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Three-dimensional Bondi-Metzner-Sachs invariant two-dimensional field theories as the flat limit of Liouville theory

Glenn Barnich*

Physique Théorique et Mathématique, Université Libre de Bruxelles and International Solvay Institutes, Campus Plaine C.P. 231, B-1050 Bruxelles, Belgium

Andrés Gomberoff

Departamento de Ciencias Físicas, Universidad Andres Bello, Avenida República 252, Santiago, Chile

Hernán A. González

Departamento de Física, Pontificia Universidad Católica de Chile, Casilla 306, Santiago 22, Chile
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In the gravitational context, Liouville theory is the two-dimensional conformal field theory that controls the boundary dynamics of asymptotically AdS_3 spacetimes at the classical level. By taking a suitable limit of the coupling constants of the Hamiltonian formulation of Liouville, we construct and analyze a BMS_3 invariant two-dimensional field theory that is likely to control the boundary dynamics at null infinity of threedimensional asymptotically flat gravity.

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I. INTRODUCTION

As a nontrivial two-dimensional conformal field theory, Liouville theory is ubiquitous in theoretical physics (see e.g. [1–3] for reviews). In particular, in the context of three-dimensional asymptotically anti–de Sitter spacetimes, and more generally the AdS/CFT correspondence (see e.g. Sec. 5.5 of [4]), Liouville theory controls the boundary dynamics [5] on the classical level: starting from the Chern-Simons formulation of anti–de Sitter gravity [6,7], it is obtained through a Hamiltonian reduction from a suitable Wess-Zumino-Witten model by taking into account gravitational boundary conditions.

For flat three-dimensional gravity, asymptotic dynamics that is as rich as the one of the anti–de Sitter case can be defined at null infinity [8–10]. It can be connected through a well-defined flat-space limit to the anti–de Sitter case [11]: the limit of the BTZ black holes are cosmological solutions whose horizon entropy can be understood from symmetry arguments [12,13] consistent with those of the anti–de Sitter case [14].

In this context of flat space holography, a natural problem is to construct the action that controls the boundary dynamics by starting from the Chern-Simons formulation of flat gravity and taking the gravitational boundary conditions into account. This will be addressed in detail elsewhere.

In this note, we take a shortcut and directly construct a candidate for such an action: by taking appropriate "flat" limits of Liouville theory, we construct two BMS_3 invariant two-dimensional field theories and work out their Poisson algebra of conserved charges. Whereas the first limit has no central extension, the second one admits a central extension of exactly the same type than in the gravitational surface charge algebra.

The constructed theories are interacting two-dimensional field theories with a symmetry group that is of the same dimension than the conformal algebra. We briefly elaborate on some of their classical properties by working out the anomalous transformations laws of their energy momentum tensors on-shell and relating them to the general solution of the field equations obtained from a suitable free field.

II. LIOUVILLE THEORY, FLAT LIMITS AND BMS_3 INVARIANCE

We start by writing the Liouville action in Hamiltonian form on the Minkowskian cylinder with time coordinate time,[1] u, angular coordinate $\phi \in [0, 2\pi)$ and metric $\eta_{\mu\nu} = \mathrm{diag}(-1, l^2)$

$$I_H[\varphi, \pi; \gamma, \mu, l] = \int du d\phi \, \mathcal{L}_H, \qquad \mathcal{L}_H = \pi\dot{\varphi} - \frac{1}{2}\pi^2 - \frac{1}{2l^2}\varphi'^2 - \frac{\mu}{2\gamma^2}e^{\gamma\varphi}. \tag{2.1}$$

In this parametrization, if L is the basic physical dimension of length, $[\varphi] = L^{\frac{1}{2}}$, $[\pi] = L^{-\frac{1}{2}}$, $[\gamma] = L^{-\frac{1}{2}}$, $[l] = L$, and $[\mu] = [L]^{-2}$. The cylinder coordinates are related to the light-cone variables through $x^{\pm} = \frac{u}{l} \pm \phi$. Under two-dimensional conformal transformations $\tilde{x}^+ = F(x^+)$,

*Also Research Director of the Fund for Scientific Research-FNRS Belgium.
gbarnich@ulb.ac.be

[1] The choice of the letter u for time is due to the fact that the time of the flat limit is a null time in the gravitational context.

$\tilde{x}^- = G(x^-)$, the Lagrangian action is invariant if the field transforms as

$$\tilde{\varphi}(\tilde{x}) = \varphi(x) - \frac{1}{\gamma}\ln F'G'. \qquad (2.2)$$

This invariance is lifted to the Hamiltonian action through

$$\tilde{\pi}(\tilde{x}) = \frac{1}{\sqrt{F'G'}}\Big(\pi(x) - \frac{1}{l}(\partial_+ + \partial_-)\varphi(x)\Big) + \frac{1}{l}\Big(\frac{1}{F'}\partial_+ + \frac{1}{G'}\partial_-\Big)\Big(\varphi(x) - \frac{1}{\gamma}\ln F'G'\Big). \qquad (2.3)$$

We are interested in two types of "flat" limits of the Hamiltonian Liouville action. The first consists in just taking $l \to \infty$ with γ, μ fixed,

$$I_H[\varphi, \pi; \gamma, \mu] = \int du d\phi \mathcal{L}_H, \qquad \mathcal{L}_H = \pi\dot{\varphi} - \frac{1}{2}\pi^2 - \frac{\mu}{2\gamma^2}e^{\gamma\varphi}. \qquad (2.4)$$

In this case, it is still possible to eliminate the momentum by its equation of motion leading to

$$I_L[\varphi; \gamma, \mu] = \int du d\phi\Big(\frac{1}{2}\dot{\varphi}^2 - \frac{\mu}{2\gamma^2}e^{\gamma\varphi}\Big). \qquad (2.5)$$

For the second limit, we first rescale the field and its momentum through a canonical transformation,

$$\varphi = l\Phi, \qquad \pi = \frac{\Pi}{l}, \qquad (2.6)$$

and then take the limit while keeping $\beta = \gamma l$, $\nu = \mu l^2$ fixed so that

$$I_H[\Phi, \Pi; \beta, \nu] = \int du d\phi \mathcal{L}_H, \qquad \mathcal{L}_H = \Pi\dot{\Phi} - \frac{1}{2}\Phi'^2 - \frac{\nu}{2\beta^2}e^{\beta\Phi}. \qquad (2.7)$$

Even though there is no local second-order version of this theory, one can in principle eliminate Φ from the action. In order to so, one has to solve the equations of motion of Φ in terms of $\dot{\Pi}$ at the price of sacrificing spatial locality. In this way, one ends up with a theory for Π that is of second order in time derivatives. For example, in the mini-superspace approximation where the canonical fields do not depend on ϕ, one ends up with

$$\mathcal{L}_L = -\frac{\dot{\Pi}}{\beta}\ln|\dot{\Pi}|. \qquad (2.8)$$

The BMS_3 group admits a realization in terms of coordinate transformations of $S^1 \times \mathbb{R}$ of the form

$$\tilde{\phi} = \tilde{\phi}(\phi), \qquad \tilde{u} = \tilde{\phi}'(u + \alpha(\phi)), \qquad (2.9)$$

where the tensor density α transforms as $\tilde{\alpha}(\tilde{\phi}) = (\alpha\tilde{\phi}')(\phi)$. It is then straightforward to check that action (2.4) is invariant under

$$\tilde{\varphi}(\tilde{u}, \tilde{\phi}) = \varphi(u, \phi) - \frac{2}{\gamma}\ln|\tilde{\phi}'|, \qquad \tilde{\pi}(\tilde{u}, \tilde{\phi}) = \frac{1}{\tilde{\phi}'}\pi(u, \phi), \qquad (2.10)$$

while action (2.7) is invariant under

$$\tilde{\Phi}(\tilde{u}, \tilde{\phi}) = \Phi(u, \phi) - \frac{2}{\beta}\ln|\tilde{\phi}'|,$$
$$\tilde{\Pi}(\tilde{u}, \tilde{\phi}) = \frac{1}{\tilde{\phi}'}\Pi(u, \phi) + \frac{1}{2}\tilde{\phi}'\Big(\frac{\partial u}{\partial\tilde{\phi}}\Big)^2\partial_u\Phi + \frac{\partial u}{\partial\tilde{\phi}}\partial_\phi\Phi - \frac{2}{\beta(\tilde{\phi}')^2}\Big(\tilde{u}'' - 2\frac{\tilde{u}'}{\tilde{\phi}'}\tilde{\phi}''\Big), \qquad (2.11)$$

by using $\frac{\partial\tilde{u}}{\partial u} = \frac{\partial\tilde{\phi}}{\partial\phi}$, $\frac{\partial\tilde{\phi}}{\partial u} = 0$, $\frac{\partial u}{\partial\tilde{u}} = \frac{\partial\phi}{\partial\tilde{\phi}}$, $\frac{\partial\phi}{\partial\tilde{u}} = 0$, $\frac{\partial u}{\partial\tilde{\phi}} = -\frac{\tilde{u}'}{(\tilde{\phi}')^2}$ and also $\frac{(\tilde{\phi}'')^2}{(\tilde{\phi}')^2} = \partial_u(\frac{\tilde{u}'\tilde{\phi}''}{(\tilde{\phi}')^2})$.

III. POISSON ALGEBRA OF CONSERVED CHARGES

A. Liouville theory

In the current set-up, if $\xi = f\partial_u + Y\partial_\phi$ is a conformal Killing vector on the cylinder

$$\partial_u f = \partial_\phi Y, \qquad \partial_u Y = \frac{1}{l^2}\partial_\phi f, \qquad (3.1)$$

or, equivalently, $f = \frac{l}{2}(Y^+ + Y^-)$ with $Y = \frac{1}{2}(Y^+ - Y^-)$, $Y^+ = Y^+(x^+)$, $Y^- = Y^-(x^-)$, the infinitesimal symmetry transformations of the field and its momentum are given by

$$-\delta_\xi\varphi = f\pi + Y\varphi' + \frac{2}{\gamma}Y', \qquad -\delta_\xi\pi = -f\frac{\mu}{2\gamma}e^{\gamma\varphi} + \Big(\frac{1}{l^2}f\varphi'\Big)' + (\pi Y)' + \frac{2}{\gamma l^2}f''. \qquad (3.2)$$

They are related to the infinitesimal versions of the finite transformations discussed in the previous section through trivial equations-of-motion symmetries chosen so as to remove the time-derivatives of the canonical variables.

Invariance of the action follows from

$$-\delta_\xi\mathcal{L}_H = \partial_\phi\Big(Y\Big[\pi\dot{\varphi} - \frac{1}{2}\pi^2 - \frac{1}{2l^2}\varphi'^2 - \frac{\mu}{2\gamma^2}e^{\gamma\varphi}\Big] - \frac{2}{\gamma l^2}Y''\varphi + \frac{1}{l^2}f(\dot{\varphi} - \pi)\varphi'\Big) + \partial_u\Big(f\Big[\frac{1}{2}\pi^2 - \frac{1}{2l^2}\varphi'^2 - \frac{\mu}{2\gamma^2}e^{\gamma\varphi}\Big] + \frac{2}{\gamma l^2}f''\varphi\Big). \qquad (3.3)$$

Writing $-\delta_\xi\mathcal{L}_H = \partial_\mu k^\mu_\xi$, the canonical Noether current is given by $-j^\mu_\xi = \frac{\partial\mathcal{L}_H}{\partial\partial_\mu\varphi}\delta_\xi\varphi + \frac{\partial\mathcal{L}_H}{\partial\partial_\mu\pi}\delta_\xi\pi + k^\mu_\xi$, or explicitly

$$\begin{aligned} j^u_\xi &= f\Big(\frac{1}{2}\pi^2 + \frac{1}{2l^2}\varphi'^2 + \frac{\mu}{2\gamma^2}e^{\gamma\varphi}\Big) - \frac{2}{\gamma l^2}f''\varphi \\ &\quad + Y\pi\varphi' + \frac{2}{\gamma}Y'\pi, \\ j^\phi_\xi &= -\frac{1}{l^2}f\pi\varphi' - Y\Big(\frac{1}{2}\pi^2 + \frac{1}{2l^2}\varphi'^2 - \frac{\mu}{2\gamma^2}e^{\gamma\varphi}\Big) \\ &\quad - \frac{2}{\gamma l^2}Y'\varphi' + \frac{2}{\gamma l^2}Y''\varphi, \end{aligned} \tag{3.4}$$

where the equations of motion have been used to eliminate time derivatives in the spatial part of the Noether current. Defining $j^\mu_\xi = -T^\mu{}_\nu\xi^\nu + \partial_\nu k^{[\nu\mu]}_\xi$ with $k^{[\phi u]}_\xi = -\frac{2}{\gamma l^2}f'\varphi + \frac{2}{\gamma l^2}f\varphi' + \frac{2}{\gamma}Y\pi$ and using again equations of motions to eliminate time-derivatives gives the symmetric and traceless energy-momentum tensor with components $T_{uu} = \mathcal{H} = \frac{1}{l^2}T_{\phi\phi}$, $T_{u\phi} = \mathcal{P}$ where

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}\pi^2 + \frac{1}{2l^2}\varphi'^2 + \frac{\mu}{2\gamma^2}e^{\gamma\varphi} - \frac{2}{\gamma l^2}\varphi'', \\ \mathcal{P} &= \pi\varphi' - \frac{2}{\gamma}\pi'. \end{aligned} \tag{3.5}$$

The associated Noether charge $Q_\xi = \int_0^{2\pi} d\phi\, j^u_\xi$ is

$$Q_\xi = \int_0^{2\pi} d\phi[f\mathcal{H} + Y\mathcal{P}]. \tag{3.6}$$

In terms of the canonical equal-time Poisson bracket, $\{\varphi(u,\phi_1), \pi(u,\phi_2)\} = \delta(\phi_1 - \phi_2)$, the charges generate the symmetry transformations (3.2) through $-\delta_\xi z^a = \{z^a, Q_\xi\}$, and the algebra of their integrands is

$$\{Q_{\xi_1}, Q_{\xi_2}\} = Q_{[\xi_1,\xi_2]_H} + K_{\xi_1,\xi_2}, \tag{3.7}$$

where

$$\begin{aligned} \hat{f} &= f_1Y_2' + Y_1f_2' - (1 \leftrightarrow 2), \\ \hat{Y} &= \frac{1}{l^2}f_1f_2' + Y_1Y_2' - (1 \leftrightarrow 2), \end{aligned} \tag{3.8}$$

and the central extension is

$$K_{\xi_1,\xi_2} = \frac{4}{\gamma^2 l^2}\int_0^{2\pi} d\phi[f_1'Y_2'' - (1 \leftrightarrow 2)]. \tag{3.9}$$

The bracket $[\xi_1, \xi_2]_H = \hat{f}\partial_u + \hat{Y}\partial_\phi$ is related to the standard Lie bracket by eliminating the time derivatives using the conformal Killing equation (3.1). Algebra (3.7) implies in particular that the charges are conserved. Indeed, $H = Q_{\partial_u}$ and conservation means that $\frac{\partial}{\partial u}Q_\xi + \{Q_\xi, H\} = 0$. This is encoded in (3.7) by choosing $\xi_1 = \xi$, $\xi_2 = \partial_u$.

In terms of Fourier modes, the conformal Killing vectors of the cylinder are given by

$$\begin{aligned} p_m &= e^{im\phi}\frac{1}{2l}[(e^{im\frac{u}{l}} + e^{-im\frac{u}{l}})l\partial_u + (e^{im\frac{u}{l}} - e^{-im\frac{u}{l}})\partial_\phi], \\ j_m &= e^{im\phi}\frac{1}{2}[(e^{im\frac{u}{l}} - e^{-im\frac{u}{l}})l\partial_u + (e^{im\frac{u}{l}} + e^{-im\frac{u}{l}})\partial_\phi]. \end{aligned} \tag{3.10}$$

If we denote the associated charges by

$$P_m = Q_{p_m}, \qquad J_m = Q_{j_m}, \tag{3.11}$$

their algebra reads

$$\begin{aligned} i\{P_m, P_n\} &= \frac{1}{l^2}(m-n)J_{m+n}, \\ i\{J_m, J_n\} &= (m-n)J_{m+n}, \\ i\{J_m, P_n\} &= (m-n)P_{m+n} + \frac{8\pi}{\gamma^2 l^2}m^3\delta_{m+n}. \end{aligned} \tag{3.12}$$

The change of basis $P_m = l^{-1}(L^+_m + L^-_{-m})$ and $J_m = L^+_m - L^-_{-m}$ transforms this algebra into two copies of the Virasoro algebra, $i\{L^\pm_m, L^\pm_n\} = (m-n)L^\pm_{m+n} + \frac{c^\pm}{12}m^3\delta_{m+n}$, $i\{L^\pm_m, L^\mp_n\} = 0$ with $c^\pm = \frac{48\pi}{\gamma^2 l}$. This is consistent with the Dirac bracket algebra of surface charges in three-dimensional asymptotically anti–de Sitter spacetimes, normalized with respect to the $M = 0 = J$ BTZ black hole, which has central charges $c^\pm = \frac{3l}{2G}$ [15]. If one uses the normalization of the action as is given in Eq. (2.1), the theory is equivalent to (2 + 1)-dimensional gravity [5], when its coupling constants are related to the gravitational ones by

$$G = \frac{\gamma^2 l^2}{32\pi}, \qquad \Lambda = -\frac{1}{l^2}, \tag{3.13}$$

where Λ is the cosmological constant and G is Newton's constant.

Written in terms of these parameters, this is precisely the Brown-Henneaux central charge,

$$c^\pm = \frac{48\pi}{\gamma^2 l} = \frac{3l}{2G}. \tag{3.14}$$

B. Gravitational results for three-dimensional asymptotically flat spacetimes

The Dirac bracket algebra of surface charges for asymptotically flat three-dimensional spacetimes at null infinity [9,10], normalized with respect to the null orbifold which is defined to have zero mass,[2] is the centrally extended $\mathfrak{bms}_3$ algebra,

[2]See also [11], where the algebra is normalized with respect to global Minkowski space. This amounts to shift P_0 by $-c_2/12$.

$$\begin{aligned} i\{P_m, P_n\} &= 0, \\ i\{J_m, J_n\} &= (m-n)J_{m+n} + \frac{c_1}{12}m^3\delta_{m+n}, \\ i\{J_m, P_n\} &= (m-n)P_{m+n} + \frac{c_2}{12}m^3\delta_{m+n}, \end{aligned} \tag{3.15}$$

with gravitational values $c_1 = 0$, $c_2 = \frac{3}{G}$.

C. Noncentrally extended limit

The first limit $l \to \infty$ simply amounts to dropping all terms involving l^{-2} in formulas (3.1)–(3.12), with the exception of (3.10). In particular, the general solution to (3.1) for $l \to \infty$ is given by

$$f = T(\phi) + uY', \qquad Y = Y(\phi), \tag{3.16}$$

for arbitrary functions T, Y of ϕ. The transformations simplify to

$$\begin{aligned} -\delta_\xi \varphi &= f\pi + Y\varphi' + \frac{2}{\gamma}Y', \\ -\delta_\xi \pi &= -f\frac{\mu}{2\gamma}e^{\gamma\varphi} + (\pi Y)', \end{aligned} \tag{3.17}$$

while the Hamiltonian density in the expression for the Noether charge (3.6) reduces to

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{\mu}{2\gamma^2}e^{\gamma\varphi}, \tag{3.18}$$

with $\mathcal{P}$ unchanged. At the same time, the components of the energy-momentum tensor are given by $-T^u{}_u = \mathcal{H} = T^\phi{}_\phi$, $-T^u{}_\phi = \mathcal{P}$, $T^\phi{}_u = 0$. In the algebra, the central extension K_{ξ_1,ξ_2} vanishes while (3.8), rewritten in terms of (T, Y), turns into

$$\hat{T} = T_1Y_2' + Y_1T_2' - (1 \leftrightarrow 2), \qquad \hat{Y} = Y_1Y_2' - (1 \leftrightarrow 2). \tag{3.19}$$

In terms of modes, which now become

$$P_m = Q_{e^{im\phi}\partial_u}, \qquad J_m = Q_{e^{im\phi}(imu\partial_u + \partial_\phi)}, \tag{3.20}$$

one then finds (3.15) with $c_1 = 0 = c_2$. In other words, while the theory defined by (2.4) is invariant under $\mathfrak{bms}_3$ transformations, the associated Poisson algebra of Noether charges has no central extension. Hence this theory is not related to asymptotically flat gravity in three dimensions, which is known to have a nonvanishing central extension in its corresponding algebra, as we discussed in Sec. III B. Another way of looking at this is to notice that this limit of vanishing cosmological constant produces $G \to \infty$. This can be seen from (3.13) when keeping γ fixed as $l \to \infty$.

D. Centrally extended limit

Since the rescaling of variables is a canonical transformation, the Poisson algebra (3.7), or equivalently (3.12), is unchanged before taking the limit. After redefining the constants and then taking the limit, the symmetry transformations reduce to

$$\begin{aligned} -\delta_\xi \Phi &= Y\Phi' + \frac{2}{\beta}Y', \\ -\delta_\xi \Pi &= -f\frac{\nu}{2\beta}e^{\beta\Phi} + (f\Phi')' + (\Pi Y)' + \frac{2}{\beta}f'', \end{aligned} \tag{3.21}$$

where now $\partial_u f = \partial_\phi Y$ and $\partial_u Y = 0$, or equivalently, (3.16) holds. Their generators can be written as in (3.6) with

$$\mathcal{H} = \frac{1}{2}\Phi'^2 + \frac{\nu}{2\beta^2}e^{\beta\Phi} - \frac{2}{\beta}\Phi'', \qquad \mathcal{P} = \Phi'\Pi - \frac{2}{\beta}\Pi'. \tag{3.22}$$

Their Poisson algebra is centrally extended, it is given by (3.7), where

$$\begin{aligned} K_{\xi_1,\xi_2} &= -\frac{4}{\beta^2}\int_0^{2\pi} d\phi[T_1Y_2''' + Y_1T_2'''] \\ &= \frac{2}{\beta^2}\int_0^{2\pi} d\phi[T_1'Y_2'' + Y_1'T_2'' - (1 \leftrightarrow 2)]. \end{aligned} \tag{3.23}$$

In terms of modes defined again by (3.20), one gets the centrally extended $\mathfrak{bms}_3$ algebra (3.15) with $c_1 = 0$, $\frac{c_2}{12} = \frac{8\pi}{\beta^2}$. Note that in this case, we may see from (3.13) that the constant G is kept finite because $\beta = \sqrt{32\pi G}$ is held fixed in the limit. The value of the central charge turns out to be precisely the gravitational one.

IV. ENERGY-MOMENTUM TENSOR, BÄCKLUND TRANSFORMATION AND GENERAL SOLUTION

A. Liouville theory

In this section, we recall the classical part of the analysis in [16–18].

When using the Hamiltonian equations of motion, the charge densities satisfy

$$\partial_u \mathcal{H} = \frac{1}{l^2}\partial_\phi \mathcal{P}, \qquad \partial_u \mathcal{P} = \partial_\phi \mathcal{H}. \tag{4.1}$$

They are thus given by

$$\mathcal{H} = \frac{4}{\gamma^2 l^2}(\Xi_{++} + \Xi_{--}), \qquad \mathcal{P} = \frac{4}{\gamma^2 l}(\Xi_{++} - \Xi_{--}), \tag{4.2}$$

with $\Xi_{++} = \Xi_{++}(x^+)$, $\Xi_{--} = \Xi_{--}(x^-)$ and the conserved charges reduce on-shell to

$$Q_\xi = \frac{4}{\gamma^2 l}\int_0^{2\pi} d\phi(Y^+\Xi_{++} + Y^-\Xi_{--}), \tag{4.3}$$

where the normalization is chosen here in order to agree with conventions used in the gravitational context.

Equivalently, one can first express the energy-momentum tensor in light-cone coordinates,

$$T_{\pm\pm} = \frac{l}{2}(l\mathcal{H} \pm \mathcal{P}) = \frac{1}{4}(l\pi \pm \varphi')^2 + \frac{\mu l^2}{4\gamma^2}e^{\gamma\varphi} \mp \frac{1}{\gamma}(l\pi \pm \varphi')', \quad (4.4)$$

so that $T_{\pm\pm} = \frac{4}{\gamma^2}\Xi_{\pm\pm}$. One recovers the more familiar form on-shell,

$$T_{\pm\pm} = (\partial_\pm\varphi)^2 - \frac{2}{\gamma}\partial_\pm^2\varphi, \qquad T_{\pm\mp} = 0. \quad (4.5)$$

Conservation is equivalent to $\partial_\mp T_{\pm\pm} = 0$ and the transformation laws follow from (2.2),

$$\tilde{T}_{++} = (F')^{-2}\Big(T_{++} + \frac{2}{\gamma^2}\{F; x^+\}\Big), \quad (4.6)$$

where $\{F; x^+\} = \frac{F'''}{F'} - \frac{3}{2}\frac{(F'')^2}{(F')^2} = (\ln F')'' - \frac{1}{2}((\ln F')')^2$ denotes the Schwarzian derivative and similarly for T_{--}.

Let us now assume $\mu \geq 0$. The Bäcklund transformation from Liouville theory to a free field ψ with momentum π_ψ is the canonical transformation determined by

$$\int_0^{2\pi} d\phi\,\pi\dot{\varphi} - H[\varphi, \pi] = \int_0^{2\pi} d\phi\,\pi_\psi\dot{\psi} - K[\psi, \pi_\psi] + \frac{d}{du}W[\varphi, \psi],$$
$$W[\varphi, \psi] = \int_0^{2\pi} d\phi\Big[\frac{1}{l}\varphi\psi' - \frac{2}{\gamma^2}\sqrt{\mu}e^{\frac{\gamma\varphi}{2}}\sinh\Big(\frac{\gamma\psi}{2}\Big)\Big]. \quad (4.7)$$

This gives the transformation equations

$$\pi = \frac{\delta W}{\delta\varphi} = \frac{1}{l}\psi' - \frac{1}{\gamma}\sqrt{\mu}e^{\frac{\gamma\varphi}{2}}\sinh\Big(\frac{\gamma\psi}{2}\Big),$$
$$\pi_\psi = -\frac{\delta W}{\delta\psi} = \frac{1}{l}\varphi' + \frac{1}{\gamma}\sqrt{\mu}e^{\frac{\gamma\varphi}{2}}\cosh\Big(\frac{\gamma\psi}{2}\Big). \quad (4.8)$$

When used in the integrand of $H[\varphi, \pi]$ one finds, after an integration by parts and another use of the last of relations (4.8), that

$$K[\psi, \pi_\psi] = \int_0^{2\pi} d\phi\Big(\frac{1}{2}\pi_\psi^2 + \frac{1}{2l^2}\psi'^2\Big), \quad (4.9)$$

which is the Hamiltonian of a free massless field in two dimensions. A useful form for its solution is

$$\psi = \frac{1}{\gamma}\ln\Big(\frac{A'}{B'}\Big), \qquad \pi_\psi = \dot{\psi}, \qquad A = A(x^+), \qquad B = B(x^-). \quad (4.10)$$

One may find the general solution φ to Liouville's equation by replacing ψ above in the second of relations (4.8),

$$e^{\gamma\varphi} = \frac{16}{l^2\mu}\frac{A'B'}{(A-B)^2} = \frac{16}{l^2\mu}\frac{C'B'}{(1+CB)^2}, \qquad A = -\frac{1}{C}. \quad (4.11)$$

Finally, this expression can be used to express the energy-momentum tensor in terms of the arbitrary functions appearing in the general solution,

$$T_{++} = -\frac{2}{\gamma^2}\{A; x^+\} = -\frac{2}{\gamma^2}\{C; x^+\},$$
$$T_{--} = -\frac{2}{\gamma^2}\{B; x^-\}. \quad (4.12)$$

B. Noncentrally extended limit

On-shell, the charge densities, and thus the components of the energy-momentum tensor, now satisfy $\partial_u\mathcal{H} = 0$ and $\partial_u\mathcal{P} = \partial_\phi\mathcal{H}$, so that they are given by

$$\mathcal{H} = \frac{2}{\sigma^2}\Theta, \qquad \mathcal{P} = \frac{2}{\sigma^2}(2\Xi + u\Theta'),$$
$$\Theta = \Theta(\phi), \qquad \Xi = \Xi(\phi), \quad (4.13)$$

for some normalization σ. On-shell, the charges reduce to

$$Q_\xi = \frac{2}{\sigma^2}\int_0^{2\pi} d\phi(T\Theta + 2Y\Xi). \quad (4.14)$$

The on-shell transformation laws for the functions determining the energy-momentum tensor can then be worked out and are given by

$$\tilde{\Theta}(\tilde{\phi}) = (\tilde{\phi}')^{-2}\Theta, \qquad \tilde{\Xi}(\tilde{\phi}) = (\tilde{\phi}')^{-2}\Big[\Xi - \frac{\alpha}{2}\Theta' - \alpha'\Theta\Big]. \quad (4.15)$$

The associated infinitesimal versions are

$$-\delta\Theta = Y\Theta' + 2Y'\Theta,$$
$$-\delta\Xi = Y\Xi' + 2Y'\Xi + \frac{1}{2}T\Theta' + T'\Theta. \quad (4.16)$$

In the Bäcklund transformations (4.7)–(4.9), the terms proportional to l^{-1}, l^{-2} drop out, so that

$$K[\psi, \pi_\psi] = \int_0^{2\pi} d\phi\,\frac{1}{2}\pi_\psi^2. \quad (4.17)$$

The free field ψ now satisfies $\ddot{\psi} = 0$ and so is given by

$$\psi = \frac{1}{\gamma}(A + uB), \qquad A = A(\phi), \qquad B = B(\phi). \quad (4.18)$$

The second equation of (4.8) now yields

$$e^{\gamma\varphi} = \frac{B^2}{\mu\cosh^2\frac{A+uB}{2}}. \quad (4.19)$$

Again, on-shell, the arbitrary functions determining the components of the energy-momentum tensor can be expressed in terms of the arbitrary functions appearing in the general solution,

$$\Theta = \frac{\sigma^2}{4\gamma^2}B^2, \qquad \Xi = \frac{\sigma^2}{4\gamma^2}A'B. \tag{4.20}$$

C. Centrally extended limit

The charge densities again satisfy $\partial_u \mathcal{H} = 0$ and $\partial_u \mathcal{P} = \partial_\phi \mathcal{H}$ on-shell, so that

$$\begin{aligned} \mathcal{H} &= \frac{2}{\beta^2}\Theta, \qquad \mathcal{P} = \frac{2}{\beta^2}(2\Xi + u\Theta'), \\ \Theta &= \Theta(\phi), \qquad \Xi = \Xi(\phi), \end{aligned} \tag{4.21}$$

with on-shell charges given by

$$Q_\xi = \frac{2}{\beta^2}\int_0^{2\pi}(T\Theta + 2Y\Xi). \tag{4.22}$$

The on-shell transformation laws for the functions determining the energy-momentum tensor are now given by

$$\begin{aligned} \tilde{\Theta}(\tilde{\phi}) &= (\tilde{\phi}')^{-2}[\Theta(\phi) + 2\{\tilde{\phi};\phi\}], \\ \tilde{\Xi}(\tilde{\phi}) &= (\tilde{\phi}')^{-2}\left[\Xi - \frac{\alpha}{2}\Theta' - \alpha'\Theta + \alpha'''\right]. \end{aligned} \tag{4.23}$$

The associated infinitesimal versions are

$$\begin{aligned} -\delta\Theta &= Y\Theta' + 2Y'\Theta - 2Y''', \\ -\delta\Xi &= Y\Xi' + 2Y'\Xi + \frac{1}{2}T\Theta' + T'\Theta - T'''. \end{aligned} \tag{4.24}$$

The normalization $\frac{2}{\beta^2}$ chosen above is conventional. The choice made here is such that the transformation laws agree with the gravitational ones. In the latter context Θ, Ξ denote the arbitrary functions that appear in the general solution to asymptotically flat gravity in three dimensions in BMS gauge (cf. Sec. 3 of [10]).

The Bäcklund transformations are now determined by

$$\begin{aligned} \int_0^{2\pi} d\phi \Pi\dot{\Phi} - H[\Phi,\Pi] &= \int_0^{2\pi} d\phi \pi_\psi \dot{\psi} - K[\psi,\pi_\psi] \\ &\quad + \frac{d}{du}W[\Phi,\psi], \\ W[\Phi,\psi] &= \int_0^{2\pi} d\phi\left[\Phi\psi' - \frac{1}{\beta}\sqrt{\nu}e^{\frac{\beta\Phi}{2}}\psi\right], \end{aligned} \tag{4.25}$$

so that

$$\begin{aligned} \Pi &= \frac{\delta W}{\delta\varphi} = \psi' - \frac{1}{2}\sqrt{\nu}e^{\frac{\beta\Phi}{2}}\psi, \\ \pi_\psi &= -\frac{\delta W}{\delta\psi} = \Phi' + \frac{1}{\beta}\sqrt{\nu}e^{\frac{\beta\Phi}{2}}. \end{aligned} \tag{4.26}$$

This gives again

$$K[\psi,\pi_\psi] = \int_0^{2\pi} d\phi \frac{1}{2}\pi_\psi^2. \tag{4.27}$$

For the solution of the free theory, we now choose

$$\psi = \frac{1}{\beta}(A + 2u(\ln B')'), \quad A = A(\phi), \quad B = B(\phi), \tag{4.28}$$

and from (4.26), one then finds the local solution

$$\begin{aligned} e^{\beta\Phi} &= \frac{4}{\nu}((\ln B)')^2, \\ \beta\Pi &= \frac{A'B - B'A}{B} + 2u\left((\ln B')'' - \frac{B''}{B}\right). \end{aligned} \tag{4.29}$$

In this case, the relation between the arbitrary functions in the energy-momentum tensor and those in the general solution is

$$\Theta = -2\{B;\phi\}, \qquad \Xi = \frac{1}{2}\frac{A'B'' - B'A''}{B'}. \tag{4.30}$$

V. CONCLUSIONS

In this note, we have taken a shortcut for constructing an action describing the boundary degrees of freedom of (2 + 1)-dimensional, asymptotically flat Einstein gravity. In order to do so, we have taken appropriate "flat" limits of Liouville, which is known to be the theory that describes the boundary dynamics in the asymptotically anti–de Sitter case. The limit may be taken in at least two different ways. Both give rise to BMS_3 invariant two-dimensional field theories. Whereas the first limit has no central extension, the second one admits a central extension of exactly the same type as in the gravitational surface charge algebra.

The constructed theories are interacting two-dimensional field theories with a symmetry group, namely BMS_3, that is of the same dimension than the conformal algebra. We have explicitly constructed the finite symmetry transformations and constructed the conserved charges in each theory. As for conformally invariant theories, these charges are related to the corresponding energy-momentum tensors, which are also given explicitly. We have constructed the most general solutions of both theories making use of the Bäcklund transformations which, as for Liouville, maps the nonlinear to a free field theory.

We have worked in the canonical formulation. It turns out that for the case with vanishing central extension, the momentum may be eliminated in the Hamiltonian action principle, leading us to a second-order, Lagrangian action. In the centrally extended case, which is the one appropriated for describing gravity, one cannot eliminate the momentum. One may, however, eliminate the original field in terms of the momentum. This gives rise to a spatially nonlocal Lagrangian.

The complete analysis, which will be carried out in follow-up work, consists in starting from the first-order Chern-Simons formulation of three-dimensional gravity and implementing the Hamiltonian reduction required by the gravitational boundary conditions on the associated

WZW theory to end up with the proposed centrally extended flat limit of Liouville theory.

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