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Quantum corrections to $AdS_5 \times S^5$ left-invariant superstring current algebra

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In this work the pure spinor formulation of the superstring is used to study quantum corrections to the left current operator product expansion algebra of the coset $PSU(2,2|4)/SO(4,1) \times SO(5)$ sigma model, which describes the superstring dynamics in the $AdS_5 \times S^5$ background. In particular, the one-loop corrections to the simple poles of the bosonic currents are computed. Unlike the case of the double poles, we show that the simple poles suffer corrections, which are important since the simple poles contribute to the four-point amplitudes. We show that the only contribution to the simple poles comes from the pure spinor Lorentz currents.

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I. INTRODUCTION

In view of the structural role anti–de Sitter (AdS) superstrings plays in the construction of viable theoretic frameworks for understanding the nonperturbative regime of Yang-Mills theory, the AdS superstrings and the AdS/CFT correspondence are topics of constant and renewed interest in the string literature. Also, on the string side, the quantization of the string in the $AdS_5 \times S^5$ space provides an important example of string quantization in a Ramond-Ramond background and it is interesting by itself.

Despite all the results concerning the $AdS_5 \times S^5$ superstring, owing to the complicated structure of the worldsheet action, the quantum properties of the string sigma model still remain elusive. In this direction, the one-loop conformal invariance was proved in Refs. [1,2] and the argument for all loop conformal invariance was presented in Ref. [3]. In Ref. [4], the one-loop effective action was computed, where it was shown that the 't Hooft coupling λ and the AdS radius are not renormalized at one loop. This result was confirmed in Ref. [5], where the double poles of the left-invariant currents were calculated at one loop.

In this work, more ingredients of the quantum properties of the $AdS_5 \times S^5$ string sigma model are presented. A basic set of operators in this two-dimensional conformal field theory is composed by the left-invariant currents. Since these currents are not holomorphic even in the classical limit, their operator product expansions (OPEs) cannot be deduced from general arguments and we need to develop a perturbative approach. Although these currents are not gauge invariant, they are invariant under global $PSU(2,2|4)$ transformations and they can be used to construct integrated massless vertex operators; also, they appear in massive unintegrated vertex operators. In the case of the pure spinor superstring, these OPEs play an important role, since the Becchi-Rouet-Stora-Tyutin (BRST) charge is written in terms of the left-invariant currents. In addition, the quantum OPEs are needed to unambiguously define the b antighost in $AdS_5 \times S^5$, taking into account normal-ordering corrections [6]. Besides these practical applications, the knowledge of this current algebra in the worldsheet may shed light on more general aspects of the theory, such as the apparent quantum integrability.[1] The tree-level OPEs of these currents were computed in Ref. [12] (see also Ref. [9]). In Ref. [5] a momentum-space perturbative approach was developed to calculate the quantum corrections and it was used to calculate the double pole corrections. Surprisingly, in the bosonic sector most of the possible one-loop corrections vanish due to spacetime supersymmetry and there is no correction. The result obtained in this reference confirms the effective action result obtained in Ref. [4], and it serves as further evidence that the relation between the 't Hooft coupling and the AdS radius is not renormalized. However, for the simple pole we do not expect this behavior, because the simple poles contribute to the four-point amplitudes.

Keeping in mind this whole framework and the motivations mentioned above, we use the perturbative techniques developed in Ref. [5] and calculate the one-loop simple poles of bosonic left-invariant currents of the $AdS_5 \times S^5$

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[1]The classical integrability of the $AdS_5 \times S^5$ sigma model was established in the Ref. [7] for the Green-Schwarz-Metsaev-Tseytlin action and in Ref. [8] for the pure spinor description. Using cohomological and algebraic renormalization techniques, Berkovits argued that the sigma model still has an infinite number of conserved charges when quantum effects are taken into account [3]. A detailed study of the transfer matrix of the worldsheet was done in Ref. [9], where it was shown to be a well-defined operator in quantum theory. The literature on this subject is very large, and we did not attempt to give a list of references. The reader can see a list of references in Refs. [10,11].

pure spinor superstring and we show that the pure spinor Lorentz currents play an important role in the result. In fact, we show that the only contribution for the simple poles comes from the pure spinor currents. We present this paper according to the following outline. In Sec. II, we present a review of the pure spinor superstring in the $AdS_5 \times S^5$ background. Section III is devoted to presenting the Feynman rules and dimensional regularization used in our calculations. In Sec. IV we present our results. The conclusions are summarized in Sec. V and is followed by the Appendix.

II. REVIEW OF PURE SPINOR $AdS_5 \times S^5$ SUPERSTRING

In the Green-Schwarz (GS) superstring, the target space supersymmetries are manifest and the superspace coordinates are treated more symmetrically with respect to the Ramond-Neveu-Schwarz formalism. So, relative to the Ramond-Neveu-Schwarz formalism, the GS formalism is more appropriate to study curved backgrounds with Ramond-Ramond fluxes. Using the GS superstring, Metsaev and Tseytlin constructed the worldsheet action for the type IIB superstring in $AdS_5 \times S^5$ from a geometrical point of view based on a super coset approach [13]. They showed that the superstring in the $AdS_5 \times S^5$ background can be described using some currents defined in the superalgebra psu(2, 2|4). Those currents are described in terms of the supervielbein $E_M{}^A$ and are defined in a left-invariant way by

$$\begin{aligned} J^A &= (g^{-1}\partial g)^A = \partial Z^M E_M^A, \\ \bar{J}^A &= (g^{-1}\bar{\partial} g)^A = \bar{\partial} Z^M E_M^A, \end{aligned} \tag{1}$$

where Z^M are the curved superspace coordinates and g an element in the coset supergroup $PSU(2,2|4)/SO(4,1) \times SO(5)$. The index A denotes $(\underline{a}, \alpha, \hat{\alpha},)$ and $a = 0, \ldots, 4$ for AdS_5, $a' = 5, \ldots, 9$ for S^5, $\alpha = 1, \ldots, 16$, $\hat{\alpha} = 1, \ldots, 16$ and $\underline{a}$ denotes both a and a'.

In spite of the target space manifest supersymmetry of the GS formalism, we encounter—already in flat space—serious difficulties once we try to covariantly quantize the GS superstring. This is because the first- and second-class constraints that appear in this formalism cannot be separated in a covariant way. A manifestly supersymmetric formalism, which can be quantized in a covariant way, was proposed by Berkovits in Ref. [14]. In this formalism, in order to have standard fermionic kinetic terms, certain ghost fields have to be introduced, satisfying pure spinor constraints. The nonphysical degrees of freedom introduced in the theory in this way are later removed through a BRST-like operator Q. Although the BRST operator used in the pure spinor formalism is not originally constructed in a traditional way, the formalism has passed all tests and nowadays is a powerful tool for understanding superstring theory.[2]

In a curved background, the pure spinor sigma-model action for the type II superstring is obtained by adding to the flat action the integrated vertex operator for supergravity massless states and then covariantizing with respect to ten-dimensional $N = 2$ superreparametrization invariance. The result of doing this is

$$\begin{aligned} S = \frac{1}{2\pi\alpha'} \int d^2z \Big(& \frac{1}{2}\partial Z^M \bar{\partial} Z^N (G_{NM} + B_{NM}) + d_\alpha \bar{\partial} Z^M E_M^\alpha \\ & + \tilde{d}_{\hat{\alpha}} \partial Z^M E_M^{\hat{\alpha}} + \lambda^\alpha \omega_\beta \bar{\partial} Z^M \Omega_{M\alpha}{}^\beta + \tilde{\lambda}^{\hat{\alpha}} \tilde{\omega}_{\hat{\beta}} \partial Z^M \Omega_{M\hat{\alpha}}{}^{\hat{\beta}} \\ & + d_\alpha \tilde{d}_{\hat{\beta}} P^{\alpha\hat{\beta}} + \lambda^\alpha \omega_\beta \tilde{d}_{\hat{\gamma}} C_\alpha{}^{\beta\hat{\gamma}} + \tilde{\lambda}^{\hat{\alpha}} \tilde{\omega}_{\hat{\beta}} d_\gamma \tilde{C}_{\hat{\alpha}}{}^{\hat{\beta}\gamma} \\ & + \lambda^\alpha \omega_\beta \tilde{\lambda}^{\hat{\alpha}} \tilde{\omega}_{\hat{\beta}} S_{\alpha\hat{\alpha}}{}^{\beta\hat{\beta}}) + S_{\text{pure}} + S_{\text{FT}}, \end{aligned} \tag{2}$$

where B_{NM} is the super two-form potential, and S_{pure} is the action for the pure spinor ghosts and is the same as in the flat-space case. The pure spinor condition means that they satisfy $\lambda^\alpha \gamma^c_{\alpha\beta} \lambda^\beta = 0$ and $\hat{\lambda}^{\hat{\alpha}} \gamma^c_{\hat{\alpha}\hat{\beta}} \hat{\lambda}^{\hat{\beta}} = 0$, where $c = 0, \ldots, 9$ is a tangent-space bosonic index. The gravitini and the dilatini fields are described by the lowest θ components of the superfields $C_\alpha{}^{\beta\bar{\gamma}}$ and $\tilde{C}_{\bar{\alpha}}{}^{\bar{\beta}\gamma}$, while the Ramond-Ramond field strengths are in the superfield $P^{\alpha\bar{\beta}}$ [16]. The dilaton is the theta-independent part of the superfield Φ which defines the Fradkin-Tseytlin term

$$S_{\text{FT}} = \frac{1}{2\pi} \int d^2z r\Phi, \tag{3}$$

where r is the worldsheet curvature.

In $AdS_5 \times S^5$ the nonzero component of the two-form B_{AB}, the five-form Ramond-Ramond field strength, and the curvature are, respectively,

$$\begin{aligned} B_{\alpha\hat{\beta}} &= \frac{1}{2}(Ng_s)^{\frac{1}{4}} \sqrt{\alpha'} \delta_{\alpha\hat{\beta}}, \quad P^{\alpha\hat{\beta}} = \frac{\delta^{\alpha\hat{\beta}}}{\sqrt{\alpha'}(Ng_s)^{\frac{1}{4}}}, \\ R_{abcd} &= -\frac{1}{R^2}\eta_{a[c}\eta_{d]b}, \quad R_{a'b'c'd'} = \frac{1}{R^2}\eta_{a'[c'}\eta_{d']b'}, \end{aligned} \tag{4}$$

where R is the radius of AdS_5 and S^5. As shown in Ref. [16], the term containing $S_{\alpha\hat{\alpha}}^{\beta\hat{\beta}}$ is related to the constant spacetime curvature, while the values of the superfields $C_\alpha^{\beta\hat{\gamma}}$ and $\tilde{C}_{\hat{\alpha}}^{\hat{\beta}\gamma}$ are zero, as well as $\Omega_M^{(s)}$ and $\tilde{\Omega}_M^{(s)}$ because they are related to derivatives of the superfield containing the dilaton, which is constant for this background. Finally, the terms containing the spin connections will lead to

$$\begin{aligned} \lambda^\alpha \omega_\beta \bar{\partial} Z^M \Omega_{M\alpha}{}^\beta &= N_{\underline{ab}} \bar{J}^{\underline{ab}}, \\ \hat{\lambda}^{\hat{\alpha}} \hat{\omega}_{\hat{\beta}} \partial Z^M \tilde{\Omega}_{M\hat{\alpha}}{}^{\hat{\beta}} &= \hat{N}_{\underline{ab}} J^{\underline{ab}}, \end{aligned} \tag{5}$$

[2]Actually, in Ref. [15] a new interpretation of the pure spinor BRST charge was proposed, which elegantly explains its origin.

where $J^{\underline{ab}} = \frac{1}{2}\partial Z^M \tilde{\Omega}_{M\underline{ab}}$, $\bar{J}^{\underline{ab}} = \frac{1}{2}\bar{\partial} Z^M \Omega_{M\underline{ab}}$ and $N^{\underline{ab}} = \frac{1}{2}(\lambda\gamma^{\underline{ab}}\omega)$, $\hat{N}^{\underline{ab}} = \frac{1}{2}(\hat{\lambda}\gamma^{\underline{ab}}\hat{\omega})$ are the pure-spinor Lorentz currents, which play an important role in the pure spinor formalism.[3] Using the definition of the currents in terms of the supervielbein and taking the $AdS_5 \times S^5$ values for the metric G_{MN}, the two-form B_{MN}, and the Ramond-Ramond flux, after using the equations of motion for d_α and scaling the fields (see Ref. [5] for details), the pure spinor superstring in $AdS_5 \times S^5$ is written in terms of the psu(2, 2|4) currents as [14]

$$S = \frac{1}{2\pi\alpha'\alpha^2}\int d^2z\Big(\frac{1}{2}J^{\underline{a}}\bar{J}^{\underline{b}}\eta_{\underline{ab}} + \delta_{\alpha\hat{\beta}}(J^\alpha\bar{J}^{\hat{\beta}} - 3J^{\hat{\beta}}\bar{J}^\alpha) + N_{\underline{ab}}\bar{J}^{[\underline{ab}]} + \hat{N}_{\underline{ab}}J^{[\underline{ab}]} - N_{ab}\hat{N}^{ab} + N_{a'b'}\hat{N}^{a'b'}\Big) + S_\lambda + S_{\hat{\lambda}}. \tag{6}$$

Note also that in Eq. (6) all J's, $\bar{J}$'s, and pure-spinor Lorentz currents have engineering dimension one. So, by choosing units in which $2\pi\alpha' = 1$, the action is given in terms of dimensionless worldsheet fields.

The currents $(J^A, \bar{J}^A)$ satisfy the Maurer-Cartan identities $\partial\bar{J}^A - \bar{\partial}J^A + [J, \bar{J}]^A = 0$, so by making a variation of the action and using those identities, we can find the equations of motion,

$$\nabla\bar{J}_2 = -[J_1, \bar{J}_1] + \frac{1}{2}[N, \bar{J}_2] - \frac{1}{2}[J_2, \bar{N}], \tag{7}$$

$$\bar{\nabla}J_2 = [J_3, \bar{J}_3] - \frac{1}{2}[J_2, \bar{N}] + \frac{1}{2}[N, \bar{J}_2], \tag{8}$$

$$\nabla\bar{J}_1 = \frac{1}{2}[N, \bar{J}_1] - \frac{1}{2}[J_1, \bar{N}], \tag{9}$$

$$\bar{\nabla}J_1 = [J_2, \bar{J}_3] + [J_3, \bar{J}_2] + \frac{1}{2}[N, \bar{J}_1] - \frac{1}{2}[J_1, \bar{N}], \tag{10}$$

$$\bar{\nabla}J_3 = \frac{1}{2}[N, \bar{J}_3] - \frac{1}{2}[J_3, \bar{N}], \tag{11}$$

$$\nabla\bar{J}_3 = -[J_2, \bar{J}_1] - [J_1, \bar{J}_2] + \frac{1}{2}[N, \bar{J}_3] - \frac{1}{2}[J_3, \bar{N}]. \tag{12}$$

The pure spinors also have equations of motion, given by $\bar{\nabla}N = \frac{1}{2}[N, \hat{N}]$ and $\nabla\hat{N} = \frac{1}{2}[N, \hat{N}]$, where $\nabla = \partial + [J_0,]$ and $\bar{\nabla} = \bar{\partial} + [\bar{J}_0,]$. We have suppressed the index A and introduced a subindex 0, 1, 2, 3 for the currents. Using this notation, the current can be written in terms of the generators of psu(2, 2|4) as follows:

$$J_0 = J^{[\underline{ab}]}M_{\underline{ab}}, \qquad J_1 = J^\alpha Q_\alpha, \qquad J_2 = J^{\underline{a}}P_{\underline{a}}, \tag{13}$$

$$J_3 = J^{\hat{\alpha}}\hat{Q}_{\hat{\alpha}}, \tag{14}$$

[3]These currents are the ghost part of a redefined Lorentz current M^{mn}, necessary for the implementation of a formalism which has manifest supersymmetry and can be covariantly quantized; see Ref. [14].

and similarly for the $\bar{J}$ currents. The structure constants different from zero from the psu(2, 2|4) algebra are

$$\begin{aligned}
f^{\underline{c}}_{\alpha\beta} &= 2\gamma^{\underline{c}}_{\alpha\beta}, \qquad f^{\underline{c}}_{\hat{\alpha}\,\hat{\beta}} = 2\gamma^{\underline{c}}_{\hat{\alpha}\,\hat{\beta}},\\
f^{[ef]}_{\alpha\hat{\beta}} &= (\gamma^{ef})_\alpha{}^\gamma\delta_{\gamma\hat{\beta}} = -(\gamma^{ef})_{\hat{\beta}}{}^{\hat{\gamma}}\delta_{\alpha\hat{\gamma}} = f^{[ef]}_{\hat{\beta}\alpha},\\
f^{[e'f']}_{\alpha\hat{\beta}} &= -(\gamma^{e'f'})_\alpha{}^\gamma\delta_{\gamma\hat{\beta}} = (\gamma^{e'f'})_{\hat{\beta}}{}^{\hat{\gamma}}\delta_{\alpha\hat{\gamma}} = f^{[e'f']}_{\hat{\beta}\alpha},\\
f^{\hat{\beta}}_{\alpha\underline{c}} &= -f^{\hat{\beta}}_{\underline{c}\alpha} = \frac{1}{2}(\gamma_{\underline{c}})_{\alpha\beta}\delta^{\beta\hat{\beta}},\\
f^{\beta}_{\hat{\alpha}\,\underline{c}} &= -f^{\beta}_{\underline{c}\,\hat{\alpha}} = -\frac{1}{2}(\gamma_{\underline{c}})_{\hat{\alpha}\,\hat{\beta}}\delta^{\beta\hat{\beta}},\\
f^{[ef]}_{cd} &= \frac{1}{2}\delta^{[e}_{c}\delta^{f]}_{d}, \qquad f^{[e'f']}_{c'd'} = -\frac{1}{2}\delta^{[e'}_{c'}\delta^{f']}_{d'},\\
f^{[\underline{gh}]}_{[\underline{cd}][\underline{ef}]} &= \frac{1}{2}(\eta_{\underline{ce}}\delta^{[\underline{g}}_{\underline{d}}\delta^{\underline{h}]}_{\underline{f}} - \eta_{\underline{cf}}\delta^{[\underline{g}}_{\underline{d}}\delta^{\underline{h}]}_{\underline{e}} + \eta_{\underline{df}}\delta^{[\underline{g}}_{\underline{c}}\delta^{\underline{h}]}_{\underline{e}} - \eta_{\underline{de}}\delta^{[\underline{g}}_{\underline{c}}\delta^{\underline{h}]}_{\underline{f}}),\\
f^{\underline{f}}_{[\underline{c}\,\underline{d}]\underline{e}} &= -f^{\underline{f}}_{\underline{e}[\underline{c}\,\underline{d}]} = \eta_{\underline{e}[\underline{c}}\delta^{\underline{f}}_{\underline{d}]},\\
f^{\beta}_{[\underline{cd}]\alpha} &= -f^{\beta}_{\alpha[\underline{cd}]} = \frac{1}{2}(\gamma_{\underline{c}\,\underline{d}})_\alpha{}^\beta,\\
f^{\hat{\beta}}_{[\underline{cd}]\hat{\alpha}} &= -f^{\hat{\beta}}_{\hat{\alpha}[\underline{c}\,\underline{d}]} = \frac{1}{2}(\gamma_{\underline{cd}})_{\hat{\alpha}}{}^{\hat{\beta}}.
\end{aligned} \tag{15}$$

This Z_4 grading for the superalgebra was noted in Ref. [1].

III. FEYNMAN RULES AND DIMENSIONAL REGULARIZATION

Now we are going to show the strategy of the calculations we intend to do. In order to calculate the simple pole corrections to the bosonic part of the algebra, we need to calculate contributions to the expectation value $\langle J^{\underline{a}}(y)J^{\underline{b}}(z)\rangle$ perturbatively, including double contractions (one loop) with contributions of one classical field. In order to do this, we perform a background field expansion as in Refs. [1,2,17], choosing a classical background given by an element g_0 in the supergroup and parametrizing the quantum fluctuations by X as $g = g_0 e^{\alpha X}$. Then, the currents can be written as

$$\begin{aligned}
J &= g^{-1}\partial g = e^{-\alpha X}J_0 e^{\alpha X} + e^{-\alpha X}\partial e^{\alpha X},\\
\bar{J} &= g^{-1}\bar{\partial} g = e^{-\alpha X}\bar{J}_0 e^{\alpha X} + e^{-\alpha X}\bar{\partial} e^{\alpha X}.
\end{aligned} \tag{16}$$

By expanding the exponentials in Eq. (16) we get

$$J = J_0 + \alpha(\partial X + [J_0, X]) + \frac{\alpha^2}{2}([\partial X, X] + [[J_0, X], X]) + \frac{\alpha^3}{3!}([[\partial X, X], X] + [[[J_0, X], X], X]) + \cdots, \tag{17}$$

where J_0 denotes the classical part of J and we have a similar expression for $\bar{J}$. The commutators can be evaluated using the structure constants of the psu(2, 2|4) Lie superalgebra given by Eq. (15). Here we are going to use the $SO(1, 4) \times SO(5)$ gauge invariance to fix $X_0 = 0$.

In addition to the expansion of the matter part of the currents, we need to expand the ghost part as follows:

$$N_{\underline{ab}} = N^{(0)}_{\underline{ab}} + \alpha N^{(1)}_{\underline{a}\,\underline{b}} + \alpha^2 N^{(2)}_{\underline{ab}}, \tag{18}$$

and similarly for $\hat{N}_{\underline{ab}}$. Now, the pure-spinor Lorentz currents have the following behavior:

$$N^{(1)}_{\underline{ab}}(y)N^{(1)}_{\underline{cd}}(z) \to \frac{\eta_{\underline{c}[\underline{b}}N^{(0)}_{\underline{a}]\underline{d}}(z) - \eta_{\underline{d}[\underline{b}}N^{(0)}_{\underline{a}]\underline{c}}(z)}{y-z}, \tag{19}$$

$$N^{(2)}_{\underline{ab}}(y)N^{(2)}_{\underline{cd}}(z) \to -3\frac{\eta_{\underline{a}[\underline{d}}\eta_{\underline{c}]\underline{b}}}{(y-z)^2}. \tag{20}$$

Replacing the expansions of the currents in Eq. (6), one can identify the kinetic piece S_p of the action,

$$S_p = \int d^2z\Big(\frac{1}{2}\partial X^{\underline{a}}\bar{\partial}X^{\underline{b}}\eta_{\underline{a}\,\underline{b}} + 4\delta_{\alpha\hat{\beta}}\partial X^{\alpha}\bar{\partial}X^{\hat{\beta}}\Big), \tag{21}$$

from which we obtain the propagators in coordinate space,

$$\begin{aligned} X^a(y)X^b(z) &\to -\eta^{ab}\ln|y-z|^2, \\ X^{a'}(y)X^{b'}(z) &\to -\delta^{a'b'}\ln|y-z|^2, \\ X^{\alpha}(y)X^{\hat{\beta}}(z) &\to -\frac{1}{4}\delta^{\alpha\hat{\beta}}\ln|y-z|^2. \end{aligned} \tag{22}$$

The reminder terms of the background expansion will provide the vertices of the theory. In order to calculate the one-loop simple pole corrections, we need the expansion of the action up to three terms in the quantum fields and one classical field. There are a lot of terms in the expansion of the action. Let us just write the terms that will contribute to the expectation value $\langle J^{\underline{a}}(y)J^{\underline{b}}(z)\rangle$. The terms with one classical field and two quantum fields are

$$\begin{aligned} S(X^2,J) = \int d^2z\Big[& \eta_{\underline{ab}}\eta_{\underline{cd}}\partial X^{\underline{a}}X^{\underline{c}}\bar{J}^{[\underline{db}]} + \eta_{\underline{ab}}\eta_{\underline{cd}}\bar{\partial}X^{\underline{a}}X^{\underline{c}}J^{[\underline{db}]} + \frac{1}{2}\partial X^{\alpha}X^{\hat{\gamma}}\bar{J}^{[\underline{cd}]}(\gamma_{\underline{cd}})_{\hat{\gamma}}{}^{\hat{\beta}}\delta_{\alpha\hat{\beta}} + \frac{3}{2}\bar{\partial}X^{\alpha}X^{\hat{\gamma}}J^{[\underline{cd}]}(\gamma_{\underline{cd}})_{\hat{\gamma}}{}^{\hat{\beta}}\delta_{\alpha\hat{\beta}} \\ & - \frac{3}{2}\partial X^{\hat{\beta}}X^{\gamma}\bar{J}^{[\underline{cd}]}(\gamma_{\underline{cd}})_{\gamma}{}^{\alpha}\delta_{\alpha\hat{\beta}} - \frac{1}{2}\bar{\partial}X^{\hat{\beta}}X^{\gamma}J^{[\underline{cd}]}(\gamma_{\underline{cd}})_{\gamma}{}^{\alpha}\delta_{\alpha\hat{\beta}}\Big] \end{aligned} \tag{23}$$

and

$$\begin{aligned} S(X^2,N) = \int d^2z\Big[& \frac{1}{2}\bar{\partial}X^cX^dN_{cd} - \frac{1}{2}\bar{\partial}X^{c'}X^{d'}N_{c'd'} + \frac{1}{2}\partial X^cX^d\hat{N}_{cd} - \frac{1}{2}\partial X^{c'}X^{d'}\hat{N}_{c'd'} - \frac{1}{2}\bar{\partial}X^{\alpha}X^{\hat{\beta}}(\gamma^{ab})_{\alpha}{}^{\gamma}\delta_{\gamma\hat{\beta}}N_{ab} \\ & + \frac{1}{2}\bar{\partial}X^{\alpha}X^{\hat{\beta}}(\gamma^{a'b'})_{\alpha}{}^{\gamma}\delta_{\gamma\hat{\beta}}N_{a'b'} - \frac{1}{2}\bar{\partial}X^{\hat{\beta}}X^{\alpha}(\gamma^{ab})_{\alpha}{}^{\gamma}\delta_{\gamma\hat{\beta}}N_{ab} + \frac{1}{2}\bar{\partial}X^{\hat{\beta}}X^{\alpha}(\gamma^{a'b'})_{\alpha}{}^{\gamma}\delta_{\gamma\hat{\beta}}N_{a'b'} \\ & - \frac{1}{2}\partial X^{\alpha}X^{\hat{\beta}}(\gamma^{ab})_{\alpha}{}^{\gamma}\delta_{\gamma\hat{\beta}}\hat{N}_{ab} + \frac{1}{2}\partial X^{\alpha}X^{\hat{\beta}}(\gamma^{a'b'})_{\alpha}{}^{\gamma}\delta_{\gamma\hat{\beta}}\hat{N}_{a'b'} \\ & - \frac{1}{2}\partial X^{\hat{\beta}}X^{\alpha}(\gamma^{ab})_{\alpha}{}^{\gamma}\delta_{\gamma\hat{\beta}}\hat{N}_{ab} + \frac{1}{2}\partial X^{\hat{\beta}}X^{\alpha}(\gamma^{a'b'})_{\alpha}{}^{\gamma}\delta_{\gamma\hat{\beta}}\hat{N}_{a'b'}\Big]. \end{aligned} \tag{24}$$

The terms with three quantum fields are

$$\begin{aligned} S(X^3) = \frac{\alpha}{4}\int d^2z[& \partial X^{\underline{a}}\bar{\partial}X^{\alpha}X^{\beta}(\gamma_{\underline{a}})_{\alpha\beta} - \partial X^{\underline{a}}\bar{\partial}X^{\hat{\alpha}}X^{\hat{\beta}}(\gamma_{\underline{a}})_{\hat{\alpha}\,\hat{\beta}} - \bar{\partial}X^{\underline{a}}\partial X^{\alpha}X^{\beta}(\gamma_{\underline{a}})_{\alpha\beta} + \bar{\partial}X^{\underline{a}}\partial X^{\hat{\alpha}}X^{\hat{\beta}}(\gamma_{\underline{a}})_{\hat{\alpha}\,\hat{\beta}} \\ & + 2X^{\underline{a}}\partial X^{\alpha}\bar{\partial}X^{\beta}(\gamma_{\underline{a}})_{\alpha\beta} - 2X^{\underline{a}}\partial X^{\hat{\alpha}}\bar{\partial}X^{\hat{\beta}}(\gamma_{\underline{a}})_{\hat{\alpha}\,\hat{\beta}}]. \end{aligned} \tag{25}$$

By integrating the first line by parts we obtain

$$S(X^3) = \alpha\int d^2z[X^{\underline{a}}\partial X^{\alpha}\bar{\partial}X^{\beta}(\gamma_{\underline{a}})_{\alpha\beta} - X^{\underline{a}}\partial X^{\hat{\alpha}}\bar{\partial}X^{\hat{\beta}}(\gamma_{\underline{a}})_{\hat{\alpha}\,\hat{\beta}}]. \tag{26}$$

The terms with three quantum fields and one classical field are

$$\begin{aligned} S(X^3,J) = \alpha\int d^2z\Big[& -\frac{1}{2}\partial X^{\alpha}X^{\beta}X^{\underline{c}}\eta_{\underline{cd}}\bar{J}^{[\underline{db}]}(\gamma_{\underline{b}})_{\alpha\beta} + \frac{1}{4}\partial X^{\delta}X^{\gamma}X^{\underline{c}}\bar{J}^{[\underline{a}\,\underline{b}]}(\gamma_{\underline{ab}}\gamma_{\underline{c}})_{\gamma\delta} + \frac{1}{2}\bar{\partial}X^{\alpha}X^{\beta}X^{\underline{c}}\eta_{\underline{cd}}J^{[\underline{db}]}(\gamma_{\underline{b}})_{\alpha\beta} \\ & - \frac{1}{4}\bar{\partial}X^{\delta}X^{\gamma}X^{\underline{c}}J^{[\underline{ab}]}(\gamma_{\underline{ab}}\gamma_{\underline{c}})_{\gamma\delta} + \frac{1}{8}\partial X^{\underline{c}}X^{\delta}X^{\gamma}\bar{J}^{[\underline{ab}]}(\gamma_{\underline{ab}}\gamma_{\underline{c}})_{\delta\gamma} - \frac{1}{8}\bar{\partial}X^{\underline{c}}X^{\delta}X^{\gamma}J^{[\underline{ab}]}(\gamma_{\underline{ab}}\gamma_{\underline{c}})_{\delta\gamma} \\ & + \frac{1}{2}\partial X^{\hat{\alpha}}X^{\hat{\beta}}X^{\underline{c}}\eta_{\underline{cd}}\bar{J}^{[\underline{db}]}(\gamma_{\underline{b}})_{\hat{\alpha}\,\hat{\beta}} - \frac{1}{4}\partial X^{\hat{\delta}}X^{\hat{\gamma}}X^{\underline{c}}\bar{J}^{[\underline{ab}]}(\gamma_{\underline{ab}}\gamma_{\underline{c}})_{\hat{\gamma}\,\hat{\delta}} - \frac{1}{2}\bar{\partial}X^{\hat{\alpha}}X^{\hat{\beta}}X^{\underline{c}}\eta_{\underline{cd}}J^{[\underline{d}\,\underline{b}]}(\gamma_{\underline{b}})_{\hat{\alpha}\,\hat{\beta}} \\ & + \frac{1}{4}\bar{\partial}X^{\hat{\delta}}X^{\hat{\gamma}}X^{\underline{c}}J^{[\underline{a}\,\underline{b}]}(\gamma_{\underline{ab}}\gamma_{\underline{c}})_{\hat{\gamma}\,\hat{\delta}} - \frac{1}{8}\partial X^{\underline{c}}X^{\hat{\delta}}X^{\hat{\gamma}}\bar{J}^{[\underline{a}\,\underline{b}]}(\gamma_{\underline{ab}}\gamma_{\underline{c}})_{\hat{\delta}\,\hat{\gamma}} + \frac{1}{8}\bar{\partial}X^{\underline{c}}X^{\hat{\delta}}X^{\hat{\gamma}}J^{[\underline{ab}]}(\gamma_{\underline{ab}}\gamma_{\underline{c}})_{\hat{\delta}\,\hat{\gamma}}\Big] \end{aligned} \tag{27}$$

and

$$
\begin{aligned}
S(X^3, N) = \alpha \int d^2z \Big[& \frac{1}{2\times 3!}\bar\partial X^{\hat\alpha} X^{\underline c}(\gamma_{\underline c})_{\hat\alpha\hat\beta}\delta^{\alpha\hat\beta} X^{\hat\gamma}(\gamma^{ab})_\alpha{}^\gamma \delta_{\gamma\hat\gamma} N_{ab} - \frac{1}{2\times 3!}\bar\partial X^{\hat\alpha} X^{\underline c}(\gamma_{\underline c})_{\hat\alpha\hat\beta}\delta^{\alpha\hat\beta} X^{\hat\gamma}(\gamma^{a'b'})_\alpha{}^\gamma \delta_{\gamma\hat\gamma} N_{a'b'} \\
& - \frac{1}{2\times 3!} X^{\hat\alpha}\bar\partial X^{\underline c}(\gamma_{\underline c})_{\hat\alpha\hat\beta}\delta^{\alpha\hat\beta} X^{\hat\gamma}(\gamma^{ab})_\alpha{}^\gamma \delta_{\gamma\hat\gamma} N_{ab} + \frac{1}{2\times 3!} X^{\hat\alpha}\bar\partial X^{\underline c}(\gamma_{\underline c})_{\hat\alpha\hat\beta}\delta^{\alpha\hat\beta} X^{\hat\gamma}(\gamma^{a'b'})_\alpha{}^\gamma \delta_{\gamma\hat\gamma} N_{a'b'} \\
& - \frac{1}{2\times 3!}\bar\partial X^{\alpha} X^{\underline c}(\gamma_{\underline c})_{\alpha\beta}\delta^{\beta\hat\beta} X^{\gamma}(\gamma^{ab})_\gamma{}^\delta \delta_{\delta\hat\beta} N_{ab} + \frac{1}{2\times 3!}\bar\partial X^{\alpha} X^{\underline c}(\gamma_{\underline c})_{\alpha\beta}\delta^{\beta\hat\beta} X^{\gamma}(\gamma^{a'b'})_\gamma{}^\delta \delta_{\delta\hat\beta} N_{a'b'} \\
& + \frac{1}{2\times 3!} X^{\alpha}\bar\partial X^{\underline c}(\gamma_{\underline c})_{\alpha\beta}\delta^{\beta\hat\beta} X^{\gamma}(\gamma^{ab})_\gamma{}^\delta \delta_{\delta\hat\beta} N_{ab} - \frac{1}{2\times 3!} X^{\alpha}\bar\partial X^{\underline c}(\gamma_{\underline c})_{\alpha\beta}\delta^{\beta\hat\beta} X^{\gamma}(\gamma^{a'b'})_\gamma{}^\delta \delta_{\delta\hat\beta} N_{a'b'} \\
& - \frac{1}{3!}\bar\partial X^\alpha X^\beta \gamma^c_{\alpha\beta} X^d \delta^{[a}_c \delta^{b]}_d N_{ab} + \frac{1}{3!}\bar\partial X^\alpha X^\beta \gamma^{c'}_{\alpha\beta} X^{d'} \delta^{[a'}_{c'} \delta^{b']}_{d'} N_{a'b'} - \frac{1}{3!}\bar\partial X^{\hat\alpha} X^{\hat\beta} \gamma^c_{\hat\alpha\hat\beta} X^d \delta^{[a}_c \delta^{b]}_d N_{ab} \\
& + \frac{1}{3!}\bar\partial X^{\hat\alpha} X^{\hat\beta} \gamma^{c'}_{\hat\alpha\hat\beta} X^{d'} \delta^{[a'}_{c'} \delta^{b']}_{d'} N_{a'b'} + \frac{1}{2\times 3!}\partial X^{\hat\alpha} X^{\underline c}(\gamma_{\underline c})_{\hat\alpha\hat\beta}\delta^{\alpha\hat\beta} X^{\hat\gamma}(\gamma^{ab})_\alpha{}^\gamma \delta_{\gamma\hat\gamma} \hat N_{ab} \\
& - \frac{1}{2\times 3!}\partial X^{\hat\alpha} X^{\underline c}(\gamma_{\underline c})_{\hat\alpha\hat\beta}\delta^{\alpha\hat\beta} X^{\hat\gamma}(\gamma^{a'b'})_\alpha{}^\gamma \delta_{\gamma\hat\gamma} \hat N_{a'b'} - \frac{1}{2\times 3!} X^{\hat\alpha}\partial X^{\underline c}(\gamma_{\underline c})_{\hat\alpha\hat\beta}\delta^{\alpha\hat\beta} X^{\hat\gamma}(\gamma^{ab})_\alpha{}^\gamma \delta_{\gamma\hat\gamma} \hat N_{ab} \\
& + \frac{1}{2\times 3!} X^{\hat\alpha}\partial X^{\underline c}(\gamma_{\underline c})_{\hat\alpha\hat\beta}\delta^{\alpha\hat\beta} X^{\hat\gamma}(\gamma^{a'b'})_\alpha{}^\gamma \delta_{\gamma\hat\gamma} \hat N_{a'b'} - \frac{1}{2\times 3!}\partial X^{\alpha} X^{\underline c}(\gamma_{\underline c})_{\alpha\beta}\delta^{\beta\hat\beta} X^{\gamma}(\gamma^{ab})_\gamma{}^\delta \delta_{\delta\hat\beta} \hat N_{ab} \\
& + \frac{1}{2\times 3!}\partial X^{\alpha} X^{\underline c}(\gamma_{\underline c})_{\alpha\beta}\delta^{\beta\hat\beta} X^{\gamma}(\gamma^{a'b'})_\gamma{}^\delta \delta_{\delta\hat\beta} \hat N_{a'b'} + \frac{1}{2\times 3!} X^{\alpha}\partial X^{\underline c}(\gamma_{\underline c})_{\alpha\beta}\delta^{\beta\hat\beta} X^{\gamma}(\gamma^{ab})_\gamma{}^\delta \delta_{\delta\hat\beta} \hat N_{ab} \\
& - \frac{1}{2\times 3!} X^{\alpha}\partial X^{\underline c}(\gamma_{\underline c})_{\alpha\beta}\delta^{\beta\hat\beta} X^{\gamma}(\gamma^{a'b'})_\gamma{}^\delta \delta_{\delta\hat\beta} \hat N_{a'b'} - \frac{1}{3!}\partial X^\alpha X^\beta \gamma^c_{\alpha\beta} X^d \delta^{[a}_c \delta^{b]}_d \hat N_{ab} \\
& + \frac{1}{3!}\partial X^\alpha X^\beta \gamma^{c'}_{\alpha\beta} X^{d'} \delta^{[a'}_{c'} \delta^{b']}_{d'} \hat N_{a'b'} - \frac{1}{3!}\partial X^{\hat\alpha} X^{\hat\beta} \gamma^c_{\hat\alpha\hat\beta} X^d \delta^{[a}_c \delta^{b]}_d \hat N_{ab} + \frac{1}{3!}\partial X^{\hat\alpha} X^{\hat\beta} \gamma^{c'}_{\hat\alpha\hat\beta} X^{d'} \delta^{[a'}_{c'} \delta^{b']}_{d'} \hat N_{a'b'} \Big].
\end{aligned} \tag{28}
$$

Finally, we need terms with quantum pure-spinor Lorentz currents. These terms will contribute to diagram 21 (see Fig. 1) and they are

$$
\begin{aligned}
S(X^2, N^{(1)}) = \int d^2z \Big[& \frac{1}{2}\bar\partial X^c X^d N^{(1)}_{cd} - \frac{1}{2}\bar\partial X^{c'} X^{d'} N^{(1)}_{c'd'} \\
& + \frac{1}{2}\partial X^c X^d \hat N^{(1)}_{cd} - \frac{1}{2}\partial X^{c'} X^{d'} \hat N^{(1)}_{c'd'} \Big].
\end{aligned} \tag{29}
$$

Terms with four quantum fields will not contribute since two-dimensional tadpoles vanish in the regularization scheme we are going to use. Given the propagators and the interactions, it is then straightforward to write down coordinate-space expressions for the Feynman rules of the diagrams that appear in the expectation value $\langle J^{\underline a}(y) J^{\underline b}(z)\rangle$. However, there are infrared and ultraviolet divergences which produce ambiguities in the coordinate-space integrals. In order to circumvent this problem, we use momentum-space Feynman rules with a prescription for worldsheet dimensional regularization [18–20]. After the calculation of the OPEs in momentum space, the results are written again in coordinate space by using an inverse Fourier transformation.

The two-dimensional prescription for dimensional regularization consists of keeping all the interactions in exactly two dimensions, but the kinetic terms (and hence the denominators of the propagators) will be in $d = 2 - 2\epsilon$ dimensions.

We are going to use the definition $d^2k = \frac{dk_x dk_y}{\pi}$. With this choice there is no π dependence in the results and the Green function $G(y, z)$ is represented as

$$
G(y,z) = \int d^2k \frac{e^{ik(y-z)+i\bar k(\bar y-\bar z)}}{k^2}. \tag{30}
$$

The momentum-space propagators look like

$$
\begin{aligned}
X^{\underline a}(k) X^{\underline b}(l) &\to \eta^{\underline{ab}} \frac{\delta^2(k+l)}{|k|^2}, \\
X^{\alpha}(k) X^{\hat\beta}(l) &\to \frac{1}{4}\delta^{\alpha\hat\beta} \frac{\delta^2(k+l)}{|k|^2}.
\end{aligned} \tag{31}
$$

To work out the corresponding expression for the OPE's in momentum space, we use the dimensional regularization prescription and include a factor $\Gamma(1-\epsilon)\times(4\pi)^{-\epsilon}(2\pi)^{2\epsilon}$ for each loop; this will remove the Euler constant (the G-scheme [21]). All the integrals we need to compute in the momentum space come from the formula

$$
\begin{aligned}
& \int d^2p \frac{p^a \bar p^b}{[|p|^2]^\alpha [|p-k|^2]^\beta} \\
& \quad = k^{a+1-\alpha-\beta} \bar k^{b+1-\alpha-\beta} \left|\frac{k^2}{\mu^2}\right|^{-\epsilon} \\
& \qquad \times \sum_{i=0}^{i=a} \binom{a}{i} \Big[\frac{\Gamma(2-\alpha-\beta+b+i-\epsilon)}{\Gamma(2-2\epsilon-\alpha-\beta+i+b)} \\
& \qquad \times \frac{\Gamma(\alpha+\beta-1-i+\epsilon)}{\Gamma(1+\epsilon)\mu^{-2\epsilon}} \Gamma(1-\epsilon-\beta+i) \Big],
\end{aligned} \tag{32}
$$

where μ is the usual mass parameter of the dimensional regularization.

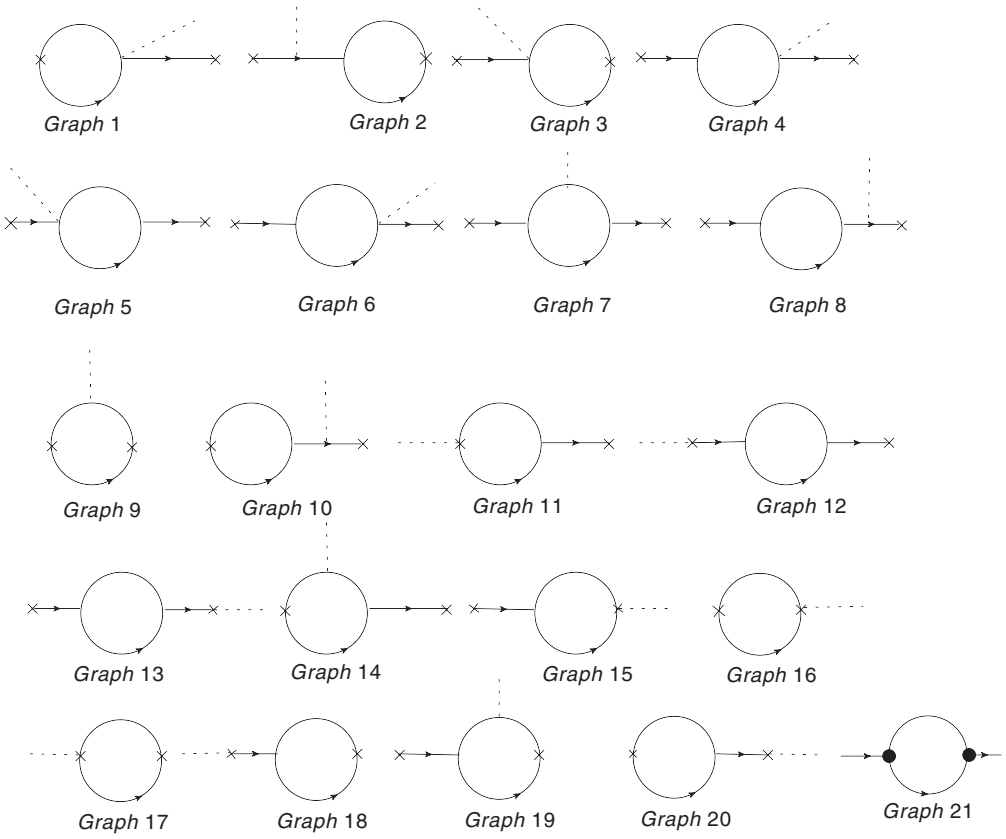

FIG. 1. Diagrams for the one-loop contribution. The full lines represent the quantum fields, and the dashed lines represent the classical field. In diagram 21, the dot vertex represents the presence of one quantum pure-spinor Lorentz current ($N^{(1)}_{\underline{ab}}$ or $\hat{N}^{(1)}_{\underline{ab}}$).

Since the main part of this paper is rather technical and we want to be as clear as possible, we end this section showing in detail the calculation of one diagram. Restricting the expansion (17) to the case with one classical current, we can write

$$
\begin{aligned}
\langle J^{\underline{a}}(y)J^{\underline{b}}(z)\rangle = {} & \alpha^2\langle\partial X^{\underline{a}}(y)\partial X^{\underline{b}}(z)\rangle + \alpha^2\langle\partial X^{\underline{a}}(y)X^{\underline{e}}(z)\rangle J^{[\underline{cd}]}\eta_{\underline{e}[\underline{c}}\delta^{\underline{b}}_{\underline{d}]} + \alpha^2\langle X^{\underline{e}}(y)\partial X^{\underline{b}}(z)\rangle J^{[\underline{cd}]}\eta_{\underline{e}[\underline{c}}\delta^{\underline{a}}_{\underline{d}]} \\
& - \alpha^3\langle\partial X^{\underline{a}}(y)\partial X^{\alpha}X^{\beta}(z)\rangle\gamma^{\underline{b}}_{\alpha\beta} - \alpha^3\langle\partial X^{\underline{a}}(y)\partial X^{\hat{\alpha}}X^{\hat{\beta}}(z)\rangle\gamma^{\underline{b}}_{\hat{\alpha}\hat{\beta}} - \frac{\alpha^3}{2}\langle\partial X^{\underline{a}}(y)X^{\gamma}X^{\beta}(z)\rangle J^{[\underline{cd}]}(\gamma_{\underline{cd}})_{\gamma}{}^{\alpha}\gamma^{\underline{b}}_{\alpha\beta} \\
& - \frac{\alpha^3}{2}\langle\partial X^{\underline{a}}(y)X^{\hat{\gamma}}X^{\hat{\beta}}(z)\rangle J^{[\underline{cd}]}(\gamma_{\underline{cd}})_{\hat{\gamma}}{}^{\hat{\alpha}}\gamma^{\underline{b}}_{\hat{\alpha}\hat{\beta}} - \alpha^3\langle\partial X^{\alpha}X^{\beta}(y)\partial X^{\underline{b}}(z)\rangle\gamma^{\underline{a}}_{\alpha\beta} - \alpha^3\langle\partial X^{\hat{\alpha}}X^{\hat{\beta}}(y)\partial X^{\underline{b}}(z)\rangle\gamma^{\underline{a}}_{\hat{\alpha}\hat{\beta}} \\
& - \alpha^3\langle\partial X^{\alpha}X^{\beta}(y)X^{\underline{e}}(z)\rangle J^{[\underline{cd}]}\gamma^{\underline{a}}_{\alpha\beta}\eta_{\underline{e}[\underline{c}}\delta^{\underline{b}}_{\underline{d}]} - \alpha^3\langle\partial X^{\hat{\alpha}}X^{\hat{\beta}}(y)X^{\underline{e}}(z)\rangle J^{[\underline{cd}]}\gamma^{\underline{a}}_{\hat{\alpha}\hat{\beta}}\eta_{\underline{e}[\underline{c}}\delta^{\underline{b}}_{\underline{d}]} \\
& - \alpha^3\langle X^{\underline{e}}(y)\partial X^{\alpha}X^{\beta}(z)\rangle J^{[\underline{cd}]}\eta_{\underline{e}[\underline{c}}\delta^{\underline{a}}_{\underline{d}]}\gamma^{\underline{b}}_{\alpha\beta} - \alpha^3\langle X^{\underline{e}}(y)\partial X^{\hat{\alpha}}X^{\hat{\beta}}(z)\rangle J^{[\underline{cd}]}\eta_{\underline{e}[\underline{c}}\delta^{\underline{a}}_{\underline{d}]}\gamma^{\underline{b}}_{\hat{\alpha}\hat{\beta}} \\
& - \frac{\alpha^3}{2}\langle X^{\gamma}X^{\beta}(y)\partial X^{\underline{b}}(z)\rangle J^{[\underline{cd}]}(\gamma_{\underline{cd}})_{\gamma}{}^{\alpha}\gamma^{\underline{a}}_{\alpha\beta} - \frac{\alpha^3}{2}\langle X^{\hat{\gamma}}X^{\hat{\beta}}(y)\partial X^{\underline{b}}(z)\rangle J^{[\underline{cd}]}(\gamma_{\underline{cd}})_{\hat{\gamma}}{}^{\hat{\alpha}}\gamma^{\underline{a}}_{\hat{\alpha}\hat{\beta}} \\
& + \alpha^4\langle\partial X^{\alpha}X^{\beta}(y)\partial X^{\hat{\gamma}}X^{\hat{\delta}}(z)\rangle\gamma^{\underline{a}}_{\alpha\beta}\gamma^{\underline{b}}_{\hat{\gamma}\hat{\delta}} + \alpha^4\langle\partial X^{\hat{\alpha}}X^{\hat{\beta}}(y)\partial X^{\gamma}X^{\delta}(z)\rangle\gamma^{\underline{a}}_{\hat{\alpha}\hat{\beta}}\gamma^{\underline{b}}_{\gamma\delta} \\
& + \frac{\alpha^4}{2}\langle\partial X^{\alpha}X^{\beta}(y)X^{\hat{\gamma}}X^{\hat{\delta}}(z)\rangle J^{[\underline{cd}]}\gamma^{\underline{a}}_{\alpha\beta}(\gamma_{\underline{cd}})_{\hat{\gamma}}{}^{\hat{\lambda}}\gamma^{\underline{b}}_{\hat{\lambda}\hat{\delta}} + \frac{\alpha^4}{2}\langle\partial X^{\hat{\alpha}}X^{\hat{\beta}}(y)X^{\gamma}X^{\delta}(z)\rangle J^{[\underline{cd}]}\gamma^{\underline{a}}_{\hat{\alpha}\hat{\beta}}(\gamma_{\underline{cd}})_{\gamma}{}^{\lambda}\gamma^{\underline{b}}_{\lambda\delta} \\
& + \frac{\alpha^4}{2}\langle X^{\hat{\gamma}}X^{\hat{\delta}}(y)\partial X^{\alpha}X^{\beta}(z)\rangle J^{[\underline{cd}]}(\gamma_{\underline{cd}})_{\hat{\gamma}}{}^{\hat{\lambda}}\gamma^{\underline{a}}_{\hat{\lambda}\hat{\delta}}\gamma^{\underline{b}}_{\alpha\beta} + \frac{\alpha^4}{2}\langle X^{\gamma}X^{\delta}(y)\partial X^{\hat{\alpha}}X^{\hat{\beta}}(z)\rangle J^{[\underline{cd}]}(\gamma_{\underline{cd}})_{\gamma}{}^{\lambda}\gamma^{\underline{a}}_{\lambda\delta}\gamma^{\underline{b}}_{\hat{\alpha}\hat{\beta}}. \qquad (33)
\end{aligned}
$$

Let us focus on the first term in Eq. (33). The diagrams that contribute to this expectation value are 4, 5, 6, 7, 8, and 21. Let us calculate the diagram 7. We can form this one-loop diagram by using the two terms on the right-hand side of Eq. (26), which will come from the expansion of the exponential of minus the action at second order. Also, we need the following:

the terms $\frac{1}{2}\partial X^\alpha X^{\hat\gamma}\bar J^{[\underline{cd}]}(\gamma_{\underline{cd}})_{\hat\gamma}{}^{\hat\beta}\delta_{\alpha\hat\beta}$, $-\frac{3}{2}\partial X^{\hat\beta}X^\gamma\bar J^{[\underline{cd}]}(\gamma_{\underline{cd}})_\gamma{}^\alpha\delta_{\alpha\hat\beta}$, $-\frac{1}{2}\bar\partial X^{\hat\beta}X^\gamma J^{[\underline{cd}]}(\gamma_{\underline{cd}})_\gamma{}^\alpha\delta_{\alpha\hat\beta}$, and $\frac{3}{2}\bar\partial X^\alpha X^{\hat\gamma}J^{[\underline{cd}]}(\gamma_{\underline{cd}})_{\hat\gamma}{}^{\hat\beta}\delta_{\alpha\hat\beta}$ of Eq. (23) (contributions of $J^{\underline{ab}}$ and $\bar J^{\underline{ab}}$); the terms $-\frac{1}{2}\bar\partial X^\alpha X^{\hat\beta}(\gamma^{ab})_\alpha{}^\gamma\delta_{\gamma\hat\beta}N_{ab}$, $\frac{1}{2}\bar\partial X^\alpha X^{\hat\beta}(\gamma^{a'b'})_\alpha{}^\gamma\delta_{\gamma\hat\beta}N_{a'b'}$, $-\frac{1}{2}\bar\partial X^{\hat\beta}X^\alpha(\gamma^{ab})_\alpha{}^\gamma\delta_{\gamma\hat\beta}N_{ab}$, $\frac{1}{2}\bar\partial X^{\hat\beta}X^\alpha(\gamma^{a'b'})_\alpha{}^\gamma\delta_{\gamma\hat\beta}N_{a'b'}$, $-\frac{1}{2}\partial X^\alpha X^{\hat\beta}(\gamma^{ab})_\alpha{}^\gamma\delta_{\gamma\hat\beta}\hat N_{ab}$, $\frac{1}{2}\partial X^\alpha X^{\hat\beta}(\gamma^{a'b'})_\alpha{}^\gamma\delta_{\gamma\hat\beta}\hat N_{a'b'}$, $-\frac{1}{2}\partial X^{\hat\beta}X^\alpha(\gamma^{ab})_\alpha{}^\gamma\delta_{\gamma\hat\beta}\hat N_{ab}$, and $\frac{1}{2}\partial X^{\hat\beta}X^\alpha(\gamma^{a'b'})_\alpha{}^\gamma\delta_{\gamma\hat\beta}\hat N_{a'b'}$ of Eq. (24) (contributions of $N_{\underline{ab}}$ and $\hat N_{\underline{ab}}$); and the two terms of Eq. (26). So, in momentum space, using the contractions (31), the contribution of diagram 7 to $\alpha^2\langle\partial X^{\underline a}(y)\partial X^{\underline b}(z)\rangle$ is

$$\begin{aligned}
D_7 = &-2i\alpha^4\frac{\bar J^{[\underline{a}\,\underline{b}]}}{\bar k^2}\int d^2p\,\frac{(p^3(\bar k-\bar p)^2+\bar p^2p(k-p)^2-2p^2\bar p(k-p)(\bar k-\bar p))}{|p|^4|k-p|^2}\\
&-2i\alpha^4\frac{J^{[\underline{ab}]}}{\bar k^2}\int d^2p\,\frac{(\bar p^3(k-p)^2+p^2\bar p(\bar k-\bar p)^2-2\bar p^2p(k-p)(\bar k-\bar p))}{|p|^4|k-p|^2}\\
&-i\alpha^4\frac{N^{ab}}{\bar k^2}\int d^2p\,\frac{(\bar p^3(k-p)^2+p^2\bar p(\bar k-\bar p)^2-2\bar p^2p(k-p)(\bar k-\bar p))}{|p|^4|k-p|^2}\\
&+i\alpha^4\frac{N^{a'b'}}{\bar k^2}\int d^2p\,\frac{(\bar p^3(k-p)^2+p^2\bar p(\bar k-\bar p)^2-2\bar p^2p(k-p)(\bar k-\bar p))}{|p|^4|k-p|^2}\\
&-i\alpha^4\frac{\hat N^{ab}}{\bar k^2}\int d^2p\,\frac{(p^3(\bar k-\bar p)^2+\bar p^2p(k-p)^2-2p^2\bar p(k-p)(\bar k-\bar p))}{|p|^4|k-p|^2}\\
&+i\alpha^4\frac{\hat N^{a'b'}}{\bar k^2}\int d^2p\,\frac{(p^3(\bar k-\bar p)^2+\bar p^2p(k-p)^2-2p^2\bar p(k-p)(\bar k-\bar p))}{|p|^4|k-p|^2}.
\end{aligned}\tag{34}$$

For this diagram, we have an overall factor of $\frac{1}{3!}$ coming from the expansion of $\exp(-S)$ at third order in S, which cancels with the factor of 6 coming from the different products of the terms in S^3. Also, we have to take into account a minus sign coming from the expansion of $\exp(-S)$ at third order in S. Therefore, using the results of the integrals summarized in the Appendix, we obtain

$$\begin{aligned}
D_7 = 4i\alpha^4\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\Bigg[&\frac{1}{\bar k}\left(J^{[\underline{ab}]}+\frac{(N^{ab}-N^{a'b'})}{2}\right)\\
&+\frac{k}{\bar k^2}\left(\bar J^{[\underline{ab}]}+\frac{(\hat N^{ab}-\hat N^{a'b'})}{2}\right)\Bigg].
\end{aligned}\tag{35}$$

IV. ONE-LOOP OPE WITH ONE CLASSICAL FIELD

In this section, we are going to present the results of the OPE's to each expectation value that appears in Eq. (33), specifying which diagrams contribute to each one. But first note that

$$\langle J^{\underline a}(y)J^{\underline b}(z)\rangle = \langle J^a(y)J^b(z)\rangle + \langle J^{a'}(y)J^{b'}(z)\rangle. \tag{36}$$

So, due to the different contributions to each type of index a and a', the results for $\langle J^a(y)J^b(z)\rangle$ and $\langle J^{a'}(y)J^{b'}(z)\rangle$ will be presented separately. We will use the notation I_n for the contribution of the nth diagram and we are going to write the contributions for each expectation value in Eq. (33). The summary of the results is presented in subsection C.

A. Results for $\langle J^a(y)J^b(z)\rangle$

To the expectation value $\alpha^2\langle\partial X^a(y)\partial X^b(z)\rangle$, the diagrams 4, 5, 6, 7, 8, and 21 contribute,

$$\begin{aligned}
I_4 &= 4i\alpha^4\left(\bar J^{[ab]}-\frac{\hat N^{ab}}{2}\right)\frac{k}{\bar k^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{\epsilon}+2\right)+4i\alpha^4\left(J^{[ab]}-\frac{N^{ab}}{2}\right)\frac{1}{\bar k}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{\epsilon}+2\right),\\
I_5 &= -2i\alpha^4\bar J^{[ab]}\frac{k}{\bar k^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{\epsilon}+2\right)-2i\alpha^4J^{[ab]}\frac{1}{\bar k}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{\epsilon}+2\right),\\
I_6 &= -2i\alpha^4\bar J^{[ab]}\frac{k}{\bar k^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{\epsilon}+2\right)-2i\alpha^4J^{[ab]}\frac{1}{\bar k}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{\epsilon}+2\right),\\
I_7 &= 2i\alpha^4(2\bar J^{[ab]}+\hat N^{ab})\frac{k}{\bar k^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}+2i\alpha^4(2J^{[ab]}+N^{ab})\frac{1}{\bar k}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon},
\end{aligned}$$

$$I_8 = 4i\alpha^4\left(\bar{J}^{[ab]} - \frac{\hat{N}^{ab}}{2}\right)\frac{k}{\bar{k}^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{\epsilon}+2\right) + 4i\alpha^4\left(J^{[ab]} - \frac{N^{ab}}{2}\right)\frac{1}{\bar{k}}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{\epsilon}+2\right),$$
$$I_{21} = \frac{3i\alpha^4}{4}\hat{N}^{ab}\frac{k}{\bar{k}^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(-\frac{4}{\epsilon}-\frac{7}{2}\right) + \frac{3i\alpha^4}{4}N^{ab}\frac{1}{\bar{k}}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(-\frac{4}{\epsilon}-\frac{7}{2}\right), \tag{37}$$

giving as a result

$$\alpha^2\langle\partial X^a(y)\partial X^b(z)\rangle = i\alpha^4\frac{k}{\bar{k}^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(4\bar{J}^{[ab]}\left(\frac{1}{\epsilon}+3\right) - \hat{N}^{ab}\left(\frac{7}{\epsilon}+\frac{69}{8}\right)\right)$$
$$+ i\alpha^4\frac{1}{\bar{k}}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(4J^{[ab]}\left(\frac{1}{\epsilon}+3\right) - N^{ab}\left(\frac{7}{\epsilon}+\frac{69}{8}\right)\right). \tag{38}$$

To the expectation value $-\alpha^3\langle\partial X^a(y)\partial X^\alpha X^\beta(z)\rangle\gamma^b_{\alpha\beta}$, the diagrams 2, 3, and 19 contribute,

$$I_2 = -2i\alpha^4(\bar{J}^{[ab]} + \hat{N}^{ab})\frac{k}{\bar{k}^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{\epsilon}+2\right) - 2i\alpha^4(J^{[ab]} - N^{ab})\frac{1}{\bar{k}}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{\epsilon}+2\right),$$
$$I_3 = i\alpha^4\left(\bar{J}^{[ab]} + \frac{\hat{N}^{ab}}{3!}\right)\frac{k}{\bar{k}^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{\epsilon}+2\right) + i\alpha^4\left(J^{[ab]} - \frac{N^{ab}}{3!}\right)\frac{1}{\bar{k}}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{\epsilon}+2\right), \tag{39}$$
$$I_{19} = -i\alpha^4\left(\bar{J}^{[ab]} + \frac{\hat{N}^{ab}}{2}\right)\frac{k}{\bar{k}^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{\epsilon}+4\right) + i\alpha^4\frac{1}{\bar{k}}\left(J^{[ab]} + \frac{N^{ab}}{2}\right)\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\frac{1}{\epsilon},$$

giving as a result

$$-\alpha^3\langle\partial X^a(y)\partial X^\alpha X^\beta(z)\rangle\gamma^b_{\alpha\beta} = -2i\alpha^4\bar{J}^{[ab]}\frac{k}{\bar{k}^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{\epsilon}+3\right) - i\alpha^4\frac{\hat{N}^{ab}}{3}\frac{k}{\bar{k}^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{7}{\epsilon}+17\right)$$
$$- 2i\alpha^4 J^{[ab]}\frac{1}{\bar{k}}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon} + i\alpha^4\frac{N^{ab}}{3}\frac{1}{\bar{k}}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{7}{\epsilon}+11\right). \tag{40}$$

To the expectation value $-\alpha^3\langle\partial X^a(y)\partial X^{\hat\alpha}X^{\hat\beta}(z)\rangle\gamma^b_{\hat\alpha\hat\beta}$, the diagrams 2, 3, and 19 contribute,

$$I_2 = 2i\alpha^4(\bar{J}^{[ab]} + \hat{N}^{ab})\frac{k}{\bar{k}^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{\epsilon}+2\right) + 2i\alpha^4(J^{[ab]} - N^{ab})\frac{1}{\bar{k}}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{\epsilon}+2\right),$$
$$I_3 = -i\alpha^4\left(\bar{J}^{[ab]} - \frac{\hat{N}^{ab}}{3!}\right)\frac{k}{\bar{k}^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{\epsilon}+2\right) - i\alpha^4\left(J^{[ab]} + \frac{N^{ab}}{3!}\right)\frac{1}{\bar{k}}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{\epsilon}+2\right), \tag{41}$$
$$I_{19} = i\alpha^4\left(\bar{J}^{[ab]} + \frac{\hat{N}^{ab}}{2}\right)\frac{k}{\bar{k}^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{\epsilon}+4\right) - i\alpha^4\frac{1}{\bar{k}}\left(J^{[ab]} + \frac{N^{ab}}{2}\right)\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\frac{1}{\epsilon},$$

giving as a result

$$-\alpha^3\langle\partial X^a(y)\partial X^{\hat\alpha}X^{\hat\beta}(z)\rangle\gamma^b_{\hat\alpha\hat\beta} = 2i\alpha^4\bar{J}^{[ab]}\frac{k}{\bar{k}^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{\epsilon}+3\right) + i\alpha^4\frac{\hat{N}^{ab}}{3}\frac{k}{\bar{k}^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{8}{\epsilon}+19\right)$$
$$+ 2i\alpha^4 J^{[ab]}\frac{1}{\bar{k}}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon} - i\alpha^4\frac{N^{ab}}{3}\frac{1}{\bar{k}}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{8}{\epsilon}+13\right). \tag{42}$$

To the expectation value $\alpha^2\langle\partial X^a(y)X^e(z)\rangle J^{[cd]}\eta_{e[c}\delta^b_{d]}$, only diagram 13 contributes,

$$\alpha^2\langle\partial X^a(y)X^e(z)\rangle J^{[cd]}\eta_{e[c}\delta^b_{d]} = I_{13} = -4i\alpha^4 J^{[ab]}\frac{1}{\bar{k}}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{\epsilon}+2\right). \tag{43}$$

To the expectation values $-\frac{\alpha^3}{2}\langle\partial X^a(y)X^\gamma X^\beta(z)\rangle J^{[cd]}(\gamma_{cd})_\gamma{}^\alpha\gamma^b_{\alpha\beta}$ and $-\frac{\alpha^3}{2}\langle\partial X^a(y)X^{\hat\gamma}X^{\hat\beta}(z)\rangle J^{[cd]}(\gamma_{cd})_{\hat\gamma}{}^{\hat\alpha}\gamma^b_{\hat\alpha\hat\beta}$ only diagram 15 contributes; however, it cancels,

$$-\frac{\alpha^3}{2}\langle\partial X^a(y)X^\gamma X^\beta(z)\rangle J^{[cd]}(\gamma_{cd})_\gamma{}^\alpha\gamma^b_{\alpha\beta}=I_{15}=0,\qquad -\frac{\alpha^3}{2}\langle\partial X^a(y)X^{\hat\gamma}X^{\hat\beta}(z)\rangle J^{[cd]}(\gamma_{cd})_{\hat\gamma}{}^{\hat\alpha}\gamma^b_{\hat\alpha\hat\beta}=I_{15}=0. \quad (44)$$

To the expectation value $-\alpha^3\langle\partial X^\alpha X^\beta(y)\partial X^b(z)\rangle\gamma^a_{\alpha\beta}$, the diagrams 1, 10, and 14 contribute,

$$\begin{aligned}
I_1 &= i\alpha^4\Big(\bar J^{[ab]}+\frac{\hat N^{ab}}{3!}\Big)\frac{k}{\bar k^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\Big(\frac{1}{\epsilon}+2\Big)+i\alpha^4\Big(J^{[ab]}-\frac{N^{ab}}{3!}\Big)\frac{1}{\bar k}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\Big(\frac{1}{\epsilon}+2\Big),\\
I_{10} &= -2i\alpha^4(\bar J^{[ab]}+\hat N^{ab})\frac{k}{\bar k^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\Big(\frac{1}{\epsilon}+2\Big)-2i\alpha^4(J^{[ab]}-N^{ab})\frac{1}{\bar k}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\Big(\frac{1}{\epsilon}+2\Big),\\
I_{14} &= -i\alpha^4\Big(\bar J^{[ab]}+\frac{\hat N^{ab}}{2}\Big)\frac{k}{\bar k^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\Big(\frac{1}{\epsilon}+4\Big)+i\alpha^4\frac{1}{\bar k}\Big(J^{[ab]}+\frac{N^{ab}}{2}\Big)\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\frac{1}{\epsilon},
\end{aligned} \quad (45)$$

giving as a result

$$\begin{aligned}
-\alpha^3\langle\partial X^\alpha X^\beta(y)\partial X^b(z)\rangle\gamma^a_{\alpha\beta} &= -2i\alpha^4\bar J^{[ab]}\frac{k}{\bar k^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\Big(\frac{1}{\epsilon}+3\Big)-i\alpha^4\frac{\hat N^{ab}}{3}\frac{k}{\bar k^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\Big(\frac{7}{\epsilon}+17\Big)\\
&\quad -2i\alpha^4J^{[ab]}\frac{1}{\bar k}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}+i\alpha^4\frac{N^{ab}}{3}\frac{1}{\bar k}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\Big(\frac{7}{\epsilon}+11\Big).
\end{aligned} \quad (46)$$

To the expectation value $-\alpha^3\langle\partial X^{\hat\alpha}X^{\hat\beta}(y)\partial X^b(z)\rangle\gamma^a_{\hat\alpha\hat\beta}$, the diagrams 1, 10, and 14 contribute,

$$\begin{aligned}
I_1 &= -i\alpha^4\Big(\bar J^{[ab]}-\frac{\hat N^{ab}}{3!}\Big)\frac{k}{\bar k^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\Big(\frac{1}{\epsilon}+2\Big)-i\alpha^4\Big(J^{[ab]}+\frac{N^{ab}}{3!}\Big)\frac{1}{\bar k}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\Big(\frac{1}{\epsilon}+2\Big),\\
I_{10} &= 2i\alpha^4(\bar J^{[ab]}+\hat N^{ab})\frac{k}{\bar k^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\Big(\frac{1}{\epsilon}+2\Big)+2i\alpha^4(J^{[ab]}-N^{ab})\frac{1}{\bar k}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\Big(\frac{1}{\epsilon}+2\Big),\\
I_{14} &= i\alpha^4\Big(\bar J^{[ab]}+\frac{\hat N^{ab}}{2}\Big)\frac{k}{\bar k^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\Big(\frac{1}{\epsilon}+4\Big)-i\alpha^4\frac{1}{\bar k}\Big(J^{[ab]}+\frac{N^{ab}}{2}\Big)\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\frac{1}{\epsilon},
\end{aligned} \quad (47)$$

giving as a result

$$\begin{aligned}
-\alpha^3\langle\partial X^{\hat\alpha}X^{\hat\beta}(y)\partial X^b(z)\rangle\gamma^a_{\hat\alpha\hat\beta} &= 2i\alpha^4\bar J^{[ab]}\frac{k}{\bar k^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\Big(\frac{1}{\epsilon}+3\Big)+i\alpha^4\frac{\hat N^{ab}}{3}\frac{k}{\bar k^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\Big(\frac{8}{\epsilon}+19\Big)\\
&\quad +2i\alpha^4J^{[ab]}\frac{1}{\bar k}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}-i\alpha^4\frac{N^{ab}}{3}\frac{1}{\bar k}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\Big(\frac{8}{\epsilon}+13\Big).
\end{aligned} \quad (48)$$

To the expectation value $\alpha^4\langle\partial X^\alpha X^\beta(y)\partial X^{\hat\gamma}X^{\hat\delta}(z)\rangle\gamma^a_{\alpha\beta}\gamma^b_{\hat\gamma\hat\delta}$ only diagram 9 contributes,

$$\alpha^4\langle\partial X^\alpha X^\beta(y)\partial X^{\hat\gamma}X^{\hat\delta}(z)\rangle\gamma^a_{\alpha\beta}\gamma^b_{\hat\gamma\hat\delta}=I_9=-2i\alpha^4\Big(\bar J^{[ab]}+\frac{\hat N^{ab}}{2}\Big)\frac{k}{\bar k^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\Big(\frac{1}{\epsilon}+3\Big)+2i\alpha^4\Big(J^{[ab]}+\frac{N^{ab}}{2}\Big)\frac{1}{\bar k}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\Big(\frac{1}{\epsilon}+1\Big). \quad (49)$$

To the expectation value $\alpha^4\langle\partial X^{\hat\alpha}X^{\hat\beta}(y)\partial X^\gamma X^\delta(z)\rangle\gamma^a_{\hat\alpha\hat\beta}\gamma^b_{\gamma\delta}$ only diagram 9 contributes,

$$\alpha^4\langle\partial X^{\hat\alpha}X^{\hat\beta}(y)\partial X^\gamma X^\delta(z)\rangle\gamma^a_{\hat\alpha\hat\beta}\gamma^b_{\gamma\delta}=I_9=-2i\alpha^4\Big(\bar J^{[ab]}+\frac{\hat N^{ab}}{2}\Big)\frac{k}{\bar k^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\Big(\frac{1}{\epsilon}+3\Big)+2i\alpha^4\Big(J^{[ab]}+\frac{N^{ab}}{2}\Big)\frac{1}{\bar k}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\Big(\frac{1}{\epsilon}+1\Big). \quad (50)$$

To the expectation value $-\alpha^3\langle\partial X^\alpha X^\beta(y)X^e(z)\rangle\gamma^a_{\alpha\beta}J^{[cd]}\eta_{e[c}\delta^b_{d]}$, only diagram 20 contributes,

$$-\alpha^3\langle\partial X^\alpha X^\beta(y)X^e(z)\rangle\gamma^a_{\alpha\beta}J^{[cd]}\eta_{e[c}\delta^b_{d]}=I_{20}=2i\alpha^4J^{[ab]}\frac{1}{\bar k}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\Big(\frac{1}{\epsilon}+2\Big). \quad (51)$$

To the expectation value $-\alpha^3\langle\partial X^{\hat{\alpha}}X^{\hat{\beta}}(y)X^e(z)\rangle\gamma^a_{\hat{\alpha}\,\hat{\beta}}J^{[cd]}\eta_{e[c}\delta^b_{d]}$, only diagram 20 contributes,

$$-\alpha^3\langle\partial X^{\hat{\alpha}}X^{\hat{\beta}}(y)X^e(z)\rangle\gamma^a_{\hat{\alpha}\,\hat{\beta}}J^{[cd]}\eta_{e[c}\delta^b_{d]} = I_{20} = -2i\alpha^4J^{[ab]}\frac{1}{\bar{k}}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{\epsilon}+2\right). \tag{52}$$

To the expectation value $\frac{\alpha^4}{2}\langle\partial X^{\alpha}X^{\beta}(y)X^{\hat{\gamma}}X^{\hat{\delta}}(z)\rangle\gamma^a_{\alpha\beta}J^{[cd]}(\gamma_{cd})_{\hat{\gamma}}{}^{\hat{\lambda}}\gamma^b_{\hat{\lambda}\,\hat{\delta}}$ only diagram 16 contributes; however, it cancels,

$$\frac{\alpha^4}{2}\langle\partial X^{\alpha}X^{\beta}(y)X^{\hat{\gamma}}X^{\hat{\delta}}(z)\rangle\gamma^a_{\alpha\beta}J^{[cd]}(\gamma_{cd})_{\hat{\gamma}}{}^{\hat{\lambda}}\gamma^b_{\hat{\lambda}\,\hat{\delta}} = I_{16} = 0. \tag{53}$$

To the expectation value $\frac{\alpha^4}{2}\langle\partial X^{\hat{\alpha}}X^{\hat{\beta}}(y)X^{\gamma}X^{\delta}(z)\rangle\gamma^a_{\hat{\alpha}\,\hat{\beta}}J^{[cd]}(\gamma_{cd})_{\gamma}{}^{\lambda}\gamma^b_{\lambda\delta}$ only diagram 16 contributes; however, it cancels,

$$\frac{\alpha^4}{2}\langle\partial X^{\hat{\alpha}}X^{\hat{\beta}}(y)X^{\gamma}X^{\delta}(z)\rangle\gamma^a_{\hat{\alpha}\,\hat{\beta}}J^{[cd]}(\gamma_{cd})_{\gamma}{}^{\lambda}\gamma^b_{\lambda\delta} = I_{16} = 0. \tag{54}$$

To the expectation value $\alpha^2\langle X^e(y)\partial X^b(z)\rangle J^{[cd]}\eta_{e[c}\delta^a_{d]}$, only diagram 12 contributes,

$$\alpha^2\langle X^e(y)\partial X^b(z)\rangle J^{[cd]}\eta_{e[c}\delta^a_{d]} = I_{12} = -4i\alpha^4J^{[ab]}\frac{1}{\bar{k}}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{\epsilon}+2\right). \tag{55}$$

To the expectation value $-\alpha^3\langle X^e(y)\partial X^{\alpha}X^{\beta}(z)\rangle\gamma^b_{\alpha\beta}J^{[cd]}\eta_{e[c}\delta^a_{d]}$, only diagram 18 contributes,

$$-\alpha^3\langle X^e(y)\partial X^{\alpha}X^{\beta}(z)\rangle\gamma^b_{\alpha\beta}J^{[cd]}\eta_{e[c}\delta^a_{d]} = I_{18} = 2i\alpha^4J^{[ab]}\frac{1}{\bar{k}}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{\epsilon}+2\right). \tag{56}$$

To the expectation value $-\alpha^3\langle X^e(y)\partial X^{\hat{\alpha}}X^{\hat{\beta}}(z)\rangle\gamma^b_{\hat{\alpha}\,\hat{\beta}}J^{[cd]}\eta_{e[c}\delta^a_{d]}$, only diagram 18 contributes,

$$-\alpha^3\langle X^e(y)\partial X^{\hat{\alpha}}X^{\hat{\beta}}(z)\rangle\gamma^b_{\hat{\alpha}\,\hat{\beta}}J^{[cd]}\eta_{e[c}\delta^a_{d]} = I_{18} = -2i\alpha^4J^{[ab]}\frac{1}{\bar{k}}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{\epsilon}+2\right). \tag{57}$$

To the expectation values $-\frac{\alpha^3}{2}\langle X^{\gamma}X^{\beta}(y)\partial X^b(z)\rangle J^{[cd]}(\gamma_{cd})_{\gamma}{}^{\alpha}\gamma^a_{\alpha\beta}$ and $-\frac{\alpha^3}{2}\langle X^{\hat{\gamma}}X^{\hat{\beta}}(y)\partial X^b(z)\rangle J^{[cd]}(\gamma_{cd})_{\hat{\gamma}}{}^{\hat{\alpha}}\gamma^a_{\hat{\alpha}\,\hat{\beta}}$ only diagram 11 contributes; however, it cancels for the two cases,

$$-\frac{\alpha^3}{2}\langle X^{\gamma}X^{\beta}(y)\partial X^b(z)\rangle J^{[cd]}(\gamma_{cd})_{\gamma}{}^{\alpha}\gamma^a_{\alpha\beta} = I_{11} = 0, \tag{58}$$

$$-\frac{\alpha^3}{2}\langle X^{\hat{\gamma}}X^{\hat{\beta}}(y)\partial X^b(z)\rangle J^{[cd]}(\gamma_{cd})_{\hat{\gamma}}{}^{\hat{\alpha}}\gamma^a_{\hat{\alpha}\,\hat{\beta}} = I_{11} = 0. \tag{59}$$

To the expectation value $\frac{\alpha^4}{2}\langle X^{\hat{\gamma}}X^{\hat{\delta}}(y)\partial X^{\alpha}X^{\beta}(z)\rangle\gamma^b_{\alpha\beta}J^{[cd]}(\gamma_{cd})_{\hat{\gamma}}{}^{\hat{\lambda}}\gamma^a_{\hat{\lambda}\,\hat{\delta}}$ only diagram 17 contributes; however, it cancels,

$$\frac{\alpha^4}{2}\langle X^{\hat{\gamma}}X^{\hat{\delta}}(y)\partial X^{\alpha}X^{\beta}(z)\rangle\gamma^b_{\alpha\beta}J^{[cd]}(\gamma_{cd})_{\hat{\gamma}}{}^{\hat{\lambda}}\gamma^a_{\hat{\lambda}\,\hat{\delta}} = I_{17} = 0. \tag{60}$$

To the expectation value $\frac{\alpha^4}{2}\langle X^{\gamma}X^{\delta}(y)\partial X^{\hat{\alpha}}X^{\hat{\beta}}(z)\rangle\gamma^b_{\hat{\alpha}\,\hat{\beta}}J^{[cd]}(\gamma_{cd})_{\gamma}{}^{\lambda}\gamma^a_{\lambda\delta}$ only diagram 17 contributes; however, it cancels,

$$\frac{\alpha^4}{2}\langle X^{\gamma}X^{\delta}(y)\partial X^{\hat{\alpha}}X^{\hat{\beta}}(z)\rangle\gamma^b_{\hat{\alpha}\,\hat{\beta}}J^{[cd]}(\gamma_{cd})_{\gamma}{}^{\lambda}\gamma^a_{\lambda\delta} = I_{17} = 0. \tag{61}$$

B. Results for $\langle J^{a'}(y)J^{b'}(z)\rangle$

The results for the expectation value $\langle J^{a'}(y)J^{b'}(z)\rangle$ are analogous to the results of subsection A. We just need to be careful with the signals of some vertices. The result is

$$\langle J^{a'}(y)J^{b'}(z)\rangle = -i\alpha^4\frac{N^{a'b'}}{\bar{k}}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{3\epsilon}-\frac{65}{24}\right)+i\alpha^4\hat{N}^{a'b'}\frac{k}{\bar{k}^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{7}{3\epsilon}+\frac{193}{24}\right). \tag{62}$$

C. Summary of the results

Finally, summing up all the expectation values presented in subsections A and B, we have

$$J^{a'}(x)J^{b'}(y)\to -i\alpha^4\frac{N^{a'b'}}{\bar{k}}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{1}{3\epsilon}-\frac{65}{24}\right)+i\alpha^4\hat{N}^{a'b'}\frac{k}{\bar{k}^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{7}{3\epsilon}+\frac{193}{24}\right), \tag{63}$$

$$J^{a}(x)J^{b}(y)\to -i\alpha^4\frac{N^{ab}}{\bar{k}}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{17}{3\epsilon}+\frac{191}{24}\right)-i\alpha^4\hat{N}^{ab}\frac{k}{\bar{k}^2}\left|\frac{k^2}{\mu^2}\right|^{-\epsilon}\left(\frac{25}{3\epsilon}+\frac{319}{24}\right). \tag{64}$$

The results can now be expressed in coordinate space using the following:

$$\frac{ik}{\bar{k}^2}\Rightarrow\frac{\bar{z}-\bar{w}}{(z-w)^2}, \tag{65}$$

$$\frac{-i}{\bar{k}}\Rightarrow\frac{1}{z-w}, \tag{66}$$

$$\frac{-i}{\bar{k}}\left[\frac{1}{\epsilon}-\ln\left|\frac{k}{\mu}\right|^2\right]\Rightarrow\ln|z-w|^2\frac{1}{z-w}, \tag{67}$$

$$\frac{-ik}{\bar{k}^2}\left[\frac{1}{\epsilon}-\ln\left|\frac{k}{\mu}\right|^2+2\right]\Rightarrow\ln|z-w|^2\frac{(\bar{z}-\bar{w})}{(z-w)^2}. \tag{68}$$

The terms (65) and (66) can be derived by a simple inverse Fourier transform. For Eq. (67) we calculate the expectation value $\langle\partial X(z)X(z)X(w)X(w)\rangle$ in momentum and coordinate space, and then compare them; the same is done for Eq. (68), calculating $\langle\partial X(z)X(z)\int d^2u\,\partial X(u)X(u)\partial X(w)X(w)\rangle$ in momentum and in coordinate space. The results in coordinate space are

$$J^{a'}(x)J^{b'}(y)\to\frac{N^{a'b'}}{(x-y)}\left(\frac{1}{3}\ln|x-y|^2-\frac{65}{24}\right)+\hat{N}^{a'b'}\frac{(\bar{x}-\bar{y})}{(x-y)^2}\left(-\frac{7}{3}\ln|x-y|^2+\frac{81}{24}\right), \tag{69}$$

$$J^{a}(x)J^{b}(y)\to\frac{N^{ab}}{(x-y)}\left(\frac{17}{3}\ln|x-y|^2+\frac{191}{24}\right)+\hat{N}^{ab}\frac{(\bar{x}-\bar{y})}{(x-y)^2}\left(\frac{25}{3}\ln|x-y|^2+\frac{81}{24}\right). \tag{70}$$

Note that the divergences in momentum-space OPEs appear as logarithms in coordinate-space OPEs, suggesting that the currents may get anomalous dimensions. This is not a surprising result since the $AdS_5 \times S^5$ left-invariant currents are not protected by any symmetry argument. However—as it was shown in Ref. [5], where the one-loop OPEs between the energy-momentum tensor and the left-invariant currents were computed—the bosonic left-invariant currents do not get anomalous dimensions. On the other hand, the fermionic currents get anomalous dimension contributions. However, it was shown that the two types of fermionic currents get the precise contributions that cancel when combined into the single operator which appears in the energy-momentum tensor. Therefore, the energy-momentum tensor still has zero anomalous dimension. This result suggests that, although the left-invariant currents have logarithmic terms in their one-loop OPEs, protected operators constructed with these currents do not have them.

V. CONCLUSION

It was shown in Ref. [5] that, at one loop, there are nontrivial cancellations in the possible corrections to the double pole of the product of the bosonic currents $J^{\underline{a}}(y)J^{\underline{b}}(z)$. Here it was shown that there are contributions to the simple poles. As the simple poles are important in the calculations of the four-point function, the results presented here are a step forward in order to get a world-sheet description of pure-spinor closed strings in $AdS_5 \times S^5$. Since these currents are not holomorphic, there are antiholomorphic terms in the OPEs. However, if we try to make the same calculation for holomorphic quantities, such as the combination $\lambda^\alpha J_\alpha$, which is used to define the BRST charge in $AdS_5 \times S^5$, we can see that it is not possible to form diagrams with one classical field and so there is no antiholomorphic contributions for these quantities.

As in the tree-level calculation of Ref. [12], the one-loop OPEs calculated here respect the Z4-grading of the psu(2, 2|4) superalgebra, reflecting the fact that all the interactions respect the Z4-automorphism of the superalgebra. It is important to note that all terms proportional to the currents $J^{\underline{ab}}$, $\bar{J}^{\underline{ab}}$ cancel and the only contribution comes from the pure-spinor Lorentz currents $N^{\underline{ab}}$ and $\hat{N}^{\underline{ab}}$. The calculations presented in this work show the way to get the complete one-loop current algebra for the pure-spinor $AdS_5 \times S^5$ string. However, owing to the huge number of vertices coming from background expansion,

we hope to develop some algebraic computational methods in order to complete such a task.

As a final remark, we would like to draw attention to Ref. [22], where the author presents another method to compute the current algebra. In this paper, this method was used to compute the current algebra at tree level. However, the author states that it can be generalized to compute quantum corrections to the algebra of the currents. It would be a good task to investigate this in a future work.

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APPENDIX

1. Table of integrals

$$\int d^d m \frac{1}{|m|^2|m-k|^2} = -\frac{2}{k\bar{k}}\left(\frac{1}{\epsilon} - \ln\frac{|k|^2}{\mu^2}\right) \tag{A1}$$

$$\int d^d m \frac{m\bar{m}}{|m|^2|m-k|^2} = 1 \tag{A2}$$

$$\int d^d m \frac{m}{|m|^2|m-k|^2} = -\frac{1}{\pi^\epsilon}\frac{1}{\bar{k}}\frac{[|k|^2]^{-\epsilon}}{\mu^{-2\epsilon}}\frac{\Gamma(1-\epsilon)^2\Gamma(\epsilon)}{\Gamma(1-2\epsilon)} = -\frac{1}{\bar{k}}\left(\frac{1}{\epsilon} - \ln\frac{|k|^2}{\mu^2}\right) \tag{A3}$$

$$\int d^d m \frac{\bar{m}}{|m|^2|m-k|^2} = -\frac{1}{\pi^\epsilon}\frac{1}{k}\frac{[|k|^2]^{-\epsilon}}{\mu^{-2\epsilon}}\frac{\Gamma(1-\epsilon)^2\Gamma(\epsilon)}{\Gamma(1-2\epsilon)} = -\frac{1}{k}\left(\frac{1}{\epsilon} - \ln\frac{|k|^2}{\mu^2}\right) \tag{A4}$$

$$\int d^d m \frac{m^2}{|m|^2|m-k|^2} = -\frac{1}{\pi^\epsilon}\frac{k}{\bar{k}}\frac{[|k|^2]^{-\epsilon}}{\mu^{-2\epsilon}}\frac{\Gamma(2-\epsilon)\Gamma(1-\epsilon)\Gamma(\epsilon)}{\Gamma(2-2\epsilon)} = -\frac{k}{\bar{k}}\left(\frac{1}{\epsilon} + 1 - \ln\frac{|k|^2}{\mu^2}\right) \tag{A5}$$

$$\int d^d m \frac{\bar{m}^2}{|m|^2|m-k|^2} = -\frac{1}{\pi^\epsilon}\frac{\bar{k}}{k}\frac{[|k|^2]^{-\epsilon}}{\mu^{-2\epsilon}}\frac{\Gamma(2-\epsilon)\Gamma(1-\epsilon)\Gamma(\epsilon)}{\Gamma(2-2\epsilon)} = -\frac{\bar{k}}{k}\left(\frac{1}{\epsilon} + 1 - \ln\frac{|k|^2}{\mu^2}\right) \tag{A6}$$

$$\int d^d m \frac{m^2\bar{m}}{|m|^2|m-k|^2} = \frac{1}{2\pi^\epsilon}k\frac{[|k|^2]^{-\epsilon}}{\mu^{-2\epsilon}}\frac{\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(2-2\epsilon)} = \frac{k}{2} \tag{A7}$$

$$\int d^d m \frac{m\bar{m}^2}{|m|^2|m-k|^2} = \frac{1}{2\pi^\epsilon}\bar{k}\frac{[|k|^2]^{-\epsilon}}{\mu^{-2\epsilon}}\frac{\Gamma(1-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(2-2\epsilon)} = \frac{\bar{k}}{2} \tag{A8}$$

$$\int d^d m \frac{m^2\bar{m}^2}{|m|^2|m-k|^2} = \frac{1}{\pi^\epsilon}k\bar{k}\frac{[|k|^2]^{-\epsilon}}{\mu^{-2\epsilon}}\frac{\Gamma(2-\epsilon)^2\Gamma(1+\epsilon)}{\Gamma(4-2\epsilon)} = \frac{k\bar{k}}{6} \tag{A9}$$

$$\int d^d m \frac{m\bar{m}}{[|m|^2]^2|m-k|^2} = -\frac{1}{\pi^\epsilon}\frac{1}{k\bar{k}}\left(1 + \frac{2}{\epsilon} - 2\ln\frac{|k|^2}{\mu^2}\right) \tag{A10}$$

$$\int d^d m \frac{m^2\bar{m}}{[|m|^2]^2|m-k|^2} = -\frac{1}{\pi^\epsilon}\frac{1}{\bar{k}}\left(\frac{1}{\epsilon} - \ln\frac{|k|^2}{\mu^2}\right) \tag{A11}$$

$$\int d^d m \frac{m^3\bar{m}}{[|m|^2]^2|m-k|^2} = -\frac{1}{\pi^\epsilon}\frac{k}{\bar{k}}\left(\frac{1}{\epsilon} + 1 - \ln\frac{|k|^2}{\mu^2}\right) \tag{A12}$$

$$\int d^d m \frac{m^2\bar{m}^2}{[|m|^2]^2|m-k|^2} = \frac{3}{2} \tag{A13}$$

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