

Fractional vortex molecules and vortex polygons in a baby Skyrme modelMichikazu Kobayashi^{1,2} and Muneto Nitta³¹*Department of Basic Science, University of Tokyo, Komaba 3-8-1, Meguro-ku, Tokyo 153-8902, Japan*²*Department of Physics, Kyoto University, Oiwake-cho, Kitashirakawa, Sakyo-ku, Kyoto 606-8502, Japan*³*Department of Physics, and Research and Education Center for Natural Sciences,**Keio University, Hiyoshi 4-1-1, Yokohama, Kanagawa 223-8521, Japan*

(Received 18 April 2013; published 7 June 2013)

We construct a molecule of fractional vortices with fractional topological lump charges as a baby Skyrmion with the unit topological lump charge in the antiferromagnetic (or XY) baby Skyrme model, that is, an $O(3)$ sigma model with a four-derivative term and an antiferromagnetic or XY-type potential term quadratic in fields. We further construct configurations with topological lump charges $Q \leq 7$ and find that bound states of vortex molecules constitute regular polygons with $2Q$ vertices as vortices, where the rotational symmetry $SO(2)$ in real space is spontaneously broken into a discrete subgroup \mathbf{Z}_Q . We also find metastable and arrayed bound states of fractional vortices for $Q = 5, 6$. On the other hand, we find for $Q = 7$ that the regular polygon is metastable and the arrayed bound state is stable. We calculate binding energies of all configurations.

DOI: [10.1103/PhysRevD.87.125013](https://doi.org/10.1103/PhysRevD.87.125013)

PACS numbers: 11.27.+d, 14.80.Hv, 12.39.Dc

I. INTRODUCTION

Vortices are topological solitons present in various physical systems, from field theory [1] and cosmological models [2] to condensed matter systems [3]. In particular, they play essential roles in condensed matter systems such as superconductors, superfluids, magnetism, quantum Hall states, nematic liquids, optics, and so on. One of the exotic aspects common in condensed matter systems but not familiar in high energy physics and cosmology are vortex molecules, which have been studied in multicomponent Bose-Einstein condensates (BECs) [4–10], multigap superconductors [11–13], superfluid ³He (as a double core vortex) [3], and nonlinear optics [14]. In the cases of BECs [4] and superconductors [15], fractional vortices in two different components with fractional topological charges constitute a meson-like bound state with the unit topological charge in total. However, a crucial difference between these two systems is that a repulsion between vortices is exponentially suppressed in superconductors due to the Higgs mechanism in the presence of a gauge field, while a repulsion between vortices is polynomially reduced in BECs [16] in the absence of a gauge field. Consequently, vortex molecules are stable and visible in BECs because of a balance between the vortex repulsion and the domain wall tension, while in superconductors they can be seen, in principle, only at temperatures above a certain critical temperature by a mechanism similar to the Berezinskii-Kosterlitz-Thouless transition [12,13]. However, stable vortex molecules in BECs are global vortices winding around a global $U(1)$ symmetry, and consequently their energies are logarithmically divergent with respect to the system size; they are infinitely heavy in infinite space, and thereby they are not very realistic in high energy physics or cosmology.

In this paper, we propose a field theoretical model admitting a vortex molecule with *finite* energy, motivated by these condensed matter systems. We consider an $O(3)$ nonlinear

sigma model on the target space S^2 in $d = 2 + 1$ dimensions, described by a unit three-vector of scalar fields $\mathbf{n}(x) = (n_1(x), n_2(x), n_3(x))$ with the constraint $\mathbf{n}^2 = 1$, which is equivalent to a CP^1 model. The $O(3)$ model admits lumps or sigma model instantons characterized by $\pi_2(S^2) \simeq \mathbf{Z}$ [17] as a relative of vortices. We consider a potential term motivated by condensed matter systems admitting vortex molecules. The potential terms make lumps unstable against shrinkage, in general, as can be inferred from the scaling argument [18], so we also consider a four-derivative (Skyrme) term, by which the lumps are stabilized to become baby Skyrmons [19]. In the context of baby Skyrmons, the potential terms of the type $V = m^2(1 - n_3)$ [19] or of the type $V = m^2(1 - n_3^2)$ [20–23] have already been studied. The latter admits two discrete vacua $n_3 = \pm 1$ and a domain wall interpolating between them [21,22,24,25], and a baby Skyrmon is in the shape of a twisted closed domain wall [20,26]. In our previous papers [26–28], we considered both types of potential terms $V = m^2(1 - n_3^2) + \beta^2 n_1$ in the regime $\beta \ll m$. In this case, a straight domain wall can absorb lumps as sine-Gordon kinks [26–28], and a baby Skyrmon is in the form of a domain wall ring with a sine-Gordon kink [26]. In condensed matter physics, the quadratic potential admitting two vacua is known as the Ising type in ferromagnets, so we may refer to this model as the Ising (or ferromagnetic) baby Skyrme model.

Here, we consider a potential of the XY type or of antiferromagnets, $V = m^2 n_3^2$ [29]. The model can be referred to as the XY (or antiferromagnetic) baby Skyrme model. The vacua characterized by $n_3 = 0$ are S^1 at the equator of the target space S^2 , as in Fig. 1(a). One lump solution is schematically drawn in Fig. 1(b), where we chose $\mathbf{n} = (1, 0, 0)$ as the vacuum at the boundary. One can find two separated half-lumps (merons) whose centers are mapped to the north and south poles $\mathbf{n} = (0, 0, \pm 1)$ of the target space S^2 . These half-lumps are separated because the

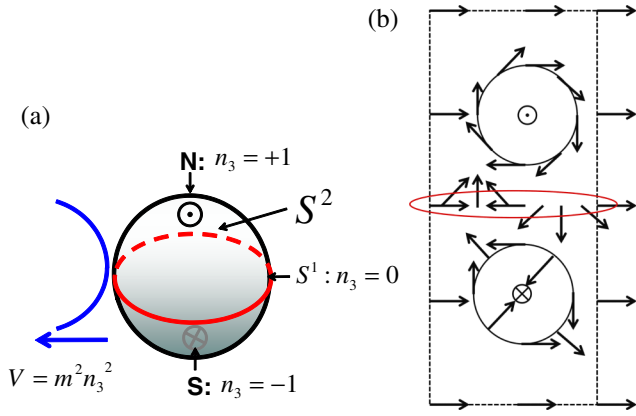


FIG. 1 (color online). (a) The target space and the potential. (b) A vortex molecule as one lump. Here, \leftarrow , \downarrow , \rightarrow and \uparrow represent $\mathbf{n} = (-1, 0, 0)$, $(0, -1, 0)$, $(1, 0, 0)$ and $(0, 1, 0)$, respectively. The north and south poles $\mathbf{n} = (0, 0, 1)$ and $(0, 0, -1)$ are denoted by \odot and \otimes , respectively.

vacua S^1 appear between them, as indicated by an ellipse in Fig. 1(b). Therefore, a lump is nonaxisymmetric, unlike those for the massless case and the Ising-type case with $\beta = 0$ in which lumps are axisymmetric. There, the vacuum $U(1)$ winds (counter)clockwise between the two half-lumps so that these half-lumps are (anti)global vortices. While isolated (anti)global vortices have logarithmically divergent energy in the infinite system size, a pair of global and antiglobal vortices have finite energy. They attract each other and collapse in the absence of the Skyrme term, while the Skyrme term forms a vortex molecule. We numerically construct fractional vortex molecules with $1 \leq Q \leq 7$ by a relaxation method. The configuration of $Q = 1$, in fact, looks as we expected. For $Q = 2$, we find that two molecules face each other with opposite orientations to constitute a regular square. Since two kinds of constituent vortices are placed at the vertices, the configuration is \mathbf{Z}_2 symmetric. For $Q = 3$, we find a regular hexagonal structure of vortex molecules, like a benzene. This structure is \mathbf{Z}_3 symmetric. The configurations with $Q = 2, 3$ resemble those recently found in a two-component BEC under rotation [7]. Furthermore, for $Q = 4, 5, 6$, we find regular octagonal, decagonal, and dodecagonal structures of fractional vortices with $\mathbf{Z}_4, \mathbf{Z}_5, \mathbf{Z}_6$ symmetries. Therefore, in general, we expect, for the topological charge Q , a polygon with $2Q$ fractional vortices at vertices with \mathbf{Z}_Q symmetry [29]. We also find metastable and arrayed bound states of fractional vortices for $Q = 5, 6$. These configurations are obtained by squeezing the corresponding stable polygons, and they have slightly higher energies. We also find that the regular polygon is metastable and the arrayed bound state is stable for $Q = 7$. Finally, we calculate the binding energies of all the configurations.

Conventional lumps in the massless $O(3)$ sigma model are invariant under a combination of the $U(1)$ rotation of the target space and the space rotation. The lumps spontaneously

break the other linear combination of the two $U(1)$ symmetries. On the other hand, our solutions spontaneously break the purely space rotation. Note that nonaxisymmetric configurations were known before for higher topological charges in the baby Skyrme model [30], while even one baby Skyrmion is nonaxisymmetric in our model. In this regard, our solutions are similar to those in Ref. [31], in which nonaxisymmetric molecular configurations were found in the model with a more complicated potential. We would like to emphasize that our potential is quite common in antiferromagnets and two-component BECs and that it is natural.

A closely related model is a $U(1)$ gauged supersymmetric CP^1 model where a rotation along the n_3 axis is gauged [32]. The gauge symmetry induces the potential $V = m^2 n_3^2$ known as the D-term potential, where supersymmetry requires m to coincide with the gauge coupling e . In this model, a lump is decomposed into two fractional gauged Bogomol'nyi-Prasad-Sommerfield (BPS) vortices, and these two vortices can be placed with arbitrary separation because no force is present between BPS vortices [33,34]. Each constituent vortex carries half of the lump charge characterized by $\pi_2(S^2)$, as in our model. A set of fractional vortices carrying the total unit charge was also found in supersymmetric gauge theories and sigma models [35].

This paper is organized as follows. After our model is explained in Sec. II, we give numerical solutions of vortex molecules in Sec. III. Section IV is devoted to a summary and discussion.

II. THE MODEL

We consider an $O(3)$ sigma model in $d = 2 + 1$ dimensions described by a three-vector of scalar fields $\mathbf{n}(x) = (n_1(x), n_2(x), n_3(x))$ with a constraint $\mathbf{n} \cdot \mathbf{n} = 1$. The Lagrangian of our model is given by

$$\mathcal{L} = \frac{1}{2} \partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n} - \mathcal{L}_4(\mathbf{n}) - V(\mathbf{n}), \quad (1)$$

with $\mu = 0, 1, 2$. Here, the four-derivative (baby Skyrme) term is expressed as

$$\begin{aligned} \mathcal{L}_4(\mathbf{n}) &= \kappa F_{\mu\nu}^2 = \kappa [\mathbf{n} \cdot (\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n})]^2 \\ &= \kappa (\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n})^2, \\ F_{\mu\nu} &\equiv \mathbf{n} \cdot (\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n}). \end{aligned} \quad (2)$$

In this paper, we take the potential term to be [29]

$$V(\mathbf{n}) = m^2 n_3^2. \quad (3)$$

This potential is known in antiferromagnets and the XY model, while the potential in the form of ferromagnets $m^2(1 - n_3^2)$ was studied before in Refs. [20–23]. The energy density of static configurations is

$$\mathcal{E} \equiv \frac{1}{2} (\partial_a \mathbf{n} \cdot \partial^a \mathbf{n}) + \mathcal{L}_4(\mathbf{n}) + V(\mathbf{n}) \quad (a = 1, 2). \quad (4)$$

Introducing the projective coordinate $u \in \mathbf{C}$ of CP^1 by

$$n_i = \phi^\dagger \sigma_i \phi, \quad \phi^T = (1, u) / \sqrt{1 + |u|^2}, \quad (5)$$

the Lagrangian (1) can be rewritten in the form of the CP^1 model with potential terms given by

$$\mathcal{L} = 2 \frac{\partial_\mu u^* \partial^\mu u}{(1 + |u|^2)^2} - 8\kappa \frac{(\partial_\mu u^* \partial^\mu u)^2 - |\partial_\mu u \partial^\mu u|^2}{(1 + |u|^2)^4} - V, \quad (6)$$

$$V = m^2 D_3^2, \quad D_3 \equiv \frac{1 - |u|^2}{1 + |u|^2} = n_3, \quad (7)$$

Here, $g_{uu^*} = 1/(1 + |u|^2)^2$ is the Kähler (Fubini-Study) metric of CP^1 , $g^{uu^*} = (1 + |u|^2)^2$ is its inverse, and $D_i = n_i$ are the moment maps (or the Killing potentials) of the $SU(2)$ isometry generated by σ_i . If we gauge the isometry generated by the generator σ_3 with gauge coupling e , the potential $V = e^2 D_3^2$ (with $m = e$) is known as the D-term potential in the supersymmetric $U(1)$ gauged CP^1 model [32].

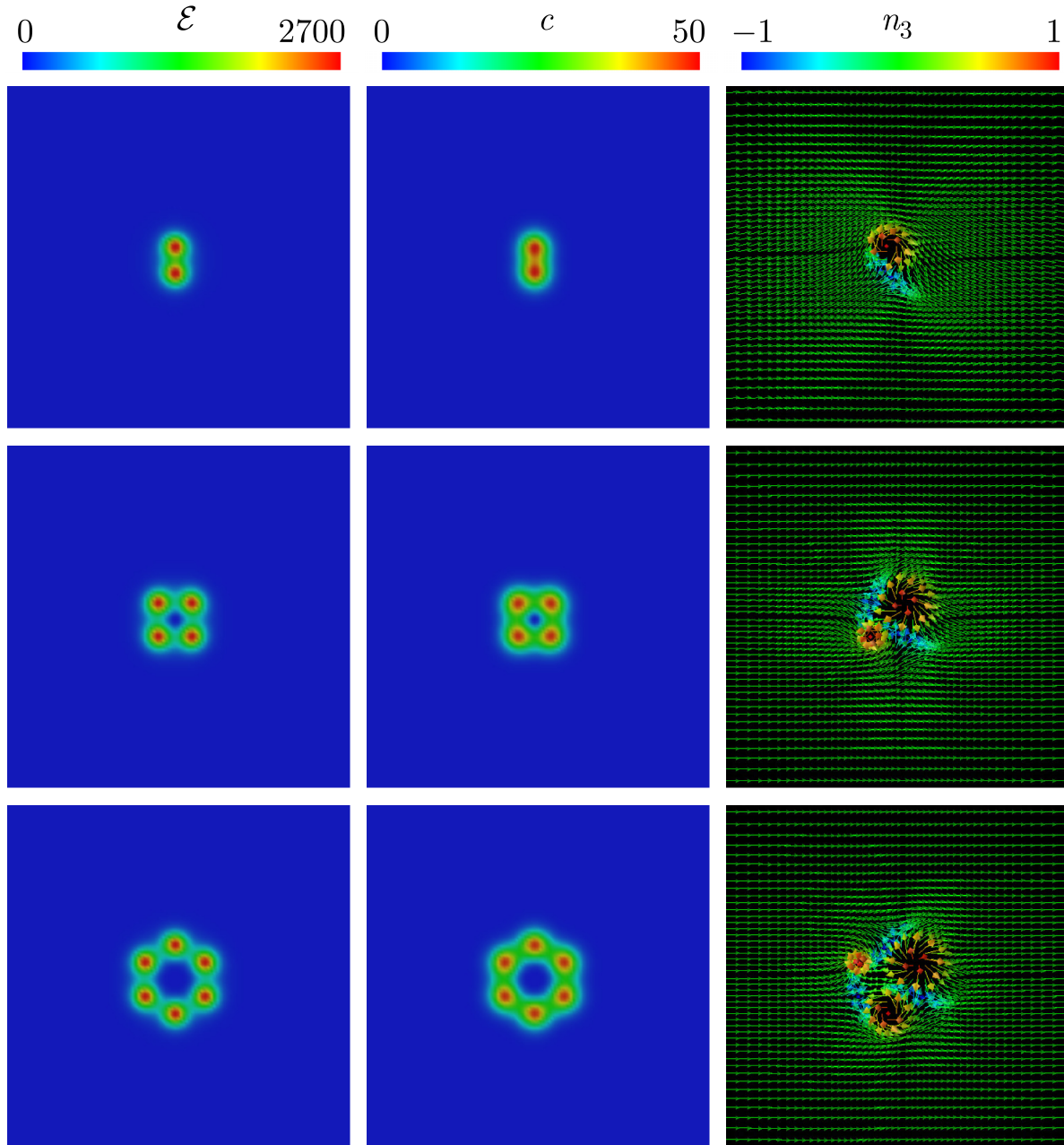


FIG. 2 (color online). Bound states of fractional vortex molecules in the region $-1.16 < x_a < 1.16$. The topological charges are $Q = 1, 2, 3$ from top to bottom. The left and middle panels represent the total energy densities $\mathcal{E}(x)$ and the topological charge densities $c(x) \equiv F_{12}/4\pi$, respectively. In the right panels, the arrows denote the scalar fields of the three-vector $\mathbf{n}(x)$, where color represents the value of n_3 from which one can find whether the constituent fractional vortices are of N or S.

III. VORTEX MOLECULES

The topological charge of the lump $\pi_2(S^2) \simeq \mathbf{Z}$ is given by

$$\begin{aligned}
 Q &= \frac{1}{4\pi} \int d^2x F_{12} \\
 &= \frac{1}{4\pi} \int d^2x \mathbf{n} \cdot (\partial_1 \mathbf{n} \times \partial_2 \mathbf{n}) \\
 &= \frac{1}{4\pi} \int d^2x \epsilon_{ijk} n_i \partial_1 n_j \partial_2 n_k \\
 &= \frac{1}{2\pi} \int d^2x \frac{i(\partial_1 u^* \partial_2 u - \partial_2 u^* \partial_1 u)}{(1 + |u|^2)^2}. \tag{8}
 \end{aligned}$$

In the presence of the potential, a lump is unstable to shrinking from the Derrick’s scaling argument [18]. It can be stabilized in the presence of the baby Skyrme term, resulting in a baby Skyrmion.

We construct numerical solutions of fractional vortex molecules with the topological lump charge $1 \leq Q \leq 7$. As the numerical parameters, we fix $\kappa = 0.002$ and $m^2 = 800$. We obtain the stationary state using the relaxation method: introducing the parameter τ and the τ dependence of $n_i(\tau)$, and finding the asymptotic solution $n_i(\tau \rightarrow \infty)$ of the equation

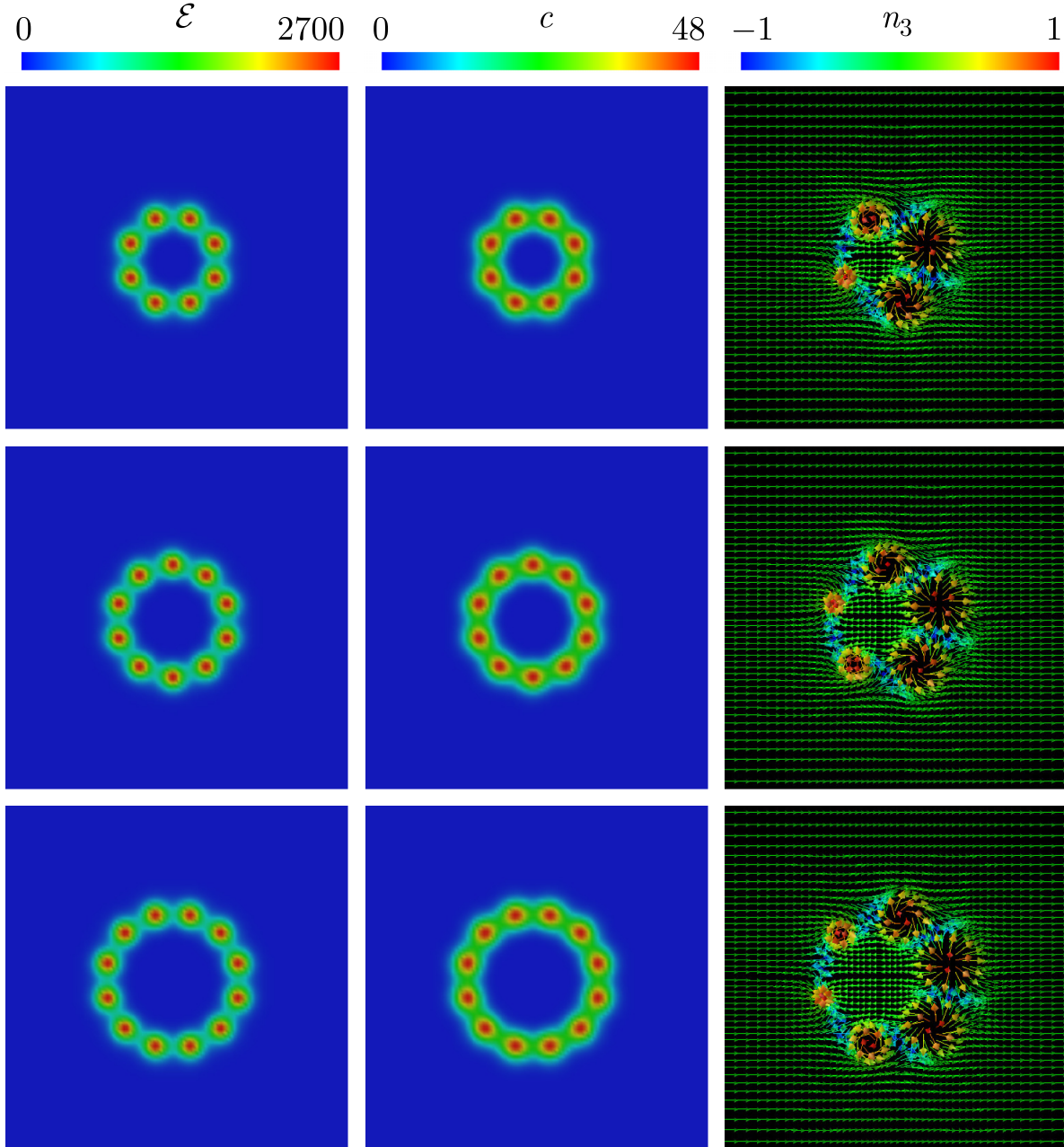


FIG. 3 (color online). Bound states of fractional vortex molecules in the region $-1.16 < x_a < 1.16$. The topological charges are $Q = 4, 5, 6$ from top to bottom. Details of the plots are the same as those shown in Fig. 2.

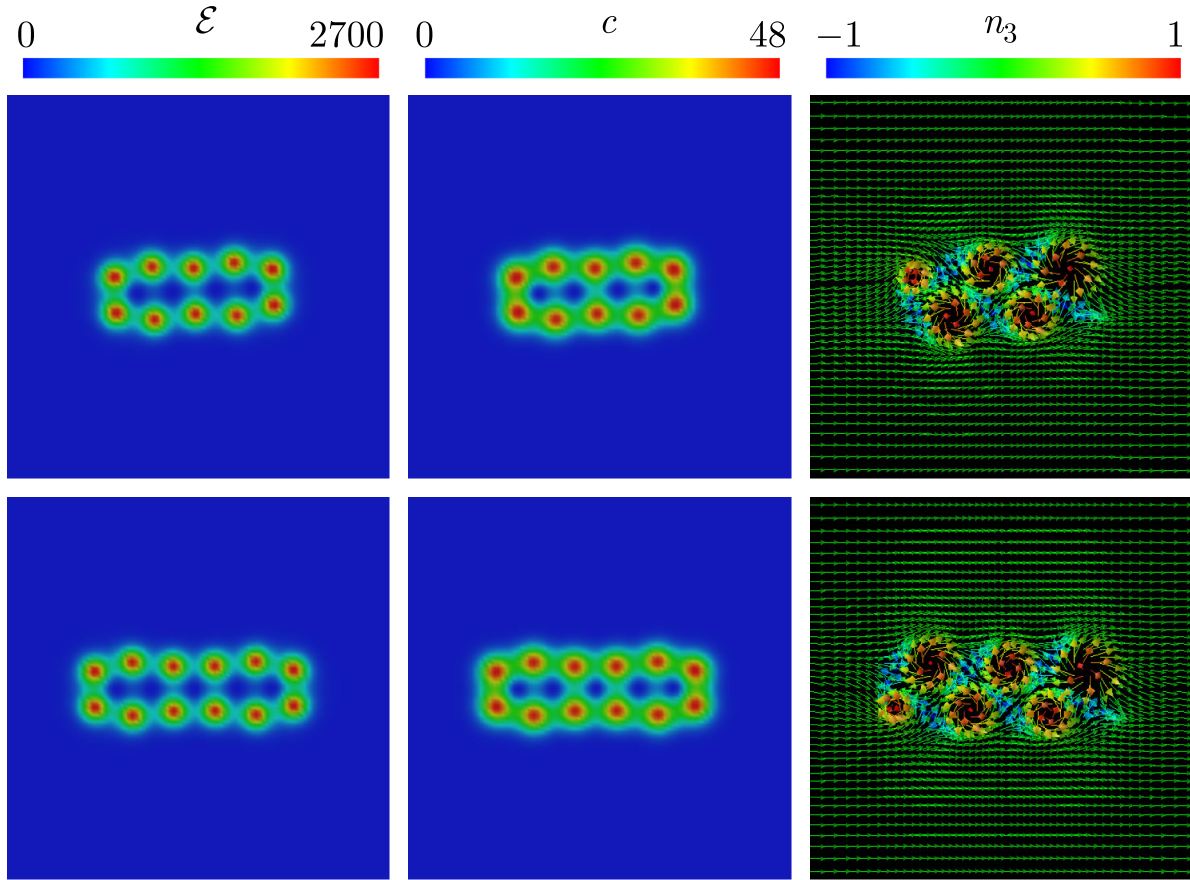


FIG. 4 (color online). Metastable and arrayed bound states of fractional vortex molecules in the region $-1.16 < x_a < 1.16$. The topological charges are $Q = 5, 6$ for the top and bottom. Details of the plots are the same as those shown in Fig. 2.

$$\frac{\partial n_i}{\partial \tau} = -\frac{\delta \mathcal{E}}{\delta n_i}, \quad (9)$$

under the constraint $n_1^2 + n_2^2 + n_3^2 = 1$. The detailed numerical procedure is shown in the Appendix. As the initial state $n_i(\tau = 0)$, we give the ansatz for $\kappa = m = 0$:

$$\begin{aligned} n_1 &= \cos f(r), & n_2 &= -\cos(Q\theta) \sin f(r), \\ n_3 &= \sin(Q\theta) \sin f(r), \end{aligned} \quad (10)$$

with the monotonically decreasing function satisfying

$$f(r \rightarrow \infty) \rightarrow 0, \quad f(r \rightarrow 0) \rightarrow \pi. \quad (11)$$

The topological charge Q given by Eq. (8) is invariant for arbitrary τ .

Our stable solutions are given in Fig. 2 for $Q = 1, 2, 3$ and Fig. 3 for $Q = 4, 5, 6$. For the unit topological charge, $Q = 1$, one can find a fractional vortex molecule, as we expected; two fractional vortices oppositely wind around the equator S^1 of the target space S^2 , and their cores are filled by the north $n_3 = +1$ and south $n_3 = -1$ poles of S^2 , as can be seen in the plot of n_3 in Fig. 2. The vacua S^1 appear between the two half-lumps separating them, as can be seen from both the plots of the energy density and topological charge density. Although all of them are

(anti)global vortices having logarithmically divergent energy, a pair of them has finite energy. They attract each other, but the Skyrme term prevents the collapse.

For $Q = 2$, two vortex molecules face each other with opposite orientations. Since molecules attract each other with these orientations, they make a bound state to constitute a square. Since the same vortices N or S are placed at each pair of diagonal corners, the configuration is \mathbf{Z}_2 axisymmetric.

For $Q = 3$, three vortex molecules with six fractional vortices constitute a hexagon with a \mathbf{Z}_3 axisymmetry. These structures of $Q = 2, 3$ resemble those in a vortex lattice recently found in a two-component BEC under rotation [7].

For $Q = 4, 5, 6$, the situation is almost the same; i.e., one finds that four, five, and six vortex molecules with eight, ten, and eleven fractional vortices constitute an octagon, decagon, and dodecagon with $\mathbf{Z}_4, \mathbf{Z}_5, \mathbf{Z}_6$ axisymmetries, respectively.

In general, for the topological charge Q , we expect $2Q$ fractional vortices to be placed on a circle in a \mathbf{Z}_Q axisymmetric way. In all the cases, one can see that the topological lump charge density is distributed around the fractional vortices, and each of them carries a half-lump charge. The rotational symmetry $SO(2)$ in the x^1 - x^2

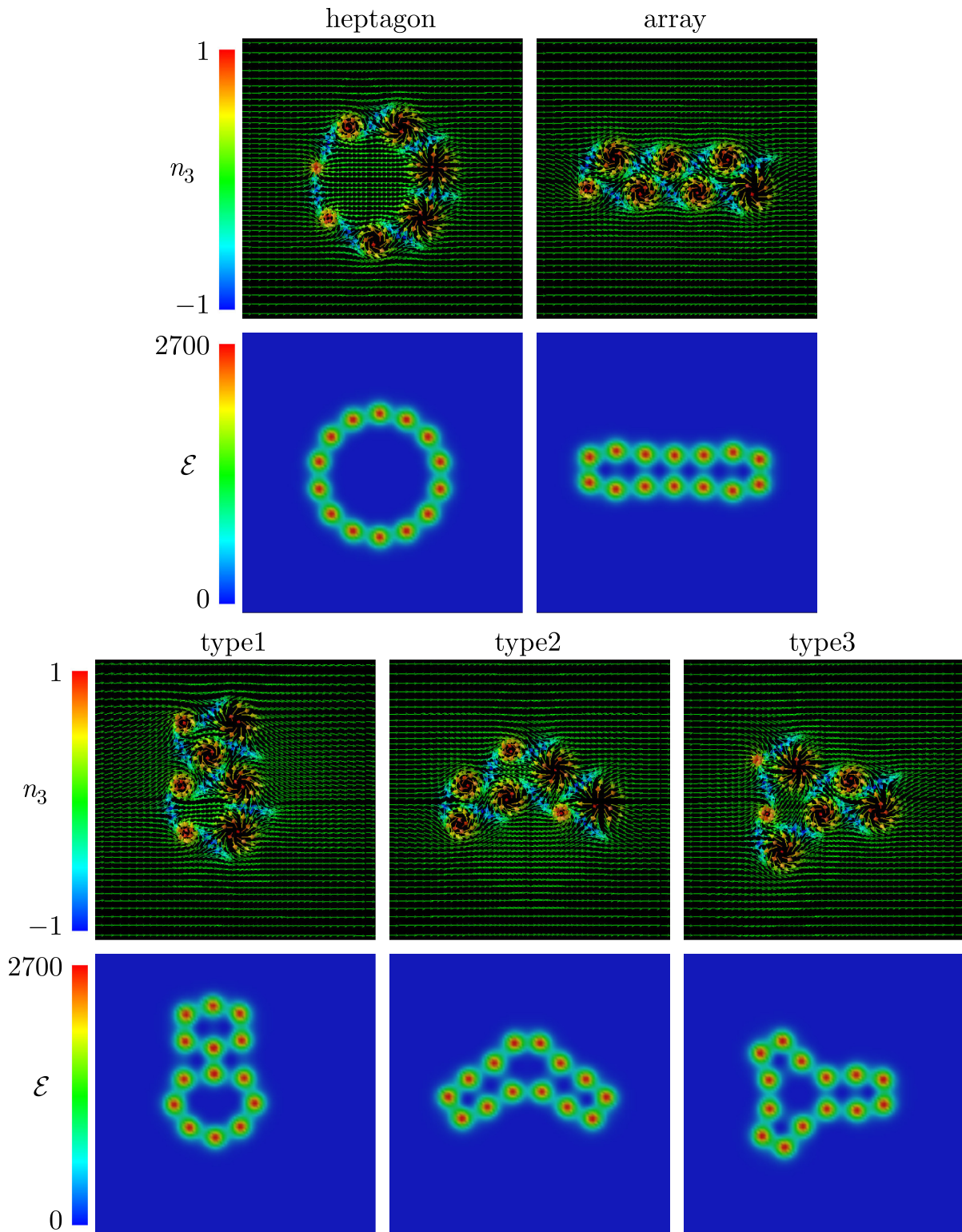


FIG. 5 (color online). Stable (arrayed) and metastable (heptagon and the other 3 types of) bound states of fractional vortex molecules for the topological charge $Q = 7$. The top and bottom panels for each configuration represent the three-vector $\mathbf{n}(x)$ and the total energy density $\mathcal{E}(x)$, respectively, in the region $-1.16 < x_a < 1.16$.

plane is spontaneously broken in all cases to a discrete subgroup \mathbf{Z}_Q .

To investigate the stability of vortex polygons, we also choose a randomly placed lump solution

$$u = \sum_{i=1}^Q \frac{e^{i\varphi_i}}{x_1 - x_{0i} + ix_2 - y_{0i}}, \quad (12)$$

as initial states $n_i(\tau = 0)$ for the relaxation. Here, the projective coordinate u is defined in Eq. (5), and x_{0i} , y_{0i} , and φ_i are the real random numbers. The initial state (12) indicates that Q single vortex molecules are placed at $(x_1, x_2) = (x_{0i}, y_{0i})$ with the angle φ_i , having the topological charge Q .

For $Q = 1, 2, 3, 4$, any initial state of Eq. (12) relaxes to vortex polygons as shown in Figs. 2 and 3, while, for $Q = 5, 6$, several initial states relax to metastable states that are different from the vortex decagon and dodecagon, i.e., vortex array states as in Fig. 4, the energy of which is larger than vortex polygon states.

We also find five (meta)stable bound states for $Q = 7$ as in Fig. 5. Unlike the cases with $Q < 7$, the arrayed bound state is at the absolute minimum. The regular polygon and the other three bound states are metastable at local minima.

Finally, we show in Table I the gradient energy E_{grad} , the Skyrme energy E_{sk} , the potential energy E_{pot} , the total energy E , and the binding energy E_{bind} between vortex molecules, defined by

$$\begin{aligned} E_{\text{grad}} &= \frac{1}{2} \int d^2x \partial_a \mathbf{n} \cdot \partial^a \mathbf{n}, & E_{\text{sk}} &= \int d^2x \mathcal{L}_4(\mathbf{n}), \\ E_{\text{pot}} &= \int d^2x V(\mathbf{n}), & E &= E_{\text{grad}} + E_{\text{sk}} + E_{\text{pot}}, \\ E_{\text{bind}} &= QE(Q = 1) - E(Q), \end{aligned} \quad (13)$$

TABLE I. Gradient energy E_{grad} , Skyrme energy E_{sk} , potential energy E_{pot} , total energy E , and binding energy E_{bind} for vortex molecules.

Q	E_{grad}	E_{sk}	E_{pot}	E	E_{bind}
1	29.04	8.840	8.829	46.71	
2	53.35	16.96	16.96	87.26	6.153
3	78.45	25.17	25.17	128.8	11.33
4	103.8	33.44	33.44	170.7	16.14
5 (decagon)	129.3	41.73	41.73	212.7	20.79
5 (array)	129.8	41.68	41.68	213.1	20.42
6 (dodecagon)	154.8	50.03	50.03	254.9	25.35
6 (array)	155.2	49.90	49.90	255.0	25.22
7 (heptagon)	180.4	58.34	58.34	297.1	29.86
7 (array)	180.7	58.14	58.13	297.0	29.97
7 (type 1)	181.3	58.29	58.28	297.9	29.09
7 (type 2)	181.2	58.22	58.22	297.6	29.31
7 (type 3)	181.6	58.32	58.32	298.3	28.68

respectively. In our choice of numerical parameters, the gradient energy is dominant in the total energy. Total energy deviates from the linear relation, i.e., $E(Q) < QE(Q = 1)$, and the difference E_{bind} between $E(Q)$ and $QE(Q = 1)$ corresponds to the binding energy between molecules, which dominates in about 10% of the total energy.

IV. SUMMARY AND DISCUSSION

We have constructed stable and metastable configurations of the fractional vortex molecules as lumps (baby Skyrmions) in the XY (or antiferromagnetic) baby Skyrme model, which has the antiferromagnetic (XY) potential $V = m^2 n_3^2$. We have found that, for the unit charge $Q = 1$, two fractional vortices whose centers are filled by the north and south poles of the target space are placed within a certain distance. We have found that, for $Q = 2, 3, 4, 5$ and 6 , bound states of two, three, four, five, and six vortex molecules constitute quadrangle, hexagonal, octagonal, decagonal, and dodecagonal vortex configurations, respectively. At least up to this topological number, \mathbf{Z}_Q symmetric vortex molecule configurations appear for the topological charge Q . Our configurations are all non-axisymmetric, and they spontaneously break the rotational symmetry in the x - y plane. While all vortex polygons are stable and at global minima, we have also found metastable and arrayed bound states of fractional vortices for $Q = 5, 6$, which are obtained by squeezing the corresponding stable polygons and have slightly higher energies. We also find for $Q = 7$ that the arrayed bound state is at the absolute minimum and the regular polygon together with the other three bound states is metastable at a local minimum, unlike the cases with $Q < 7$. Finally, we have calculated the binding energies of all the configurations.

As denoted in the Introduction, similar configurations of vortex molecules are present in condensed matter systems, such as two-component BECs, described by two condensations (scalar fields) $\phi_1(x)$ and $\phi_2(x)$ with the internal coherent (Josephson) coupling $\beta^2 \phi_1^* \phi_2 + \text{c.c.}$ [4–7,9], where a four-derivative term is not present. In these cases, the vortex molecules are global vortices; that is, they wind around a global $U(1)$ symmetry. If we gauge the common phase of the two components, they become semilocal vortices. If we send the scalar coupling and gauge coupling to infinity, keeping the internal coherent coupling, we obtain our model, except for the Skyrme term, where the Josephson coupling $\beta^2 \phi_1^* \phi_2 + \text{c.c.}$ reduces to $\beta^2 n_1$, which we did not consider in this paper. We expect that semilocal vortices, keeping the couplings, can make a stable vortex molecule if one properly adds a four-derivative term.

In the CP^1 model with a ferromagnetic potential, a Q -lump solution is known [36], in which a $U(1)$ Nambu-Goldstone mode associated with the spontaneously broken $U(1)$ internal symmetry of the rotation in the n_1 - n_2 plane in the target space is rotating in time. In our case, there is a Nambu-Goldstone mode associated with the

spontaneously broken rotational symmetry in real space, instead of an internal $U(1)$ Nambu-Goldstone mode. Consequently, we may have a Q lump in our case as a spinning molecule.

If we promote our configuration linearly in $d = 3 + 1$ dimensions, it becomes a bound state of two cosmic strings. As usual, the solution breaks translational symmetries in two transverse directions, resulting in two translational zero modes which propagate along the string. One added feature of our solutions is the existence of a Nambu-Goldstone mode associated with the spontaneously broken rotational symmetry along the string, resulting in a twisting wave propagating along the string. This is known as a “twiston” in two-gap superconductors [37]. In cosmology, our solutions can be regarded as some exotic cosmic strings with an internal structure. For instance, it is an interesting question whether or not two such strings reconnect to each other when they collide.

We have found that, for the topological charge Q , two kinds of vortices are placed at $2Q$ vertices of a regular polygon, on which a \mathbf{Z}_Q symmetry acts. In this regard, \mathbf{Z}_n symmetric vortex configurations were studied in the Abelian-Higgs model [38]. This is equivalent to the vortices on the orbifold \mathbf{C}/\mathbf{Z}_n studied recently [39]. Vortex polygons have also been studied in hydrodynamics for a long time [40]. Vortex polygons with less than seven vortices as vertices are shown to be stable, while those with more than seven are unstable. One example realized in nature is a vortex hexagon found by Cassini in the north poles of Saturn [41]. In our case too, we have found vortex polygons up to $Q = 6$ to be stable, which may be interesting compared with hydrodynamics.

In this paper, we have found a two vortex molecule, that is, a vortex dimer with unit topological charge in the CP^1 model with the antiferromagnetic potential. Stable three or N vortex molecules, that is, vortex trimers or N -omers, are present in three or N component BECs [9,10]. Since two components ϕ_1, ϕ_2 with a constraint $|\phi_1|^2 + |\phi_2|^2 = 1$ divided by $U(1)$ imply the CP^1 model, the same procedure for N components yields a CP^{N-1} model with a certain potential term. While a CP^{N-1} generalization of the antiferromagnetic-type potential was considered before [42] and was found to admit parallel multiple domain walls [43] or domain wall junctions or networks [44,45], depending on parameters in the potential, a CP^{N-1} generalization of the ferromagnetic (XY-type) potential has not been studied thus far in the CP^{N-1} model. Skyrme-like terms in the CP^{N-1} model were studied before without [46,47] and with [48–50] supersymmetry. A CP^2 or CP^{N-1} generalization of the baby Skyrme model with an antiferromagnetic potential will admit vortex trimers or N -omers, respectively.

ACKNOWLEDGMENTS

M.N. thanks Minoru Eto for collaborations which motivated this work. This work is supported in part by

Grant-in-Aid for Scientific Research [Grants No. 22740219 (M.K.) and No. 23740198 and No. 25400268 (M.N.)], and the work of M.N. is also supported in part by the “Topological Quantum Phenomena” Grant-in-Aid for Scientific Research on Innovative Areas (No. 23103515 and No. 25103720) from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan.

APPENDIX: DETAILED NUMERICAL PROCEDURE

Equation (9) can be solved as the steepest descent method:

$$n_i(\tau + \Delta\tau) = n_i(\tau) - \Delta\tau \frac{\delta\mathcal{E}_\lambda(\tau)}{\delta n_i(\tau)}, \quad (\text{A1})$$

where we omit the spatial dependence of n_i . $\mathcal{E}_\lambda(\tau)$ is defined as

$$\mathcal{E}_\lambda(\tau) = \mathcal{E}(\tau) - \lambda(\tau)\{n_1(\tau)^2 + n_2(\tau)^2 + n_3(\tau)^2\}. \quad (\text{A2})$$

Here, the Lagrange multiplier $\lambda(\tau)$ is fixed to satisfy $n_1(\tau + \Delta\tau)^2 + n_2(\tau + \Delta\tau)^2 + n_3(\tau + \Delta\tau)^2 = 1$.

For the space, to approximately consider the infinite space, we use the following scaling transformation:

$$x_a = L \tanh^{-1} X_a \quad (\text{A3})$$

for $-1 < X_a < 1$, and consider the dependence of n_i on X_a instead of x_a , where L is the scaling parameter. We use the square with the $(N + 1)^2$ grid points. On the l th grid point in the x_a direction, the value of $n_i(\{(l)_a\})$ is defined as

$$n_i(\{(l)_a\}) \equiv n_i(\{(\cos(l\pi/N))_a\}), \quad (\text{A4})$$

where $n_i(\{X_a\})$ is the value of n_i at $\{X_a\} \equiv (X_1, X_2)$. We omit the τ dependence on n_i here. For $l = 0$ or N , which corresponds to infinity, the value of $n_i(\{(l)_a\})$ is fixed to the ground state:

$$\begin{aligned} n_1(\{(0)_a\}) &= n_1(\{(N)_a\}) = 1, \\ n_2(\{(0)_a\}) &= n_2(\{(N)_a\}) = 0, \\ n_3(\{(0)_a\}) &= n_3(\{(N)_a\}) = 0. \end{aligned} \quad (\text{A5})$$

To calculate the spatial derivative of n_i , we use the spectral collocation method. We expand n_i in the Chebyshev polynomials:

$$\begin{aligned} \tilde{n}_i(\{(j)_a\}) &= \frac{1}{N} n_i(\{(0)_a\}) + \frac{2}{N} \sum_{l=1}^{N-1} n_i(\{(l)_a\}) \cos\left(\frac{l j \pi}{N}\right) \\ &\quad + \frac{(-1)^j}{N} n_i(\{(N)_a\}), \end{aligned} \quad (\text{A6})$$

where $\tilde{n}_i(\{(j)_a\})$ ($0 \leq j \leq N$) is the j th coefficient of the Chebyshev expansion in the x_a direction. The first and second spatial derivatives $\partial_a n_i$ and $\partial_a^2 n_i$ can be calculated as

$$\begin{aligned}
 \partial_a n_i &= \frac{1 - X_a^2}{L} \frac{\partial n_i}{\partial X_a}, \\
 \partial_a^2 n_i &= \frac{(1 - X_a^2)^2}{L^2} \frac{\partial^2 n_i}{\partial X_a^2} - \frac{2X_a(1 - X_a^2)}{L^2} \frac{\partial n_i}{\partial X_a}, \\
 \frac{\partial n_i(\{(l)_a\})}{\partial X_a} &= \frac{1}{N} \tilde{m}_i(\{(0)_a\}) + \frac{2}{N} \sum_{j=1}^{N-1} \tilde{m}_i(\{(j)_a\}) \cos\left(\frac{l j \pi}{N}\right) + \frac{(-1)^l}{N} \tilde{m}_i(\{(N)_a\}), \\
 \frac{\partial^2 n_i(\{(l)_a\})}{\partial X_a^2} &= \frac{1}{N} \tilde{s}_i(\{(0)_a\}) + \frac{2}{N} \sum_{j=1}^{N-1} \tilde{s}_i(\{(j)_a\}) \cos\left(\frac{l j \pi}{N}\right) + \frac{(-1)^l}{N} \tilde{s}_i(\{(N)_a\}).
 \end{aligned} \tag{A7}$$

Coefficients $\tilde{m}_i(\{(j)_a\})$ and $\tilde{s}_i(\{(j)_a\})$ satisfy the following recurrence relations:

$$\begin{aligned}
 \tilde{m}_i(\{(j)_a\}) &= \tilde{m}_i(\{(j+2)_a\}) + 2(j+1)\tilde{n}_i(\{(j+1)_a\}), \\
 \tilde{m}_i(\{(N)_a\}) &= 0, \\
 \tilde{s}_i(\{(j)_a\}) &= \tilde{s}_i(\{(j+2)_a\}) + 2(j+1)\tilde{m}_i(\{(j+1)_a\}), \\
 \tilde{s}_i(\{(N)_a\}) &= 0.
 \end{aligned} \tag{A8}$$

Equations (A6) and (A7) can be calculated by the fast Fourier transform algorithm.

In this paper, we fix $L = 0.5$, $N = 256$, $\Delta\tau = 10^{-6}$.

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