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# Hamiltonian mass of asymptotically Schwarzschild–de Sitter space-times

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We derive the Hamiltonian mass for general relativistic initial data sets with asymptotically Schwarzschild–de Sitter ends.

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## I. INTRODUCTION

There is growing astrophysical evidence that space-times with positive cosmological constant should be given serious consideration. Large families of such noncompact, vacuum, general-relativistic models can be constructed using singular solutions of the Yamabe problem (see [1,2] and references therein). In particular one thus obtains initial data sets with one or more ends of cylindrical type, in which the metric becomes periodic when one recedes to infinity along half-cylinders, approaching the Schwarzschild–de Sitter metric in the limit, with the extrinsic curvature tensor approaching zero. This construction can be carried out in any number of space dimensions $n \geq 3$. (See [3–7] for further families of vacuum initial data sets with various ends of cylindrical type.) This raises the question of existence of a natural notion of mass in this context. The object of this work is to show that the numerical value of a natural Hamiltonian $\mathcal{H}$ for a class of such metrics is proportional to the parameter $m$ appearing in the asymptotic metric. We further prove that the contribution to the Hamiltonian from each asymptotically Schwarzschild–de Sitter end can be calculated as

$$\mathcal{H} = \lim_{x_0 \to \infty} \frac{1}{2\gamma} \int_{x=x_0} (k\nu - k_0 \nu_0) \lambda d^{n-1} x. \tag{1.1}$$

Here we assume that the space metric is asymptotic to the space part of a Birmingham metric on $[0, \infty) \times \mathring{M}$ as in (A1) and (A2), for a compact Riemannian $(n-1)$-dimensional Einstein manifold $(\mathring{M}, \mathring{h})$; $\nu$ is the lapse function as in (2.3); $k$ is the mean curvature of $\{x = x_0\}$ as defined in (2.11); and $\lambda$ is the $(n-1)$-volume element on $\{x = x_0\}$. The fields $\nu_0$ and $k_0$ are the corresponding quantities for the Birmingham metric with vanishing mass (the "de Sitter solution"); see (3.23) and (3.24). Finally $\gamma$ is a dimension-dependent coupling constant; see (D2) in Appendix D below, related to the "$(n+1)$-dimensional Newton constant" as in (D6).

We note that a Hamiltonian is always defined up to a constant. Our choice in (1.1) is precisely what is needed for positivity of $\mathcal{H}$; compare Theorem III.1 below.

See [8–10] and references therein for alternative approaches to a definition of mass in the presence of a positive cosmological constant.

## II. THE BASIC VARIATIONAL FORMULA

In order to present our results some notation is needed. Let $\mathcal{S}$ be a smooth spacelike hypersurface in an $(n+1)$-dimensional space-time $(\mathcal{M}, g)$, $n \geq 2$. Consider a space-time domain $\Omega$ with smooth timelike boundary such that $V := \Omega \cap \mathcal{S}$ is compact. Let $x^n$ be a coordinate such that $x^n$ is constant on $\partial V$, and let $(x^a) = (x^0, x^A)$ be local coordinates on $\partial\Omega$ such that $x^0$ is constant on $\mathcal{S}$. Let $L_{ab}$ denote the extrinsic curvature tensor of $\partial\Omega$,

$$L_{ab} = -\frac{1}{\sqrt{g^{nn}}} \Gamma^n_{ab}, \tag{2.1}$$

and let $Q^{ab}$ be its "Arnowitt-Deser-Misner (ADM) counterpart,"

$$Q^{ab} := \sqrt{|\det g_{cd}|}(L\hat{g}^{ab} - L^{ab}), \tag{2.2}$$

where $\hat{g}^{ab}$ is the $n$-dimensional inverse with respect to the induced metric $g_{ab}$ on the world tube $\partial\Omega$. Let $\nu$ and $\nu^A$ denote the "lapse" and the "shift" in the $n$-dimensional geometry $g_{ab}$ of the boundary of the world tube $\partial\Omega$,

$$\nu := \frac{1}{\sqrt{|\hat{g}^{00}|}}, \qquad \nu^A := \tilde{\tilde{g}}^{AB} g_{0B}, \tag{2.3}$$

where $\tilde{\tilde{g}}^{AB}$ is the $(n-1)$-dimensional metric on $\partial V$, inverse with respect to the induced metric $g_{AB}$. We have the identity

$$g_{00} = -\nu^2 + \nu_A \nu^A. \tag{2.4}$$

One can define the following $(n-1)$-dimensional objects on $\partial V$: a scalar density

$$\mathbf{Q} := \nu Q^{00}, \tag{2.5}$$

and a covector density

$$\mathbf{Q}_A := Q^0{}_A. \tag{2.6}$$

It is further useful to introduce the field

$$\overset{\perp}{Q}{}^{AB} := Q_{CD}\tilde{\tilde{g}}^{CA}\tilde{\tilde{g}}^{DB}. \tag{2.7}$$

The $n$-dimensional Lorentzian metric $g_{ab}$ on $\partial\Omega$ can be parametrized as

$$g_{ab} = \begin{bmatrix} -\nu^2 + \nu^A \nu_A & \nu_A \\ \nu_A & g_{AB} \end{bmatrix}. \tag{2.8}$$

The corresponding inverse metric reads

$$\hat{g}^{ab} = \begin{bmatrix} -\frac{1}{\nu^2} & \frac{\nu^A}{\nu^2} \\ \frac{\nu^A}{\nu^2} & \tilde{\tilde{g}}^{AB} - \frac{\nu^A\nu^B}{\nu^2} \end{bmatrix}. \tag{2.9}$$

We also have

$$L^{00} = L_{ab}\hat{g}^{0a}\hat{g}^{0b} = \frac{1}{\nu^4}(L_{00} - 2L_{0A}\nu^A + L_{AB}\nu^A\nu^B), \tag{2.10}$$

with the trace $L$ of $L_{ab}$ being equal to

$$\begin{aligned} L &= L_{ab}\hat{g}^{ab} = L_{00}\hat{g}^{00} + 2L_{0A}\hat{g}^{0A} + L_{AB}\hat{g}^{AB} \\ &= -\frac{1}{\nu^2}L_{00} + 2L_{0A}\frac{\nu^A}{\nu^2} + L_{AB}\left(\tilde{\tilde{g}}^{AB} - \frac{\nu^A\nu^B}{\nu^2}\right) \\ &= -\nu^2 L^{00} + L_{AB}\tilde{\tilde{g}}^{AB} = a - k, \end{aligned}$$

where

$$k := -L_{AB}\tilde{\tilde{g}}^{AB}$$

(for the Birmingham metrics of Appendix A 3, $k$ is the signed length of the extrinsic curvature vector), and where we use the symbol

$$a := -\nu^2 L^{00}$$

to denote the curvature ("acceleration") of the world lines which are geodesic within $\partial\Omega$ and orthogonal to $\partial\Omega \cap \mathcal{S}$. It holds that

$$\begin{aligned} \mathbf{Q} &= \nu Q^{00} = \nu^2\lambda(L\hat{g}^{00} - L^{00}) \\ &= \lambda(-L - \nu^2 L^{00}) = \lambda k, \end{aligned} \tag{2.11}$$

$$\begin{aligned} \overset{\perp}{Q}{}^{AB} g_{AB} &= Q_{CD}\tilde{\tilde{g}}^{CA}\tilde{\tilde{g}}^{DB}g_{AB} = Q_{CD}\tilde{\tilde{g}}^{CD} \\ &= \nu\lambda(Lg_{AB} - L_{AB})\tilde{\tilde{g}}^{AB} \\ &= \nu\lambda((n-1)a - (n-2)k). \end{aligned} \tag{2.12}$$

Let $P^{ij}$ be the usual ADM momentum on $V$. Denote by

$$\lambda = \sqrt{\det g_{AB}}$$

the $(n-1)$-volume element on $\partial V$. Let $\alpha$ be the hyperbolic angle between $\partial\Omega$ and $V$: in the adapted coordinates above,

$$\alpha := \sinh^{-1}\left(\frac{g^{0n}}{\sqrt{|g^{00}g^{nn}|}}\right).$$

In [11] the following variational formula has been proved for Ricci-flat Lorentzian metrics in dimension 3 + 1:

$$\begin{aligned} 0 = \frac{1}{2\gamma}\int_V (\dot{P}^{kl}\delta g_{kl} - \dot{g}_{kl}\delta P^{kl}) + \frac{1}{\gamma}\int_{\partial V}(\dot{\lambda}\delta\alpha - \dot{\alpha}\delta\lambda) \\ + \frac{1}{2\gamma}\int_{\partial V}(2\nu\delta\mathbf{Q} - 2\nu^A\delta\mathbf{Q}_A + \overset{\perp}{Q}{}^{AB}\delta g_{AB}), \end{aligned} \tag{2.13}$$

with $\gamma = 8\pi$. It can be checked that the formula remains true for vacuum metrics, possibly with a cosmological constant, in any space dimension $n \geq 2$, with a constant $\gamma$ which depends upon dimension; see Appendix D for a discussion. In fact, several terms proportional to $(n-3)$ appear when generalizing the calculations in [11], but they end up giving no contribution to (2.13).

We will not dwell upon the Hamiltonian interpretation of this identity; the reader is referred to [11–13] for details.

In the nonvacuum case (2.13) has to be supplemented by terms involving variations of the matter fields and their momenta. Nevertheless, the formula (1.1) for the Hamiltonian remains valid for a large class of matter models [11] without any further explicit contributions from the matter sources. (Obviously, there is an implicit contribution of the sources via the constraint equations.)

## III. THE MASS OF ASYMPTOTICALLY BIRMINGHAM METRICS

We consider (2.13) for metrics which, as $x$ tends to infinity, asymptote to

$$\mathring{g} = -f(x)dt^2 + \phi^2(x)(dx^2 + \mathring{h}_{AB}dx^A dx^B). \tag{3.1}$$

Similarly we will assume that the derivatives of the metric $g$ asymptote to those of the metric $\mathring{g}$. The coordinate $x^n$ of the calculations above will be taken to be equal to $x$, and the boundary $\partial V \approx \mathring{M}$ in (2.13) will be assumed to be given by the equation $x = x_0$ for a constant $x_0$. We will let $x_0$ tend to infinity; this implies

$$\begin{aligned} L_{ab}dx^a dx^b &= -\frac{1}{\sqrt{g^{xx}}}\Gamma^x_{ab}dx^a dx^b \\ &\to -\frac{1}{2}\phi^{-1}\partial_x f dt^2 + \partial_x\phi \mathring{h}_{AB}dx^A dx^B, \end{aligned} \tag{3.2}$$

$$L = g^{ab}L_{ab} \to \frac{\partial_x f}{2\phi f} + (n-1)\frac{\partial_x \phi}{\phi^2}, \tag{3.3}$$

$$a = -\nu^2 L^{00} \to \frac{\partial_x f}{2\phi f}, \tag{3.4}$$

$$\nu \to \sqrt{f}, \qquad \lambda \to \phi^{n-1}\sqrt{\det \mathring{h}_{AB}}, \qquad \nu_A \to 0, \tag{3.5}$$

$$\mathbf{Q} = \nu Q^{00} \to -(n-1)\phi^{n-3}\sqrt{\det \mathring{h}_{AB}}\,\partial_x\phi, \tag{3.6}$$

$$k = \lambda^{-1}\mathbf{Q} \to -(n-1)\phi^{-2}\partial_x\phi, \tag{3.7}$$

$$\mathbf{Q}_A \to 0, \tag{3.8}$$

$$\overset{\perp}{Q}{}^{AB} \to \sqrt{f}\phi^{n-3}\sqrt{\det \mathring{h}_{AB}}\left(\frac{\partial_x f}{2\phi f} + (n-2)\frac{\partial_x\phi}{\phi^2}\right)\mathring{h}^{AB}$$
$$= \sqrt{f}\phi^{n-4}\sqrt{\det \mathring{h}_{AB}}\,\frac{\partial_x(\sqrt{f}\phi^{n-2})}{\sqrt{f}\phi^{n-2}}\,\mathring{h}^{AB}. \tag{3.9}$$

Above, and in what follows, we assume that $\partial_x$ is pointing outwards from the region $V$ of the previous section; some sign adjustments are needed otherwise. Using these formulas, the last line in (2.13) approaches

$$\frac{(n-1)}{2\gamma}\int_{\mathring{M}}\sqrt{f}\phi^{n-3}\sqrt{\det \mathring{h}_{AB}}\left(\frac{\partial_x(f\phi^2)}{\phi^2 f}\delta\phi - 2\partial_x\delta\phi\right). \tag{3.10}$$

Let us assume that $f$ and $\partial_x\phi$ take the Birmingham form (A1) [14],

$$f = \beta - \frac{2m}{\phi^{n-2}} - \frac{\phi^2}{\ell^2}, \qquad \partial_x\phi = \phi\sqrt{f}, \tag{3.11}$$

where $\beta \in \{0, \pm 1\}$ is related to the scalar curvature, assumed to be constant, of the metric $\mathring{h}$ [see (A5)]. Finally, $\ell^{-2}$ is related to the cosmological constant as in (C17). When $\mathring{h}$ is the unit round metric on the sphere, then $\beta = 1$ and one recovers the familiar Schwarzschild–de Sitter metrics. Equation (3.11) allows us to express $\partial_x\delta\phi = \delta\partial_x\phi$ in terms of $\delta\phi$ and $\delta m$. Perhaps surprisingly, all the $\delta\phi$ terms cancel out and (3.10) becomes

$$\frac{(n-1)}{\gamma}\int_M \sqrt{\det \mathring{h}_{AB}} \times \delta m. \tag{3.12}$$

Setting

$$\mathcal{H} = \frac{(n-1)}{\gamma}\int_{\mathring{M}} \sqrt{\det \mathring{h}_{AB}} \times m, \tag{3.13}$$

we conclude that for any family of metrics which asymptote to Birmingham metrics as the variable $x$ recedes to infinity it holds that

$$-\delta\mathcal{H} = \frac{1}{2\gamma}\int_S (\dot{P}^{kl}\delta g_{kl} - \dot{g}_{kl}\delta P^{kl}). \tag{3.14}$$

This is the first main result of this work.

We wish, next, to provide a geometric formula for the Hamiltonian $\mathcal{H}$. The integrand of the boundary term in (2.13),

$$2\nu\delta\mathbf{Q} - 2\nu^A\delta\mathbf{Q}_A + \overset{\perp}{Q}{}^{AB}\delta g_{AB}, \tag{3.15}$$

can be rearranged using the identity

$$2\nu\delta\mathbf{Q} = 2\nu\delta(\lambda k) = \delta(\lambda\nu k) + \lambda\nu^2\delta\left(\frac{k}{\nu}\right) + \nu k\delta\lambda. \tag{3.16}$$

Using

$$\nu k\delta\lambda = \frac{1}{2}\lambda\nu k\tilde{\tilde{g}}^{AB}\delta g_{AB}$$

we can write

$$2\nu\delta\mathbf{Q} + \overset{\perp}{Q}{}^{AB}\delta g_{AB} = \delta(\lambda\nu k) + \underbrace{\lambda\nu^2\delta\left(\frac{k}{\nu}\right) + \mathbf{Q}^{AB}\delta g_{AB}}_{(*)}, \tag{3.17}$$

where

$$\mathbf{Q}^{AB} := \overset{\perp}{Q}{}^{AB} + \frac{1}{2}\lambda\nu k\tilde{\tilde{g}}^{AB}. \tag{3.18}$$

As before, we assume that the metric asymptotes to a Birmingham metric as $x$ tends to infinity, similarly for first derivatives. We then have

$$\frac{k}{\nu} + \frac{(n-1)}{\phi}\xrightarrow[x\to\infty]{} 0, \tag{3.19}$$

$$\overset{\perp}{Q}{}^{AB} - \nu\phi^{n-3}\left(a - \frac{n-2}{n-1}k\right)\sqrt{\det \mathring{h}_{CD}}\,\mathring{h}^{AB}\xrightarrow[x\to\infty]{} 0,$$
$$\mathbf{Q}^{AB} - \nu\phi^{n-3}\left(a - \frac{n-3}{2(n-1)}k\right)\sqrt{\det \mathring{h}_{CD}}\,\mathring{h}^{AB}\xrightarrow[x\to\infty]{} 0,$$
$$a - \frac{\partial_\phi f}{2\nu}\xrightarrow[x\to\infty]{} 0. \tag{3.20}$$

Inserting those relations into the underbraced terms in (3.17) one finds

$$(*)\xrightarrow[x\to\infty]{} (n-1)\phi^{n-3}\sqrt{\det \mathring{h}_{CD}}[\phi\partial_\phi f + (n-2)f]\delta\phi$$
$$= (n-1)\delta\left[\sqrt{\det \mathring{h}_{CD}}\left(\beta\phi^{n-2} - \frac{1}{\ell^2}\phi^n\right)\right]. \tag{3.21}$$

We thus obtain the following formula for the Hamiltonian:

$$\mathcal{H} = \lim_{x_0\to\infty}\frac{1}{2\gamma}\int_{x=x_0}\left(\nu k + (n-1)\left(\frac{\beta}{\phi} - \frac{\phi}{\ell^2}\right)\right)\lambda d^{n-1}x. \tag{3.22}$$

Here $\phi$ should be viewed as a function of $\lambda$, and hence of the metric:

$$\phi = \left(\frac{\lambda}{\sqrt{\det \mathring{h}_{AB}}}\right)^{\frac{1}{n-1}} = \left(\frac{\sqrt{\det g_{AB}}}{\sqrt{\det \mathring{h}_{AB}}}\right)^{\frac{1}{n-1}}. \tag{3.23}$$

Choose a constant $m_0 \in \mathbb{R}$ and set

$$f_0 = \beta - \frac{2m_0}{\phi^{n-2}} - \frac{\phi^2}{\ell^2}, \qquad \nu_0 = \sqrt{f_0}, \\ k_0 = -\frac{(n-1)}{\phi}\sqrt{f_0}. \tag{3.24}$$

This leads to the following rewriting of (3.22):

$$\mathcal{H} = \lim_{x_0 \to \infty} \frac{1}{2\gamma} \int_{x=x_0} (k\nu - k_0\nu_0)\lambda d^{n-1}x + \frac{(n-1)|\mathring{M}|_{\mathring{h}}}{\gamma} m_0, \tag{3.25}$$

where $|\mathring{M}|_{\mathring{h}}$ is the volume of the set $\{x = x_0\}$ in the metric $\mathring{h}$. This is the second main result of this work.

Note that the parameter $m_0$ has been introduced only to define the reference fields $k_0$ and $\nu_0$, and that the left-hand side is independent of $m_0$.

See Appendix B for an alternative derivation of (3.25).

One can simply disregard the last term in (3.25), or use reference fields associated with the solution equal to $m_0 = 0$ there. Here one should keep in mind that a Hamiltonian analysis always defines a Hamiltonian up to a constant, and the choice of this constant is equivalent to the decision of which field configuration (if any) has zero energy. As such, the subtraction of the term $k_0\nu_0$ can be viewed as a comparison term, where one compares the given field configuration with that time-independent solution which is determined by the parameter $m_0$.

One could argue that reference fields corresponding to the solution with $m_0 = 0$ make no sense because, in the $\mathring{M} = S^{n-1}$ case, the initial data surface is *compact*, so comparing with a solution with asymptotically periodic ends is unnatural from a Hamiltonian perspective. However, one can adopt the point of view that energy in general relativity is not assigned to a *volume* $V$ but rather to a *surface* $\partial V$. Given a level set of $r$ in a Schwarzschild–de Sitter solution, we can find a surface with identical induced metric in the de Sitter solution [15], $m_0 = 0$, and use the corresponding values $\nu_0$ and $k_0$ in (3.25).

In any case, somewhat surprisingly, the choice of the value of $m_0$ is irrelevant, in that the numerical value of $\mathcal{H}$ as given by (3.25) does not depend upon that choice. This is related to the fact that the mass parameter $m$ is the (unique) "constant of motion" for the spherically symmetric Yamabe equation; cf. (C16).

We note that the time-symmetric Birmingham metrics lead to the periodic metrics (3.1) with a strictly positive parameter $m$; see the discussion in Appendix A. This leads to the following trivial observation:

*Theorem III.1* ("*positive energy theorem*") For all asymptotically periodic metrics as above, the numerical value of the Hamiltonian $\mathcal{H}$ given by (3.25) is positive.

Now, Theorem III.1 requires neither positivity of matter-energy nor regularity of initial data (in particular, interior boundaries are allowed without any geometric restrictions), and is based purely on asymptotic properties of the solutions. As such it does not carry much nontrivial information: the positivity of the mass has been built into the hypotheses on the asymptotic behavior of the metric.

### A. Several ends, black hole boundaries

So far we have assumed that the initial data manifold is the union of a compact manifold without boundary and an asymptotically cylindrical end. The generalization of our analysis to a finite number of asymptotically flat, asymptotically cylindrical and asymptotically hyperboloidal ends is straightforward: in such a case each end contributes its respective Hamiltonian mass (as defined here for asymptotically Birmingham ends, and as defined in e.g. [13,16–18] and references therein for the remaining ones) to the total Hamiltonian of the system.

Yet another generalization is of interest, that to manifolds with *horizon boundaries*. For this purpose, suppose that the boundary of the domain $\Omega$ of Sec. II consists of a timelike "world tube" $S^+$ and of a null hypersurface $S^-$. Accordingly, the boundary $\partial V$ of the intersection $V$ of the Cauchy surface $\mathcal{S}$ with $\Omega$ is composed of two disjoint manifolds, $\partial V^+ = V \cap S^+$, and $\partial V^- = V \cap S^-$, assumed to be compact, each of them contributing to the boundary terms in variational formula (2.13). Assume that the spacetime metric asymptotes to a Birmingham metric as the "external" boundary $\partial V^+$ recedes to infinity. The corresponding contribution to (2.13) is handled as in the previous section. The contribution to (2.13) from the null component $S^-$ was calculated in [19] in considerable generality. However, for the sake of simplicity, we restrict attention to stationary solutions with Killing horizons, as arising in a thermodynamical analysis of stationary black holes. Then the volume term in (2.13) vanishes identically (since the time derivatives vanish) and the entire formula reduces to [see Eq. (4.2) in [19]]

$$\delta\mathcal{H} = \frac{s}{\gamma}\int_{\partial V^-}(\kappa\delta\lambda + \nu^A\delta\mathcal{W}_A), \tag{3.26}$$

where the right-hand side is the (only remaining) boundary term [20] corresponding to the cross section $\partial V^-$ of the horizon $S^-$. Here $\mathcal{H}$ is our Hamiltonian (3.25), $s = \pm 1$ is a constant which depends upon the time orientation of the Killing vector so that $-s\kappa$ is the surface gravity in usual circumstances [one should also keep in mind further negative signs in (3.26) which might arise from the orientation

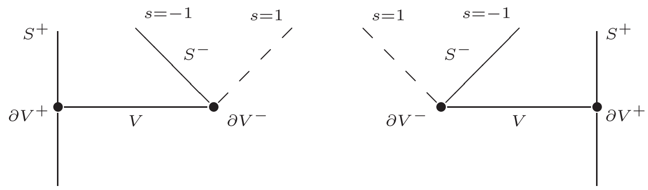


FIG. 1. The orientation of $\partial V^-$.

of the boundary; see Fig. 1]. The field $\mathcal{W}_A$ is defined on the horizon by the formula

$$\mathcal{W}_A = -\lambda dx^0(\nabla_A K),$$

where $K$ is a Killing vector field which is null on a horizon, assuming that the horizon is located at $x^n = \text{const}$, and that $x^0$ is a coordinate on the horizon satisfying

$$dx^0(K) = 1.$$

It is conceivable that the only such vacuum black hole space-times which are asymptotic to the Birmingham metrics are the Birmingham metrics themselves, in which case the "thermodynamical identity" (3.26) can be derived by the trivial calculation of Appendix A 4. However, this is not clear, and unlikely in higher dimensions in any case.

As already emphasized, the positive energy theorem III.1 remains valid in the black hole setting.

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## APPENDIX A: BIRMINGHAM METRICS

Consider an $(n+1)$-dimensional metric, $n \geq 3$, of the form

$$g = -f dt^2 + \frac{dr^2}{f} + r^2 \underbrace{\mathring{h}_{AB}(x^C) dx^A dx^B}_{:=\mathring{h}}, \qquad \text{(A1)}$$

where $\mathring{h}$ is a Riemannian metric on a compact manifold $\mathring{M}$ with constant scalar curvature $\mathring{R}$; we denote by $x^A$ local coordinates on $\mathring{M}$. As discussed in [21], for any $m \in \mathbb{R}$ and $\ell \in \mathbb{R}^*$ the function

$$f = \frac{\mathring{R}}{(n-1)(n-2)} - \frac{2m}{r^{n-2}} - \frac{r^2}{\ell^2} \qquad \text{(A2)}$$

leads to a vacuum metric,

$$R_{\mu\nu} = \frac{n}{\ell^2} g_{\mu\nu}; \qquad \text{(A3)}$$

thus $\ell$ is a constant related to the cosmological constant as in (C17) below. (Clearly, the case $n = 2$ would require separate considerations, and we will therefore ignore this dimension in our work.) The multiplicative factor 2 in front of $m$ is convenient in dimension 3 when $\mathring{h}$ is a unit round metric on $S^2$, and we will keep this factor regardless of topology and dimension of $\mathring{M}$.

There is a rescaling of the coordinate $r = b\bar{r}$, with $b \in \mathbb{R}^*$, which leaves (A1) and (A2) unchanged (up to "adding bars") if moreover

$$\bar{\mathring{h}} = b^2 \mathring{h}, \qquad \bar{m} = b^{-n} m, \qquad \bar{t} = bt. \qquad \text{(A4)}$$

We can use this to achieve

$$\beta := \frac{\mathring{R}}{(n-1)(n-2)} \in \{0, \pm 1\}, \qquad \text{(A5)}$$

which will be assumed from now on. The set $\{r = 0\}$ corresponds to a singularity when $m \neq 0$. Except in the case $m = 0$ and $\beta = -1$, by an appropriate choice of the sign of $b$ we can always achieve $r > 0$ in the regions of interest. This will also be assumed from now on.

For reasons which should be clear from the main text, we will now be seeking functions $f$ which, after a suitable extension of the space-time manifold and metric, lead to spatially periodic solutions.

### 1. Cylindrical solutions

Consider, first, the case where $f$ has no zeros. Since $f$ is negative for large $|r|$, $f$ is negative everywhere. It therefore makes sense to rename $r$ to $\tau > 0$, $t$ to $x$, and $-f$ to $F > 0$, leading to the metric

$$g = -\frac{d\tau^2}{F(\tau)} + F(\tau) dx^2 + \tau^2 \mathring{h}. \qquad \text{(A6)}$$

The level sets of the time coordinate $\tau$ are infinite cylinders with topology $\mathbb{R} \times \mathring{M}$, with a product metric. Note that the extrinsic curvature of those level sets is never zero because of the $\tau^2$ term in front of $\mathring{h}$, except possibly for the $\{\tau = 0\}$ slice in the case $\beta = -1$ and $m = 0$.

Assuming that $m \neq 0$, the region $r \equiv \tau \in (0, \infty)$ is a "big bang–big freeze" space-time with cylindrical spatial sections. A $(\tau, x)$-projection diagram (in the sense of [22]) is an infinite horizontal strip with a singular spacelike boundary at $\tau = 0$, and a smooth conformal spacelike boundary at $\tau = \infty$; see Fig. 2.

In the case $m = 0$ and $\beta = 0$ the spatial sections are again cylindrical, with the boundary $\{\tau = 0\}$ being now at infinite temporal distance: indeed, setting $T = \ln \tau$, when $m = 0$ and $\beta = 0$ we can write

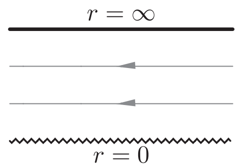


FIG. 2. The $(t, r)$-projection diagram when $m < 0$ and $f$ has no zeros.

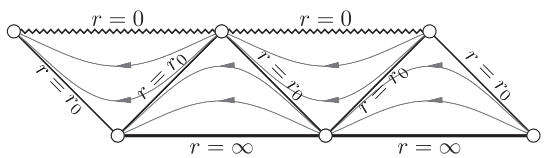


FIG. 3. The $(t, r)$-projection diagram for (suitably extended) Birmingham metrics with $f \leq 0$, and $f$ vanishing precisely at $r_0$.

$$g = -\ell^2 \frac{d\tau^2}{\tau^2} + \frac{\tau^2}{\ell^2} dx^2 + \tau^2 \mathring{h} \;\; = -\ell^2 dT^2 + e^{2T}\Big(\tfrac{dx^2}{\ell^2} + \mathring{h}\Big).$$

When $\mathring{h}$ is a flat torus, this is one of the forms of the de Sitter metric ([23], p. 125).

The next case which we consider is $f \leq 0$, with $f$ vanishing precisely at one positive value $r = r_0$. This occurs if and only if $\beta = 1$ and

$$r_0 = \sqrt{\frac{n-2}{n}}\ell, \qquad m = \frac{r_0^n}{(n-2)\ell^2}. \tag{A7}$$

A $(r = \tau, t = x)$-projection diagram can be found in Fig. 3.

No nontrivial, periodic, time-symmetric $(K_{ij} = 0)$ spacelike hypersurfaces occur in all space-times above. Periodic spacelike hypersurfaces with $K_{ij} \not\equiv 0$ arise, but a Hamiltonian analysis of initial data asymptotic to such hypersurfaces goes beyond the scope of this work.

From now on we assume that $f$ has positive zeros.

### 2. Spheres and naked singularities

Assuming that $m = 0$ but $\beta \neq 0$, we must have $\beta = 1$ in view of our hypothesis that $f$ has positive zeros. For $r \geq 0$ the function $f$ has exactly one zero, $r = \ell$. The boundaries $\{r = 0\}$ correspond either to regular centers of symmetry, in which case the level sets of $t$ are $S^n$'s or their quotients, or to conical singularities. See Fig. 4.

If $m < 0$ the function $f$: $(0, \infty) \rightarrow \mathbb{R}$ is monotonously decreasing, tending to minus infinity as $r$ tends to zero, where a naked singularity occurs, and to minus infinity when $r$ tends to $\infty$; hence $f$ has then precisely one zero. The $(t, r)$-projection diagram can be seen again in Fig. 4.

No spatially periodic time-symmetric spacelike hypersurfaces occur in the space-times above.

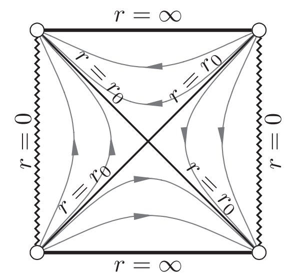


FIG. 4. The $(t, r)$-projection diagram for a maximal extension of the Birmingham metrics with $m < 0$, $\beta \in \mathbb{R}$, or $m = 0$ and $\beta = 1$, with $r_0$ defined by the condition $f(r_0) = 0$. The set $\{r = 0\}$ is a singularity unless the metric is the de Sitter metric ($\mathring{M} = S^{n-1}$ and $m = 0$), or a suitable quotient thereof so that $\{r = 0\}$ corresponds to a center of (possibly local) rotational symmetry.

### 3. Spatially periodic time-symmetric initial data

We continue with the remaining cases, that is, $f$ having zeros and $m > 0$. [When $\beta = 1$ this implies $0 < m \leq \frac{1}{n}(1 - \frac{2}{n})^{\frac{n}{2}-1}\ell^{n-2}$.] The function $f$: $(0, \infty) \rightarrow \mathbb{R}$ is then concave and thus has precisely two first-order zeros, except when $m$ attains its maximal allowed value, a case already discussed [see (A7)]. A projection diagram for a maximal extension of the space-time, for the cases with two first-order zeros, is provided by Fig. 5. The level sets of $t$ within each of the diamonds in that figure can be smoothly continued across the bifurcation surfaces of the Killing horizons to smooth spatially periodic Cauchy surfaces.

Observe that for $\beta = 1$ and $0 < \frac{m}{\ell^{n-2}} < \frac{1}{n}(1 - \frac{2}{n})^{\frac{n}{2}-1}$ the roots $r_0^{(a)}$, $a = 1, 2$, satisfy

$$r_0^{(a)} \in (0, \ell). \tag{A8}$$

To see this, note that the equation $f(r_0) = 1 - \frac{2m}{r_0^{n-2}} - \frac{r_0^2}{\ell^2} = 0$ is equivalent to

$$W_n(x) := x^{n-2}(1 - x)(1 + x) = \frac{2m}{\ell^{n-2}},$$

where $x = r_0/\ell$. The polynomials $W_n$ are positive precisely on (0, 1), which implies the result. Compare Fig. 6.

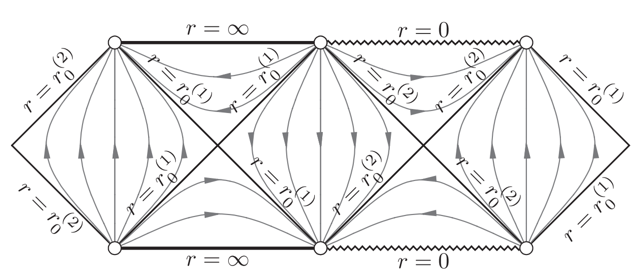


FIG. 5. The $(t, r)$-projection diagram for suitably extended Birmingham metrics with exactly two first-order zeros of $f$. The symbols $r_0^{(a)}$, $a = 1, 2$, denote zeros of $f$.

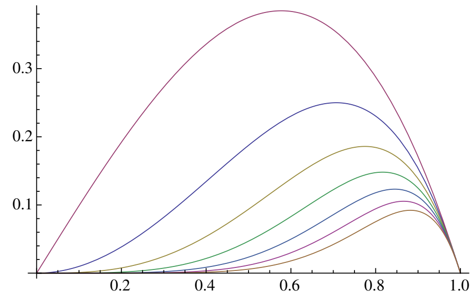


FIG. 6 (color online). The polynomials $W_n$ for $3 \leq n \leq 9$.

### 4. Killing horizons

The locations of Killing horizons of the Birmingham metrics are defined, in space dimension $n$, by the condition

$$f(r_0) = \beta - \frac{2m}{r_0^{n-2}} - \frac{r_0^2}{\ell^2} = 0.$$

Thus, variations of the metric on the horizons satisfy

$$0 = \delta f|_{r=r_0} = \left[ (\partial_r f)\delta r - \frac{2}{r^{n-2}}\delta m \right] \Big|_{r=r_0}; \quad \text{(A9)}$$

equivalently

$$\delta m = \frac{1}{2(n-1)}(\partial_r f)\delta(r^{n-1}) = \frac{1}{(n-1)\sigma_{n-1}} \frac{(\partial_r f)}{2} \Big|_{r=r_0} \delta A, \quad \text{(A10)}$$

where $r^{n-1}\sigma_{n-1}$ is the $\mathring{h}$-volume of the cross section of the horizon.

Let us check that $\kappa := \frac{(\partial_r f)}{2}|_{r=r_0}$ coincides with the surface gravity of the horizon, defined through the usual formula

$$\nabla_K K = -\kappa K, \quad \text{(A11)}$$

where $K$ is the Killing vector field which is null on the horizon. For this, we rewrite the space-time metric (A1) in the familiar form

$$g = -f du^2 - 2du dr + r^2 \mathring{h},$$

where $du = dt - f^{-1}dr$. The Killing field $K = \partial_u = \partial_t$ is indeed tangent to the horizon and null on it. Formula (A11) implies that

$$\kappa = -\Gamma^u_{uu} = -\frac{1}{2} g^{u\lambda}(2g_{\lambda u,u} - g_{uu,\lambda}). \quad \text{(A12)}$$

The inverse metric equals

$$g^\sharp = -2\frac{\partial}{\partial u}\frac{\partial}{\partial r} + f\left(\frac{\partial}{\partial r}\right)^2 + r^{-2}\mathring{h}^\sharp,$$

whence $g^{u\lambda} = -\delta^\lambda_r$, and

$$\kappa = -\frac{1}{2} g_{uu,r} = \frac{(\partial_r f)}{2} \Big|_{r=r_0},$$

as claimed. We conclude that on Killing horizons it holds that

$$\delta m = \frac{1}{(n-1)\sigma_{n-1}} \kappa \Big|_{r=r_0} \delta A. \quad \text{(A13)}$$

### 5. Singularities

Consider a metric of the form

$$g = -e^{-2\chi(\tau)}d\tau^2 + e^{2\chi(\tau)}dx^2 + \tau^2 \mathring{h},$$

with $h$ as before. For $A = 1, \ldots, n$ let $\mathring{\theta}^A$ be an orthonormal (ON)-coframe for $\mathring{h}$,

$$\mathring{h} = \sum_{A=1}^{n-1} \mathring{\theta}^A \otimes \mathring{\theta}^A,$$

and let $\mathring{\omega}_{AB}$ and $\mathring{\Omega}_{AB}$ be the associated connection and curvature forms, as in the Cartan structure equations:

$$0 = d\mathring{\theta}^A + \mathring{\omega}^A{}_B \wedge \mathring{\theta}^B, \qquad \mathring{\Omega}^A{}_B = d\mathring{\omega}^A{}_B + \mathring{\omega}^A{}_C \wedge \mathring{\omega}^C{}_B.$$

Let $\theta^\mu$ be the following $g$-ON coframe:

$$\theta^0 = e^{-\chi}d\tau, \qquad \theta^A = \tau\mathring{\theta}^A, \qquad \theta^n = e^\chi dx.$$

The condition of vanishing of torsion is solved by setting

$$\omega^n{}_A = 0, \qquad \omega^n{}_0 = e^\chi \dot{\chi}\theta^n = \frac{1}{2}(\dot{e^{2\chi}})dx,$$

$$\omega^A{}_0 = e^\chi \mathring{\theta}^A, \qquad \omega^A{}_B = \mathring{\omega}^A{}_B.$$

This gives the following curvature two-forms:

$$\Omega^0{}_n = \frac{1}{2}(\ddot{e^{2\chi}})\delta^0_{[\mu}g_{\nu]n}\theta^\mu \wedge \theta^\nu,$$

$$\Omega^0{}_A = \frac{1}{2}(\dot{e^{2\chi}})\tau^{-1}\delta^0_{[\mu}g_{\nu]A}\theta^\mu \wedge \theta^\nu,$$

$$\Omega^n{}_A = \frac{1}{2}(\dot{e^{2\chi}})\tau^{-1}\delta^n_{[\mu}g_{\nu]A}\theta^\mu \wedge \theta^\nu,$$

$$\Omega^A{}_B = \frac{1}{2}\tau^{-2}(\mathring{\Omega}^A{}_{BCD} + 2e^{2\chi}\delta^A_{[C}\delta_{D]B})\theta^C \wedge \theta^D.$$

Suppose that $g$ is a Birmingham metric with $m = 0$; thus

$$e^{2\chi} = -\beta + \frac{\tau^2}{\ell^2}$$

for a constant $\beta$, and then

$$\frac{1}{2}(\ddot{e^{2\chi}}) = \frac{1}{2}(\dot{e^{2\chi}})\tau^{-1} = \tau^{-2}(e^{2\chi} + \beta) = \frac{1}{\ell^2}.$$

If $\mathring{h}$ is a space-form, with

$$\mathring{\Omega}^A{}_{BCD} = 2\beta\delta^A_{[C}\delta_{D]B},$$

consistently with (A5), we obtain

$$R_{\mu\nu\rho\sigma} = \frac{2}{\ell^2} g_{\mu[\rho}g_{\sigma]\nu}.$$

If, however, $\mathring{h}$ is *not* a space-form, we have

$$\mathring{\Omega}^A{}_{BCD} = 2\beta\delta^A_{[C}\delta_{D]B} + r^A{}_{BCD},$$

for some nonidentically vanishing tensor $r^A{}_{BCD}$, with all traces zero. Hence

$$R_{\mu\nu\rho\sigma} = \frac{2}{\ell^2} g_{\mu[\rho}g_{\sigma]\nu} + \tau^{-2} r_{\mu\nu\rho\sigma},$$

where the functions $r_{\mu\nu\rho\sigma}$ are $\tau$ independent in the current frame, and vanish whenever one of the indices is 0 or $n$. This gives

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{2n(n+1)}{\ell^4} + r^{\mu\nu\rho\sigma}r_{\mu\nu\rho\sigma} = \frac{2n(n+1)}{\ell^4} + \tau^{-4}\sum_{A,B,C,D=1}^{n-1}(r_{ABCD})^2,$$

which is singular at $\tau = 0$.

## APPENDIX B: A CONTROL-RESPONSE CALCULATION

To give our considerations a precise Hamiltonian meaning we need to define explicitly the family of metrics considered, as well as the time parameter with respect to which the Hamiltonian will be determined. The latter is closely related to a choice of the lapse function.

Here we will consider two distinct settings: (a) a boundary $\partial V$ at finite distance with prescribed induced metric there, and (b) a family of metrics which asymptote, along the asymptotically periodic ends, to Birmingham metrics.

At the boundary, or asymptotically, we make the following choice of the lapse function:

$$\frac{k}{\nu}\frac{\nu_0}{k_0}\bigg|_{\partial V} = 1 \quad \text{or} \quad \frac{k}{\nu}\frac{\nu_0}{k_0} \to 1; \tag{B1}$$

as already mentioned, this corresponds to a choice of the boundary time, or asymptotic time. The choice is motivated by the fact that (B1) holds for all metrics in the Birmingham family; see (B4)–(B7) below.

In the case of a boundary at finite distance, we choose an $(n-1)$-dimensional metric $r^2\mathring{h}$ on $\partial V$, as in (A1), and consider *the collection of all initial metrics which induce* $r^2\mathring{h}$ on $\partial V$.

In the asymptotic case, we choose a compact Riemannian Einstein manifold $(\mathring{M}, \mathring{h})$ and consider *the collection of all metrics which asymptote to the associated Birmingham solutions* along the cylindrical end.

It should be mentioned that the definition of a phase space requires describing also the space of canonical momenta. In the finite-boundary case this issue will be ignored in this work. Concerning asymptotically cylindrical metrics, we will only consider asymptotically vanishing extrinsic curvature tensors $K_{ij}$. We plan to return to asymptotically periodic tensors $K_{ij}$ in future work.

In view of (B1), when $\mathbf{Q}_A = 0$ and $\overset{\perp}{Q}{}^{AB}$ is pure trace at $\partial V$, it is useful [using (2.12) and (3.16)] to rewrite the boundary form (3.15) as

$$\begin{aligned} 2\nu\delta\mathbf{Q} - 2\nu^A\delta\mathbf{Q}_A + \overset{\perp}{Q}{}^{AB}\delta g_{AB} &= \delta(\lambda\nu k) + \lambda\nu^2\delta\left(\frac{k}{\nu}\right) + \nu\left(2a - \frac{n-3}{n-1}k\right)\delta\lambda \\ &= \delta[\lambda(\nu k - \nu_0 k_0)] + \lambda\nu^2\frac{k_0}{\nu_0}\delta\left(\frac{k}{\nu}\frac{\nu_0}{k_0}\right) + \psi, \end{aligned} \tag{B2}$$

where

$$\psi := \delta(\lambda\nu_0 k_0) - \lambda\nu^2\frac{k_0}{\nu_0}\frac{k}{\nu}\delta\left(\frac{\nu_0}{k_0}\right) + \nu\left(2a - \frac{n-3}{n-1}k\right)\delta\lambda, \tag{B3}$$

whereas $k_0$ and $\nu_0$ are the corresponding quantities calculated on a "reference configuration" corresponding to $m = m_0$. For configurations without boundaries with

$$\mathbf{Q}_A \underset{x\to\infty}{\longrightarrow} 0,$$

the above equalities should be understood in the limit $x \to \infty$.

For the Birmingham metrics we have

$$f = \beta - \frac{2m}{\phi^{n-2}} - \frac{\phi^2}{\ell^2}; \qquad f_0 = \beta - \frac{2m_0}{\phi^{n-2}} - \frac{\phi^2}{\ell^2}, \tag{B4}$$

$$\nu = \sqrt{f}; \qquad \nu_0 = \sqrt{f_0}, \tag{B5}$$

$$k = -\frac{n-1}{\phi}\sqrt{f}; \qquad k_0 = -\frac{n-1}{\phi}\sqrt{f_0}, \tag{B6}$$

$$\lambda = \phi^{n-1}\sqrt{\det\mathring{h}}; \qquad \nu a = \frac{(n-2)m}{\phi^{n-1}} - \frac{\phi}{\ell^2}. \tag{B7}$$

This implies that $\psi$ vanishes identically on $\partial V$ with the above boundary conditions, so that the entire boundary form reduces to

$$(2n\delta\mathbf{Q} - 2n^A\delta\mathbf{Q}_A + \overset{\perp}{Q}{}^{AB}\delta g_{AB}) = \delta[\lambda(\nu k - \nu_0 k_0)] + \lambda\nu^2\frac{k_0}{\nu_0}\delta\left(\frac{k}{\nu}\frac{\nu_0}{k_0}\right). \tag{B8}$$

The last term vanishes because of the time gauge (B1), whereas the first term represents the variation of mass: indeed, for all Birmingham metrics we have

$$\lambda(\nu k - \nu_0 k_0) = 2(n-1)(m - m_0)\sqrt{\det \mathring{h}}. \tag{B9}$$

Hence

$$\int_{x=x_0} \lambda(k\nu - k_0\nu_0) + 2(n-1)|\partial V|_{\mathring{h}} m_0 = 2(n-1)|\partial V|_{\mathring{h}} m, \tag{B10}$$

when the boundary data on $\partial V$ are as above and where, as before, $|\partial V|_{\mathring{h}}$ denotes the volume of $\partial V$ in the metric $\mathring{h}$. In particular the integrand is independent of $x_0$. Similarly,

$$\lim_{x_0\to\infty} \int_{x=x_0} \lambda(k\nu - k_0\nu_0) + 2(n-1)|\partial V|_{\mathring{h}} m_0 = 2m(n-1)|\partial V|_{\mathring{h}}, \tag{B11}$$

along each asymptotically periodic end.

## APPENDIX C: THE YAMABE EQUATION ON CYLINDERS

In this section we relate the parameter $m$ appearing in the Schwarzschild–de Sitter metrics to a Hamiltonian for the spherically symmetric Yamabe equation. The reader should note that the Hamiltonian here is a Hamiltonian for the dynamics in $x$, not to be confused with that for the dynamics in time, as used elsewhere in this work.

Let

$$g_{ij} = \varphi^{\frac{4}{n-2}} \tilde{g}_{ij}. \tag{C1}$$

Recall the vacuum Lichnerowicz equation with cosmological constant $\Lambda$, in space dimension $n$,

$$\Delta_{\tilde{g}}\varphi - \frac{n-2}{4(n-1)}\tilde{R}\varphi = -\tilde{\sigma}^2\varphi^{(2-3n)/(n-2)} + \tilde{\beta}\varphi^{\frac{n+2}{n-2}}, \tag{C2}$$

where

$$\tilde{\sigma}^2 := \frac{n-2}{4(n-1)}|\tilde{L}|^2_{\tilde{g}}, \quad \tilde{\beta} := \frac{n-2}{4n}\tau^2 - \frac{n-2}{2(n-1)}\Lambda. \tag{C3}$$

Here $\tilde{L}_{ij}$ is $\tilde{g}$-transverse traceless, and $\tau$ is the trace of the extrinsic curvature tensor $\tau = g^{ij}K_{ij}$, assumed to be constant, with $K_{ij}$ obtained from $\tilde{L}_{ij}$ by the usual formula.

Suppose that

$$\tilde{g} = dx^2 + \mathring{h}, \tag{C4}$$

where $\mathring{h}$ is as in (A1). We then have $\tilde{R} = \mathring{R}$, and when $\tau$ is a constant we can seek an $x^A$-independent solution of (C2) with $\tilde{L}_{ij} = 0$:

$$\frac{d^2\varphi}{dx^2} - \frac{n-2}{4(n-1)}\mathring{R}\varphi = \tilde{\beta}\varphi^{\frac{n+2}{n-2}}. \tag{C5}$$

Equation (C5) has a usual first integral: setting

$$H = \frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 - \frac{n-2}{8(n-1)}\mathring{R}\varphi^2 - \frac{(n-2)\tilde{\beta}}{2n}\varphi^{\frac{2n}{n-2}}, \tag{C6}$$

we have

$$\frac{dH}{dx} = 0.$$

We apply the above to the Birmingham metrics with $f \geq 0$; as discussed in Appendix A, the metrics with $f \leq 0$ do not occur as asymptotic models in our context. We only consider regions where $f > 0$; the final formulas remain valid at $f = 0$ by continuity.

The field of unit normals $N$ to the static slices $t = \text{const}$, which we denote by $\mathcal{S}_t$, is given by

$$N = \frac{1}{\sqrt{f}}\partial_t. \tag{C7}$$

For those slices we have $\tau = 0$.

The volume form $d\mu_{\mathring{M}}$ on the submanifolds of constant $t$ and $x$ reads

$$d\mu_{\mathring{M}} = \lambda d\mathring{\mu}_{\mathring{M}}, \quad \text{with} \quad d\mathring{\mu}_{\mathring{M}} = \sqrt{\det \mathring{h}_{AB}}\, d^{n-1}x, \tag{C8}$$

and where

$$\lambda = r^{n-1} = \varphi^{\frac{2(n-1)}{n-2}}, \tag{C9}$$

with $\varphi$ as in (C1) and (C2):

$$\gamma := \frac{dr^2}{f} + r^2\mathring{h} = \varphi^{\frac{4}{n-2}}(dx^2 + \mathring{h}). \tag{C10}$$

The last equation implies

$$\varphi = r^{\frac{n-2}{2}}, \qquad \frac{dr}{dx} = \varphi^{\frac{2}{n-2}}\sqrt{f} = r\sqrt{f}, \tag{C11}$$

$$\sqrt{\det\gamma} = \frac{r^{n-1}\sqrt{\det \mathring{h}}}{\sqrt{f}} \quad \text{or} \quad \varphi^{\frac{2n}{n-2}}\sqrt{\det \mathring{h}}. \tag{C12}$$

Let $\mathbf{m}$ denote the field of unit normals to the level sets of $r$ within $\mathcal{S}_t$, and let $k$ denote the extrinsic curvature, within $\mathcal{S}_t$, of those level sets. We have

$$\mathbf{m} = \sqrt{f}\partial_r = \varphi^{-\frac{2}{n-2}}\partial_x, \tag{C13}$$

$$k = \frac{1}{\sqrt{\det\gamma}}\partial_r\left(\sqrt{\det\gamma}\,\mathbf{m}^r\right) = \frac{(n-1)}{r}\sqrt{f} = \varphi^{-\frac{2n}{n-2}}\partial_x\left(\varphi^{\frac{2(n-1)}{n-2}}\right) = \frac{2(n-1)}{n-2}\varphi^{-\frac{n}{n-2}}\partial_x\varphi. \tag{C14}$$

It follows that

$$\partial_x \varphi = \frac{(n-2)\varphi^{\frac{n}{n-2}}}{2r}\sqrt{f} = \frac{(n-2)}{2} r^{\frac{n-2}{2}}\sqrt{f}, \tag{C15}$$

and that the constant of motion $H$ of (C6) equals

$$\begin{aligned} H &= \frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 - \frac{n-2}{8(n-1)}\mathring{R}\varphi^2 - \frac{(n-2)\tilde{\beta}}{2n}\varphi^{\frac{2n}{n-2}} \\ &= \frac{(n-2)^2}{8} r^{n-2}\left[f - \frac{\mathring{R}}{(n-2)(n-1)} - \frac{4\tilde{\beta}}{n(n-2)} r^2\right] \\ &= -\frac{(n-2)^2}{4} m, \end{aligned} \tag{C16}$$

provided that

$$\frac{(n-2)^2}{8\ell^2} = -\frac{(n-2)\tilde{\beta}}{2n} = \frac{(n-2)}{2n} \times \frac{(n-2)}{2(n-1)}\Lambda,$$

which will be the case if

$$\frac{1}{\ell^2} = \frac{2\Lambda}{n(n-1)}. \tag{C17}$$

## APPENDIX D: EINSTEIN EQUATIONS IN $n+1$ DIMENSIONS

Hamiltonian dynamics is usually derived from a Lagrangian. The latter is determined by the equations of the theory up to a multiplicative constant. One therefore needs a prescription which determines this constant. For this, we decree that for a point particle of rest mass $m_0$ moving on a timelike curve $\Gamma$ the Lagrangian is, independently of dimension,

$$\mathcal{L}_{m_0} = -m_0 \int_{\mathbb{R}} \sqrt{g(\dot{\Gamma}, \dot{\Gamma})} dt. \tag{D1}$$

Equivalently, the energy-momentum tensor $T_{\mu\nu} := \partial \mathcal{L}_{m_0}/\partial g^{\mu\nu}$ of such a particle is

$$T_{\mu\nu} = m_0 u_\mu u_\nu \delta_\Gamma,$$

where $\delta_\Gamma$ is the distribution acting on functions as

$$\langle \delta_\Gamma, f\rangle = \int_{\mathbb{R}} (f \circ \Gamma)(t)\sqrt{|g(\dot{\Gamma}, \dot{\Gamma})|} dt.$$

The Einstein equations in $n+1$ dimensions, which we write in the form

$$G_{\mu\nu} = \gamma T_{\mu\nu}, \tag{D2}$$

where $\gamma$ is a dimension-dependent constant, are compatible with (D2) if

$$\mathcal{L} = \frac{1}{2\gamma}\int R\mu_g - \mathcal{L}_{m_0}. \tag{D3}$$

We emphasize that the considerations here are *not* supposed to be rigorous. The aim is to give a heuristic justification of the choice of the constants involved, and the questions of convergence of the integrals, or consistency of the scheme, are completely irrelevant for our purposes.

In order to relate the value of $\gamma$ to physics in $n+1$ dimensions we consider the "Newtonian" limit of (D2): we assume that the metric is time independent, and takes the form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

where $\eta_{\mu\nu}$ is the Minkowski metric. We suppose that all expressions quadratic in the $h_{\mu\nu}$'s and their derivatives can be neglected in the calculations that follow. Taking $T_{\mu\nu}$ of the form $\rho\delta^0_\mu\delta^0_\nu$, and a harmonic gauge

$$\partial_\mu\underbrace{\left(h^{\mu\nu} - \frac{1}{2}\eta^{\alpha\beta}h_{\alpha\beta}\eta^{\mu\nu}\right)}_{=:\hbar^{\mu\nu}} = 0$$

(with all indices raised and lowered with the Minkowski metric), a standard calculation in which time derivatives are also neglected leads to

$$-\frac{1}{2}\Delta_e \hbar_{\mu\nu} = \rho\delta^0_\mu\delta^0_\nu, \tag{D4}$$

where $\Delta_e$ is the Laplace operator of the Euclidean metric.

Recall the identity, in space dimension $n \geq 3$,

$$\Delta_e \frac{1}{r^{n-2}} = -(n-2)\omega_{n-1}\delta_0,$$

where $\omega_d$ denotes the volume of a unit, round $d$-dimensional sphere. The solution of (D4) for a point distribution with total mass $M$ therefore takes the form

$$\hbar_{\mu\nu} = \frac{2\gamma M}{(n-2)\omega_{n-1}r^{n-2}}\delta^0_\mu\delta^0_\nu. \tag{D5}$$

Consider an approximate geodesic of the form $(t, \vec{x}(t))$. Assuming that all terms quadratic in $\dot{\vec{x}}$ and its derivatives can be neglected, the coordinate acceleration vector $\vec{a}$ equals

$$a^k = \ddot{x}^k \approx -\Gamma^k_{00} \approx \frac{1}{2}\partial_k h_{00} =: -\partial_k\varphi,$$

where $\varphi$ is the Newtonian potential. From (D5) we have

$$h_{00} = \frac{2\gamma M}{(n-1)\omega_{n-1}r^{n-2}},$$

leading to

$$\varphi = -\frac{\gamma M}{(n-1)\omega_{n-1}r^{n-2}}.$$

This makes it clear how $\gamma$ is related to the $(n+1)$-dimensional Newton constant $G_n$:

$$\begin{aligned} \vec{F} \equiv m_0\vec{a} = -G_n m_0 M\frac{\vec{x}}{r^{n-1}} = -m_0\nabla\varphi \Longleftrightarrow \\ \gamma = \frac{(n-1)\omega_{n-1}}{n-2}G_n. \end{aligned} \tag{D6}$$

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