

Creation of neutral fermions with anomalous magnetic moments from a vacuum by inhomogeneous magnetic fields

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A consistent nonperturbative approach (based on QFT) to neutral fermion creation (due to their magnetic moments) in strong inhomogeneous magnetic fields is considered. It is demonstrated that quantization in terms of neutral particles and antiparticles is possible in terms of the states with well-defined spin polarization. Such states are localizable and can form wave packets in a given asymptotic region. In this case, the problem can be technically reduced to the problem of charged-particle creation by an electric step. In particular, the relation to the Schwinger method of an effective action is established. As an example, we calculate neutral fermion creation from the vacuum by a linearly growing magnetic field. We show that the total number and the vacuum-to-vacuum transition probability of created pairs depend only on the gradient of the magnetic field, but not on its strength, and this fact does not depend on the spacetime dimension. We show that the created flux aimed in one of the directions is formed from fluxes of particles and antiparticles of equal intensity and with the same magnetic moments parallel to the external field. In such a flux, particle and antiparticle velocities that are perpendicular to the plane of the magnetic moment and flux direction are essentially depressed. The creation of neutral fermions with anomalous magnetic moments leads to a smoothing of the initial magnetic field, which in turn prevents the appearance of superstrong constant magnetic fields. Our estimations show that the vacuum instability with respect to the creation of neutrinos and even neutrons in strong magnetic fields of the magnetars and fields generated during a supernova explosion has to be taken into account in the astrophysics. In particular, it may be of significance for dark matter studies.

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I. INTRODUCTION

Usually, particle creation from the vacuum by strong electromagnetic fields is associated with the creation of charged particles by strong electric-like fields. Acting on virtual charged particles, an electric-like field can produce a work and materialize them on the mass shell as real particles. Nevertheless, if a neutral particle has an anomalous magnetic moment, an inhomogeneous magnetic field acting on such a particle can also change its kinetic energy (produce a work). This mechanism can provide neutral-particle creation from the vacuum by strong inhomogeneous magnetic fields. In this respect, one can speak about two candidates among the known elementary particles: neutrons and neutrinos. It is known that the neutron has a negative magnetic moment given by $\mu_n = -1.9130427(5)\mu_N$, where μ_N is the nuclear magneton, $\mu_N = e/2m_N$. It is also possible that neutrinos have magnetic moments (in general, effective magnetic moments which take into account neutrino mixing and the oscillations) acquired through quantum loop effects; for a review see Refs. [1–3]. The recent experimental constraints on the neutrino magnetic moments are in the range $\sim 10^{-11}\mu_B$

($\mu_\nu < 2.9 \times 10^{-11}\mu_B$ for electron neutrino) [4], where $\mu_B = e/2m_e$ is the Bohr magneton. Astrophysical constraints on the magnetic moment of the Dirac neutrino can be even stronger, $\mu_\nu < 1.1 \times 10^{-12}\mu_B$ [5]. Note that in order to satisfy $m_\nu \lesssim 1$ eV, the theory argues that a more natural scale for the Dirac neutrino would be $\mu_\nu \lesssim 10^{-14}\mu_B$ [6].

The discovery of neutrino masses suggests the likely existence of the light sterile neutrinos that appear in the low-energy effective theory in most extensions of the standard model, and in principle can have any mass, in particular, in the 1 eV mass range. The sterile neutrinos with masses of several keV can account for cosmological dark matter, e.g., see Refs. [7,8] for a recent review, and references therein. It is possible that due to some new physics the neutrino magnetic moment is big. Various observational constraints on the magnetic moment μ of a dark matter particle for masses M in the range 1 keV to 100 MeV have been considered in Refs. [9,10]. The strongest limits on μ emerge at the lightest mass scales. For example, if $M = m_e/10$ then $|\mu| < 3.4 \times 10^{-5}\mu_B$ due to precision electroweak measurements. It is noted [10] that a variety of astrophysical constraints can be significantly weakened by the candidate particle's mass and the above-mentioned constraints can be weakened by other means as well.

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The effect under discussion can be observed in inhomogeneous magnetic fields that have to be very strong in a certain domain. Such fields can exist in nature. It has been suggested that magnetic fields of order 10^{15} – 10^{16} G or stronger, up to 10^{18} G, can probably be generated during a supernova explosion or in the vicinity of the special group of neutron stars known as magnetars; see, for example, Ref. [11]. For magnetar cores made of quark matter the interior field can be estimated to reach values of $B \sim 10^{20}$ G [12]. The possibility to create a strong quasiuniform magnetic field with the strength of the hadronic scale $B \sim 10^{19}$ G—or even higher in heavy-ion collisions at the RHIC and LHC, when the matter in the central region is presumably in the quark-gluon plasma phase—was recently shown [13]. Superconducting cosmic strings—if they exist—could generate fields of more than 10^{30} G in their vicinities [14].

Recently, the Schwinger effective action approach [15] was formally applied to calculate the probability for the vacuum to remain a vacuum in a linearly growing magnetic field for neutral fermions of spin 1/2 with anomalous magnetic moment. The same problem in 2 + 1 dimensions was considered in Ref. [16], and in 3 + 1 dimensions in Ref. [17]. It is difficult to accept the results presented in Ref. [17], which, in particular, admit neutral-particle creation in a homogeneous magnetic field. This means that formal calculations *à la* Schwinger, without any theoretical justification based on quantum field theory (QFT), can lead to mistakes. The results of Ref. [16] seem to be reasonable, but essentially use specific gamma matrices in 2 + 1 dimensions, and cannot provide a complete description of the effect.

It should be noted that until now a consistent description of particle creation in the framework of QFT (due to their magnetic moments) in strong inhomogeneous magnetic fields was unknown. To provide such a description is a part of the present paper. In Secs. II and III, we demonstrate that in specific cases, the problem can be technically reduced to the problem of charged-particle creation by an electric field given by a step scalar potential and all the information about the problem can be extracted from exact solutions of the corresponding Dirac equation. We analyze the latter problem once again in the framework of QFT and derive all the necessary expressions for the probabilities of particle creation. As for the Dirac equation, here we find a complete set of mutually commuting integrals of motion, separate variables, and show that the energy spectrum of a neutral fermion that interacts with an inhomogeneous magnetic field due to an anomalous magnetic moment is real and consists of two branches separated by a gap. In Sec. IV we calculate all the characteristics of neutral fermion creation from the vacuum by a linearly growing magnetic field. These results and some of their astrophysical implications are discussed in Sec. V.

II. DIRAC-PAULI EQUATION WITH A CONSTANT MAGNETIC FIELD

In 3 + 1 dimensions (dim.), the relativistic neutral fermions of spin 1/2 and mass m with anomalous magnetic moment μ (without an electric dipole moment) in an external electromagnetic field $F_{\lambda\nu}$ are described by the Dirac-Pauli equation; see Refs. [18,19]. Such an equation has the form¹

$$\left(\gamma^\lambda \hat{p}_\lambda - m - \frac{1}{2} \mu \sigma^{\lambda\nu} F_{\lambda\nu} \right) \psi(x) = 0, \quad (1)$$

$$\hat{p}_\nu = i\partial_\nu, \quad \sigma^{\lambda\nu} = \frac{i}{2} [\gamma^\lambda, \gamma^\nu],$$

where $F_{\lambda\nu}(x)$ is the field tensor, $\psi(x)$ is a four spinor, $x = (x^0 = t, \mathbf{r})$, $\mathbf{r} = (x, y, z)$, and $\gamma^\nu = (\gamma^0, \boldsymbol{\gamma})$ are Dirac matrices.

Let the external field be a constant nonuniform magnetic field \mathbf{B} that is directed along the z axis and depends on the coordinate y only, $\mathbf{B}(y) = (0, 0, B_z(y))$ such that the only nonzero components of the field tensor are $F_{21}(y) = -F_{12}(y) = B_z(y)$. In addition, we suppose that $B_z(y)$ takes constant values as $y \rightarrow \pm\infty$, such that

$$\partial_y B_z(y) \xrightarrow{y \rightarrow \pm\infty} 0.$$

Moreover, we suppose that for $y < y_L$ (the region $S_L = (-\infty, y_L]$) and $y > y_R$ (the region $S_R = [y_R, \infty)$) the field $B_z(y)$ is already uniform and its values are $B_z(y) = B_z(-\infty)$ and $B_z(y) = B_z(+\infty)$, respectively. Thus, the magnetic field under consideration is constant and uniform (or zero) at spatial infinities and, in fact, represents either a potential barrier or step for the magnetic moment μ . With such an external field, Eq. (1) takes the form

$$i\partial_0 \psi(t, \mathbf{r}) = \hat{H} \psi(t, \mathbf{r}), \quad \hat{H} = \gamma^0 \gamma^3 \hat{p}^3 + \gamma^0 \Sigma_z \hat{\Pi}_z,$$

$$\hat{\Pi}_z = \Sigma_z \boldsymbol{\gamma} \hat{\mathbf{p}}_\perp + m \Sigma_z - \mu B_z(y), \quad \hat{\mathbf{p}}_\perp = (\hat{p}^1, \hat{p}^2, 0). \quad (2)$$

In the case under consideration, the operators \hat{p}^0 , \hat{p}^1 , \hat{p}^3 , and $\hat{\Pi}_z$ are mutually commuting integrals of motion (all these operators commute with the Hamiltonian \hat{H}). The integral of motion $\hat{\Pi}_z$ is a generalization of the z component of a spin polarization tensor for a uniform magnetic field; see Ref. [19].

It is useful to use an additional spin operator \hat{R} , which is also an integral of motion commuting with the previous ones,

$$\hat{R} = \hat{H} \hat{\Pi}_z^{-1} [1 + (\hat{p}^3 \hat{\Pi}_z^{-1})^2]^{-1/2}. \quad (3)$$

A complete set of solutions of Eq. (2) can be written in the form

$$\psi_n(t, \mathbf{r}) = \exp(-ip_0 t + ip_x x + ip_z z) \psi_n(y), \quad (4)$$

¹Here we are using the natural system of units $\hbar = c = 1$.

where $\psi_n(y)$ are eigenvectors of the equations

$$\begin{aligned} s\omega\sqrt{1+(p_z/\omega)^2}R\psi_n(y) &= p_0\psi_n(y) \Rightarrow R\psi_n(y) = s\psi_n(y), \\ p_0 &= \omega\sqrt{1+(p_z/\omega)^2}, \\ R &= [1+(p_z/\omega)^2]^{-1/2}(s\gamma^0\gamma^3p_z/\omega + \gamma^0\Sigma_z), \end{aligned} \quad (5)$$

$$\begin{aligned} [\hat{\Pi}_z(p_x, y) - s\omega]\psi_n(y) &= 0, \\ s &= \pm 1, \\ \hat{\Pi}_z(p_x, y) &= \hat{\pi}_z - \mu B_z(y), \\ \hat{\pi}_z &= \Sigma_z(\gamma^1 p_x + \gamma^2 \hat{p}^2) + m\Sigma_z, \end{aligned} \quad (6)$$

and $n = (p_x, p_z, \omega, s)$ is the set of quantum numbers from a complete set of numbers that will be specified below. Choosing $\psi_n(y)$ as

$$\psi_n(y) = \frac{1}{2}(1 + sR)\Phi(y),$$

where $\Phi(y)$ is an arbitrary spinor, we obey Eq. (5). It should be particularly emphasized that the real continuous quantum number ω can be positive and negative and determines the transversal part of the full energy, $\omega^2 = p_0^2 - p_z^2$, that is, it determines the full energy of a particle moving in the xy plane. We see that the energy spectrum of the neutral fermion with anomalous momentum is real and consists of positive and negative branches, similarly to the spectrum of the charged fermion in a time-independent electric field.

Then solutions of Eq. (6) can be represented as

$$\psi_n(y) = \frac{1}{2}(1 + sR)[\hat{\pi}_z + \mu B_z(y) + s\omega]\phi_n(y), \quad (7)$$

where the spinors $\phi_n(y)$ satisfy the following equation:

$$\begin{aligned} \{-\partial_y^2 + m^2 + p_x^2 - \mu\gamma^1\partial_y B_z(y) \\ - [\omega + s\mu B_z(y)]^2\}\phi_n(y) = 0. \end{aligned} \quad (8)$$

It is convenient to represent the spinor $\phi_n(y)$ in the form

$$\phi_n(y) = \varphi_{n,\chi}(y)\frac{1}{2}(1 + i\chi\gamma^1)v, \quad (9)$$

where it is selected that either $\chi = +1$ or $\chi = -1$, v is an arbitrary constant spinor, and the scalar functions $\varphi_{n,\chi}(y)$ are solutions of the equation

$$\begin{aligned} \{-\partial_y^2 + m^2 + p_x^2 + i\chi\mu\partial_y B_z(y) \\ - [\omega + s\mu B_z(y)]^2\}\varphi_{n,\chi}(y) = 0. \end{aligned} \quad (10)$$

In what follows, we suppose that v is normalized as $v^\dagger v = 1$. In addition, vv^\dagger is the identity 4×4 matrix, $vv^\dagger = I$. Thus, the spinor structure of the solutions (7) is defined completely. One can easily verify that the solutions (7) that differ by values of χ only are linearly dependent; this is an effect which the projection operator [...] in the representation (7) produces. Because of this, it is enough to

work with solutions corresponding to one of two possible values for χ . This is why the superscript χ will sometimes disappear from solutions, but in such cases it is supposed that χ is fixed in a certain way that is the same for all solutions under consideration.

Using the freedom inherent in the solutions of Eq. (10), we construct two (in general different) sets $\{\zeta\psi_n(t, \mathbf{r})\}$ and $\{\xi\psi_n(t, \mathbf{r})\}$ of independent solutions, $\zeta = \pm$, satisfying the specific boundary conditions $y \rightarrow -\infty$ or $y \rightarrow +\infty$. The first set contains states $\zeta\psi_n(t, \mathbf{r})$ with definite real values p^L of the y component of the momentum, such that ζ defines the sign of the momentum,

$$-i\partial_{y\zeta}\psi_n(t, \mathbf{r}) = p^L_\zeta\psi_n(t, \mathbf{r}), \quad \zeta = \text{sgn } p^L, \quad y \rightarrow -\infty. \quad (11)$$

The second set contains states $\xi\psi_n(t, \mathbf{r})$ with definite real values p^R of the y component of the momentum, and again ξ defines the sign of the momentum,

$$-i\partial_{y\xi}\psi_n(t, \mathbf{r}) = p^R_\xi\psi_n(t, \mathbf{r}), \quad \xi = \text{sgn } p^R, \quad y \rightarrow +\infty. \quad (12)$$

We are interested in the nondecaying solutions of Eq. (10) as $y \rightarrow \pm\infty$. In this case both p^L and p^R are real. We believe that for any given quantum numbers n both sets $\{\zeta\psi_n(t, \mathbf{r})\}$ and $\{\xi\psi_n(t, \mathbf{r})\}$ represent complete sets of non-decaying solutions. In fact, this is the above-mentioned supposition about the form of the field $B_z(y)$.

It should be noted that the time independence of the magnetic field under consideration is an idealization. In fact, it is supposed that a field inhomogeneity was switched on in a time instant t_{in} , which then acts as the constant field during a large time T , and was switched off in a time instant $t_{\text{out}} = t_{\text{in}} + T$, and one can ignore the effects of its switching on and off. This is a kind of regularization, which could—under certain conditions—be replaced by periodic boundary conditions in t . Namely, by analogy with periodic boundary conditions in space—which are usually imposed as the volume regularization—here we impose periodic (with the period T) boundary conditions in time t . Thus, we consider a theory in a big three-dimensional spacetime box that has a volume $V_y = TS_{xz}$, $S_{xz} = L_x \times L_z$, where L_x , L_z , and T are macroscopically large, $L_x, L_z \rightarrow \infty$ and $T \rightarrow \infty$.

It is convenient to use the inner product on the time-like hyperplane $y = \text{const}$, which has the form

$$(\psi, \psi')_y = \int_{V_y} \psi^\dagger(t, \mathbf{r})\gamma^0\gamma^2\psi'(t, \mathbf{r})dtdxdz. \quad (13)$$

The integration in Eq. (13) is fulfilled in the limits from $-L_x/2$ to $+L_x/2$, $-L_z/2$ to $+L_z/2$, and from $-T/2$ to $+T/2$ in the time t . It is supposed that all the functions ψ are periodic under translations from one box to another. Under these assumptions, the inner product (13) does not

depend on y . We note that the quantity (13) for $\psi' = \psi$ represents the particle current via the hyperplane $y = \text{const}$.

By using the inner product (13), we obtain

$$(\psi_n, \psi'_k)_y = V_y \delta_{n,k} \psi_n^\dagger(y) \gamma^0 \gamma^2 \psi'_n(y).$$

Thus, the current density in the y direction in the state $\psi_n(t, \mathbf{r})$ is

$$I_n = \psi_n^\dagger(y) \gamma^0 \gamma^2 \psi_n(y). \quad (14)$$

Using the structure (7), we rewrite the combination $\psi_n^\dagger(y) \gamma^0 \gamma^2 \psi'_n(y)$ as follows:

$$\begin{aligned} \varphi_{n,\chi}^*(y) \text{tr} \left\{ [-\Sigma_z (\gamma^1 p_x + i \vec{\partial}_y \gamma^2) - m \Sigma_z \right. \\ \left. - \mu B_z(y) - s\omega] \gamma^0 \gamma^2 \frac{1}{2} [1 + sR] [-\Sigma_z (\gamma^1 p_x - i \vec{\partial}_y \gamma^2) \right. \\ \left. - m \Sigma_z - \mu B_z(y) - s\omega] \frac{1}{2} (1 + \chi i \gamma^1) \right\} \varphi'_{n,\chi}(y), \end{aligned}$$

where $\text{tr}\{ \dots \}$ is the trace in the space of 4×4 matrices. Calculating this trace, we obtain

$$\begin{aligned} \psi_n^\dagger(y) \gamma^0 \gamma^2 \psi'_n(y) = (1 + (p_z/\omega)^2)^{-1/2} \varphi_{n,\chi}^*(y) (i \vec{\partial}_y - i \vec{\partial}_y) \\ \times (\omega + s\mu B_z(y) + s\chi i \vec{\partial}_y) \varphi'_{n,\chi}(y). \end{aligned} \quad (15)$$

As was already mentioned, we supposed that $B_z(y)$ tends to some constant values as $y \rightarrow \pm\infty$. Let us suppose for the sake of definiteness that the derivative $\partial_y B_z(y)$ has a definite sign, let us say $\partial_y B_z(y) \geq 0$, $\forall y$, and let $\mu < 0$. Note that there are no bound states in this case. To simplify the consideration, we also suppose that

$$\begin{aligned} U &= U_R - U_L > 0, \\ U_L &= -\mu B_z(-\infty) < 0, \\ U_R &= -\mu B_z(+\infty) > 0. \end{aligned}$$

For asymptotic (as $|y| \rightarrow \infty$) states with real values p^L and/or p^R , we have

$$\begin{aligned} {}_\zeta \varphi_{n,\chi}(y) &= {}_\zeta \mathcal{N} \exp(ip^L y), \\ {}^\xi \varphi_n(x) &= {}^\xi \mathcal{N} \exp(ip^R y), \end{aligned} \quad (16)$$

respectively, where ${}_\zeta \mathcal{N}$ and ${}^\xi \mathcal{N}$ are normalization factors. We introduce the notation

$$\begin{aligned} E_s(L/R) &= \pi_s(L/R) \sqrt{1 + [p_z/\pi_s(L/R)]^2}, \\ \pi_s(L/R) &= \omega - sU_{L/R}, \quad \pi_x = \sqrt{p_x^2 + m^2}, \end{aligned} \quad (17)$$

and in their terms we stress the existence of the following relations:

$$\pi_s(L) = \pi_s(R) + sU, \quad (18)$$

$$\begin{aligned} (p^L)^2 &= [E_s(L)]^2 - \pi_x^2 - p_z^2, \\ (p^R)^2 &= [E_s(R)]^2 - \pi_x^2 - p_z^2, \end{aligned} \quad (19)$$

where Eq. (19) holds due to Eq. (10). We see that $|E_s(L)|$ and $|E_s(R)|$ are the asymptotic values of the kinetic energy, while $|\pi_s(L)|$ and $|\pi_s(R)|$ are the asymptotic values of its transversal part, respectively.

Note that the case of the uniform magnetic field is realized when $U_R \rightarrow U_L = -\mu B_z$; then, asymptotic regions coincide and coincide with the whole space as well, $\pi_s(L) = \pi_s(R) = \omega + s\mu B_z$, and $p^L = p^R = p_y$. It follows from Eqs. (19) that

$$\begin{aligned} p_x^2 + p_y^2 + m^2 &= (\omega + s\mu B_z)^2 \\ \Rightarrow \omega + s\mu B_z &= \pm \sqrt{p_x^2 + p_y^2 + m^2}, \end{aligned}$$

and we see that $|\omega + s\mu B_z|$ is the transversal part of the kinetic energy. Thus, using standard second quantization, we can construct the Fock space of fermions with conserved spin polarization s , where $\omega + s\mu B_z \geq m$ for particles, and $\omega + s\mu B_z \leq -m$ for antiparticles. One can see that in contrast to the statement of Ref. [17]—which is the result of an improper treatment of naive spectra that, in fact, are valued for the case of a weak field; see Ref. [19]—the energy spectrum of neutral fermions interacting with a uniform magnetic field due to an anomalous magnetic moments is real and a level crossing and vacuum instability is absent. In fact, this Fock space is equivalent to the Fock space of free particles.

Then, using the asymptotic conditions (11) and (12), and the result (15), we can subject the introduced sets $\{ {}_\zeta \psi_n(t, \mathbf{r}) \}$ and $\{ {}^\xi \psi_n(t, \mathbf{r}) \}$ to the following orthonormality conditions:

$$\begin{aligned} ({}_\zeta \psi_n, {}_{\zeta'} \psi_{n'})_x &= \zeta \eta_L \delta_{\zeta, \zeta'} \delta_{n, n'}, \\ ({}^\xi \psi_n, {}^{\xi'} \psi_{n'})_x &= \zeta \eta_R \delta_{\zeta, \zeta'} \delta_{n, n'}, \end{aligned} \quad (20)$$

where

$$\eta_L = \text{sgn} \pi_s(L), \quad \eta_R = \text{sgn} \pi_s(R).$$

In deriving Eq. (20), it was taken into account that for asymptotic (as $|y| \rightarrow \infty$) states with real values p^L and p^R , the relations

$$|\pi_s(L)| > |p^L|, \quad |\pi_s(R)| > |p^R|$$

hold due to Eq. (19), respectively. This is why the sign of the quantity (15) with the operator $[\pi_s(L/R) + s\chi i \vec{\partial}_y]$ is due to the sign of the $\pi_s(L/R)$. The normalization factors in Eq. (16) are as follows:

$$\begin{aligned} {}_\zeta \mathcal{N} &= {}_\zeta CY, \quad {}^\xi \mathcal{N} = {}^\xi CY, \quad Y = (1 + (p_z/\omega)^2)^{1/4} V_y^{-1/2}, \\ {}_\zeta C &= [2|p^L| |\pi_s(L) - s\chi p^L|]^{-1/2}, \\ {}^\xi C &= [2|p^R| |\pi_s(R) - s\chi p^R|]^{-1/2}. \end{aligned} \quad (21)$$

In the limit of infinite volume of the normalization (continuous momenta p_0 , p_x , and p_z) one has to substitute $\delta_{n,n'}$ into the normalization conditions (20) by $\delta_{s,s'}\delta(p_0 - p'_0)\delta(p_x - p'_x)\delta(p_z - p'_z)$. In this case, $V_y^{-1/2} \rightarrow (2\pi)^{-3/2}$ in Eqs. (21).

It is supposed that for any given quantum numbers n , both sets $\{\zeta \psi_n(t, \mathbf{r})\}$ and $\{\zeta' \psi_n(t, \mathbf{r})\}$ represent complete sets of nondecaying solutions of Eq. (2). Then their mutual decompositions have the form

$$\begin{aligned} \eta_L^\zeta \psi_n(t, \mathbf{r}) &= +\psi_n(t, \mathbf{r})g(+|\zeta) - -\psi_n(t, \mathbf{r})g(-|\zeta), \\ \eta_R^{\zeta'} \psi_n(t, \mathbf{r}) &= +\psi_n(t, \mathbf{r})g(+|\zeta') - -\psi_n(t, \mathbf{r})g(-|\zeta'), \end{aligned} \quad (22)$$

where the decomposition coefficients g are defined by the relations

$$(\zeta \psi_n, \zeta' \psi_{n'})_y = \delta_{nn'} g(\zeta'|\zeta'), \quad g(\zeta'|\zeta) = g(\zeta|\zeta')^*. \quad (23)$$

Using the orthonormality conditions (20), we derive the following relations for the decomposition coefficients:

$$\begin{aligned} g(\zeta'|\zeta)g(+|\zeta) - g(\zeta'|\zeta)g(-|\zeta) &= \zeta \eta_L \eta_R \delta_{\zeta, \zeta'}, \\ g(\zeta'|\zeta)g(+|\zeta) - g(\zeta'|\zeta)g(-|\zeta) &= \zeta \eta_L \eta_R \delta_{\zeta, \zeta'}. \end{aligned} \quad (24)$$

In particular, these relations imply that

$$\begin{aligned} |g(-|+)|^2 &= |g(+|-)|^2, \\ |g(+|+)|^2 &= |g(-|-)|^2, \\ \frac{g(+|-)}{g(-|-)} &= \frac{g(+|+)}{g(-|+)}. \end{aligned} \quad (25)$$

Thus, one can see that all these coefficients can be expressed via only two of them, e.g., via $g(+|+)$ and $g(+|-)$. However, even these coefficients are not completely independent; they are related as follows:

$$|g(+|-)|^2 - |g(+|+)|^2 = -\eta_L \eta_R. \quad (26)$$

III. CREATION OF NEUTRAL FERMIONS

It is useful to make a preliminary qualitative analysis of the behavior of particles and antiparticles in the fields under consideration. It should be noted that here there exist two principally different cases: the first one corresponds to $U < 2m$, whereas the second one (we call it the creation case, or C-case) corresponds to $U > 2m$. In the first case, there exist only a scattering of neutral fermions by the magnetic field without additional particle creation from the vacuum. This case can be treated in the framework of one-particle relativistic quantum mechanics. The quantum number s gives the spin polarization for both particles and antiparticles. Choosing the magnetic moment of the particle as μ , we have the magnetic moment of the antiparticle as $-\mu$. Note that we fix $\mu = -|\mu|$. Then, according to the standard particle-antiparticle identification of wave functions, the asymptotic kinetic energy (at $y \rightarrow \pm\infty$) of the

particle moving in the xy plane is $\pi_s(L/R) > 0$, while it is $-\pi_s(L/R) > 0$ for the antiparticle. One can see from Eq. (17) that the particle potential energy $s|\mu|B_z(y)$ decreases along the y axis for $s = -1$ and increases for $s = +1$. At the same time, the antiparticle potential energy $-s|\mu|B_z(y)$ increases along the y axis for $s = -1$ and decreases for $s = +1$. This means that the field $B_z(y)$ accelerates particles with $s = -1$ and antiparticles with $s = +1$ along the y axis. Respectively, antiparticles with $s = -1$ and particles with $s = +1$ are accelerated by the field in the opposite direction. The same observation holds in the case $U > 2m$.

We note that real particles are described by some wave packets localized in the spacetime, such that we have to study the motion of such packets in the external field (obviously, it is enough to speak about a localization in the y direction). Let us denote by S_{int} the region where the magnetic field is inhomogeneous. In the region S_L , situated to the left of S_{int} , and in the region S_R to the right of S_{int} , the magnetic field is homogeneous. For big enough differences U between the initial and final potential energies, particles and antiparticles with any initial kinetic momenta along the y axis get final kinetic momenta that is always in the same direction as their acceleration by the magnetic field. This is

what we have in the case $U > 2\sqrt{p_x^2 + p_z^2 + m^2}$ for all partial waves with given p_x and p_z of a wave packet. Because particles and their antiparticles with a given s have opposite directions of acceleration, there exists a state polarization out of the region S_{int} . The final particles with $s = +1$ and antiparticles with $s = -1$ are situated in the region S_L , and final antiparticles with $s = +1$ and particles with $s = -1$ are situated in the region S_R .

From the physical point of view, there is a similarity between the two cases—one where neutral fermions with an anomalous magnetic moment are placed in an inhomogeneous magnetic field $B_z(y)$ with $\partial_y B_z(y) > 0$, and another where charged fermions are placed in a constant electric field directed along y and given by a scalar potential $A_0(y)$. In both cases external fields produce a work which implies an acceleration of the corresponding particles in the y direction. From the QFT point of view if such a work is greater than $2m$ (C-case), particle creation from the vacuum is possible. In fact, this analogy allows in both cases formally to use the same techniques of calculation. It turns out that the problem of neutral fermion creation in a strong inhomogeneous magnetic field can be technically reduced to the problem of charged-particle creation by an electric potential step. Some heuristic exact calculations of the particle creation by potential steps in the framework of the relativistic quantum mechanics were presented by Nikishov [20,21], further developed in Ref. [22], and used in numerous works in the framework of semiclassical considerations; for a review see Refs. [23,24].

In such a way it seems that we could use the known results to find the mean number of neutral

particle-antiparticle pairs created. However, a closer consideration shows that the particle-antiparticle and causal identification of wave functions ${}^{\zeta}\psi_n(t, \mathbf{r})$ and ${}^{\xi}\psi_n(t, \mathbf{r})$ given by Nikishov [20,21] does not coincide with that given by Hansen and Ravndal [22] for the C-case; see the discussion in Ref. [25]. Within the WKB approximation this difficulty can be bypassed, but the question remains. Trying to resolve this contradiction, we have realized that at that time no justification for quantum mechanical calculations from the QFT point of view were elaborated. Such a justification can be obtained in the framework of a strict QFT formulation of particle creation by potential steps; see our forthcoming work [26]. Here for our specific purposes it is enough to use the solution presented above, taking into account some necessary physical considerations.

In the C-case, there exists a range $2\sqrt{\pi_x^2 + p_z^2} < U$ of the momentum p_x and p_z of the fermions, such that particle creation is possible. This case is described by the wave functions (7) with quantum numbers from the range Ω , where ω , p_x , and p_z are restricted by the inequalities

$$\Omega: s\pi_s(L) \geq \pi_x, \quad s\pi_s(R) \leq -\pi_x, \quad 2\sqrt{\pi_x^2 + p_z^2} < U. \quad (27)$$

If we treat this case using the identification of a wave function by an analogy with one-particle scattering theory, there appears an analog of the Klein paradox for charged relativistic particles in an electric field [27]. This is an indication that one has to use an appropriate many-particle description given by QFT to treat the problem correctly.

In the first stage of the canonical quantization of the field $\psi(t, \mathbf{r})$ one establishes that the corresponding quantum field is the Heisenberg field operator $\Psi(t, \mathbf{r})$ that satisfies the equal-time anticommutation relations

$$\begin{aligned} [\Psi(t, \mathbf{r}), \Psi(t, \mathbf{r}')]_+ &= [\Psi(t, \mathbf{r})^\dagger, \Psi(t, \mathbf{r}')^\dagger]_+ = 0, \\ [\Psi(t, \mathbf{r}), \Psi(t, \mathbf{r}')^\dagger]_+ &= \delta(\mathbf{r} - \mathbf{r}') \end{aligned} \quad (28)$$

and the Dirac-Pauli equation (2). The formal expressions for the Hamiltonian $\hat{\mathcal{H}}$ of the quantized fermion field and the corresponding magnetic momentum operator $\hat{\mathcal{M}}$ can be easily constructed,

$$\begin{aligned} \hat{\mathcal{H}} &= \int \Psi(t, \mathbf{r})^\dagger \hat{H} \Psi(t, \mathbf{r}) d\mathbf{r}, \\ \hat{\mathcal{M}} &= \frac{\mu}{2} \int [\Psi(t, \mathbf{r})^\dagger, \Psi(t, \mathbf{r})]_- d\mathbf{r}. \end{aligned} \quad (29)$$

To perform quantization in terms of particles and antiparticles, we define the inner product

$$(\psi, \psi')_t = \int_t \psi^\dagger(t, \mathbf{r}) \psi'(t, \mathbf{r}) d\mathbf{r} \quad (30)$$

between two solutions of the the Dirac-Pauli equation on a $t = \text{const}$ hyperplane. This inner product does not depend

on the choice of such a hyperplane if the spinors $\psi(t, \mathbf{r})$ obey certain boundary conditions that allow one to integrate by parts in Eq. (30), neglecting boundary terms. Since physical states are wave packets that vanish on the remote boundaries, the above assumption holds true and the inner product (30) is time independent for such states. Considering plane waves instead of natural wave packets, one has to impose corresponding periodic boundary conditions on the corresponding wave functions and the external field to keep the inner product (30) time independent. However, in the case under consideration the external field with different asymptotics at $y \rightarrow \pm\infty$ cannot be adapted to any periodic boundary conditions in the y direction without changing its physical content. To provide time independence of the inner product, one has to redefine the inner product itself. This modification is applied to the integration over y in the expression (30) and is described below.

Let $\psi_n(t, \mathbf{r})$ and $\psi'_{n'}(t, \mathbf{r})$ be wave functions (7) and the integral over the variable y in the infinite limits be regularized by large positive numbers L_1 and L_2 . Integrating over the variables x , z , and using representation (9), we obtain

$$\begin{aligned} (\psi_n, \psi'_{n'})_t &= \delta_{n,n'} S_{xz} \mathcal{R}, \quad \mathcal{R} = \int_{-L_1}^{L_2} Q dy, \\ Q &= (\varphi_{n,\chi}(y))^* [\pi_x^2 + (\omega + s\mu B_z(y) + s\chi i\partial_y)^2] \varphi'_{n,\chi}(y), \end{aligned} \quad (31)$$

where the orthogonality for $n \neq n'$ follows as $L_1, L_2 \rightarrow \infty$.

We represent the regularized integral \mathcal{R} as

$$\mathcal{R} = \int_{-L_1}^{y_L} Q dy + \int_{y_L}^{y_R} Q dy + \int_{y_R}^{L_2} Q dy, \quad (32)$$

where only the second term—the integral over the region S_{int} —depends on the derivative $\partial_y B_z(y)$. The smoothness of the $\partial_y B_z(y)$ allows us to believe that this integral is finite as $L_1, L_2 \rightarrow \infty$. The first and the third terms are calculated as integrals over the regions where $\partial_y B_z(y) = 0$. Then their values are determined by the asymptotics (16) in the following form:

$$\begin{aligned} \mathcal{R}_L &= \int_{-L_1}^{y_L} Q_L dy, \\ \mathcal{R}_R &= \int_{y_R}^{L_2} Q_R dy, \\ Q_{L/R} &= (\varphi_{n,\chi}(y))^* [\pi_x^2 + (\pi_s(L/R) + s\chi i\partial_y)^2] \varphi'_{n,\chi}(y). \end{aligned} \quad (33)$$

Q_L and Q_R are constant then $\mathcal{R}_L \sim L_1$ and $\mathcal{R}_R \sim L_2$. We see that only \mathcal{R}_L and \mathcal{R}_R make a contribution to \mathcal{R} in Eq. (32) as $L_1, L_2 \rightarrow \infty$,

$$\mathcal{R} \xrightarrow{L_1, L_2 \rightarrow \infty} \mathcal{R}_L + \mathcal{R}_R.$$

There exist two independent solutions with a given quantum number n from the range Ω . In spite of the fact that these solutions are obtained in the constant external field we believe that they represent asymptotic forms of some unknown solutions of the Dirac-Pauli equation with an external field $\partial_y B_z(t, y)$ that is switched on and off at $t \rightarrow \pm\infty$ and the effects of the switching from on to off are negligible. Since the inner product (30) does not depend on t for such solutions, we believe that orthogonal pairs of solutions that describe alternative particle/antiparticle states at the initial and the final time instants remain orthogonal at an arbitrary instant of time. Therefore we have to find out which solutions among those we have introduced before are such orthogonal pairs. Taking into account the relations (22), one can show that

$$({}_\zeta\psi_n, {}^\zeta\psi_n)_t = 0, \quad n \in \Omega, \quad (34)$$

if we assume that L_1 and L_2 satisfy the relation

$$L_1 \left| \frac{\pi_s(L)}{p^L} \right| - L_2 \left| \frac{\pi_s(R)}{p^R} \right| = O(1). \quad (35)$$

Condition (35) guarantees that the wave functions ${}_\zeta\psi_n(t, \mathbf{r})$ and ${}^\zeta\psi_n(t, \mathbf{r})$ for $n \in \Omega$ correspond to alternative physical states. Note that condition (35) is unique to guarantee that all the wave functions with any n of the complete set corresponding to alternative physical states are orthogonal with respect to the inner product (30); for details see our forthcoming work [26]. In fact, such a condition has to be considered as a part of the definition of the inner product (30).

Consider the quantities $\mathcal{R}_{L/R}$ (33) defined by the functions ${}_\zeta\varphi_n(x)$ and ${}^\zeta\varphi_n(x)$ with quantum numbers n from the range Ω . In this case we attribute the corresponding index ζ to these quantities as follows: $\mathcal{R}_{L/R} \rightarrow {}_\zeta\mathcal{R}_{L/R}$ or $\mathcal{R}_{L/R} \rightarrow {}^\zeta\mathcal{R}_{L/R}$. Using Eqs. (16) and (21) and retaining only leading terms in the limit $L_1, L_2 \rightarrow \infty$, we obtain

$${}_\zeta\mathcal{R}_L = Y^2 L_1 \left| \frac{\pi_s(L)}{p^L} \right|, \quad {}^\zeta\mathcal{R}_R = Y^2 L_2 \left| \frac{\pi_s(R)}{p^R} \right|. \quad (36)$$

To calculate the quantities ${}_\zeta\mathcal{R}_R$ and ${}^\zeta\mathcal{R}_L$, we use the relations (22). Again, retaining only leading terms in the limit $L_1, L_2 \rightarrow \infty$ (neglecting in particular oscillating terms) and taking into account Eqs. (16) and (21), we find

$$\begin{aligned} {}_\zeta\mathcal{R}_R &= Y^2 L_2 \left| \frac{\pi_s(R)}{p^R} \right| \left[|g({}_\zeta|^{+})|^2 + |g({}_\zeta|^{-})|^2 \right], \\ {}^\zeta\mathcal{R}_L &= Y^2 L_1 \left| \frac{\pi_s(L)}{p^L} \right| \left[|g(+|^\zeta)|^2 + |g(-|^\zeta)|^2 \right]. \end{aligned} \quad (37)$$

Note that ${}^\zeta\mathcal{R}_L > {}^\zeta\mathcal{R}_R$ and ${}_\zeta\mathcal{R}_R > {}_\zeta\mathcal{R}_L$ due to $|g(+|^{-})|^2 > 1$. Taking the unitarity relations (26) and the condition (35) into account, we obtain the following orthonormality relations:

$$\begin{aligned} ({}_\zeta\psi_n, {}_\zeta\psi_{n'})_t &= \delta_{n,n'} C_t, & ({}^\zeta\psi_n, {}^\zeta\psi_{n'})_t &= \delta_{n,n'} C_t, \\ C_t &= 2 \frac{L_2}{T} \left| \frac{\pi_s(R)}{p^R} \right| |g(+|^{-})|^2. \end{aligned} \quad (38)$$

One can see that the following symmetry occurs: particles with opposite values of s have opposite accelerations; the same is valid for antiparticles. This is why the cases $s = +1$ and $s = -1$ differ only by opposite directions of all the motions, and respectively by the opposite dispositions of all the asymptotic ranges. The probabilities of all the processes are equal in both the cases. This is why it is enough to consider only one case, let us say $s = +1$.

It is supposed that we know the complete set of the solutions of the Dirac-Pauli equation, parametrized by a set of quantum numbers n , on the hyperplane $t = \text{const}$. Then we can decompose the quantum Heisenberg field operator $\Psi(t, \mathbf{r})$ and its Hermitian conjugate $\Psi^\dagger(t, \mathbf{r})$ in this complete set using the inner product (30). Assuming that both sets $\{+ \psi_n(t, \mathbf{r}), + \psi_n(t, \mathbf{r})\}$ and $\{- \psi_n(t, \mathbf{r}), - \psi_n(t, \mathbf{r})\}$ represent the complete set of non-decaying solutions in the range Ω , we introduce the notation $\Psi_n(t, \mathbf{r})$ for the component of the quantum field operator that can be expanded via either $+ \psi_n(t, \mathbf{r})$, $+ \psi_n(t, \mathbf{r})$ or $- \psi_n(t, \mathbf{r})$, $- \psi_n(t, \mathbf{r})$. Operator coefficients in such decompositions do not depend on spacetime coordinates because both quantum field operators and classical solutions obey the same Pauli-Dirac equation. For example, for $s = +1$, we can decompose the $\Psi_n(t, \mathbf{r})$ and $\Psi_n^\dagger(t, \mathbf{r})$ as follows:

$$\begin{aligned} \Psi_n(t, \mathbf{r}) &= C_t^{-1/2} [a_n(\text{out})^+ \psi_n(t, \mathbf{r}) + b_n^\dagger(\text{out})_+ \psi_n(t, \mathbf{r})], \\ \Psi_n^\dagger(t, \mathbf{r}) &= C_t^{-1/2} [a_n^\dagger(\text{out})^+ \psi_n^\dagger(t, \mathbf{r}) + b_n(\text{out})^+ \psi_n^\dagger(t, \mathbf{r})], \end{aligned} \quad (39)$$

and

$$\begin{aligned} \Psi_n(t, \mathbf{r}) &= C_t^{-1/2} [a_n(\text{in})^- \psi_n(t, \mathbf{r}) + b_n^\dagger(\text{in})_- \psi_n(t, \mathbf{r})], \\ \Psi_n^\dagger(t, \mathbf{r}) &= C_t^{-1/2} [a_n^\dagger(\text{in})^- \psi_n^\dagger(t, \mathbf{r}) + b_n(\text{in})_- \psi_n^\dagger(t, \mathbf{r})]. \end{aligned} \quad (40)$$

In what follows, we interpret all a and b as annihilation and all a^\dagger and b^\dagger as creation operators, all a and a^\dagger as describing particles and b and b^\dagger as describing antiparticles, and all the operators labeled by the argument ‘‘in’’ are in-operators, whereas all the operators labeled by the argument ‘‘out’’ are out-operators. It can be shown that these creation and annihilation operators obey canonical anti-commutation relations,

$$\begin{aligned}
[a_n(\text{in}), a_k^\dagger(\text{in})]_+ &= [a_n(\text{out}), a_k^\dagger(\text{out})]_+ = [b_n(\text{in}), b_k^\dagger(\text{in})]_+ = [b_n(\text{out}), b_k^\dagger(\text{out})]_+ = \delta_{n,k}, \\
[a_n(\text{out}), a_k(\text{out})]_+ &= [b_n(\text{out}), b_k(\text{out})]_+ = [a_n(\text{out}), b_k(\text{out})]_+ = [a_n(\text{out}), b_k^\dagger(\text{out})]_+ = 0, \\
[a_n(\text{in}), a_k(\text{in})]_+ &= [b_n(\text{in}), b_k(\text{in})]_+ = [a_n(\text{in}), b_k(\text{in})]_+ = [a_n(\text{in}), b_k^\dagger(\text{in})]_+ = 0,
\end{aligned} \tag{41}$$

due to relation (28). In such an interpretation, the in-vacuum $|0, \text{in}\rangle$ and out-vacuum $|0, \text{out}\rangle$ are defined by the conditions

$$\begin{aligned}
a_n(\text{in})|0, \text{in}\rangle &= b_n(\text{in})|0, \text{in}\rangle = 0, \quad \forall n, \\
a_n(\text{out})|0, \text{out}\rangle &= b_n(\text{out})|0, \text{out}\rangle = 0, \quad \forall n.
\end{aligned} \tag{42}$$

Let us consider the magnetic momentum operator,

$$\hat{\mathcal{M}}_\Omega = \frac{\mu}{2} \int [\Psi_\Omega(t, \mathbf{r})^\dagger, \Psi_\Omega(t, \mathbf{r})]_- d\mathbf{r}, \tag{43}$$

and the operator of the kinetic energy of the quantum Dirac field $\Psi_\Omega(t, \mathbf{r})$ in the domain Ω ,

$$\begin{aligned}
\hat{\mathcal{H}}_\Omega^{\text{kin}} &= \int \Psi_\Omega(t, \mathbf{r})^\dagger [\hat{\Pi}_z + \mu B_z(y)] \\
&\times \sqrt{1 + \left[\frac{\hat{p}^3}{\hat{\Pi}_z + \mu B_z(y)} \right]^2} \Psi_\Omega(t, \mathbf{r}) d\mathbf{r} - H_\Omega^0, \tag{44}
\end{aligned}$$

where $\Psi_\Omega(t, \mathbf{r}) = \sum_{n \in \Omega} \Psi_n(t, \mathbf{r})$ and $H_\Omega^0 = \langle 0, \text{in} | \hat{\mathcal{H}}_\Omega^{\text{kin}} | 0, \text{in} \rangle$ is the constant term corresponding to the energy of vacuum fluctuations. Using relations (36), (28), (22), and (26), one can represent these operators in equivalent diagonal forms as follows:

$$\begin{aligned}
\hat{\mathcal{M}}_\Omega &= \mu \sum_{n \in \Omega} [a_n^\dagger(\text{in})a_n(\text{in}) - b_n^\dagger(\text{in})b_n(\text{in})] \\
&= \mu \sum_{n \in \Omega} [a_n^\dagger(\text{out})a_n(\text{out}) - b_n^\dagger(\text{out})b_n(\text{out})], \\
\hat{\mathcal{H}}_\Omega^{\text{kin}} &= \sum_{n \in \Omega} [-\mathcal{E}_n a_n^\dagger(\text{in})a_n(\text{in}) - {}_-\mathcal{E}_n b_n^\dagger(\text{in})b_n(\text{in})] \\
&= \sum_{n \in \Omega} [{}^+\mathcal{E}_n a_n^\dagger(\text{out})a_n(\text{out}) - {}_+\mathcal{E}_n b_n^\dagger(\text{out})b_n(\text{out})], \tag{45}
\end{aligned}$$

where

$$\begin{aligned}
{}_\zeta \mathcal{E}_n &= E_{+1}(R) + \frac{1}{2} U |g_{(+|-)}|^{-2}, \\
{}^\xi \mathcal{E}_n &= E_{+1}(L) - \frac{1}{2} U |g_{(+|-)}|^{-2}
\end{aligned}$$

(see details in our forthcoming work [26]). We suppose that

$${}^\xi \mathcal{E}_n > 0, \quad {}_\zeta \mathcal{E}_n < 0 \tag{46}$$

in the external field under consideration, so that the signs of the energies ${}^\xi \mathcal{E}_n$ and ${}_\zeta \mathcal{E}_n$ are determined by the signs of $\pi_{+1}(R/L)$. In known solvable cases the inequalities (46) hold true; for example, see Refs. [20,21,25]. Thus, the operator $\hat{\mathcal{H}}_\Omega^{\text{kin}}$ is positively defined. This fact provides a

consistent quantization in terms of particles and antiparticles in the range Ω .

Kinetic energy must be positive for any wave packets of both particles and antiparticles. This is why particle wave packets are situated in the region S_L and antiparticle wave packets are situated in the region S_R , that is, there is a total reflection from S_{int} for both particles and antiparticles. This is consistent with the physical meaning. Note that the expressions $({}^\xi \psi_n, {}^{\xi'} \psi_{n'})_x$ and $(-1)({}^\xi \psi_n, {}^{\xi'} \psi_{n'})_x$, given by Eq. (20), are the probability currents of particles and antiparticles through the surface $y = \text{const}$, respectively. The particle and antiparticle currents are positive for $\zeta = -1$ and negative for $\zeta = +1$. Thus, we see that for $s = +1$ the functions ${}^+ \psi_n(t, \mathbf{r})$ and ${}_+ \psi_n(t, \mathbf{r})$ describe outgoing particles and antiparticles, while the functions ${}^- \psi_n(t, \mathbf{r})$ and ${}_- \psi_n(t, \mathbf{r})$ describe incoming particles and antiparticles, respectively. The particle-antiparticle and causal identification of the wave functions (7) is unique in the framework of QFT.

The vacuum corresponds to the absence of incoming particles and antiparticles. In such a case the presence of outgoing particles and antiparticles indicates particle creation from the vacuum. The effect of particle creation implies constant currents of outgoing particles and antiparticles. These currents are equal in the regions S_L and S_R .

Then, taking into account Eqs. (39) and (40), we obtain direct and inverse linear canonical transformations between the ‘‘in’’ and ‘‘out’’ creation and annihilation operators (Bogolyubov transformations),

$$\begin{aligned}
a_n(\text{out}) &= g(-|_+)^{-1} g(+|_+) a_n(\text{in}) - g(-|_+)^{-1} b_n^\dagger(\text{in}), \\
b_n^\dagger(\text{out}) &= g(-|_+)^{-1} a_n(\text{in}) + g(-|_+)^{-1} g(+|_+) b_n^\dagger(\text{in}), \\
a_n(\text{in}) &= g(+|_-)^{-1} g(-|_-) a_n(\text{out}) + g(+|_-)^{-1} b_n^\dagger(\text{out}), \\
b_n^\dagger(\text{in}) &= -g(+|_-)^{-1} a_n(\text{out}) + g(+|_-)^{-1} g(-|_-) b_n^\dagger(\text{out}).
\end{aligned} \tag{47}$$

These transformations are similar to that used by Nikishov in the problem of charged-particle scattering on an electric step [20,21].

With the help of the transformations (47), we calculate the differential mean number of created particles and antiparticles,

$$\begin{aligned}
N_n^{(+)} &= \langle 0, \text{in} | a_n^\dagger(\text{out}) a_n(\text{out}) | 0, \text{in} \rangle = |g(-|_+)|^{-2}, \\
N_n^{(-)} &= \langle 0, \text{in} | b_n^\dagger(\text{out}) b_n(\text{out}) | 0, \text{in} \rangle = |g(+|_-)|^{-2}.
\end{aligned} \tag{48}$$

The relations (25) imply the equality

$$N_n^{(+)} = N_n^{(-)} = N_n,$$

which allows us to treat N_n as the differential mean number of created pairs. The total number N of created pairs is the sum

$$N = \sum_{n \in \Omega} N_n. \quad (49)$$

The elementary relative probability amplitudes of particle creation, annihilation, and scattering are defined as follows:

$$\begin{aligned} c_v &= \langle 0, \text{out} | 0, \text{in} \rangle, \\ w(+|+)_{n'n} &= c_v^{-1} \langle 0, \text{out} | a_{n'}(\text{out}) a_n^\dagger(\text{in}) | 0, \text{in} \rangle, \\ w(-|-)_{nn'} &= c_v^{-1} \langle 0, \text{out} | b_{n'}(\text{out}) b_n^\dagger(\text{in}) | 0, \text{in} \rangle, \\ w(0|-+)_{nn'} &= c_v^{-1} \langle 0, \text{out} | b_{n'}^\dagger(\text{in}) a_n^\dagger(\text{in}) | 0, \text{in} \rangle, \\ w(+ - |0)_{n'n} &= c_v^{-1} \langle 0, \text{out} | a_{n'}(\text{out}) b_n(\text{out}) | 0, \text{in} \rangle, \end{aligned} \quad (50)$$

where c_v is the vacuum-to-vacuum transition amplitude. One can see that the amplitudes (50) are diagonal,

$$\begin{aligned} w(+|+)_{n'n} &= \delta_{n,n'} w_n(+|+), \\ w(-|-)_{nn'} &= \delta_{n,n'} w_n(-|-), \\ w(0|-+)_{nn'} &= \delta_{n,n'} w_n(0|-+), \\ w(+ - |0)_{n'n} &= \delta_{n,n'} w_n(+ - |0), \end{aligned} \quad (51)$$

and can be expressed via the coefficients $g(\xi' | \xi)$ as follows:

$$\begin{aligned} w_n(+|+) &= g(+|-)g(-|-)^{-1} = g(+|-)g(+|+)^{-1}, \\ w_n(-|-) &= g(-|+)g(-|-)^{-1} = g(-|+)g(+|+)^{-1}, \\ w_n(+ - |0) &= g(+|+)^{-1}, \quad w_n(0|-+) = -g(-|-)^{-1}, \end{aligned} \quad (52)$$

where the transformations (47) are used.

One can express the probabilities of particle scattering and pair creation for quantum numbers $n \in \Omega$ and the probability for the vacuum to remain a vacuum via the differential mean numbers N_n as follows:

$$\begin{aligned} P(+|+)_{nn'} &= |\langle 0, \text{out} | a_n(\text{out}) a_n^\dagger(\text{in}) | 0, \text{in} \rangle|^2 \\ &= \delta_{n,n'} \frac{1}{1 - N_n} P_v, \\ P(- + |0)_{nn'} &= |\langle 0, \text{out} | b_n(\text{out}) a_{n'}(\text{out}) | 0, \text{in} \rangle|^2 \\ &= \delta_{n,n'} \frac{N_n}{1 - N_n} P_v, \\ P_v &= |c_v|^2 = \exp \left\{ \sum_{n \in \Omega} \ln(1 - N_n) \right\} \end{aligned} \quad (53)$$

(see details in our forthcoming work [26]). The probabilities for the antiparticle scattering and the pair annihilation

are described by the same expressions, $P(+|+)$ and $P(- + |0)$, respectively.

IV. QUASILINEAR MAGNETIC FIELD

Here, we consider a specific case of an inhomogeneous magnetic field, namely a field linearly growing on an interval L_y . More exactly, the field has the form

$$B_z(y) = \begin{cases} B_0, & y < 0, \\ B_0 + B'y, & y \in [0, L_y], \\ B_0 + B'L_y, & y > L_y, \end{cases}$$

where $B' > 0$ and $B_0 = -B'L_y/2$. Let us call such a field a quasilinear magnetic field. Consider the case given by the condition

$$\sqrt{|\mu B'|} L_y \gg \max \left\{ 1, m/\sqrt{|\mu B'|} \right\}, \quad (54)$$

which implies that there is particle creation in a wide enough range Ω of momenta given by condition (27). One can demonstrate, similar to the case considered in Ref. [28] (see also Ref. [29]), that leading contributions to the differential mean numbers N_n of created pairs do not depend on L_y in the limit $L_y \rightarrow \infty$. This is why it is enough to consider the case of a linearly growing magnetic field. Equation (10) in the latter field for the function $\varphi_{n,\chi}(y)$ given by Eq. (9) can be written as

$$\begin{aligned} \left(\frac{d^2}{d\xi^2} + \xi^2 - \lambda + i\chi \right) \varphi_{n,\chi}(y) &= 0, \\ \xi &= \sqrt{|\mu|B'} [y + (|\mu|B')^{-1} (|\mu|B_0 - \omega)], \\ \lambda &= \frac{m^2 + p_x^2}{|\mu B'|}. \end{aligned} \quad (55)$$

Solutions of this equation, obeying the boundary conditions (11) and (12), have the form

$$\begin{aligned} \bar{\varphi}_{n,\chi}(y) &= \mathcal{N}_\chi D_{-\nu-1} [\pm(1+i)\xi], \\ \pm \varphi_{n,\chi}(y) &= \mathcal{N}_\chi D_\nu [\pm(1-i)\xi], \end{aligned} \quad (56)$$

where $D_\nu(z)$ are Weber parabolic cylinder (WPC) functions, $\nu = -(i\lambda + 1 + \chi)/2$. With the help of an asymptotic expansion of WPC functions, one can verify the validity of the boundary conditions (11) and (12). Using the solutions (56), we construct the sets $\{\xi \psi_n(t, \mathbf{r})\}$ and $\{\xi \bar{\psi}_n(t, \mathbf{r})\}$ of solutions of the Dirac-Pauli equation.

The obtained form of solutions formally coincide with the one found in Refs. [20,21,29] for the case of charged-particle creation by a constant uniform electric field (compare with Ref. [28]). Note that our identification of wave functions is in agreement with one given by Nikishov for such a special case. This allows us to use these calculations to find differential mean numbers of created pairs given by Eq. (48).

In the limit $\sqrt{|\mu B'|}L_y \gg K$, where K is a given arbitrary number $K \gg \max\{1, m/\sqrt{|\mu B'|}\}$, and if ω and p_z satisfy the condition

$$|\omega| < \omega_{\max}, \quad |p_z| < \omega_{\max},$$

$$\omega_{\max} = |\mu B'|L_y/2 - \sqrt{|\mu B'|}K,$$

we obtain

$$N_n = e^{-\pi\lambda}. \quad (57)$$

Following the idea of finite work regularization presented in Ref. [28], one can show that an exact expression for N_n is rapidly decreasing as $|\omega| \rightarrow \infty$ due to the finite work of this field, $|\mu B'|L_y$, that is, ω_{\max} is an effective maximum value of the quantum number $|\omega|$ for the quasilinear field under consideration. The maximum value for $|p_z|$ from the range Ω follows from condition (27). One can check that the mean numbers do not depend on the sign of $\mu B'$ and on the spin polarization s . Note, however, that unlike the case of particle creation due to the electric potential step, the neutral particles (antiparticles) created with different s form fluxes aimed in opposite directions. The leading approximation given by expression (57) does not depend on the quantum numbers ω and p_z . Although the result (57) has been derived for a $B' = \text{const}$ field, it can be applicable to a spatially slowly varying $B'(y)$ as a good approximation if its gradient variation is sufficiently small in comparison with the mean value \bar{B}' on the interval $[-L_y/2, L_y/2]$, $\bar{B}'^{-1}\partial_y B'(y)L_y \ll 1$.

Let us calculate the total number \mathcal{N}_s of created pairs with given s defined by Eq. (49). To do this we go over from the sum to an integral,

$$\sum_{p_x, p_z, p_0} (\dots) \Rightarrow \frac{L_x L_z T}{(2\pi)^3} \int (\dots) dp_x dp_z dp_0.$$

Taking into account that the exact distribution N_n plays the role of a cutoff factor in the integral over ω , p_x , and p_z we represent the total number \mathcal{N}_s in the form

$$\mathcal{N}_s = 2 \int_0^{\omega_{\max}} dp_z \mathcal{N}_{s, p_z}, \quad (58)$$

$$\mathcal{N}_{s, p_z} = \frac{L_x L_z T}{(2\pi)^3} \int dp_x \int_0^{\omega_{\max}} \frac{N_n d\omega^2}{\sqrt{\omega^2 + p_z^2}},$$

where the relation $p_0 = \omega\sqrt{1 + (p_z/\omega)^2}$ from Eq. (5) is used. We obtain the leading contribution in Eq. (58) as follows:

$$\mathcal{N}_{s, p_z} = L_x L_z T n_{s, p_z},$$

$$n_{s, p_z} = \frac{\sqrt{|\mu B'|}}{4\pi^3} \exp\left(-\frac{\pi m^2}{|\mu B'|}\right) \left(\sqrt{\omega_{\max}^2 + p_z^2} - |p_z|\right). \quad (59)$$

From Eq. (59), we see that the leading term of the density n_{s, p_z} is a linear function of the length L_y for sufficiently small momentum p_z , $|p_z| \ll \omega_{\max}$, that is, the density of the particles created per unit spacetime volume, $n_{s, p_z}/L_y$, is uniform. Of course, this is not the case when $|p_z|$ is not small. Thus, we see a complete similarity between the case of particle creation due to a quasiuniform electric field and a quasilinear magnetic field for small momenta p_z only. Using Eq. (58), we obtain the total number \mathcal{N}_s of created pairs with a given s in the form

$$\mathcal{N}_s = \frac{\sqrt{2} - 1 + \ln(1 + \sqrt{2})}{16\pi^3}$$

$$\times TL_x L_z L_y^2 |\mu B'|^{5/2} \exp\left(-\frac{\pi m^2}{|\mu B'|}\right). \quad (60)$$

The total number of created pairs with both $s = \pm 1$ is $\mathcal{N} = N_{+1} + N_{-1}$.

The vacuum-to-vacuum transition probability defined in Eq. (53) can be calculated in the same way. Then we express it via the total number N as follows:

$$P_v = \exp(-\beta N), \quad (61)$$

$$\beta = \sum_{l=0}^{\infty} (l+1)^{-3/2} \exp\left(-\frac{l\pi m^2}{|\mu B'|}\right).$$

V. DISCUSSION

It should be noted that the particle creation in the linearly growing magnetic field represents a wide class of physical situations where the gradient of magnetic fields is slowly varying in big enough but restricted areas. One can also see that the leading contribution to differential mean numbers of created pairs in such fields does not depend on the asymptotic behavior of the magnetic field as the size of the heterogeneity tends to infinity. This allows one to make some general conclusions from the obtained results.

First of all, in 3 + 1 dimensions, both the total number N of created pairs and the vacuum-to-vacuum transition probability P_v given by Eqs. (60) and (61), respectively, depend only on the gradient of the magnetic field, but not on its strength, similarly to what happens in 2 + 1 dimensions [16]. Both quantities are finite for the finite spacetime volume of field inhomogeneity. In particular, it seems that the level crossing discovered in Ref. [17] for the system of neutral fermions interacting with a strong uniform magnetic field due to an anomalous magnetic moment is a result of the improper treatment of the weak-field case spectrum. The arbitrarily strong uniform magnetic field is stable with respect to the creation of neutral fermions with an anomalous magnetic moment and this fact does not depend on the spacetime dimension.

Secondly, due to the nonperturbative consideration in the framework of QFT, some results could emerge that can

be difficult to expect when remaining in the framework of one-particle quantum mechanics. In particular, in the case under consideration of neutral particle creation, we have to stress the following nontrivial peculiarities.

- (a) In contrast to the case of charged particles that are accelerated by an electric field in directions that are defined by their charges, both the neutral particles and antiparticles with opposite values of the conserved spin polarization s have—due to the Pauli interaction—opposite directions of acceleration. For this reason, only states with a definite s are localizable and can form wave packets in the asymptotic regions. In fact, in the problem under consideration, it is convenient to speak about two different species of particles and antiparticles that are labeled by the sign of s . For each kind s there exist “in” and “out” sets of solutions of the Dirac-Pauli equation that in QFT define the corresponding “in” and “out” states. Note that neutral particles (antiparticles) that are created by the external field with different spin polarization s form fluxes directed in opposite directions. In a sense this explains the fact that quantization in terms of neutral particles and antiparticles in $d \geq 3 + 1$ dimensions is possible only in terms of exact solutions with definite spin integrals of motion [in the case under consideration, this integral of motion is the operator \hat{R} given by Eq. (3)]. This means that in models with a non-minimal interaction with an external field and with $d \geq 3 + 1$ the formal second quantization similar to QED may not work.
- (b) At a certain stage, calculations of the creation of neutral fermions from the vacuum by inhomogeneous magnetic fields are technically reduced to the calculations of the creation of charge particles from the vacuum by corresponding electric fields. This allowed us to use some technical results obtained earlier in QED regarding charged Dirac particles. However, this does not mean that physically both effects are similar. For example—in contrast to the case of charged-particle creation in a constant electric field—in the case of the neutral fermion creation, the total number N of created pairs and $\ln P_v^{-1}$ are not linear in all length scales of an accelerating field. This peculiarity is due to the different form of the area in the phase space where particle creation occurs.

It is known that the Schwinger method of an effective action [15] is convenient for semiclassical calculations of pair creation from vacuum due to an electric-like field [24]. In this approach, one calculates the probability for the vacuum to remain a vacuum using the following Schwinger representation:

$$P_v = e^{-2 \text{Im}W}, \tag{62}$$

where W is the one-loop effective action of the corresponding QFT model. The worldline approaches to QED are suitable for realistic backgrounds [30,31]. In particular, for the case of the creation of neutral fermions with an anomalous magnetic moment, representation (62) was used in Refs. [16,17]. One can find a relation between our results—obtained in the framework of canonically quantized field theory—and the latter approach. To this end we present the quantity (62) as an infinite product,

$$P_v = \prod_{n \in \Omega} e^{-2 \text{Im}W_n}, \tag{63}$$

where the quantum numbers $n = (p_x, p_z, \omega, s) \in \Omega$ (eigenvalues of the corresponding integrals of motion) are used for parametrization, so that the effective action W is written as a sum, $W = \sum_n W_n$. Then, $e^{-2 \text{Im}W_n}$ is the vacuum-persistence probability in a cell of the space of quantum numbers n . Using an exact expression for P_v in terms of the differential mean values N_n , given by Eq. (53), we obtain the following relation:

$$2 \text{Im}W_n = -\ln(1 - N_n). \tag{64}$$

As was noted above, the creation of neutral fermions with given quantum numbers n is reduced to the problem of charged-particle creation from vacuum by a corresponding electric step. Then relation (64), well known for the case of a constant electric field [20,21], also holds for the creation of neutral fermions in a linearly growing magnetic field. This means that the Schwinger method works for the case under consideration, provided we have a suitable parametrization. However, we see that the total quantities N (and $\ln P_v^{-1}$) in $3 + 1$ dimensions are quadratic in L_y . This is a consequence of the fact that the number of states with all possible ω and p_z excited by the field B' is quadratic in the kinetic momentum $|\mu B'|L_y$. This is also the reason why the density of created pairs and the density of $\text{Im}W$ per unit of length L_y are not constant. In this case the divergence of the effective action W as $L_y \rightarrow \infty$ is not linear and it is quite difficult to invent a reliable method of regularization of W for a linearly growing magnetic field in the framework of the Schwinger approach, if the parametrization is not appropriately chosen, as was done above. We believe that ignoring this fact was the main cause of the questionable results in Ref. [17]. On the other hand, in $2 + 1$ dimensions, there is only one spin polarization and the integration over p_z is absent, that is, the calculation of the quantities N (and $\ln P_v^{-1}$) for created neutral fermions by a linearly growing magnetic field is completely reduced to the problem of charged-particle creation from vacuum by a constant electric field. Then the expression for P_v obtained in Ref. [16] is in agreement with our result for N_n , given by Eq. (57). Note that our techniques in the framework of QFT can be used to separate the divergent term of $\text{Im}W$ as $L_y \rightarrow \infty$ in the framework of the effective action

techniques and to relate it to pair creation, cf. Ref. [32]. It means that recent computational developments [30,31] can also be extended to calculate the effects of particle creation with an anomalous magnetic moment.

The cases with opposite values of the spin polarization s differ only in that they have opposite directions of all the motions and all the asymptotic regions with respect to a nonzero-gradient region of the magnetic field. Then, the neutral particles (antiparticles) created with different s form fluxes that are moving in opposite directions. The probabilities of all the processes are equal for different values of s . We see that the created flux aimed in one of the directions is formed from fluxes of particles and antiparticles of equal intensity and with the same magnetic moments parallel to the external field. In such a flux, particle and antiparticle velocities that are perpendicular to the plane of the magnetic moment and flux direction are essentially depressed. This is a typical property of neutral fermions created by inhomogeneous magnetic fields that can be used to observe their effects in astrophysical situations.

As follows from the obtained results, the effective creation of neutral fermions from vacuum starts when there exists a big enough difference between the asymptotic magnetic fields, i.e., $U > 2m$. Let us suppose that the magnetic field under consideration achieves its maximal value $|B_{\max}|$ inside of a finite region and is absent outside this region. In this case, the minimal value of the quantity $|B_{\max}|$ which provides the effective particle creation is $|B_{\max}| \sim B_{\text{cr}} = 2m/|\mu|$. It is convenient to express the magnetic moment μ in terms of the Bohr magneton, $|\mu| = 2c_\mu \mu_B$, $\mu_B = \frac{e}{2m_e}$, and the particle mass m in terms of the electron mass m_e , $m = c_m m_e$, such that c_μ and c_m are the corresponding dimensionless quantities. Then the characteristic magnetic field B_{cr} in the problem under consideration is

$$B_{\text{cr}} = 2B_{\text{QED}} \frac{c_m}{c_\mu},$$

$$B_{\text{QED}} = m_e^2/e = m_e^2 c^3 / e \hbar \simeq 4.4 \cdot 10^{13} \text{ G},$$

where B_{QED} is the characteristic magnetic field value above which the nonlinearity of QED becomes actual. There are two species of neutral fermions among the known elementary particles: the neutron and the active neutrino. For the neutrons $c_m/c_\mu \sim 10^6$ which implies $B_{\text{cr}}^{(n)} \sim 10^{20}$ G. In the active neutrino case the optimistic estimation is $c_\mu \sim 10^{-12}$. Cosmological constraints indicate that the total active neutrino mass is below 0.3 eV [33]. Then, supposing that the mass of the active neutrino is of the order $m_\nu \sim 0.1$ eV, i.e., $c_m \sim 10^{-7}$, we obtain $\sim 10^5$ for the factor c_m/c_μ , which implies that the critical value is $B_{\text{cr}}^{(\nu)} \sim 10^{19}$ G. However, it should be noted that if the active neutrino mass is essentially less than 0.1 eV (which is theoretically admissible) and, at the same time, its

magnetic moment is not significantly less than $10^{-12} \mu_B$, then it is possible that $B_{\text{cr}}^{(\nu)} \ll 10^{19}$ G. None of the neutrino models are currently universally accepted, such that we do not have any theoretical estimation of their masses and magnetic moments. We do not certainly know whether neutrinos are Dirac or Majorana particles. Moreover, the neutrino magnetic moment and therefore the ratio c_m/c_μ can depend on the strength of a strong magnetic field; see, for example, Ref. [34]. This is why at present it is difficult to give a more exact estimation for $B_{\text{cr}}^{(\nu)}$.

Taking into account the possible existence of the light sterile neutrinos with masses M in the range of 1 keV [7,8] and weak observational constraints on their magnetic moment μ [9,10], we propose the new scenario in which pairs of sterile neutrinos and antineutrinos could be produced from their coupling to an inhomogeneous magnetic field. For example, if $M = m_e/10$ then $|\mu| < 3.4 \times 10^{-5} \mu_B$ while if $M = m_e/100$ then $|\mu| \lesssim 10^{-4} \mu_B$ due to precision electroweak measurements [10]. In the latter case, the most optimistic estimation is $c_m/c_\mu \sim 10^2$ which implies that the critical value $B_{\text{cr}}^{(s\nu)} \sim 10^{16}$ G. Sterile neutrinos with masses of several keV are a dark matter candidate. Thus, we have an estimation of the critical value $B_{\text{cr}}^{(s\nu)}$ that is relevant for dark matter. These constraints can be weakened by the mechanism of compositeness and a variety of astrophysical constraints can be significantly weakened by the candidate particle's mass. In this situation, one can use, for example, the direct limits on $|\mu|$, which would follow from the nonobservance of Faraday rotation at a given sensitivity, and see that $|\mu| \lesssim \mu_B$ [10]. If $M = m_e/100$ then such a weak limit implies $B_{\text{cr}}^{(s\nu)} \sim 10^{12}$ G.

One can see from the discussion presented in the Introduction that the magnetic field in the magnetar cores made of quark matter can likely reach the critical value $B_{\text{cr}}^{(n)}$ which is enough to create neutron-antineutron pairs. Magnetic fields generated during a supernova explosion or in the vicinity of magnetars are of the order 10^{15} – 10^{16} G or even stronger, up to 10^{18} G. Such fields cannot create neutron-antineutron pairs from the vacuum but are strong enough to create neutrino-antineutrino pairs. In any case the vacuum instability with respect to the creation of neutrinos and even neutrons in strong magnetic fields has to be taken into account in the astrophysics. In particular, it may be of significance for dark matter studies.

It follows from Eq. (60) that the intensity of fluxes of created pairs turns out to be essential when the gradient B' is sufficiently large, $|B'| \sim |B_{\max}|/L_y \sim m^2/|\mu|$, and the condition of applicability of the model of the linearly growing magnetic field is valid, $\sqrt{|\mu B'|} L_y \gg 1$. This implies the following estimation for $|B_{\max}|$:

$$|B_{\max}| \sim L_y m B_{\text{cr}},$$

where $L_y m \gg 1$. Thus, considering astrophysical objects, one has to take into account the backreaction due to the vacuum instability in magnetic fields with $|B_{\max}| \gg B_{\text{cr}}$. The magnetic moments of created pairs are antiparallel in opposite asymptotic regions; the corresponding induced magnetic field has a gradient that is opposite to the gradient of the external magnetic field. Thus, neutral particle creation leads to a smoothing of the initial magnetic field, which in turn prevents the appearance of superstrong constant magnetic fields. In any case, background magnetic fields greater than $B_{\text{cr}}^{(\nu)}$, $B_{\text{cr}}^{(s\nu)}$, and $B_{\text{cr}}^{(n)}$ may create effects of the vacuum instability due to the above considered mechanism. In particular, magnetic fields with $|B_{\max}| \gg B_{\text{cr}}^{(s\nu)}$ can produce fluxes of pairs of sterile neutrinos and anti-neutrinos, which could escape the star with an anisotropy equal to the anisotropy in their production.

We hope that by applying similar approaches to quantum massive neutral fermionic fields, interacting with external backgrounds [35], we can study the creation of Dirac and (probably) Majorana massive neutrinos from the vacuum by an inhomogeneous background matter.

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