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# Castillejo, Dalitz, and Dyson ambiguity in field theoretic models of scattering

R. J. McLeod*

*School of Physics, University of Melbourne, Victoria 3010, Australia*
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We examine the numerical solutions of a field theoretic model to better understand the role played by the poles associated with the ambiguity identified by Castillejo, Dalitz, and Dyson (CDD). By analytically analyzing the numerical solutions to the crossing-symmetric Chew-Low model previously found, we show that the solutions are unique. This is done by requiring a second solution that would reproduce the original threshold and resonance. This uniqueness applies to the family of solutions, found by varying the parameters, that produces a resonance. The $T$ matrix for this model is not a generalized $R$ function. and no analogy can be made to the CDD analysis. Adding a CDD pole to the solution would constitute a new model, so there is no CDD ambiguity. None of the family of solutions from this new model will reproduce the original threshold and resonance. We also find that one cannot treat an individual channel as an $R$ function, nor can one add a CDD pole to an individual channel alone. Both a pole and a crossed pole must be added to all channels to maintain crossing symmetry. Crossing symmetry analytically connects all of the channels. The Chew-Low model being phenomenological is more pertinent to modern field theoretic models.

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## I. INTRODUCTION

The use of dispersion relations is a popular way to describe scattering. Quite often the authors will include poles such as those specified in the study by Castillejo, Dalitz, and Dyson [1]; the acronym CDD is used throughout. In recent work, for example, one can find CDD poles in pion-nucleon scattering [2–6], nucleon-nucleon scattering [7–9], pion-pion scattering [10], and scattering with other mesons [2,11–13].

CDD poles come from early studies [1] of the Low equation [14], which revealed that for any solution it could be possible to produce an additional infinite number of other solutions which contain arbitrary adjustable parameters. The question of the analytic structure of the results of the CDD study [1] and its the application herein is of importance. The model used in CDD had only $s$-wave scattering. With it an exact solution was found with amplitudes that are generalized $R$ functions. Such amplitudes allow additional solutions built on an infinite number of adjustable parameters. The ambiguity is accomplished by adding poles, CDD poles, to a denominator function, which produce zeros in the amplitude [15].

For other field theoretic models, exact solutions are not known. We will investigate this ambiguity for the numerical solution of the crossing-symmetric Chew-Low model. This model, while being physically more realistic, has only $p$-wave scattering, and thus there is a finite number of partial waves. Numerical solutions have been produced for the complete model with and without coupling to the inelastic channels [16,17]. In this work, we analytically analyze these numerical solutions and find them to be unique. This uniqueness precludes additional zeros of the amplitudes and as such precludes the CDD ambiguity. This uniqueness is for the total $T$ matrix. Because the partial-wave projection is not holonomic [18,19], there is no justification for calling an individual channel a generalized $R$ function.

In his earlier work on scattering, Wigner developed the analytic properties of his $R$ functions [20–22]. These functions are meromorphic with $\mathrm{Im}[R\{\mathrm{Im}(z)>0\}]>0$ and $\mathrm{Im}[R\{\mathrm{Im}(z)<0\}]<0$. Some of their properties [21] are: 1) A linear combination of $R$ functions with real positive coefficients is an $R$ function; 2) The function is real on the real axis, and only on the real axis; 3) The derivative of $R$ is positive on the real axis; 4) All poles are on the real axis with negative residues; 5) If $S=-1/R$, then $S$ is an $R$ function. From this last property, it can be seen that poles in $R$ correspond to zeros in $S$, and zeros in $R$ (which can only be on the real axis) correspond to poles in $S$. Because of the positive derivative of $R$ on the real axis, the phase of $R$ in going from $-\infty \to +\infty$ along the real axis can only increase and must increase by $\pi$ for each zero. In between each pair of zeros, the phase must pass through a resonance.

The two models investigated by CDD were both a scalar model for pion-nucleon scattering, one for a field of charged pions and one for a neutral pion field. These models have a fixed source nucleon (no recoil) and only allow for $s$-wave scattering. The Hamiltonian for the models has translational invariance and conserves momentum. The Hamiltonians are crossing symmetric and lead to a nonlinear equation for the $T$ matrix, a Low equation. Direct iteration of the Low equation for the models considered does not lead to a finite result. By using an $N/D$ method (dispersion relations), one can generate exact solutions for

*rmcleod@unimelb.edu.au

 

the models. The denominator functions used, $D$, were shown to have the property that $\mathrm{Im}[D\{\mathrm{Im}(z) > 0\}] > 0$ and $\mathrm{Im}[D\{\mathrm{Im}(z) < 0\}] < 0$ in the cut complex plane. An analogy can be made between the $D$ functions and Wigner's $R$ functions where the $D$ functions are called generalized $R$ functions. Because of this analogy, one could add as many poles, zeros for the amplitude, as one wants and still produce a solution to the Low equation. These poles could be added anywhere; however, physical constraints precluded them from being added below threshold or at low energy where the bound states and phase shifts are known. By using nearest singularity arguments, it can be reasoned that one could always add as many CDD poles as one wishes at high enough energy without effecting the low-energy phases. Thus, the low-energy phases do not set the analytic structure of the model since the number of CDD poles to be included is ambiguous. Later, these poles were connected to unknown particles [23,24], since each pair of zeros would require the phases between the zeros to rise through a resonance, and resonances are associated with particles.

While the scalar model is exactly solvable, it is not applicable to low-energy $\pi - n$ scattering nor photomeson production [15]. The Chew-Low model [25], although simple, is not exactly solvable. However, it is applicable to low-energy $\pi - n$ scattering and photomeson production [15]. It is considered the nonrelativistic limit of the $\gamma_5$ theory of the $\pi - n$ interaction [15,26]. Being a more realistic model, it would offer a better guide for modern field theoretic models. The Chew-Low Hamiltonian is invariant under space rotation, isospin rotation, time inversion, and space inversion. The model has an infinite nucleon mass (static), conserves momentum, and only has $p$-wave scattering. All the partial waves are analytically connected by the crossing cut. The crossing symmetric Chew-Low model will be reviewed in Sec. II, along with the analytic properties of the partial-wave amplitudes. The energy dependence of the total $T$ matrix will be developed in Sec. III. This will include the analytical properties of the $T$ matrix. We investigate the Chew-Low model and give a general prescription for uniqueness of the solutions. These solutions are a family of solutions for continuous values of the parameters. For fixed values of the threshold and the resonance, we will prove a solution is unique in Sec. IV. We find this uniqueness is related to the total $T$ matrix. In the case of the Chew-Low model, the total $T$ matrix is not a generalized $R$ function. This uniqueness precludes the addition of a CDD poles, i.e., a zero in the amplitude, and shows that the numerical solutions to the crossing symmetric Chew-Low model found in Refs. [16,17] are unique, i.e., cannot be duplicated if one adds a CDD pole. Furthermore, we find that one cannot apply the criterion of a generalized $R$ function to an individual partial wave and that adding a CDD pole to any partial wave requires that the pole must be added to all partial waves in order to maintain crossing. We interpret the results in Sec. V, where we examine the results as related to the original work of CDD. The partial-wave expansion is not holonomic; as such, one cannot apply the CDD analysis to individual channels. We conclude that one cannot apply the criterion of a generalized $R$ function to an individual channel that is analytically tied to other partial waves by a crossing cut. We make other concluding remarks in Sec. VI. Appendix A examines the phase of a polynomial along the x axis, used in the proof. Appendix B explicitly shows that adding a CDD pole to the $P_{33}$ channel requires adding pairs of CDD poles to all channels in order to maintain crossing symmetry. This results in two overall zeros in the total $T$ matrix and would constitute a new model, and thus is not allowed.

## II. THE CHEW-LOW MODEL

In 1955 Low [14] published his paper on the crossing symmetric Low equation. Chew and Low [25] then applied the Low equation to a simple field theoretic model for a pseudo-scalar pion scattering from an infinitely massive nucleon. As such, it only involves $p$-wave scattering. This model, which can be considered the nonrelativistic limit of the relativistic $\gamma_5$ theory [15,26], is invariant under rotations in space and in isospin space as well as under time and space inversions [15]. However, exact solutions for the model are not known. Direct iteration of the Low equation does not lead to a finite result. The first attempt at a numerical solution [27] used an $N/D$ method and the one-pion approximation. While they were able to produce crossing symmetric solutions in the weak coupling limit, their solutions for physical values of the coupling, that produced the $P_{33}$ resonance, did not satisfy their original crossing-symmetric Low equation. Subsequently, Ernst and Johnson [28] extended that approach to include coupling to the inelastic channels. However, the solutions of their $N/D$ method also did not satisfy their original Low equation for physical values of the coupling. Later, numerical solutions were calculated [16] that do satisfy the original Low equation for the Chew-Low model as extended by Ernst and Johnson and, even later [17], for the original Chew-Low model in the one-pion approximation.

Herein, we examine the numerical solutions to the Chew-Low model [25] as extended by Ernst and Johnson [28]. We write the $T$ matrix for the model as

$$\langle \mathbf{p}'|T(z)|\mathbf{p}\rangle = \sum_{\mathbf{n}} \int d\omega\, \delta(\omega - E_{\mathbf{n}}) \left\{ \frac{\langle \mathbf{p}'|T^\dagger|\mathbf{n}\rangle\langle \mathbf{n}|T|\mathbf{p}\rangle}{\omega - z} + \frac{\langle \mathbf{p}|T^\dagger|\mathbf{n}\rangle\langle \mathbf{n}|T|\mathbf{p}'\rangle}{\omega + z} \right\}, \quad (1)$$

where $\mathbf{n}$ runs over a complete set of intermediate states, $\omega = (p^2 + 1)^{1/2}$ is the pion center-of-mass energy for the momentum $\mathbf{p}$ with the pion mass set equal to one, and $z$ is the analytically continued real energy ($\omega$) into the complex plane. The $T$ matrix is decomposed by projecting it onto a complete orthogonal set of spin and isospin states so that it can be expressed by

$$\langle \mathbf{p}'|T(\omega)|\mathbf{p}\rangle = \frac{4\pi v(p')v(p)}{(4\omega'_p\omega_p)^{1/2}}\sum_\alpha P_\alpha(\mathbf{p}',\mathbf{p})h_\alpha(\omega), \quad (2)$$

where the $P_\alpha(\mathbf{p}',\mathbf{p})$ are the projectors onto the spin, isospin state $\alpha = (2I, 2J)$, where $J$ is the total angular momentum, and $I$ is the total isospin. $v(p)$ is the cutoff function, the momentum space transformation of $v(r)$, the extended interaction region. Without inelasticities the 13 and 31 channels are identical and therefore can be treated as a single channel. Although the inelasticities split them, they are usually averaged to leave the channels degenerate so again they can be treated as a single channel. Developing the partial-wave equations for this model leads to the nonlinear Low equations for the $T$ matrix $h_\alpha(z)$,

$$h_\alpha(z) = \frac{\lambda_\alpha}{z} + \frac{1}{\pi}\int_1^\infty d\omega' \frac{\mathrm{Im}[h_\alpha(\omega'+i\eta)]}{\omega'-z} + \frac{1}{\pi}\int_1^\infty d\omega' \frac{\mathrm{Im}[h_\alpha(-\omega'-i\eta)]}{\omega'+z}, \quad (3)$$

where

$$\lambda_\alpha = \frac{2}{3}f^2(-4,-1,2) \quad \text{for } \alpha = [11, (13 \text{ and } 31), 33], \quad (4)$$

with $f^2$ the renormalized coupling constant. Its value can be considered as an adjustable parameter in the model. These functions $h_\alpha(z)$ are related to the partial-wave phase shifts on the physical cut ($z \to \omega + i\eta$) by

$$h_\alpha(\omega) = \frac{\hat{\eta}_\alpha(p)e^{i\hat{\delta}_\alpha(p)}\sin\hat{\delta}_\alpha(p)}{p^3v^2(p)}, \quad (5)$$

where $\hat{\delta}$ and $\hat{\eta}$ are the $T$-matrix phases and inelasticities, respectively. The inelasticities are taken from experiment. With $\hat{\eta} = 1$, this model reduces to the original Chew-Low model in the one-pion approximation.

To solve these equations for $h_\alpha$, the renormalized coupling constant was set to its physical value $f^2 = 0.081$, and a Gaussian cutoff function,

$$v(p) = e^{-p^2/\beta^2}, \quad (6)$$

was used, where $\beta$ is an adjustable parameter. The weak coupling limit occurs when $\beta$ is small and the amplitudes $h_\alpha$ are pole dominant. Solutions for this case were generated that solved the original Low equations for both $\hat{\eta} = 1$ [27] and for $\hat{\eta} \neq 1$ [28]; however, the solutions did not have the dominant $P_{33}$ resonance. With strong coupling, when $\beta$ was increased to produce the physical resonance, the solutions found by using the $N/D$ method did not satisfy the original Low equations. Later work did find crossing symmetric solutions in the strong coupling limit that did produce the $P_{33}$ resonance and satisfy the Low equations [16,17]. This was done by allowing zeros to develop in the complex plane.

Besides having cuts from $+1 \to +\infty$ and from $-\infty \to -1$, the functions $h_\alpha(\omega)$ have the following properties:

(1) $h_\alpha(z)$ has a simple pole at $\omega = 0$ with residue $\lambda_\alpha$;
(2) $h_\alpha(z)$ behaves as $z^{-1}$ at infinity;
(3) $h^*_\alpha(z) = h_\alpha(z^*)$, a result of the Schwarz reflection principle;
(4) $h_\alpha(-z) = A_{\alpha\beta}h_\beta(z)$;
(5) $\mathrm{Im}[h_\alpha\{-1 < \mathrm{Re}(z) < 1; \mathrm{Im}(z) = 0\}] = 0$;
(6) $\mathrm{Im}[h_\alpha\{\omega \to +\infty\}]$ behaves as $v^2$;
(7) in the $P_{33}$ channel only: $\mathrm{Im}[h_{33}\{\mathrm{Im}(z) > 0\}] > 0$.

Property 7 is one of the defining properties of an $R$ function. It is necessary to have a generalized $R$ function as defined in CDD [1]. The other two channels do not have this property. Property 4 above is crossing symmetry in which the crossing matrix, $A_{\alpha\beta}$, is

$$A_{\alpha\beta} = \frac{1}{9}\begin{pmatrix} 1 & -8 & 16 \\ -2 & 7 & 4 \\ 4 & 4 & 1 \end{pmatrix}. \quad (7)$$

## III. THE REDUCED $T$ MATRIX

In this section, we set up the reduced $T$ matrix that will be used to show uniqueness. From Eq. (2) we reconstruct the $T$ matrix using the form [15,26]

$$\langle \mathbf{p}'|T(\omega)|\mathbf{p}\rangle = \sum_\alpha P_\alpha(\mathbf{p}',\mathbf{p})\langle p'|T(\omega)|p\rangle. \quad (8)$$

The $T$ matrix is summed over initial spin and isospin states and averaged over final spin and isospin states. We also integrate over the azimuthal angle $\phi$ to find the energy dependence of the $T$ matrix given by

$$T(\omega,\theta) = \frac{3}{2}\cos(\theta)\sin(\theta)p^2v^2(p)H_T(\omega), \quad (9)$$

with

$$H_T(\omega) = \frac{1}{9}(h_{11} + 2h_{13} + 2h_{31} + 4h_{33}). \quad (10)$$

Using the crossing relation, property 4 listed below, we analytically continue $H_T$ into the complex plane and use Cauchy's integral formulas to obtain the nonlinear Low equation for $H_T(z)$,

$$H_T(z) = \frac{1}{\pi}\int_1^\infty d\omega' \frac{\mathrm{Im}[H_T(\omega'+i\eta)]}{\omega'-z} + \frac{1}{\pi}\int_1^\infty d\omega' \frac{\mathrm{Im}[H_T(\omega'+i\eta)]}{\omega'+z}. \quad (11)$$

From the properties of the $h_\alpha(\omega)$, and the Low equation, it can be shown that the functions $H_T(z)$ have a cut from $+1 \to +\infty$ and one from $-\infty \to -1$ with the following properties:

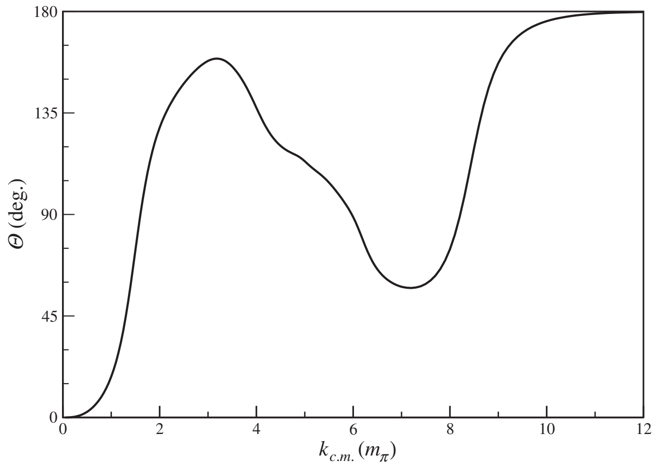


FIG. 1. The phase of $H_T(z \to \omega + i\eta)$, $\theta(\omega)$, as a function of $k_{c.m.}$ for the solution of the Chew-Low model with $\beta =$ 752 MeV as found in Ref. [16].

(1) $H_T(z)$ has no pole at $\omega = 0$;
(2) $H_T(z)$ behaves as $z^{-2}$ at infinity;
(3) $H_T^*(z) = H_T(z^*)$, again a result of the Schwarz reflection principle;
(4) $H_T(-z) = H_T(z)$, the crossing symmetry [15];
(5) $\mathrm{Im}[H_T\{\mathrm{Re}(z) > 0; \mathrm{Im}(z) > 0\}] > 0$;
(6) $\mathrm{Im}[H_T\{\mathrm{Re}(z) < 0; \mathrm{Im}(z) > 0\}] < 0$;
(7) $\mathrm{Im}[H_T\{-1 < \mathrm{Re}(z) < 1; \mathrm{Im}(z) = 0\}] = 0$;
(8) $\mathrm{Re}[H_T(\omega + i\eta \to 1)] \to 0^+$;
(9) $\mathrm{Re}[H_T(\omega + i\eta \to \infty)] \to 0^-$;
(10) $\mathrm{Im}[H_T\{\mathrm{Im}(z) = 0\}] = 0$.

Consequently, the phase of $H_T(z \to \omega + i\eta)$ must start at $0^+$ and go to $(2n+1)\pi^-$ at infinity. Properties 3 and 4 infer that to every zero on the positive real axis, there must be a corresponding zero on the negative real axis. Likewise, for every zero on the positive imaginary axis, there must be a corresponding zero on the negative imaginary axis. In order for $H_T(z)$ to be a generalized $R$ function, it would have to have a property $\mathrm{Im}[H_T\{\mathrm{Im}(z > 0)\}] > 0$. Properties 5 and 6 for $H_T(z)$ do not allow this.

We parametrize $H_T(z)$ as

$$H_T(z) = R(z)e^{i\theta(z)} \tag{12}$$

to apply the above formulation to the solution found previously [16] with $\hat{\eta} \neq 1$. The real phases $\theta(\omega)$ as a function of $k_{c.m.}$ are shown in Fig. 1. The dominant $P_{33}$ resonance can be seen at $k_{c.m.}$ approximately equal to $1.6m_\pi$, where the phases go up through 90 deg. This resonance was set by the value of the adjustable parameter $\beta$ found to reproduce the $P_{33}$ experimental phase shifts [16]. For $\hat{\eta} = 1$, the phases would have the same features.

## IV. PROOF OF UNIQUENESS

In this section, we shall construct a proof of uniqueness for the Chew-Low model. First we have to define what we mean by a unique solution. For the model, we have two adjustable parameters $f^2$ and $\beta$. By adjusting these two parameters, an infinite family of solutions can be found. To set the parameters, we use the physically found features that we wish to reproduce. By setting $f^2$ to the experimental value, and the fact that $v(p \to 0) \to 1$, we get the experimental threshold. $\beta$ is adjusted to give the position of the $P_{33}$ resonance. In Fig. 1, we have the constructed phases $\theta$ for the solution of the Chew-Low model of McLeod and Ernst [16]. In Ref. [16] only three channels ($\alpha = [11, 31 = 31, 33]$) were used. A more comprehensive explanation is given in the literature [16,28]. In Fig. 1, the dominance of the resonance on the low-energy phases is evident (the phase passes through 90 deg at about $k_{cm}$ of $1.6\mu$). Also, at high energies the phases approach 180 deg from below, as required by the listed properties, 5 and 9, for $H_T(z)$.

For the proof, we shall consider the threshold [29] and the position and the width of $\theta$ at resonance to be fixed. Then we ask the question: *Is there another family of curves that also contains a solution with this same threshold and the same position and width of $\theta$ at resonance?* This is not too much to ask for since we are concerned with the possibility of adding a CDD pole at such a high energy as to leave the low-energy phases the same or to be able to adjust $f^2$, $\beta$, and the parameters of the CDD pole in order to maintain the same low-energy phases. It has been found that one will get a good fit to the width of the $P_{33}$ resonance with the physical value of $f^2$ if one uses a reasonable model for the interaction [30,31]. Thus, we require that any additional solution be able to reproduce the threshold and the position and width of the resonance found in the original solution.

To proceed, we assume that there are two solutions, $H_T^{(1)}(z)$ and $H_T^{(2)}(z)$. We next construct a function $g(z)$ that is the difference of the two solutions:

$$g(z) = H_T^{(2)}(z) - H_T^{(1)}(z). \tag{13}$$

The second solution will have all the same properties as the first. We can choose the difference so that $\mathrm{Im}[g(z \to 1 + i\eta)] > 0$ is maintained. We can write $g(z)$ in the same form as Eq. (12),

$$g(z) = R_g(z)e^{i\theta_g(z)}, \tag{14}$$

and this phase $\theta_g$ must go to $n\pi$ as $\omega$ goes to $+\infty$ where $n$ is an integer. Furthermore, since $g$ has the same behavior as $H_T$, $g(z \to \infty)$ behaves as $z^{-2}$.

Next, we define a denominator function $\mathcal{D}(z, \theta_g)$ by [32]

$$\mathcal{D}(z, \theta_g) = \exp\left(-\frac{z}{\pi}\int_1^\infty d\omega' \frac{\theta_g(\omega' + i\eta)}{\omega'(\omega' - z)} + \frac{z}{\pi}\int_1^\infty d\omega' \frac{\theta_g(\omega' + i\eta)}{\omega'(\omega' + z)}\right), \tag{15}$$

where we have used the crossing symmetry of $H_T$. This function has all the symmetries of $H_T(z)$ in the complex plane. In addition, by construction, $\mathcal{D}(z, \theta_g)$ is a once subtracted dispersion with the following properties:

(1) $\mathcal{D}(z, \theta_g) \neq 0$;
(2) $\mathcal{D}(z = 0, \theta_g) = 1$;
(3) $\mathcal{D}(z \to \infty, \theta_g) \to 0$.

Let

$$\Phi(z) = g(z)\mathcal{D}(z, \theta_g). \tag{16}$$

As the only singularities in $g(z)$ are the two cuts on the real axis and they are the same two cuts in $\mathcal{D}^{-1}(z, \theta_g)$, $\Phi(z)$ must not contain any singularities. Therefore, it is a polynomial—a polynomial that is subject to the same symmetries as the functions $H_T(z)$ constructed in the last section. This polynomial can have a number of zeros $\phi$. Since $\mathcal{D}$ cannot have any zeros, by construction, these must be the same as the zeros of $g$.

We now put limits on the minimum and maximum number of zeros this polynomial can have. For the maximum number of zeros, we look at the behavior of $g$ and $\mathcal{D}$ as $\omega \to \infty$. $g$ has the same behavior as $H_T$, so it goes as $\omega^{-2}$. From Ref. [32], we have that $\mathcal{D}$ behaves as $\omega^q$, where

$$q = \frac{1}{\pi}[\theta_g(+\infty) + \theta_g(-\infty)], \tag{17}$$

where $\theta_g(\pm\infty)$ is the phase of $g$ as $\omega \to \pm(\infty + i\eta)$. Using the results of Appendix A, we find

$$q = \frac{1}{\pi}[2m\pi] = 2m, \tag{18}$$

where $m$ is the number of zeros on the positive (negative) real axis. Therefore, $\Phi(\omega \to \infty)$ goes as $\omega^{2m-2}$ and

$$\phi \leq 2m - 2. \tag{19}$$

Next we count the known zeros to put a lower limit on $\phi$. From Appendix A, we assumed that there are $m$ zeros on the positive real axis, or a total of $2m$, where $m \geq 2$ in order to account for the threshold and resonance. Since these are the only zeros that we know we must have, we find

$$\phi \geq 2m. \tag{20}$$

Clearly, these two conditions are in contradiction, so $g(z) \equiv 0$ and the assumption that there exists a second solution is incorrect. The numerical solution found [16] must be unique in that one cannot find another with the same threshold and resonance for $H_T(z)$. Thus, the functions $h_\alpha(z)$ are unique as well. The same analyses can be applied to the case with $\hat{\eta} = 1$; thus, the solutions [17] will also be unique, i.e., only have one solution that reproduces the threshold and resonance.

## V. INTERPRETATION OF RESULTS

We have analytically analyzed the previously found numerical solution to the Chew-Low model and showed that this solution is unique. That is, the adjustable parameters would produce a family of solutions however, only one solution would have the physical threshold and resonance. Furthermore, it is not possible to add a CDD pole at any energy and reproduce the results at threshold and resonance. This is somewhat surprising since adding such a pole at a very high energy would add a zero to the amplitude and would have two more adjustable parameters, the strength and location of the pole. This may be related to the numerical difficulty encountered in solving the Chew-Low model in Ref. [17] as discussed below.

Usually, to analyze scattering amplitudes, the $T$ matrix is projected onto partial waves. But that projection is not holonomic [18,19]. Crossing is important as it analytically ties the scattering channels. Thus, one cannot use the analytic structure of the individual partial wave alone to analyze the scattering in the channel. For a proper analysis, the analytic structure of the original $T$ matrix, or a holonomic decomposition of it, must be used. Further the nature of each model form used must be examined, for only then can one determine if it is subject to multiple solutions by connection to a generalized $R$ function.

### A. Crossing-symmetric Chew-Low model

The failures of Salzman and Salzman [27] and of Ernst and Johnson [28] to find a numerical solution with strong coupling is related to the $N/D$ method employed. They constructed the function $g_\alpha(z)$, as suggested by Chew and Low [25], that did not allow for zeros to develop in the amplitude. We use this function, with added zeros, in Appendix B to illustrate what is needed to maintain crossing symmetry. In the strong coupling limit, McLeod and Ernst [16,17] found that a pair of zeros developed off of the negative real axis in the $P_{13} = P_{31}$ channel. This clearly shows that the crossing matrix and the crossing of the scattering cuts are dominant with strong coupling. Also, to calculate crossing-symmetric solutions to the desired degree of accuracy, the negative high-energy point at which the $\mathrm{Re}[h_2(z)]$ comes back through zero gave an essential delta function contribution to the calculation [17]. With strong coupling, the high-energy part of the amplitude affects the low-energy part, in agreement with a phenomenological model [33], which found that the high-energy region affects the position and width of the $P_{33}$ resonance.

In this study, the solution found for strong coupling that produced the 1232 resonance is unique insofar as giving another solution to reproduce the threshold and the resonance. From the analytic properties of $H_T(z)$, $\mathrm{Im}[H_T\{\mathrm{Im}(z) > 0\}] \not> 0$, and the total $T$ matrix is not a generalized $R$ function. This is self-consistent, since one cannot add zeros to the amplitude because it would violate the uniqueness. There are two conclusions we can draw. First, the values for $f^2$ and $\beta^2$ that were fixed are the only ones that will produce the threshold and resonance found. This is consistent with the numerical results of Refs. [16,17]. Second, it is not possible to add a CDD pole to the solution for the total $T$ matrix without affecting the threshold and resonance. Hence, there is no CDD ambiguity. This inability to add a CDD pole and being able to reproduce the threshold and dominant resonance may be related to the numerical difficulty already mentioned. In Ref. [17] it was found that, where $\mathrm{Re}[h_2]$ went through zero at a large value of $\omega$ on the negative axis gave a $\delta$-function-like contribution to the amplitude. When adding a CDD pole at a large value of $\omega$, the amplitude will be forced up through a resonance. This could give a $\delta$-function-like contribution to the amplitude, and it may not be possible to counter the effects by adjusting the parameters.

In the past, some authors have claimed that the $P_{33}$ channel is a generalized $R$ function. These claims are incorrect in the crossing-symmetric case. One cannot split off the $P_{33}$ channel separately, since the partial-wave projection used is not holonomic. In Appendix B, we have shown that any attempt to add a CDD pole to the $P_{33}$ amplitude will not maintain crossing symmetry unless the zero in the amplitude is included in all partial waves. This would mean adding a zero to the total $T$ matrix, something that is forbidden by uniqueness, i.e., it would constitute a different model for the new solution. Crossing ties the analytic structure of all the spin-isospin channels together. One cannot use the CDD analysis to justify adding CDD poles to any spin-isospin channel of the model. This does not contradict CDD, because the total $T$ matrix they used was a generalized $R$ function, as shown below, while the Chew-Low total $T$ Matrix is not.

### B. Chew-Low model without crossing

In the Chew-Low model under a no-crossing approximation, the channels become disconnected, such that each channel can be considered a model for that spin-isospin channel. In the $P_{33}$ channel, the solution should be a generalized $R$ function since one has property 6 for $h_{33}(z)$ as required for a generalized $R$ function. Thus, with this model, one should be able to find multiple solutions by adding CDD poles, as pointed out by CDD [34]. However, the $P_{11}$ channel is quite different. No solutions of this channel can be found for physical values of the coupling in the no-crossing approximation [35,36]. If we consider the two channels as being the spin-isospin projection of the no-crossing total Chew-Low $T$ matrix, then the completely different analytical structures of the two channels shows that the projection is not holonomic.

### C. CDD models

CDD analyzed two models: the first of charged pions and the second of neutral pions, scattering from an infinite mass nucleon. Both are scalar field models for $s$-wave scattering, so no partial-wave projection was involved. For the second model, there was only the neutral pion channel. It was self-crossing, no projection onto channels, and the analysis was of the total $T$ matrix. For this model, CDD showed that the total $T$ matrix was a generalized $R$ function and thus had an infinite number of solutions.

In the first model, the analysis was done on two charge channels, and each was found to be a generalized $R$ function. Thus, an infinite number of zeros can be added to the amplitudes. These zeros must be added in such a way as to maintain crossing. We can deduce two things from their results. First, their total $T$ matrix is a generalized $R$ function. This can be seen since the sum of two $R$ functions is itself an $R$ function [21], listed as property 1 in the introduction. Second, breaking the total $T$ matrix up into charge channels for this model is holonomic. The analytic structure of the total $T$ matrix is preserved in each channel. As such, there is no contradiction to the results of this work with the original work of CDD.

### D. Other models

When building a model for any interaction, one is free to choose how the model is to be constructed. We look only a the field theoretic constraints that our results put on such a construction. Each model must be considered independently. When examining for a generalized $R$-function solution, one must use the total $T$ matrix or channels that were found by a holonomic process. However, failing to identify the solutions as a generalized $R$ function does not mean that the solution would be unique, since adding CDD zeros may still be possible. For models that contain crossing, this crossing would tie together all the partial waves. Adding any CDD poles would require adding the pole to all the partial waves. We find that adding a CDD pole would constitute a new model. Based on our findings, it should be expected that only one model will reproduce the low-energy features.

## VI. CONCLUSIONS

In this work, we have analyzed the previously found numerical solutions to the crossing-symmetric Chew-Low model. This model is the simplest $\pi - n$ model that is phenomenological. Therefore, this analysis is more pertinent to modern field theoretic models than the previous analysis of nonphenomenological models.

There are three main conclusions that can be drawn from this analytical analysis. First, the amplitudes found are not generalized Wigner $R$ functions. No analogy can be made to the work of Wigner nor the work of CDD. This includes individual partial waves, since the partial wave projection is not holonomic. Crossing ties the analytic structure of the partial waves together. Second, the solutions are unique. One cannot find another solution that will have the same threshold and dominant resonance. Since the solution found fit the physical threshold and resonance, it is the only solution that will have the physical threshold and resonance. This does not preclude the possibility of adding a CDD-like pole and finding another family of solutions. However, this new family of solutions must be completely different from the family of solutions generated by adjusting the parameters $f^2$ and $\beta$ in the original model, i.e., no solution from the new family can have the same low-energy phases as any solution from the original family that produces a resonance. Third, it is not possible to add a CDD pole to only one partial wave. Adding such a pole to one partial wave will require that it is added to all waves in order to maintain crossing.

These results are not in conflict with the original CDD analysis since the total $T$ matrix used in their models was generalized $R$ functions, and the method used to project onto the charge channels was holonomic. However, the models used by CDD are not phenomenological.

The importance of crossing symmetry cannot be over emphasized. With strong coupling, the crossing cut influences the scattering amplitude, even at low energies. Today's models of two-body strong interactions are geared toward higher energies. Such models do not always include crossing symmetry in setting values of parameters and in the application to scattering. Since crossing symmetry becomes more important with energy, it is imperative to include its effects on the interaction.

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## APPENDIX A

In this appendix, we shall examine the effects of zeros on the phase of $\Phi$ and, as such, on $g$. $\phi$ is the number of zeros of $\Phi$. We will always make choices that maximize our $\phi_{\min}$ and to minimize our $\phi_{\max}$ used in the proof, i.e., increase the phase at $\infty$ and decrease the total number of zeros. All zeros are assumed to be at a finite distance from the origin. We will examine the change in the phase of $\Phi$ in going from $-\infty$ to $+\infty$ on the real axis defined as $\Delta_L \arg[\Phi(z)]$. This change in phase can be related to the change in phase of $g$. By construction we have

$$\Phi(z) = g(z)\mathcal{D}(z, \theta_g). \tag{A1}$$

The change in phase of $\Phi$ is then related to the change of phases of $g$ and $\mathcal{D}$ along the x axis by

$$\Delta_L \arg[\Phi(z)] = \Delta\theta_g(\omega + i\eta) + \Delta\theta_{\mathcal{D}}(\omega + i\eta), \tag{A2}$$

where $\Delta\theta_{g(\mathcal{D})}(\omega + i\eta)$ is the change in phase of $g(\mathcal{D})$ as $\omega$ varies from $-\infty \to +\infty$. Since $\mathcal{D}$, by construction, cannot be zero, has a value of 1 at $\omega = 0$ and a value 0 at $\omega = \pm\infty$, $\Delta\theta_{\mathcal{D}}(\omega + i\eta) = 0$. So we have

$$\Delta_L \arg[\Phi(z)] = \Delta\theta_g(\omega + i\eta). \tag{A3}$$

First consider the effects of zeros off the real axis. We assume that there are no zeros on the imaginary axis. Because $\Phi$ will have the same symmetries as $H_T$, we must have the same number of zeros in each quadrant. We assume there are $k$ zeros in the first quadrant and a total of $4k$ zeros. The change in the phase of $\Phi$ is given by [37]

$$p = \frac{1}{2}\left(n + \frac{1}{\pi}\Delta_L \arg[\Phi(z)]\right), \tag{A4}$$

where $p$ is the number of zeros above the x axis and $n$ is the total number of zeros. We find in our case with $p = 2k$ and $n = 4k$, that

$$\Delta_L \arg[\Phi(z)] = \pi(2p - n) = 0. \tag{A5}$$

So the zeros off the real axis do not effect the change in phase $\Delta_L \arg[\Phi(z)]$.

Next, we consider the effects of zeros on the real axis. Consider having only one zero at threshold on the positive real axis. With just this threshold zero, the phases of $g$ are expected to start at $0^+$ and go to $\pi$ as $\omega \to +\infty$. With just one zero on the x axis, we can have

$$\Delta_L \arg[\Phi(z)] = \pm\pi, 0. \tag{A6}$$

We will choose the zero so that $\Delta_L \arg[\Phi(z)] = \pi$. This is the only type of zero on the real axis we shall consider [38]. This is the same type of zero that is produced by a CDD pole. We now assume that there are $m$ such zeros on the positive real axis where $m \geq 2$ to account for threshold and the resonance. Then we will have

$$\Delta_L \arg[\Phi(z)] = 2m\pi = \Delta\theta_g(\omega + i\eta). \tag{A7}$$

So we find, using the notation of Frye and Warnock [32],

$$\theta_g(+\infty) + \theta_g(-\infty) = 2m\pi. \tag{A8}$$

## APPENDIX B

Herein, we add a CDD pole to the $P_{33}$ channel and maintain crossing symmetry. We limit the inelasticities to $\hat{\eta} = 1$ and, for brevity, consider only the three-channel case. Because of the crossing matrix, Eq. (7), only in the $P_{33}$ channel ($\alpha = 3$) can the criterion

$$\mathrm{Im}[h_3\{\mathrm{Im}(z) > 0\}] > 0 \tag{B1}$$

be met assuredly. Thus, only this channel can be considered to be a generalized $R$ function. In the solution found in

Ref. [16], the $\alpha = 2$ channel had the added analytical structure of a pair of zeros off of the negative real axis. Thus, $\mathrm{Im}[h_2\{\mathrm{Im}(z) > 0\}]$ is both positive and negative. We follow the usual procedure [25,27] for treating the $h_\alpha(z)$ by defining the function

$$g_\alpha(z) = \frac{\lambda_\alpha}{z h_\alpha(z)}. \tag{B2}$$

The functions $h_\alpha(z)$ that correspond to these $g_\alpha(z)$ will not have any zeros unless poles are explicitly added to the $g_\alpha(z)$. For these $g_\alpha(z)$, crossing is given by [27,28]

$$\frac{1}{g_\alpha(z)} = B_{\alpha\beta}\frac{1}{g_\beta(-z)}, \tag{B3}$$

where the matrix, $B_{\alpha\beta}$, is

$$B_{\alpha\beta} = \frac{1}{9}\begin{pmatrix} -1 & 2 & 8 \\ 8 & -7 & 8 \\ 8 & 2 & -1 \end{pmatrix}. \tag{B4}$$

In the $\alpha = 1, 3$ channels one can write a Low equation for $g_\alpha(z)$ as

$$g_\alpha(z) = 1 - \frac{z}{\pi}\int_1^\infty d\omega' \frac{p'^3 v^2(p')}{\omega'^2}\left(\frac{\lambda_\alpha}{\omega' - z} + \frac{G_\alpha(\omega')}{\omega' + z}\right), \tag{B5}$$

with [27,28]

$$G_\alpha(z) = -\left|\sum_\gamma \frac{B_{\alpha\gamma}}{g_\gamma(z)}\right|^{-2}\sum_\beta \frac{B_{\alpha\beta}\lambda_\beta}{|g_\beta(z)|^2}. \tag{B6}$$

The zeros in the $\alpha = 2$ channel require that we add poles to the $g_2(z)$ so as to reproduce the zeros in $h_2(z)$. So for $g_2(z)$ we write

$$g_2(z) = 1 - \frac{z}{\pi}\int_1^\infty d\omega' \frac{p'^3 v^2(p')}{\omega'^2}\left(\frac{\lambda_2}{\omega' - z} + \frac{G_2(\omega')}{\omega' + z}\right) - \frac{S_2}{Z_2 - z} - \frac{S_2}{Z_2^* - z}, \tag{B7}$$

where the $Z_2$ and $Z_2^*$ are the locations of the symmetric pair of zeros and the $S_2$ is related to the inverse of the slope at the zero. All of the $g_\alpha(z)$ will go to zero as $z$ goes to infinity.

We now add a CDD pole and a crossed CDD pole [1] to the $P_{33}$ channel and use crossing to modify the $g_\alpha(z)$ to maintain crossing symmetry [39]. The pole will be added at a large value of $\omega = \omega_3$. We choose $\omega_3$ at an energy that is large enough that the contribution from the integrals to the $g_\alpha(z)$ is small and the values of $g_\alpha(z)$ are close to one. For $g_3(z)$ we have

$$g_3(z) = 1 - \frac{z}{\pi}\int_1^\infty d\omega' \frac{p'^3 v^2(p')}{\omega'^2}\left(\frac{\lambda_3}{\omega' - z} + \frac{G_3(\omega')}{\omega' + z}\right) - \frac{R_3}{\omega_3 - z} - \frac{R_3}{\omega_3 + z}, \tag{B8}$$

where $R_3$ is an adjustable constant and $\omega_3$ is a high real energy where the pole is located. The first added term puts a zero in $h_3(z)$ (at a high positive energy) while the second is an attempt at keeping the $g_\alpha(z)$, and thus the $h_\alpha(z)$, crossing symmetric. Substituting Eqs. (B5), (B7), (B8), and (B4) into Eq. (B3) gives

$$\begin{aligned}
&\left[1 - \frac{z}{\pi}\int_1^\infty d\omega' \frac{p'^3 v^2(p')}{\omega'^2}\left(\frac{\lambda_3}{\omega' - z} + \frac{G_3(\omega')}{\omega' + z}\right) - \frac{R_3}{\omega_3 - z} - \frac{R_3}{\omega_3 + z}\right]^{-1} \\
&\quad = \frac{8}{9}\left[1 + \frac{z}{\pi}\int_1^\infty d\omega' \frac{p'^3 v^2(p')}{\omega'^2}\left(\frac{\lambda_1}{\omega' + z} + \frac{G_1(\omega')}{\omega' - z}\right)\right]^{-1} + \frac{2}{9}\left[1 + \frac{z}{\pi}\int_1^\infty d\omega' \frac{p'^3 v^2(p')}{\omega'^2}\left(\frac{\lambda_2}{\omega' + z} + \frac{G_2(\omega')}{\omega' - z}\right)\right. \\
&\qquad \left. - \frac{S_2}{Z_2 - z} - \frac{S_2}{Z_2^* - z}\right]^{-1} - \frac{1}{9}\left[1 + \frac{z}{\pi}\int_1^\infty d\omega' \frac{p'^3 v^2(p')}{\omega'^2}\left(\frac{\lambda_3}{\omega' + z} + \frac{G_3(\omega')}{\omega' - z}\right) - \frac{R_3}{\omega_3 + z} - \frac{R_3}{\omega_3 - z}\right]^{-1}.
\end{aligned} \tag{B9}$$

This gives the crossing equation for $g_3(z)$. Interest is in this crossing relation at the CDD pole of $z = \omega_3$. At this point, the CDD pole term in the denominator of the lhs of the equation is infinite, which forces the lhs to be zero. It should be noted that, without the CDD term, since we are assuming that $\omega_3$ is at a high energy, the lhs would be approximately equal to 1 with a small correction. For the rhs, the first term will be $\frac{8}{9}$ plus a small correction and the second term $\frac{2}{9}$ plus a small correction. The third term will be zero since we included the crossed CDD pole when we added the CDD pole. Otherwise it would be $-\frac{1}{9}$ plus a small correction. Clearly then, Eq. (B9) would be approximately one as it would be with no CDD poles at all. With just the CDD pole, however, the lhs is zero while the rhs would still be approximately one. With the crossed CDD pole, the third term on the rhs is forced to be zero. Thus, the crossed CDD pole needs to be added to $g_1(z)$ and $g_2(z)$ in order for rhs = lhs in Eq. (B9). Adding the crossed CDD term to $g_1(z)$ and $g_2(z)$ to have crossing symmetry for $g_3(z)$, we find

$$\left[1-\frac{z}{\pi}\int_1^\infty d\omega'\frac{p'^3v^2(p')}{\omega'^2}\left(\frac{\lambda_3}{\omega'-z}+\frac{G_3(\omega')}{\omega'+z}\right)-\frac{R_3}{\omega_3-z}-\frac{R_3}{\omega_3+z}\right]^{-1}$$
$$=\frac{8}{9}\left[1+\frac{z}{\pi}\int_1^\infty d\omega'\frac{p'^3v^2(p')}{\omega'^2}\left(\frac{\lambda_1}{\omega'+z}+\frac{G_1(\omega')}{\omega'-z}\right)-\frac{R_3}{\omega_3-z}\right]^{-1}+\frac{2}{9}\left[1+\frac{z}{\pi}\int_1^\infty d\omega'\frac{p'^3v^2(p')}{\omega'^2}\left(\frac{\lambda_2}{\omega'+z}+\frac{G_2(\omega')}{\omega'-z}\right)\right.$$
$$\left.-\frac{S_2}{Z_2-z}-\frac{S_2}{Z_2^*-z}-\frac{R_3}{\omega_3-z}\right]^{-1}-\frac{1}{9}\left[1+\frac{z}{\pi}\int_1^\infty d\omega'\frac{p'^3v^2(p')}{\omega'^2}\left(\frac{\lambda_3}{\omega'+z}+\frac{G_3(\omega')}{\omega'-z}\right)-\frac{R_3}{\omega_3+z}-\frac{R_3}{\omega_3-z}\right]^{-1}. \tag{B10}$$

Both sides of Eq. (B10) now are zero at $z=\omega_3$.

Checking the crossing relation for $g_1(z)$, we have

$$\left[1-\frac{z}{\pi}\int_1^\infty d\omega'\frac{p'^3v^2(p')}{\omega'^2}\left(\frac{\lambda_1}{\omega'-z}+\frac{G_1(\omega')}{\omega'+z}\right)-\frac{R_3}{\omega_3+z}\right]^{-1}$$
$$=-\frac{1}{9}\left[1+\frac{z}{\pi}\int_1^\infty d\omega'\frac{p'^3v^2(p')}{\omega'^2}\left(\frac{\lambda_1}{\omega'+z}+\frac{G_1(\omega')}{\omega'-z}\right)-\frac{R_3}{\omega_3-z}\right]^{-1}+\frac{2}{9}\left[1+\frac{z}{\pi}\int_1^\infty d\omega'\frac{p'^3v^2(p')}{\omega'^2}\left(\frac{\lambda_2}{\omega'+z}+\frac{G_2(\omega')}{\omega'-z}\right)\right.$$
$$\left.-\frac{S_2}{Z_2-z}-\frac{S_2}{Z_2^*-z}-\frac{R_3}{\omega_3-z}\right]^{-1}+\frac{8}{9}\left[1+\frac{z}{\pi}\int_1^\infty d\omega'\frac{p'^3v^2(p')}{\omega'^2}\left(\frac{\lambda_3}{\omega'+z}+\frac{G_3(\omega')}{\omega'-z}\right)-\frac{R_3}{\omega_3+z}-\frac{R_3}{\omega_3-z}\right]^{-1}. \tag{B11}$$

At $z=\omega_3$, the value of the lhs of Eq. (B11) is approximately one while that of its rhs is zero. To maintain crossing, we have to add the CDD pole term to $g_1(z)$ and similarly to $g_2(z)$. The final results for the $g_\alpha(z)$ are

$$g_1(z)=1-\frac{z}{\pi}\int_1^\infty d\omega'\frac{p'^3v^2(p')}{\omega'^2}\left(\frac{\lambda_1}{\omega'-z}+\frac{G_1(\omega')}{\omega'+z}\right)-\frac{R_3}{\omega_3-z}-\frac{R_3}{\omega_3+z}, \tag{B12}$$

$$g_2(z)=1-\frac{z}{\pi}\int_1^\infty d\omega'\frac{p'^3v^2(p')}{\omega'^2}\left(\frac{\lambda_2}{\omega'-z}+\frac{G_2(\omega')}{\omega'+z}\right)-\frac{R_3}{\omega_3-z}-\frac{R_3}{\omega_3+z},-\frac{S_2}{Z_2+z}-\frac{S_2}{Z_2^*+z} \tag{B13}$$

and

$$g_3(z)=1-\frac{z}{\pi}\int_1^\infty d\omega'\frac{p'^3v^2(p')}{\omega'^2}\left(\frac{\lambda_3}{\omega'-z}+\frac{G_3(\omega')}{\omega'+z}\right)-\frac{R_3}{\omega_3-z}-\frac{R_3}{\omega_3+z}. \tag{B14}$$

This will also result in a pair of zeros on the real axis in $H_T(z)$.

We started by adding a CDD pole to the $\alpha=3$ channel assuming that it could be considered a generalized $R$ function. However, because of crossing, the zero must be added to all three channels in pairs. Thus, we had to add CDD poles to the two channels that clearly are not generalized $R$ functions. This result is completely general and does not depend on the strength of the coupling. Also, looking at the crossing matrix, Eq. (B4), all the rows sum to one. One would not expect to produce a zero by adding the terms together as in the rhs of Eq. (B10). Indeed, if one includes the CDD pole in the $P_{33}$ channel, then the real parts of the three terms generated by the crossing matrix have the same sign and cannot cancel. Although the analysis was made for the $P_{13}=P_{31}$ model at high energy with $\hat{\eta}=1$, the results should be considered completely general.

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