Charged black hole with a scalar hair in (2 + 1) dimensions

Wei Xu* and Liu Zhao†

School of Physics, Nankai University, Tianjin 300071, China

(Received 28 January 2013; revised manuscript received 27 March 2013; published 11 June 2013)

We obtain and analyze an exact solution to Einstein-Maxwell-scalar theory in (2 + 1) dimensions, in which the scalar field couples to gravity in a nonminimal way, and it also couples to itself with the self-interacting potential solely determined by the metric ansatz. A negative cosmological constant naturally emerges as a constant term in the scalar potential. The metric is static and circularly symmetric and contains a curvature singularity at the origin. The conditions for the metric to contain 0, 1, and 2 horizons are identified, and the effects of the scalar and electric charges on the size of the black hole radius are discussed. Under proper choices of parameters, the metric degenerates into some previously known solutions in (2 + 1)-dimensional gravity.

DOI: 10.1103/PhysRevD.87.124008 PACS numbers: 04.20.Jb, 04.40.Nr, 04.70.−s

I. INTRODUCTION

Gravity in (2 + 1)-dimensional spacetime has been a fascinating area of theoretical investigations during the last few decades. Such studies were initiated in the early 1980s [1–4]. It was once believed [4] that there is no black hole solutions in (2 + 1) dimensions in the absence of a matter source because there is no propagating degrees of freedom. However, since the discovery of Bañados-Teitelboim-Zanelli (BTZ) [5] Martinez-Teitelboim-Zanelli [6] black holes and the asymptotic conformal symmetry [7,8], it became increasingly clear that gravity in (2 + 1) dimensions is much more interesting in its own right, not only because black hole solutions exist but also because such theories are ideal theoretical laboratories for studying AdS/CFT(condensed matter theory) [7–18], gravity-fluid dual [19], (holographic) phase transitions [20,21], etc. Moreover, the study of gravity in (2 + 1) dimensions is also expected to shed some light on the understanding of more realistic or complicated cases of four- and higher-dimensional gravities.

Recently, (2 + 1)-dimensional gravity with a matter source has attracted considerable interest. Besides the standard Maxwell source [6,22,23], the inclusion of extra scalar field(s) [8,10,12–15,20,23–46], higher rank tensor fields [47–52], higher curvature terms [53–60], and/or gravitational Chern-Simons terms [1,2,61] are also intensively studied. Unlike gravities in four and higher dimensions, it is possible to include a finite number of higher rank tensor fields in (2 + 1) dimensions [47–52]. The inclusion of gravitational Chern-Simons terms will bring in propagating degrees of freedom in (2 + 1) dimensions [62,63]. Moreover, it is often much easier to obtain and analyze black hole solutions in (2 + 1) dimensions than in other dimensions.

In this paper, we aim to study the black hole solution in an Einstein-Maxwell-scalar gravity with a nonminimally coupled scalar field in (2 + 1) dimensions. Gravity coupled with a scalar field is not a new idea. Black hole solutions in such theories are known as hairy black holes, and there is already a huge amount of literature on this subject [8,10,12–15,20,23–46,64–70], and the spacetime is not only restricted to be (2 + 1)-dimensional [64–70]. The scalar field ϕ may be coupled either minimally [8,12–14,23,24,34–40] or nonminimally [9–11,41–46,65–70] to gravity, and it may or may not couple to itself through a self-interacting potential $U(\phi)$. In the model with which we shall be dealing, ϕ couples to gravity in a nonminimal way, and it also couples to itself via a self-potential $V(\phi)$. The action reads

$$I = \frac{1}{2}\int \mathrm{d}^3x\sqrt{-g}\Big[R - g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \xi R\phi^2 - 2V(\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\Big], \tag{1}$$

where ξ is a constant signifying the coupling strength between gravity and the scalar field, and we have set the gravitational constant κ equal to unity. A similar action in four-dimensional spacetime was studied in Ref. [69].

In the absence of the scalar potential $V(\phi)$ and the Maxwell field, the constant value $\xi = \frac{1}{8}$ will make the coupling between gravity and the free scalar field ϕ conformally invariant. In the presence of $V(\phi)$, however, the conformal symmetry will, in general, be broken. Nonetheless, the special value $\xi = \frac{1}{8}$ for the coupling constant will greatly simplify the solution. So, we will stick to this particular value of the gravity-scalar coupling. Notice that we did not include explicitly a cosmological constant term in the action; however, it will turn out that, as far as a black hole solution is concerned, a negative cosmological constant will automatically emerge.

The rest of the paper is organized as follows. In Sec. II, we describe the exact solution to the field equations, which follow from the action (1) as well as the associated scalar

*xuweifuture@mail.nankai.edu.cn
†Corresponding author.
lzhao@nankai.edu.cn

potential. Some of the basic properties of the scalar potential are discussed. Meanwhile, the basic geometric properties of the metric are also outlined. In Sec. III, we describe some special degenerated cases of the solution, some of which are already known in the literature. Section IV is devoted to the analysis of the metric, in particular, the conditions for the metric to behave as an asymptotic AdS_3 spacetime with a naked singularity, as an extremal charged hairy black hole and as a nonextremal charged hairy black hole are identified. In Sec. V, we discuss the effect of the scalar and electric charges on the size of the black hole horizons. Finally, in Sec. VI we make some discussions and outline some of the open problems that we intend to solve in subsequent investigations.

II. SCALAR POTENTIAL AND THE STATIC, CIRCULARLY SYMMETRIC SOLUTION

By a straightforward variational process and discarding all possible boundary terms, we can write down the field equations associated with the action (1) as follows:

$$G_{\mu\nu} - T^{[\phi]}_{\mu\nu} - T^{[A]}_{\mu\nu} + V(\phi)g_{\mu\nu} = 0, \tag{2}$$

$$\Box\phi - \xi R\phi - V_\phi = 0, \tag{3}$$

$$\partial_\nu(\sqrt{-g}F^{\mu\nu}) = 0, \tag{4}$$

where

$$\begin{aligned} V_\phi &= \partial_\phi V(\phi), \\ \Box &\equiv g^{\mu\nu}\nabla_\mu\nabla_\nu, \\ T^{[\phi]}_{\mu\nu} &= \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\nabla^\rho\phi\nabla_\rho\phi \\ &\quad + \xi(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + G_{\mu\nu})\phi^2, \\ T^{[A]}_{\mu\nu} &= \frac{1}{2}\left(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right). \end{aligned}$$

As mentioned earlier, we take $\xi = \frac{1}{8}$ throughout this paper. Note that we did not explicitly specify the scalar potential. Actually, it will be determined uniquely by the form of the metric ansatz to be given below. The same phenomenon also happens in the study of four-dimensional hairy black holes [65–67].

A. Circularly symmetric solution

We are interested in static, circularly symmetric solutions. To obtain such a solution, we assume that the metric takes the following form:

$$\mathrm{d}s^2 = -f(r)\mathrm{d}t^2 + \frac{1}{f(r)}\mathrm{d}r^2 + r^2\mathrm{d}\psi^2, \tag{5}$$

where the coordinate ranges are given by $-\infty < t < \infty$, $r \geq 0$, and $-\pi \leq \psi \leq \pi$. We also assume that both the scalar field ϕ and the Maxwell field A_μ depend only on the radial coordinate r. Under such assumptions, the Maxwell equation (4) gives

$$A_\mu dx^\mu = -Q\ln\left(\frac{r}{r_0}\right)dt, \tag{6}$$

where Q and r_0 are integration constants, $Q \in \mathbb{R}$ corresponds to the electric charge, and $r_0 > 0$ corresponds to the radial position of the zero electric potential surface, which can (but not necessarily) be set equal to $+\infty$.

Equation (2) has only three nontrivial components, i.e., the (tt), (rr), and $(\psi\psi)$ components. The (tt) and (rr) components together give rise to

$$3\left(\frac{d}{dr}\phi(r)\right)^2 - \phi(r)\frac{d^2}{dr^2}\phi(r) = 0.$$

Solving this equation, we get

$$\phi(r) = \pm\frac{1}{\sqrt{kr+b}}, \tag{7}$$

even without providing a concrete form for the scalar potential $V(\phi)$. For $\phi(r)$ not to be singular at finite nonzero r, we require $k \geq 0$, $b \geq 0$, and k and b cannot be simultaneously zero. Note that the special choice $b \neq 0$, $k = 0$ corresponds to constant ϕ.

In this paper, we are interested in solutions with a nonconstant scalar field, so we will be considering only the $k \neq 0$ branch of solutions. Inserting Eq. (7) into the remaining field equations, Eqs. (2) and (3), we can obtain explicit solutions for $f(r)$ and $V(\phi(r))$ as a function of r. However, the result is much too complicated. In particular, $f(r)$ contains terms that are proportional to the product of two logarithm functions and terms proportional to the special function dilog(r) defined as

$$\mathrm{dilog}\,(x) = \int_1^x \frac{\ln(t)}{1-t}\,\mathrm{d}t.$$

It does not make sense to reproduce the complicated result here. Significant simplifications arise if we take the choice $k = \frac{1}{8B}$ and $b = \frac{1}{8}$. In this case, the scalar field becomes

$$\phi(r) = \pm\sqrt{\frac{8B}{r+B}}, \tag{8}$$

and the metric function reads

$$\begin{aligned} f(r) = \left(3\beta - \frac{Q^2}{4}\right) + \left(2\beta - \frac{Q^2}{9}\right)\frac{B}{r} \\ - Q^2\left(\frac{1}{2} + \frac{B}{3r}\right)\ln(r) + \frac{r^2}{\ell^2}, \end{aligned} \tag{9}$$

where β and ℓ are integration constants. In order that the above $f(r)$ is a solution, $V(\phi(r))$ as a function of r must take a very special form. We can invert $\phi(r)$ for r and insert the result in $V(\phi(r))$ to get the scalar potential $V(\phi)$:

$$V(\phi) = -\frac{1}{\ell^2} + \frac{1}{512}\left(\frac{1}{\ell^2} + \frac{\beta}{B^2}\right)\phi^6 - \frac{1}{18432}\left(\frac{Q^2}{B^2}\right)(192\phi^2 + 48\phi^4 + 5\phi^6) + \frac{1}{3}\left(\frac{Q^2}{B^2}\right) \times \left[\frac{2\phi^2}{(8-\phi^2)^2} - \frac{1}{1024}\phi^6 \ln\left(\frac{B(8-\phi^2)}{\phi^2}\right)\right]. \quad (10)$$

The set of Eqs. (6)–(10) constitute a full set of exact solutions to the system defined by the action (1), which has not been seen in the literature before. The metric contains four parameters B, β, ℓ, and Q. Among these, B and Q have already appeared in the solution for the scalar and Maxwell fields, respectively. $\Lambda \equiv -\frac{1}{\ell^2}$ appears in $V(\phi)$ as a constant term, which plays the role of a (bare) cosmological constant. In principle, the constant Λ can either be positive, zero, or negative. However, if we wish to interpret the solution as a black hole solution, Λ will be necessarily negative, because in $(2+1)$ dimensions, smooth black hole horizons can exist only in the presence of a negative cosmological constant [71]. That is why we adopted the notation $\Lambda = -\frac{1}{\ell^2}$ from the very beginning. The last parameter β is related to the black hole mass M via

$$\beta = \frac{1}{3}\left(\frac{Q^2}{4} - M\right) \quad (11)$$

as will become clear in the degenerate case of a charged BTZ black hole [5], which corresponds to the case of $B = 0$. At this point, we do not seem to have any principle to determine the allowed range for β. However, the forthcoming physical analysis will make it clear that the β value has to be subjected to some constraints; otherwise, the solution will become physically unacceptable.

B. Scalar potential

Since $\lim_{\phi\to 0} V(\phi) = -\frac{1}{\ell^2}$, we may split $V(\phi)$ into a sum

$$V(\phi) = -\frac{1}{\ell^2} + U(\phi),$$

where $U(\phi)$ encodes the true self-interaction of the scalar field ϕ. Apart from the bare cosmological constant term and the following ϕ^6 term, all the other terms are proportional to $\frac{Q^2}{B^2}$, which implies that the scalar self-interaction is in a subtle balance with the Coulomb charge, although ϕ does not couple directly with the Maxwell field. The seemingly complicated form of the potential ensures that as $\phi \to 0$, the leading term in the power series expansion of $U(\phi)$ behaves as $O(\phi^6)$, i.e.,

$$U(\phi) \simeq \frac{1}{512}\left\{\frac{1}{\ell^2} + \frac{\beta}{B^2} + \frac{1}{9}\left(\frac{Q^2}{B^2}\right)\left[1 - \frac{3}{2}\ln\left(\frac{8B}{\phi^2}\right)\right]\right\}\phi^6 + \left(\frac{Q^2}{B^2}\right)O(\phi^8), \quad (12)$$

where only even powers of ϕ are present and the coefficients of all the $O(\phi^8)$ terms are positive. Some discussions are due here:

(i) If $Q = 0$, then $U(\phi)$ degenerates into a ϕ^6 potential, with the coefficient $\frac{1}{512}(\frac{1}{\ell^2} + \frac{\beta}{B^2})$. If, in addition, $\beta = -\frac{B^2}{\ell^2}$, then the self-coupling of the scalar field vanishes. If $\beta < -\frac{B^2}{\ell^2}$, the potential has a single extremum at $\phi = 0$, which is a maximum, implying that the scalar potential is unbounded from below and the system is unstable under small perturbations in ϕ. If $\beta > -\frac{B^2}{\ell^2}$, the single extremum becomes a minimum, which is stable against small perturbations in ϕ. Thus, stability against small perturbations in ϕ requires $\beta > -\frac{B^2}{\ell^2}$.

(ii) If $Q \neq 0$, $U(\phi)$ will possess more than one extremum. This is more easily seen in the expanded form (12). It is obvious that $\phi = 0$ remains an extremum when $Q \neq 0$. Moreover, the term proportional to $\phi^6 \ln(\phi^2)$ possesses two minima at some small nonzero ϕ. The inclusion of the power series terms may change the location of these minima, but the qualitative behavior of $U(\phi)$ remains unchanged, i.e., it has two minima at $\phi = \pm\phi_{\min} \neq 0$ and one maximum at $\phi = 0$.

(iii) From either the original potential (10) or its expanded form (12), it seems that we cannot take $B = 0$. However, this observation is completely superficial because the scalar field ϕ also depends on B. If we substitute the value of $\phi(r)$ into $V(\phi)$ and then look at the resulting expression, it will be clear that the potential is perfectly regular at $B = 0$.

In order to have more intuitive feelings about the scalar potential at $Q \neq 0$, we present a plot of $U(\phi)$ as a function of ϕ as well as a function of r. These are given in Figs. 1 and 2, respectively. If ϕ takes its value at the local maxima, i.e., $\phi = 0$, the scalar field equation is automatically satisfied, and the scalar potential is exactly zero. The corresponding solution is an Einstein-Maxwell-AdS black hole. One may tends to think that if ϕ takes its value at any of the minima $\phi = \pm\phi_{\min}$, then the corresponding solution would correspond to the true vacuum of the system, with an effective cosmological constant $\Lambda_{\text{eff}} = -\frac{1}{\ell^2} + U(\pm\phi_{\min})$ that is more negative than $-\frac{1}{\ell^2}$. However, this is not the case. If we take $\phi = \pm\phi_{\min} \neq 0$, then the field equation (3) will force the Ricci scalar R to be constant, which in turn requires $Q = 0$. But when $Q = 0$, the shape of the scalar potential $U(\phi)$ changes drastically, and the two minima at nonconstant ϕ totally disappear. Despite this, it is still an important observation that when $Q \neq 0$, the scalar potential $V(\phi)$ possesses two minima which are *smaller* than the cosmological constant $-\frac{1}{\ell^2}$. The physical explanation for these minima remains open. Because of the very complicated form of the potential $U(\phi)$, we are unable to find the location of the two minima of $U(\phi)$

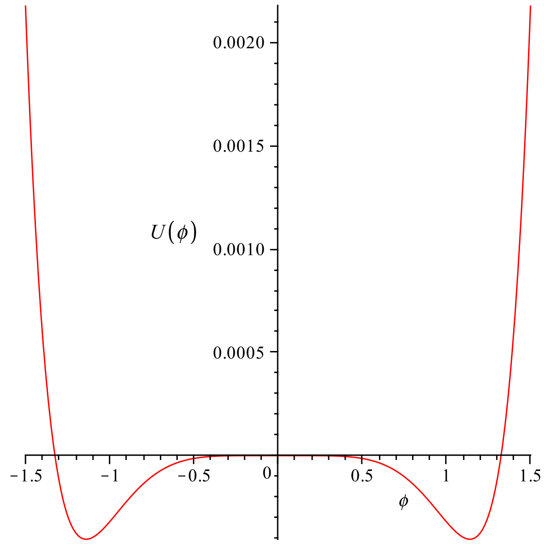

FIG. 1 (color online). Plot of $U(\phi)$ vs ϕ, with $B = 1$, $\ell = 1$, $\beta = -1$, and $Q = 1$.

analytically. Nevertheless, it is easy to find the minima of $U(\phi)$ numerically if the parameters B, Q, ℓ, and β were given numeric values. For instance, setting $B = Q = \ell = 1$, $\beta = -1$, we find that the minima of $U(\phi)$ are located at

$$\phi = \pm\phi_{\min}, \qquad \phi_{\min} \simeq 1.139824,$$

with the approximate value

$$U(\pm\phi_{\min}) \simeq -0.000405$$

C. Some geometric properties of the solution

To further characterize the geometry of the solution, we need to calculate some of the associated geometric

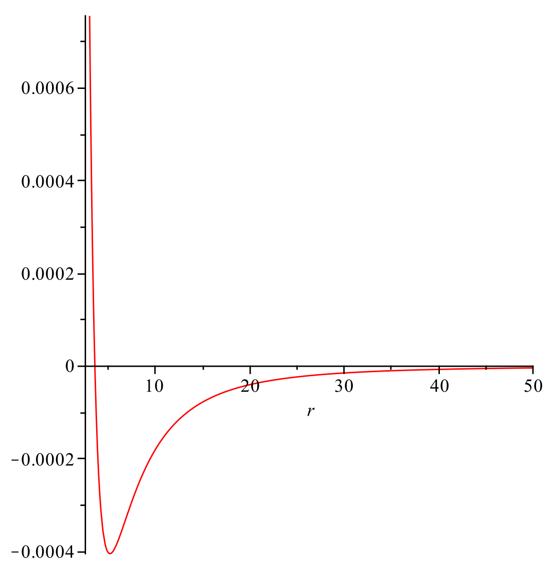

FIG. 2 (color online). Plot of $U(\phi(r))$ vs r, with $B = 1$, $\ell = 1$, $\beta = -1$, and $Q = 1$.

quantities. First of all, the Ricci scalar contains a curvature singularity at $r = 0$ if $Q \neq 0$,

$$R = -\frac{36r^3 - 3rQ^2\ell^2 + 2BQ^2\ell^2}{6\ell^2 r^3}.$$

Higher-order curvature invariants such as $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ are also singular at $r = 0$, even if $Q = 0$. However, the expressions for these invariants are much more complicated and unillustrative, so we do not reproduce them here. The Cotton tensor

$$C_{abc} = \nabla_c R_{ab} - \nabla_b R_{ac} + \frac{1}{4}(\nabla_b R g_{ac} - \nabla_c R g_{ab})$$

is nonvanishing if either $B > 0$ or $Q \neq 0$,

$$C_{trt} = -C_{ttr} = -\frac{1}{4}f(r)\frac{d^3}{dr^3}f(r),$$

$$C_{\psi r\psi} = -C_{\psi\psi r} = -\frac{1}{4}\left(\frac{d^3}{dr^3}f(r)\right)r^2.$$

In $(2 + 1)$ dimensions, the nonvanishing Cotton tensor signifies that the metric is nonconformally flat [72]. Thus, the hairy $(B > 0)$ and charged $(Q \neq 0)$ solutions are geometrically quite different from the case of the static uncharged BTZ black hole [5].

III. SPECIAL CASES

Before going into detailed analysis of our solution, we would like to point out some of the degenerated cases. Some of the degenerated cases have already been found in the literature.

A. Charged BTZ black hole

When $B = 0$, the scalar field ϕ vanishes, and the system becomes the Einstein-Maxwell-AdS theory. The solution degenerates into the already known static charged BTZ black hole [5]:

$$f(r) = -M - \frac{Q^2}{2}\ln(r) + \frac{r^2}{\ell^2},$$

$$A_\mu dx^\mu = -Q\ln\left(\frac{r}{r_0}\right)dt,$$

$$V(\phi) = -\frac{1}{\ell^2}, \qquad \phi(r) = 0,$$

where M is the mass of the BTZ black hole. As mentioned earlier, although this solution may be stable in the Einstein-Maxwell-AdS theory, it is an unstable solution in the full theory (1).

B. Uncharged hairy AdS black hole

The Maxwell field can be removed by setting $Q = 0$. In this case, we get an uncharged hairy AdS black hole solution:

$$f(r) = \left(3 + \frac{2B}{r}\right)\beta + \frac{r^2}{\ell^2}, \qquad A_\mu dx^\mu = 0,$$
$$\phi(r) = \pm\sqrt{\frac{8B}{r+B}},$$
$$V(\phi) = -\frac{1}{\ell^2} + \frac{1}{512}\left(\frac{1}{\ell^2} + \frac{\beta}{B^2}\right)\phi^6.$$

Using Eq. (11), we can change β into $-\frac{M}{3}$ everywhere in the solution. In this case, the Ricci scalar becomes constant,

$$R = -\frac{6}{\ell^2} = 6\Lambda.$$

However, higher-curvature invariants are still singular at $r = 0$. This solution has already appeared in Sec. 5.1 of Ref. [28]. As will become clear later, we need $\beta \in [-\frac{B^2}{\ell^2}, 0]$ in order for this solution to be physically well behaving.

C. Conformally dressed black hole

In addition to setting $Q = 0$, we can choose $\beta = -\frac{B^2}{\ell^2}$ in the meantime. Under such conditions, we reproduce the conformally dressed black hole in $(2+1)$ dimensions [10]:

$$f(r) = -\left(3 + \frac{2B}{r}\right)\frac{B^2}{\ell^2} + \frac{r^2}{\ell^2},$$
$$A_\mu dx^\mu = 0,$$
$$\phi(r) = \pm\sqrt{\frac{8B}{r+B}},$$
$$V(\phi) = -\frac{1}{\ell^2}.$$

Note that although the scalar field is still present, the self-interaction potential $U(\phi)$ vanishes, thus making the scalar field a "free" massless field.

D. Special charged hairy AdS black hole in three dimensions

We may also choose $\beta = -\frac{B^2}{\ell^2}$ while keeping Q nonvanishing. Then, we get a special charged hairy AdS black hole:

$$f(r) = -\left(\frac{3B^2}{\ell^2} + \frac{Q^2}{4}\right) - \left(\frac{2B^2}{\ell^2} + \frac{Q^2}{9}\right)\frac{B}{r} - Q^2\left(\frac{1}{2} + \frac{B}{3r}\right)\ln(r) + \frac{r^2}{\ell^2},$$
$$\phi(r) = \pm\sqrt{\frac{8B}{r+B}}, \qquad A_\mu dx^\mu = -Q\ln\left(\frac{r}{r_0}\right)dt,$$
$$V(\phi) = -\frac{1}{\ell^2} - \frac{1}{18432}\left(\frac{Q^2}{B^2}\right)(192\phi^2 + 48\phi^4 + 5\phi^6) + \frac{1}{3}\left(\frac{Q^2}{B^2}\right)\left[\frac{2\phi^2}{(8-\phi^2)^2} - \frac{1}{1024}\phi^6\ln\left(\frac{B(8-\phi^2)}{\phi^2}\right)\right].$$

The merit of this special case is that the scalar potential $U(\phi)$ is not fine-tuned with the cosmological constant.

IV. HORIZON STRUCTURES

Among the three fields $g_{\mu\nu}$, A_μ, and ϕ, the latter two are easily understandable. A_μ is just the standard Coulomb potential in $(2+1)$ dimensions, ϕ is a radially distributed scalar, which takes its maximum $\phi_{\max} = \sqrt{8}$ at $r = 0$ and decreases gradually to zero as $r \to +\infty$ when $B > 0$.

The metric $g_{\mu\nu}$ is more involved to be able to understand. This is because of the complicated form (9) of the metric function $f(r)$. As usual, the zeros of $f(r)$ (if any) will correspond to horizons in the metric. So, we need to find (at least the condition for the existence of) the zeros of $f(r)$.

A. $Q = 0$

When $Q = 0$, the functions $f(r)$ and $f'(r)$ are simplified drastically:

$$f(r) = \beta\left(3 + \frac{2B}{r}\right) + \frac{r^2}{\ell^2}, \qquad f'(r) = -\frac{2B\beta}{r^2} + \frac{2r}{\ell^2}.$$

We may divide the solution in three subcases:

(i) If $\beta > 0$, $f(r)$ will always be positive, implying that the metric corresponds to an asymptotically AdS_3 spacetime containing a naked singularity. This is a physically uninteresting case.

(ii) If $\beta = 0$, then the metric degenerates into the $(2+1)$-dimensional empty topological AdS spacetime.

(iii) If, instead, $\beta < 0$, then $f'(r)$ will remain positive for $r \in [0, +\infty)$. Meanwhile, $f(r) \to -\infty$ as $r \to 0$, $f(r) \to +\infty$ as $r \to +\infty$. This implies $f(r)$ increases monotonically and, hence, contains exactly one zero. The zero of $f(r)$ corresponds to the event horizon of a neutral AdS black hole with a scalar hair.

Combining with the analysis made in Sec. II B, we see that the physically acceptable range for the parameter β is $\beta \in [-\frac{B^2}{\ell^2}, 0]$ at $Q = 0$.

B. $Q \neq 0$

The form of $f(r)$ with nonvanishing Q is much more complicated than the $Q = 0$ case. To find whether $f(r)$ has some zeros, we need to know the asymptotic behavior and the number of the extrema of $f(r)$.

From Eq. (9), it can be seen that $f(r)$ is dominated by the term $\frac{r^2}{\ell^2}$ in the far region, so $f(r) \to +\infty$ as $r \to +\infty$. On the other hand, as $r \to 0$, $f(r)$ is dominated by the term $-\frac{Q^2 B}{3}\frac{\ln(r)}{r}$ if $B > 0$ or by the term $-\frac{Q^2}{2}\ln(r)$ if $B = 0$. So, $f(r)$ always approaches $+\infty$ as $r \to 0$. Remember that we have excluded the possibility of choosing $B < 0$ in order that $\phi(r)$ is not singular at finite nonzero r.

Combining the asymptotic behaviors at both ends, we see that for $Q \neq 0$, $f(r)$ will approach $+\infty$ at both ends. Therefore, $f(r)$ has to have some extrema, and the total number of extrema must be odd. Let us denote the location of the extrema of $f(r)$ by r_X. If $f(r)$ has more than one extremum, an extra index may be adopted to distinguish these different extrema if necessary. Clearly, the position r_X of every extremum of $f(r)$ obeys $f'(r_X) = 0$, or, more conveniently,

$$(r_X)^2 f'(r_X) = 0, \tag{13}$$

where $r^2 f'(r)$ is given by the following expression:

$$r^2 f'(r) = 2B\left(\frac{B^2}{9} + \beta\right) + \frac{1}{3}Q^2 B \ln(r) - \frac{1}{2}Q^2 r + \frac{2r^3}{\ell^2}. \tag{14}$$

However, the converse needs not to be true: If $r^2 f'(r)$ happens to be zero at some of its extrema r_i (not to be confused with r_X), then r_i will correspond to an inflection point of $f(r)$, rather than an extremum.

In order to determine the number of roots for $f(r)$, we need to consider two distinct cases, i.e., $B = 0$ and $B > 0$. For $B = 0$, it is easy to get the location of the extremum of $f(r)$ using Eqs. (13) and (14). The only real positive extremum of $f(r)$ in this case is located at $r = r_X = \frac{1}{2}|Q|\ell$. Inserting the value of r_X and $B = 0$ into the solution (9), we get the following:

(i) If $\beta < \frac{Q^2}{6}\ln(r_X)$, we have $f(r_X) < 0$. So, $f(r)$ will have two zeros, each corresponding to a black hole horizon. Among these, the outer horizon is the event horizon. This case corresponds to a charged nonextremal AdS black hole without the scalar hair.

(ii) If $\beta = \frac{Q^2}{6}\ln(r_X)$, we have $f(r_X) = 0$. It is evident that r_X is the only root of $f(r)$, which corresponds to the horizon of a charged extremal AdS black hole without the scalar hair. In this particular case, it may be better to replace r_X with r_{ex}, which stands for the radius of the extremal black hole.

(iii) If $\beta > \frac{Q^2}{6}\ln(r_X)$, then $f(r_X) > 0$; there will be no zeros for the function $f(r)$, so the metric becomes an asymptotically AdS spacetime with a naked singularity at the origin, which is a physically uninteresting case.

The case with $B > 0$ is much more complicated compared to the $B = 0$ case. It is impossible to solve Eq. (13) analytically to get r_X, so we turn to look at the extrema of $r^2 f'(r)$. We introduce the following polynomial function:

$$g(r) \equiv 6\ell^2 r \frac{d}{dr}[r^2 f'(r)] = 36r^3 - 3Q^2\ell^2 r + 2BQ^2\ell^2. \tag{15}$$

Every real positive root of $g(r)$ will correspond to an extremum or an inflection point of $r^2 f'(r)$, and the collection of signs of $r^2 f'(r)$ at its extrema will determine the number of roots thereof. Fortunately, the function $g(r)$ is simple enough so that its roots can be found analytically. In the appendix, we shall present the details about the roots of $g(r)$.

According to the appendix, the number of real positive roots of $g(r)$ will change when the value of the parameter B crosses $\frac{|Q|\ell}{6}$. The significance of this change in the number of real positive roots will be best illustrated if we look at the extremum of $g(r)$. Taking the first derivative of $g(r)$ with respect to r and finding the root of the resulting expression, we find that $g(r)$ has only one real positive extremum located at $r = \frac{|Q|\ell}{6}$. Clearly this is a minimum. Substituting this value of r into $g(r)$ itself, we find the minimum value of $g(r)$, which reads

$$g_{\min} = 2BQ^2\ell^2 - \frac{1}{3}Q^3\ell^3. \tag{16}$$

If $B > \frac{|Q|\ell}{6}$, then $g_{\min}$ is positive, which implies that $g(r)$ has no real positive root, which in turn implies that $r^2 f'(r)$ has no extrema for $r > 0$, so that $f'(r)$ has only one root, i.e., $f(r)$ has only one extremum. If $B = \frac{|Q|\ell}{6}$, then $g(r)$ is zero at its extremum. This means that $g(r)$ has only one root that is located at its minimum. This implies that the minimum of $g(r)$ corresponds to an inflection point rather than an extremum of $r^2 f'(r)$. So, in the end, $r^2 f'(r)$ still has no extremum for $r > 0$, resulting in the conclusion that $f(r)$ has only a single extremum for $r > 0$.

The problem becomes more complicated when $0 < B < \frac{|Q|\ell}{6}$. In this case, $g(r)$ has three roots, two of which are positive. Therefore, $r^2 f'(r)$ will also have two extrema for $r > 0$. Among these, the extremum at r_1 is a minimum, and that at r_2 is a maximum, and we have $r_2 < r_1$.

Now, let us assume $0 < B < \frac{|Q|\ell}{6}$. Since r_1 and r_2 are both real positive roots of $g(r)$, we have

$$r_i^3 = \frac{Q^2\ell^2 r_i}{12} - \frac{BQ^2\ell^2}{18}, \qquad i = 1, 2.$$

Inserting this into Eq. (14), we get

$$r_i^2 f'(r_i) = 2B(w(r_i) - \beta), \tag{17}$$

where

$$w(r_i) \equiv \frac{Q^2}{6B}(B\ln(r_i) - r_i - B).$$

Equation (17) gives the value that $r^2 f'(r)$ must take at any of its extrema. In particular, the analysis made in the appendix implies $w(r_1) < w(r_2)$.

Depending on the value of the parameter β, $r_i^2 f'(r_i)$ will take different signs:

(i) If $\beta = w(r_1)$ or $\beta = w(r_2)$, we have $r_1^2 f'(r_1) = 0$ or $r_2^2 f'(r_2) = 0$, i.e., $r^2 f'(r)$ is zero at one of its extrema. Besides this accidental root, $r^2 f'(r)$ has another root that is not at its extrema. So, totally,

$r^2 f'(r)$ will have two roots; one of these (the normal root) corresponds to the extremum of $f(r)$, and the other (the accidental root at the extremum) corresponds to an inflection point of $f(r)$. So, in the end, $f(r)$ will have only one extremum in this case.

(ii) If $\beta < w(r_1)$ or $\beta > w(r_2)$, we have either $r_2^2 f'(r_2) > r_1^2 f'(r_1) > 0$ or $0 > r_2^2 f'(r_2) > r_1^2 f'(r_1)$. In both cases, all the extrema of $r^2 f'(r)$ have the same signs, so there is only one positive root of $r^2 f'(r)$. Consequently, $f(r)$ will have only a single extremum for $r > 0$.

(iii) If $w(r_1) < \beta < w(r_2)$, we find $r_2^2 f'(r_2) > 0 > r_1^2 f'(r_1)$. The two positive extrema of $r^2 f'(r)$ have different signs, indicating that the curve for $r^2 f'(r)$ will cross the horizontal axes three times. Therefore, there are three positive roots for $r^2 f'(r)$, each corresponding to an extremum of $f(r)$.

Summarizing the above discussions, we make the following conclusion on the number of extrema for the metric function $f(r)$:

(i) If $B = 0$ or $B \geq \frac{|Q|\ell}{6}$, $f(r)$ has only a single extremum.

(ii) If $0 < B < \frac{|Q|\ell}{6}$, the number of extrema for $f(r)$ depends on the value of the parameter β. Explicitly,
  (1) if $\beta \leq w(r_1)$ or $\beta \geq w(r_2)$, $f(r)$ still has only a single extremum;
  (2) if $w(r_1) < \beta < w(r_2)$ will have three extrema.

Therefore, the horizon structure of our solution at $Q \neq 0$ will depend crucially on the range of parameters.

Since at any extremum r_X of $f(r)$ we have $f'(r_X) = 0$, an arbitrary multiple of $f'(r_X)$ can be added to $f(r_X)$ to yield a simplified expression for $f(r_X)$. Specifically, we take the following combination:

$$p(r_X) \equiv \frac{B}{r_X + B}\left(f(r_X) + f'(r_X)\frac{r_X(9r_X + 6B)}{6B}\right) = \frac{B}{r_X + B} f(r_X) = 36(r_X)^3 - 9Q^2\ell^2 r_X - 4BQ^2\ell^2. \tag{18}$$

Clearly, $p(r_X)$ and $f(r_X)$ always have the same sign, so the collection of signs of $p(r_X)$ at all the extrema r_X of $f(r)$ will determine the number of roots of $f(r)$.

Let us consider the "extremal case" defined via $f(r_{ex}) = p(r_{ex}) = 0$ and $f'(r_{ex}) = 0$. In this case, $f(r)$ happens to be zero at one of its extrema. Since the zeros of $f(r_{ex})$ and $p(r_{ex})$ always coincide, we can try to find the zeros of $p(r_{ex})$ to get the value of r_{ex}.

Since we have $B \geq 0$, we may assume

$$B = \frac{a}{4}|Q|\ell, \qquad r_{ex} = \rho|Q|\ell, \tag{19}$$

so that the equation $p(r_{ex}) = 0$ becomes

$$36\rho^3 - 9\rho - a = 0. \tag{20}$$

The condition $a \geq 0$ implies $\rho \geq \frac{1}{2}$, where the lower bound for ρ corresponds to $a = 0$, i.e., $B = 0$, as discussed previously. Generically, Eq. (20) can have three zeros. However, only one of these is real positive and lies in the range $\rho \in [\frac{1}{2}, +\infty)$, which is given in terms of a as

$$\rho = \frac{z}{6} + \frac{1}{2z}, \qquad z = \sqrt[3]{3a + 3\sqrt{a^2 - 3}}, \qquad a \geq 0. \tag{21}$$

The detailed analysis on the solution to Eq. (20) can be carried out in exactly the same way as is done in the appendix for the similar Eq. (A1).

Now, substituting Eqs. (11) and (19) into the equation $f'(r_{ex}) = 0$, we get

$$\beta = \frac{Q^2}{6}\ln(r_{ex}) = \frac{Q^2}{6}\ln(\rho|Q|\ell).$$

This is the same condition that appeared in the $B = 0$ case. The only difference lies in the fact that ρ is fixed at the value $\frac{1}{2}$ when $B = 0$, while for $B \neq 0$, its value is given by Eq. (21).

We have already made it clear that when $B = 0$ or $B \geq \frac{|Q|\ell}{6}$, or when $0 < B < \frac{|Q|\ell}{6}$ with $\beta \leq w(r_1)$ or $\beta \geq w(r_2)$, $f(r)$ always has only a single minimum. So, for these parameter ranges, the zero r_{ex} of $f(r)$ as described in Eqs. (19) and (21) is the only zero and minimum of $f(r)$. So, these cases correspond to extremal black holes with horizon radius

$$r_{ex} = \exp\left(\frac{6\beta}{Q^2}\right).$$

If at the extremum r_X of $f(r)$, $p(r_X)$ fails to be zero, then the corresponding solution will not correspond to an extremal black hole. Let us now consider such cases in more detail.

For all $r_X \geq r_{ex} = \rho|Q|\ell$, we have

$$p'(r_X) = 9(12r_X^2 - Q^2\ell^2) \geq p'(r_{ex}) = 9(3Q^2\ell^2 - Q^2\ell^2) = 18Q^2\ell^2 > 0,$$

i.e., $p(r_X)$ increases monotonically for $r_X \geq r_{ex}$. So, if $p(r_X) < 0$, then r_X must be located to the left of r_{ex}, i.e., $r_X < r_{ex}$. However, if $p(r_X) > 0$, we cannot deduce from above that $r_X > r_{ex}$.

Consider the following identity:

$$\left(1 - \frac{B}{r_X + B}\right) f(r_X) = 3\beta - \frac{Q^2}{2}\ln(r_X).$$

This is equivalent to

$$\beta = \frac{Q^2}{6}\ln(r_X) + \frac{r_X}{3(r_X + B)} f(r_X). \tag{22}$$

If $f(r_X) < 0$, then

$$\beta < \frac{Q^2}{6}\ln(r_X) < \frac{Q^2}{6}\ln(r_{ex}). \tag{23}$$

In Eq. (23), r_X may be extremely close to r_{ex}, while still keeping $r_X < r_{ex}$. So, we may think of

$$\beta < \frac{Q^2}{6}\ln(r_{ex}) \tag{24}$$

to be the condition that must be imposed on the parameter β in order for the metric to have two disjoint horizons, to behave as a nonextremal black hole. If $f(r_X) > 0$, then

$$\beta > \frac{Q^2}{6}\ln(r_X). \tag{25}$$

Under this condition, the metric contains no horizons and corresponds to an asymptotically AdS_3 spacetime with a naked singularity.

What remains untouched is the real troublesome case with $0 < B < \frac{|Q|\ell}{6}$ and $w(r_1) < \beta < w(r_2)$. In this parameter range, we have to be careful about how many zeros there are for $f(r)$ because $f(r)$ has three extrema. Let us denote the location of the three extrema by r_{X1}, r_{X2}, and r_{X3}, respectively. Let r_{X1}, r_{X2}, and r_{X3} be ordered such that $r_{X1} < r_{X2} < r_{X3}$. At these points, $r^2 f'(r)$ vanishes, and its two positive extrema locate between these zeros, i.e.,

$$r_{X1} < r_2 < r_{X2} < r_1 < r_{X3}.$$

From the appendix, we have $\frac{|Q|\ell}{6} < r_1 < \frac{\sqrt{3}|Q|\ell}{6} < \frac{|Q|\ell}{2}$, so

$$r_{X2} < \frac{|Q|\ell}{2}.$$

Therefore, according to Eq. (18), we have

$$\begin{aligned} p(r_{X2}) &= \frac{B}{r_{X2}+B} f(r_{X2}) \\ &= 36(r_{X2})^3 - 9Q^2\ell^2 r_{X2} - 4BQ^2\ell^2 \\ &= 36 r_{X2}\left(r_{X2}^2 - \frac{Q^2\ell^2}{4}\right) - 4BQ^2\ell^2 < 0, \end{aligned}$$

i.e., $f(r_{X2}) < 0$. Since r_{X2} lies in between r_{X1} and r_{X3}, it corresponds to the *local maximum* of $f(r)$, so the above result implies that $f(r)$ is negative at all its three extrema. Combining with the asymptotic behavior, we deduce that $f(r)$ have precisely two zeros in this case, so whenever $f(r)$ has three extrema, the solution corresponds to a nonextremal charged hairy black hole with AdS asymptotics. Please note that, for this case, we have

$$\begin{aligned} \beta < w(r_2) &= \frac{Q^2}{6B}(B\ln(r_2) - r_2 - B) < \frac{Q^2}{6}\ln(r_2) \\ &< \frac{Q^2}{6}\ln\left(\frac{|Q|\ell}{6}\right) < \frac{Q^2}{6}\ln\left(\frac{|Q|\ell}{2}\right) \\ &\le \frac{Q^2}{6}\ln(\rho|Q|\ell) \\ &= \frac{Q^2}{6}\ln(r_{ex}). \end{aligned}$$

Putting all the cases together, we see that the physically acceptable upper bound for β at $Q \neq 0$ is

$$\beta \le \frac{Q^2}{6}\ln(r_{ex}).$$

Depending on the value of r_{ex} determined by Eqs. (19) and (21), this upper bound can be either positive or negative. Meanwhile, for $Q \neq 0$, the scalar potential $U(\phi)$ is dominated by the $O(\phi^8)$ terms when ϕ is big enough, so we do not need to worry about the existence of lower bound for the scalar potential. Accordingly, there is no lower bound for the parameter β.

V. EFFECTS OF SCALAR AND ELECTRIC CHARGES ON THE SIZE OF THE BLACK HOLE

In this section, we shall restrict ourselves to the cases in which (a) black hole horizon(s) exist(s). The primitive goal of this section is to understand the effects of the scalar and electric charges on the size of the black hole. Of course, the best way to do this is to consider the effect of each charge independently.

A. Effect of the scalar charge

The parameter B originates solely from the scalar field ϕ, but it is also carried by the black hole solution, so this parameter may be regarded as a scalar charge of the hole. In order to understand the effect of the scalar charge B on the size of the black hole, we need to separate the metric function $f(r)$ into B-independent and B-dependent parts. According to Eq. (9), $f(r)$ depends on B only linearly, so

$$f(r) = f(r)|_{B=0} + f_B(r),$$

where

$$f_B(r) = B\frac{df(r)}{dB} = B\left[\left(2\beta - \frac{Q^2}{9}\right)\frac{1}{r} - \frac{Q^2}{3r}\ln(r)\right]. \tag{26}$$

Notice that the dominant term $\frac{r^2}{\ell^2}$ as $r \to +\infty$ is contained in the $f(r)|_{B=0}$. Solving $\ln(r)$ out of the horizon condition $f(r) = 0$ and substituting in Eq. (26), we get

$$f_B(r) = \frac{B(Q^2\ell^2 - 6r^2)}{\ell^2(6B + 9r)},$$

where r must be understood as the horizon radius. It is clear that $f_B(r) < 0$, provided $B > 0$ and $r > \frac{|Q|\ell}{\sqrt{6}}$. We already know from the previous section that the horizon radius for the extremal black hole is $r_{\rm ex} = \rho|Q|\ell$ with $\rho \geq \frac{1}{2} > \frac{1}{\sqrt{6}}$, and, in general, the radius r_+ for the outer horizon nonextremal black hole is bigger yet than $r_{\rm ex}$; we see that the effect of inclusion of the $f_B(r)$ term in $f(r)$ is to make $f(r)$ more negative. Therefore, to compensate for this extra negative contribution, the $f(r)|_{B=0}$ term must become more positive in order to make a zero for $f(r)$. In other words, the size of the black hole must increase as B increases.

B. Effect of the electric charge

We can also separate $f(r)$ into its Q^2-independent and Q^2-dependent parts, i.e.,

$$f(r) = f(r)|_{Q=0} + f_{Q^2}(r), \tag{27}$$

where

$$f_{Q^2}(r) = Q^2\frac{df(r)}{d(Q^2)} = Q^2\left[-\frac{1}{4} - \frac{B}{9r} - \left(\frac{1}{2} + \frac{B}{3r}\right)\ln(r)\right]. \tag{28}$$

The horizon condition is a proper balance between the $f(r)|_{Q=0}$ and the $f_{Q^2}(r)$ terms. In Sec. IVA, it was shown that $f(r)|_{Q=0} \to -\infty$ as $r \to 0$ and increases monotonically with r. On the other hand, from the above expression, we see that $f_{Q^2}(r) \to +\infty$ as $r \to 0$ and *decreases monotonically* with r. Moreover, in the "near end" ($r \to 0$), the term $-\frac{1}{r}\ln(r)$ in $f_{Q^2}(r)$ takes over the term $-\frac{1}{r}$ in $f(r)|_{Q=0}$, and in the far region ($r \to +\infty$), the term $\frac{r^2}{\ell^2}$ in $f(r)|_{Q=0}$ dominates, while $f_{Q^2}(r)$ becomes increasingly negative. In effect, the inclusion of the $f_{Q^2}(r)$ term in $f(r)$ results in two different consequences: In the near region, $f(r)$ develops a novel zero, which is the inner horizon for the charged black hole; in the far region, the original zero of $f(r)|_{Q=0}$ (now being the outer horizon of the charged black hole) gets increased with the inclusion of the $f_{Q^2}(r)$ term, and as Q^2 increases, the radius of the outer horizon also increases monotonically.

VI. DISCUSSIONS

In this paper, we obtained an exact solution for the Einstein-Maxwell-scalar theory in (2 + 1) dimensions, in which the scalar field couples to gravity nonminimally and also couples to itself in a peculiar way. The solution is static circularly symmetric, and the scalar self-potential is completely determined by staticness and the circular symmetry of the solution. In particular, a negative cosmological constant naturally emerges as a constant term in the scalar potential if we require that, for certain ranges of parameters, the solution represents a charged hairy black hole. Under the proper choices of parameter values, our solution degenerates into some already known (2 + 1)-dimensional black hole solutions.

When the electric charge Q is nonzero, the scalar potential possesses three extrema, one maximum at $\phi = 0$ and two minima at $\phi \neq 0$. The scalar field ϕ cannot stay at the constant value $\phi = \pm\phi_{\rm min} \neq 0$; otherwise, the field equations will not be satisfied.

We also identified the conditions for the metric to behave as a charged extremal black hole, as an asymptotically AdS_3 spacetime with a naked singularity at the origin, and as a charged nonextremal black hole. When black hole horizons exist, it is shown that the size of the (outer) horizon increases monotonically with both the scalar charge and the electric charge.

Some of the related properties and duality relations will become instantly interesting further tasks to be worked out. These are:

(i) the thermodynamic quantities and the associated laws of thermodynamics;
(ii) the properties of the boundary CFT, if any;
(iii) that these days, the fluid dual of AdS gravity is an active area of study and it will be interesting to ask whether there is fluid dual of the black hole solution given in this paper;
(iv) that the solution considered in this paper is only static and circularly symmetric, and it would be interesting to ask whether one can find more complicated solutions to the same for instance, it is interesting to ask whether one can find rotationally symmetric solutions and determine the scalar self-interaction solely by the form of the metric ansatz);
(v) that the scalar field in model of this paper is neutral and does not couple to the electromagnetic field, and it would also be interesting to allow the scalar field to become complex and thus couple directly to the electromagnetic field.

We leave the answer to all these problems for future work.

APPENDIX: ROOTS OF THE FUNCTION $g(r)$

In this appendix, we shall solve the root of the function $g(r)$ given by Eq. (15), i.e., the root of the equation

$$36r^3 - 3Q^2\ell^2 r + 2BQ^2\ell^2 = 0. \tag{A1}$$

With the aid of a computer algebra system like Maple, it is easy to find that the roots of the above equation are given by

$$r_1 = \frac{1}{6}\left(z + \frac{Q^2\ell^2}{z}\right),$$
$$r_2 = \frac{1}{12}\left[-z - \frac{Q^2\ell^2}{z} + i\sqrt{3}\left(z - \frac{Q^2\ell^2}{z}\right)\right],$$
$$r_3 = \frac{1}{12}\left[-z - \frac{Q^2\ell^2}{z} - i\sqrt{3}\left(z - \frac{Q^2\ell^2}{z}\right)\right],$$

where

$$z = \sqrt[3]{-6BQ^2\ell^2 + \sqrt{36BQ^4\ell^4 - Q^6\ell^6}}. \quad \text{(A2)}$$

Among the three roots of $g(r)$, only the real positive roots are relevant to our problem. So we need to identify which and how many of the roots are real positive.

It is evident that if $B \geq \frac{|Q|\ell}{6}$, then z is real. It follows that if $B > \frac{|Q|\ell}{6}$, then r_1 is real but negative, while both r_2 and r_3 are complex. If $B = \frac{|Q|\ell}{6}$, then r_1 and r_2 will become degenerate, and both are real positive. In this case, r_3 is negative. If $B < \frac{|Q|\ell}{6}$, z becomes complex. In this case, it is not too difficult to see that the modulus of z is equal to $|Q|\ell$, so we can denote z as

$$z = |Q|\ell(\cos\theta + i\sin\theta).$$

Comparing this expression with the original definition (A2), we see that θ must take value in the range

$$\frac{\pi}{6} < \theta < \frac{\pi}{3}$$

and the three roots of $g(r)$ can be reparametrized as

$$r_1 = \frac{|Q|\ell}{3}\cos(\theta), \quad \text{(A3)}$$

$$r_2 = -\frac{|Q|\ell}{3}\cos\left(\theta + \frac{\pi}{3}\right), \quad \text{(A4)}$$

$$r_3 = -\frac{|Q|\ell}{3}\cos\left(\theta - \frac{\pi}{3}\right). \quad \text{(A5)}$$

It is easy to see that

$$r_1 \in \left(\frac{|Q|\ell}{6}, \frac{\sqrt{3}}{6}|Q|\ell\right), \quad \text{(A6)}$$

$$r_2 \in \left(0, \frac{|Q|\ell}{6}\right), \quad \text{(A7)}$$

$$r_3 \in \left(-\frac{\sqrt{3}}{6}|Q|\ell, -\frac{|Q|\ell}{6}\right). \quad \text{(A8)}$$

All three roots are real; however, only r_1 and r_2 are positive. Clearly, r_1 is the bigger root of $g(r)$.

[1] S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. (N.Y.) **140**, 372 (1982).
[2] S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982).
[3] S. Deser and R. Jackiw, Ann. Phys. (N.Y.) **153**, 405 (1984).
[4] S. Giddings, J. Abbott, and K. Kuchar, Gen. Relativ. Gravit. **16**, 751 (1984).
[5] M. Bañados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992).
[6] C. Martinez, C. Teitelboim, and J. Zanelli, Phys. Rev. D **61**, 104013 (2000).
[7] J. Brown and M. Henneaux, Commun. Math. Phys. **104**, 207 (1986).
[8] M. Henneaux, C. Martinez, R. Troncoso, and J. Zanelli, Phys. Rev. D **65**, 104007 (2002).
[9] M. Hasanpour, F. Loran, and H. Razaghian, Nucl. Phys. **B867**, 483 (2013).
[10] C. Martinez and J. Zanelli, Phys. Rev. D **54**, 3830 (1996).
[11] M. Natsuume, T. Okamura, and M. Sato, Phys. Rev. D **61**, 104005 (2000).
[12] F. Correa, C. Martinez, and R. Troncoso, J. High Energy Phys. 01 (2011) 034.
[13] F. Correa, C. Martinez, and R. Troncoso, arXiv:1112.6198.
[14] F. Correa, A. Faundez, and C. Martinez, arXiv:1211.4878.
[15] D. F. Zeng, arXiv:0903.2620.
[16] E. Ó. Colgáin and H. Samtleben, J. High Energy Phys. 02 (2011) 031.
[17] B. Chen, Z. Xue, and J. ju Zhang, J. High Energy Phys. 03 (2013) 102.
[18] M. Blagojević, B. Cvetković, O. Miskovic, and R. Olea, arXiv:1301.1237.
[19] R.-G. Cai, L. Li, Z.-Y. Nie, and Y.-L. Zhang, Nucl. Phys. **B864**, 260 (2012).
[20] Y. S. Myung, Phys. Lett. B **663**, 111 (2008).
[21] M. Eune and W. Kim, arXiv:1301.0395.
[22] M. Kamata and T. Koikawa, Phys. Lett. B **353**, 196 (1995).
[23] C. Martinez and R. Troncoso, Phys. Rev. D **74**, 064007 (2006).
[24] C. Martinez, R. Troncoso, and J. Zanelli, Phys. Rev. D **70**, 084035 (2004).
[25] J. D. Bekenstein, arXiv:gr-qc/9605059.
[26] K. C. K. Chan, Phys. Rev. D **55**, 3564 (1997).
[27] L. R. Abramo, L. Brenig, E. Gunzig, and A. Saa, Mod. Phys. Lett. A **18**, 1043 (2003).
[28] M. Nadalini, L. Vanzo, and S. Zerbini, Phys. Rev. D **77**, 024047 (2008).
[29] A. Acena, A. Anabalon, and D. Astefanesei, arXiv:1211.6126.
[30] J. Tafel, arXiv:1112.2687.
[31] D. Pugliese and J. A. V. Kroon, Gen. Relativ. Gravit. **45**, 1247 (2013).
[32] J. Aparicio, D. Grumiller, E. Lopez, I. Papadimitriou, and S. Stricker, arXiv:1212.3609.
[33] O. J. C. Dias, G. T. Horowitz, and J. E. Santos, J. High Energy Phys. 07 (2011) 115.

[34] M. Cadoni, S. Mignemi, and M. Serra, arXiv:1107.5979.
[35] T. Kolyvaris, G. Koutsoumbas, E. Papantonopoulos, and G. Siopsis, Gen. Relativ. Gravit. **43**, 163 (2010).
[36] Y. Degura, K. Sakamoto, and K. Shiraishi, Gravitation Cosmol. **7**, 153 (2001).
[37] M. Bañados and S. Theisen, Phys. Rev. D **72**, 064019 (2005).
[38] G. de Berredo-Peixoto, Classical Quantum Gravity **20**, 4305 (2003).
[39] H. J. Schmidt and D. Singleton, Phys. Lett. B **721**, 294 (2013).
[40] Y. Kwon, S. Nam, and J.-D. Park, Nucl. Phys. **B869**, 189 (2013).
[41] E. Radu and E. Winstanley, Phys. Rev. D **72**, 024017 (2005).
[42] C. Martinez, R. Troncoso, and J. Zanelli, Phys. Rev. D **67**, 024008 (2003).
[43] M. Jamil, D. Momeni, and R. Myrzakulov, Eur. Phys. J. C **72**, 2075 (2012).
[44] P. I. Kuriakose and V. C. Kuriakose, arXiv:0805.4554.
[45] G. Dotti, R. J. Gleiser, and C. Martinez, Phys. Rev. D **77**, 104035 (2008).
[46] T. Kolyvaris, G. Koutsoumbas, E. Papantonopoulos, and G. Siopsis, arXiv:1111.0263.
[47] A. Perez, D. Tempo, and R. Troncoso, arXiv:1301.0847.
[48] D. Anninos, T. Hartman, and A. Strominger, arXiv:1108.5735.
[49] M. Ammon, M. Gutperle, P. Kraus, and E. Perlmutter, arXiv:1106.4788.
[50] B. Chen, J. Long, and Y.-N. Wang, arXiv:1212.6593.
[51] J. R. David, M. Ferlaino, and S. P. Kumar, arXiv:1210.0284.
[52] B. Chen, J. Long, and Y.-N. Wang, arXiv:1209.6185.
[53] E. A. Bergshoeff, O. Hohm, and P. K. Townsend, Phys. Rev. Lett. **102**, 201301 (2009).
[54] G. Gabadadze, G. Giribet, and A. Iglesias, arXiv:1212.6279.
[55] O. Hohm, A. Routh, P. K. Townsend, and B. Zhang, arXiv:1208.0038.
[56] B. Chen, J. Long, and J. dong Zhang, arXiv:1204.3282.
[57] N. Ohta, arXiv:1109.4458.
[58] A. Bagchi, S. Lal, A. Saha, and B. Sahoo, J. High Energy Phys. 10 (2011) 150.
[59] W. Li, W. Song, and A. Strominger, J. High Energy Phys. 04 (2008) 082.
[60] J. Oliva, arXiv:1210.4123.
[61] Y. Brihaye, E. Radu, and D. H. Tchrakian, Phys. Rev. Lett. **106**, 071101 (2011).
[62] G. W. Moore and N. Seiberg, Phys. Lett. B **220**, 422 (1989).
[63] S. Elitzur, G. W. Moore, A. Schwimmer, and N. Seiberg, Nucl. Phys. **B326**, 108 (1989).
[64] Y. Brihaye and B. Hartmann, arXiv:1112.6315.
[65] A. Anabalon and H. Maeda, Phys. Rev. D **81**, 041501 (2010).
[66] A. Anabalon, arXiv:1211.2765.
[67] A. Anabalon and A. Cisterna, arXiv:1201.2008.
[68] C. Charmousis, T. Kolyvaris, and E. Papantonopoulos, Classical Quantum Gravity **26**, 175012 (2009).
[69] C. Martinez, J. P. Staforelli, and R. Troncoso, Phys. Rev. D **74**, 044028 (2006).
[70] U. Nucamendi and M. Salgado, Phys. Rev. D **68**, 044026 (2003).
[71] D. Ida, Phys. Rev. Lett. **85**, 3758 (2000).
[72] A. Garcia, F. W. Hehl, C. Heinicke, and A. Macias, Classical Quantum Gravity **21**, 1099 (2004).