the linearized perturbations. The CDW instabilities can be embedded into top down settings. For example, the model in Sec. III is associated with Romans theory and hence type IIB and $D = 11$ supergravity (see the discussion in Sec. 2 of [9]).

Constructing the fully backreacted black hole solutions will require solving nonlinear partial differential equations. The simplest black hole solutions will depend on two variables, the radius r and the coordinate x_1, being static and translationally invariant in the remaining spatial coordinates. However, since at the linearized level we can superimpose the static normalizable modes, there will also be more elaborate black hole solutions that depend on all of the spatial coordinates. Only a detailed analysis will reveal which periodic structure is thermodynamically preferred. It would be particularly interesting to construct the black hole solutions all the way down to zero temperature so that ground states of the system can be identified. While it is possible that the spatial modulation disappears at zero temperature it seems much more likely that the generic ground states will be spatially modulated CDWs.

The results of this paper, combined with those of Ref. [25], also suggest other constructions of novel spatially modulated holographic ground states. We suppose the $AdS_2 \times \mathbb{R}^{D-2}$ solution still arises as the IR limit of a domain-wall solution interpolating from some holographic behaviour in the UV. But now we assume that the $AdS_2 \times \mathbb{R}^{D-2}$ solution is stable, without any modes violating the BF bound. In addition we assume[5] that the $AdS_2 \times \mathbb{R}^{D-2}$ IR fixed-point solution does not have any relevant operators with $k = 0$, but does have relevant operators with $k \neq 0$. If we now consider deforming the UV fixed point of the domain wall by a spatially homogeneous and isotropic deformation, the IR fixed point will be stable under RG flow, because of the absence of $k = 0$ relevant modes in the IR. However, if we switch on a suitable spatially modulated deformation in the UV (such as a spatially modulated chemical potential) the RG flow will be destabilized due to the relevant IR operators with $k \neq 0$. It seems likely that

[5] A concrete example is provide by the model (2.19) with $0 < \gamma < 1$.

such constructions will also lead to rich classes of spatially modulated ground states. As in Ref. [25], these constructions should include metal-insulator transitions where the metallic phase is described by the $AdS_2 \times \mathbb{R}^{D-2}$ IR behavior and the insulating phase by the putative spatially modulated ground state. The spatial modulation of the charge density suggests that such insulating phases can be interpreted as holographic Mott insulators.

The holographic CDW phases that we have found in this paper were all in the context of models in which the unbroken-phase black holes have a zero-temperature limit containing an AdS_2 factor in the IR. This was purely a technical simplification and similar spatially modulated phases should also occur in many other situations. For example, within Einstein-Maxwell-dilaton theory there can be black holes whose zero-temperature limit interpolates between some UV behavior and a hyperscaling-violating ground state [26,35,36] in the far IR and these should manifest similar phases, leading to even more spatially modulated ground states at zero temperature. Support for the conjecture made in Ref. [21] that the generic holographic states at finite density and/or in a magnetic field are spatially modulated continues to accumulate.

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APPENDIX: $D = 5$ MASS MATRIX

For Einstein-Maxwell-dilaton theory in D spacetime dimensions the conditions (2.4), (2.5), and (2.6) are still applicable. The relevant perturbation about the $AdS_2 \times \mathbb{R}^3$ solution is

$$\begin{aligned}
\delta g_{tt} &= L^2 r^2 e^{-i\omega t} h_{tt}(r)\cos(kx_1),\\
\delta g_{x_i x_i} &= L^2 e^{-i\omega t} h_{x_i x_i}(r)\cos(kx_1),\\
\delta g_{tx_1} &= L^2 e^{-i\omega t} h_{tx_1}(r)\sin(kx_1),\\
\delta A_t &= -E e^{-i\omega t} a_t(r)\cos(kx_1),\\
\delta A_{x_1} &= -E e^{-i\omega t} a_{x_1}(r)\sin(kx_1),\\
\delta\varphi &= e^{-i\omega t} h(r)\cos(kx_1),
\end{aligned} \tag{A1}$$

with $h_{x_2x_2} = h_{x_3x_3}$. The analysis is very similar to that of the $D = 4$ case. Using the same definition of the three scalar AdS_2 fields given in Eq. (2.13) we obtain the mass matrix

$$M^2 = \begin{pmatrix} 2 + 2\tau_1^2 + k^2 & -4k^2 & 2\tau_1(2 - k^2 - \tau_2 - v_2) \\ -2/3 & k^2 & -4\tau_1/3 \\ -\tau_1 & 0 & k^2 + v_2 + \tau_2 \end{pmatrix}. \tag{A2}$$

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