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Quasilocal first law for black hole thermodynamics

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We first show that stationary black holes satisfy an extremely simple quasilocal form of the first law, $\delta E = \frac{\bar{\kappa}}{8\pi}\delta A$, where the (quasilocal) energy $E = A/(8\pi\ell)$ and (local) surface gravity $\bar{\kappa} = 1/\ell$, with A the horizon area and ℓ is a proper length characterizing the distance to the horizon of a preferred family of quasilocal observers suitable for thermodynamical considerations. Our construction is extended to the more general framework of isolated horizons. The local surface gravity is universal. This has important implications for semiclassical considerations of black hole physics as well as for the fundamental quantum description arising in the context of loop quantum gravity.

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I. INTRODUCTION

Hawking's semiclassical calculations [1] imply that large black holes (BHs) produced by gravitational collapse behave like perfect black bodies at Hawking temperature T_H proportional to their surface gravity once they have reached their stationary equilibrium state. Moreover, different neighboring stationary states are related by the first law of BH mechanics from which black holes can be assigned an entropy $S = A/4\ell_p^2$, where $\ell_p = \hbar^{1/2}$ (in units $G = c = 1$) is the Planck length and A is the classical area of the event horizon.

A complete statistical mechanical account of the thermal properties of BHs from quantum degrees of freedom remains an important challenge for all candidate theories of quantum gravity. Statistical entropy has been calculated in string theory [2] and loop quantum gravity [3], yet in both cases significant gaps remain to be filled.

An important difficulty in dealing with black holes in quantum gravity is that, as they evaporate, the usual definition based on global structure of spacetime is ill posed. This has been recently clearly illustrated in the context of two-dimensional models [4]. Nevertheless, one would expect that the physical notion of a large black hole radiating very little and, thus, remaining close to equilibrium for a long time could be characterized in a suitable way and that such a characterization should help in studying the appropriate semiclassical regime of the underlying quantum theory.

Such quasilocal characterization of black holes is provided by *isolated horizons* [5]. Isolated horizons (IHs) capture the main local features of BH event horizons while being of a quasilocal nature itself. In particular, isolated horizons satisfy a quasilocal version of the first law [6],

$$\delta E_{\rm IH} = \frac{\kappa_{\rm IH}}{8\pi}\delta A + \Omega_{\rm IH}\delta J_{\rm IH} + \Phi_{\rm IH}\delta Q_{\rm IH}, \qquad (1)$$

where $E_{\rm IH}$, $J_{\rm IH}$, and $Q_{\rm IH}$ are suitable quasilocal energy, angular momentum, and charge functions, while $\kappa_{\rm IH}$, $\Omega_{\rm IH}$, and $\Phi_{\rm IH}$ are local notions of IH surface gravity, angular velocity, and electrostatic potential. The previous equation comes from the requirement that time evolution which respects the IH boundary conditions be Hamiltonian [6]. The first law implies that the IH energy $E_{\rm IH}$ must be function $E(A, J_{\rm IH}, Q_{\rm IH})$. The integrability conditions for $E_{\rm IH}$ stemming from the previous phase space identity imposes restrictions on the "intensive" quantities. Beyond these conditions the first law of IH does not give a preferred notion of energy of the horizon: this is a limitation for statistical mechanical descriptions of quantum BHs.

In this paper we show that the above indeterminacy disappears if one fully develops the quasilocal perspective from which IH were defined in the first place. In fact, when studied by stationary observers at proper distance ℓ from the horizon, stationary BHs (and more generally IHs) satisfy the quasilocal first law,

$$\delta E = \frac{\bar{\kappa}}{8\pi}\delta A, \qquad (2)$$

where $E_{\rm IH} = E$, $\bar{\kappa} = \ell^{-1}$ with $\ell^2 \ll A$ a proper length intrinsic to our analysis. The previous equation can be "integrated," thus providing a notion of horizon energy $E = \frac{A}{8\pi\ell}$ which is precisely the one to be used in statistical mechanical considerations by quasilocal observers at distance ℓ. We first show the validity of (2) for stationary black holes and later extend the proof for IHs. The area as a notion of energy has been evoked on several occasions in the context of BH models in loop quantum gravity [7,8]. The results of this paper put these analysis on firm ground.

The results here presented are of a very simple nature and might produce the impression that they must have been somehow well studied in the past. After all the subject of black hole thermodynamics has a vast and long history where the most important and influential results have been published in the 1970s and 1980s. Local aspects such as the notion of local temperature, measured by stationary observers close to the horizon, have been studied in [9]. An earlier reference is the review of Carter in [10]. There is of course the famous membrane paradigm [11] where a similar quasilocal perspective is explored in detail. Finally, the manipulations of [12] are very closely related to the ones found here although the interpretation sought there is quite different from ours. Therefore, despite the long history of the subject the key point that we are making here is new and has not been appropriately stressed before.

II. A QUASILOCAL FIRST LAW

A. Stationary black holes

We first study the thermodynamic properties of Kerr-Newman BHs as seen by a family of stationary observers $\mathcal{O}$, surrounding the horizon at a small proper distance $\ell^2 \ll A$. They follow integral curves of the Killing vector field,

$$\chi = \xi + \Omega \psi = \partial_t + \Omega \partial_\phi, \tag{3}$$

where ξ and ψ are the Killing fields associated with the stationarity and axisymmetry of Kerr-Newman spacetime, respectively, while Ω is the horizon angular velocity,

$$\Omega = \frac{a}{r_+^2 + a^2}, \tag{4}$$

where $a = J/M$. The four-velocity of $\mathcal{O}$ is given by

$$u^a = \frac{\chi^a}{\|\chi\|}. \tag{5}$$

These observers are the unique stationary ones that coincide with the *locally nonrotating observers* of [13] or ZAMOs of [11] as $\ell \to 0$. As a result, the angular momentum of these observers is not exactly zero, but $o(\ell)$. Thus, they are at rest with respect to the horizon: this makes them the preferred observers for studying thermodynamical issues from a quasilocal perspective.

Notice that the family of observers that we have introduced here defines a two-sphere of stationary observers around the horizon (in spacetime their history is represented by a three-dimensional world sheet). Thus, the word "local" is here used in the sense that only the near horizon geometry of the BH will play a role as it will become clear below. In order to avoid confusion from now on we use the term "quasilocal".

Standard arguments lead to the so-called first law of BH mechanics that relates different nearby stationary BH spacetimes of Einstein-Maxwell theory,

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega \delta J + \Phi \delta Q, \tag{6}$$

where M, J, and Q are respectively the total mass, angular momentum, and charge of the spacetime, A is the horizon area, $\Phi = -\chi^a A_a$ is the horizon electric potential where A_a is the Maxwell field produced by the electric charge Q of the BH, and κ is the surface gravity. Note that many of these quantities are defined for an asymptotic observer or have a global meaning—this is clear for M, J, and Q; Φ can be interpreted as the difference in electrostatic potential between the horizon and infinity, Ω is the angular velocity of the horizon as seen from infinity, and κ (if extrapolated from the nonrotating case) is the acceleration of the stationary observers as they approach the horizon as seen from infinity.

The aim of this paper is to construct a quasilocal form of the first law of black hole mechanics. For this it will be crucial to describe physics from the viewpoint of our family of observers $\mathcal{O}$.

1. Thought experiment: Throwing a test particle

The first situation that we will consider involves the process of absorption of a test particle by the BH. More precisely, one throws a test particle of unit mass and charge q from infinity to the horizon. The geometry is stationary and axisymmetric as well as the electromagnetic field, namely $\mathcal{L}_\xi g_{ab} = \mathcal{L}_\psi g_{ab} = \mathcal{L}_\xi A_a = \mathcal{L}_\psi A_a = 0$. The particle satisfies the Lorentz force equation,

$$w^a \nabla_a w_b = q F_{bc} w^c, \tag{7}$$

with four-velocity w^a. The conserved energy of the particle is $\mathcal{E} \equiv -w^a \xi_a - qA^a \xi_a$ while the conserved angular momentum is $L \equiv w^a \psi_a + qA^a \psi_a$. As the particle gets absorbed, the black hole settles down to a new state with $\delta M = \mathcal{E}$, $\delta J = L$, and $\delta Q = q$. Equation (6) then implies

$$\frac{\kappa}{8\pi}\delta A = \mathcal{E} - \Omega L - \Phi q. \tag{8}$$

For our observers having four-velocity u^a the local energy of the particle is

$$\mathcal{E}_{\rm loc} = -w^a u_{a.} \tag{9}$$

Using (5), the definitions of $\mathcal{E}$, L, and $\Phi \equiv -\chi^a A_a$ we find

$$\mathcal{E}_{\rm loc} = -\frac{w^a \xi_a + \Omega w^a \psi_a}{\|\chi\|} = \frac{\mathcal{E} - \Omega L - q\Phi}{\|\chi\|}. \tag{10}$$

Finally from (8)

$$\mathcal{E}_{\rm loc} = \frac{\bar{\kappa}}{8\pi}\delta A, \quad \text{where } \bar{\kappa} \equiv \frac{\kappa}{\|\chi\|}. \tag{11}$$

From the point of view of our quasilocal observers, the horizon has absorbed a particle of energy $\mathcal{E}_{\rm loc}$. The change in energy of the system E as seen by $\mathcal{O}$ must be $\delta E = \mathcal{E}_{\rm loc}$. All this implies a quasilocal version of the first law,

$$\delta E = \frac{\bar{\kappa}}{8\pi}\delta A. \tag{12}$$

Direct calculations show that

$$\bar{\kappa} \equiv \frac{\kappa}{\|\chi\|} = \frac{1}{\ell} + o(\ell). \tag{13}$$

In other words the local surface gravity measured by the locally nonrotating stationary observers is universal, i.e., independent of the mass M, angular momentum J, and charge Q of the Kerr-Newman black hole (for a different local definition of surface gravity see [14]). From (12) we get the quasilocal notion of energy,

$$E = \frac{A}{8\pi\ell}, \tag{14}$$

as the above quantity [defined up to a constant and in the approximation where $o(\ell)$ corrections are neglected] leads to the quasilocal first law when varied. This provides a natural quasilocal notion of horizon energy relevant for thermodynamical considerations: a thermal analog of internal energy. Its physical interpretation is restricted to the realm of small changes close to equilibrium. Equation (14) provides a bookkeeping devise that accounts for the energy exchanges with the BH horizon as seen by our quasilocal observers at distance ℓ in a way that becomes exact when $\ell \to 0$. This is the meaning of $o(\ell)$. Our formula could in principle be modified by the addition of an unknown constant. However, as such constant plays no role in the thermodynamical considerations or equilibrium statistical mechanics, we have removed it from our definition. Far from equilibrium our framework simply breaks down.

The idea is to associate the above energy and first law to the horizon itself by taking our ℓ as small as possible without being zero. An effective quantum gravity formulation where thermodynamics makes sense suggests that ℓ should be of the order of the Planck scale [8] but this is not really essential for the analysis presented here.

2. Refined thought experiment: The field theoretical version

A stronger (and local) field theoretic version of the previous arguments goes as follows: Let the matter falling into the stationary BH (with bifurcate horizon) be described by a small perturbation of the energy-momentum tensor δT_{ab} whose backreaction to the geometry will be accounted for in the linearized approximation of Einstein's equations around the stationary black hole background. The current $J^a = \delta T^a{}_b \chi^b$ is conserved, $\nabla_a J^a = 0$. Applying Gauss's law to the spacetime region bounded by the BH horizon $\mathcal{H}$ and the timelike world sheet of the observers $\mathcal{O}$ ($W_{\mathcal{O}}$), we get

$$\int_{\mathcal{H}} dV dS \delta T_{ab} \chi^a k^b = \int_{W_{\mathcal{O}}} J_b N^b, \tag{15}$$

where N^a is the inward normal of $W_{\mathcal{O}}$ and $k = \partial_V$ a null geodesic normal on $\mathcal{H}$, with V an affine parameter along the generators of the horizon. The origin $V = 0$ is chosen to coincide with the bifurcation horizon, see Fig. 1. We have also assumed that δT_{ab} vanishes in the far past and far future of the considered region. Using the fact that $\chi^a = \kappa V k^a$ on $\mathcal{H}$, the previous identity takes the form

$$\kappa \int_{\mathcal{H}} dV dS V \delta T_{ab} k^a k^b = \int_{W_{\mathcal{O}}} \|\chi\| \delta T_{ab} u^a N^b. \tag{16}$$

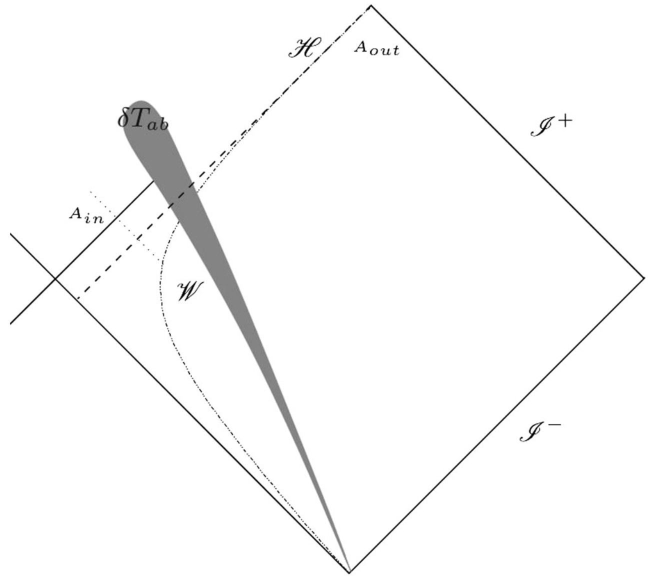

FIG. 1. Conformal diagram representing the perturbation of an initially stationary black hole with a bifurcate horizon. The dashed line represents the true BH horizon, the stationary observers world sheet is denoted by $\mathcal{W}$. The quantity $A_{\rm in}$ is the area of the initially stationary background while $A_{\rm out}$ is the final area of the BH horizon.

Notice that the integral on the right is closely related to the energy flux associated to the observers, which is equal to δE. Now, the Raychaudhuri equation in the linear approximation is

$$\frac{d\theta}{dV} = -8\pi \delta T_{ab} k^a k^b, \tag{17}$$

where θ is the expansion of the null generators k^a. Finally, using the fact that $\|\chi\|$ is constant up to first order on the right-hand side of (16), we obtain

$$\int_{\mathcal{H}} dV dS V \frac{d\theta}{dV} = -\frac{8\pi\|\chi\|}{\kappa} \delta E, \tag{18}$$

where we have neglected terms of the form $o(\ell)\delta$ which are higher order in our treatment. By an integration by parts the integral on the left is equal to $-\delta A$.[1]

[1]Explicitly,

$$-\int_{V_1}^{\infty} dV \int dS V \frac{d\theta}{dV} = \overbrace{\int_{V_1}^{\infty} dV \int dS \theta(V)}^{A_{\rm out} - A(V_1)} + \overbrace{\int dS V_1 \theta(V_1)}^{A_{\rm in} - A(V_1)}$$
$$= A_{\rm out} - A_{\rm in} = \delta A, \tag{19}$$

where in the last term we dropped the boundary contribution at $V = \infty$ using that $\theta(\infty) = 0$. The Raychaudhuri equation implies that the previous quantity is independent of V_1 when it is prior to the start of matter infall. The proof of the above equations follows from this fact, the evaluation of (19) at $V_1 = 0$, and the use of $A(0) = A_{\rm in}$ (see Fig. 1).

Finally, using $\bar{\kappa} \equiv \kappa/\|\chi\|$ we get the desired result,

$$\delta E = \frac{\bar{\kappa}}{8\pi} \delta A. \tag{20}$$

The previous local field theoretical argument will be generalized to include IHs at the end of this paper.

B. Isolated horizons

Here we prove the validity of the quasilocal form of the first law (20) in the more general framework of IHs. Moreover, the first law so derived is dynamical in character, i.e., changes in the area and energy of the system can really be seen as the consequence of the absorption of matter fields by the horizon along its history.

Isolated horizons are equipped with an equivalence class of null normal $[\chi]$ where equivalence is defined up to constant scalings. The generators χ_a are geodesic and define a notion of isolated horizon surface gravity $\kappa_{\rm IH}$ through the equation $\nabla_\chi \chi_a = \kappa_{\rm IH}\chi_a$. It is clear that $\kappa_{\rm IH}$ is not defined in $[\chi]$ because it gets rescaled when χ is rescaled. The near horizon geometry is described (in terms of Bondi-like coordinates) by the metric [15],

$$g_{ab} = 2dv_{(a}dr_{b)} - 2(r-r_0)[2dv_{(a}\omega_{b)} - \kappa_{\rm IH} dv_a dv_b] + q_{ab} + o[(r-r_0)^2], \tag{21}$$

where $\chi = \partial_v$ is the extension to the vicinity of $\mathcal{H}$ of the null generators of the IH through the flow of a natural null vector $n^a = \partial_r^a$ [i.e., $\mathcal{L}_n(\chi) = 0$], $q_{ab}n^a = q_{ab}\chi^a = 0$, and ω_a is a one form intrinsic to IHs with the important property that $\omega(\chi)|_{\mathcal{H}} = \kappa_{\rm IH}$. Also one has [16]

$$\mathcal{L}_\chi g_{ab}|_{\mathcal{H}} = 0. \tag{22}$$

Thus, χ can be used to define the observers $\mathcal{O}$ as in (5). The proper distance ℓ to the horizon from a point with coordinate r along a curve normal both to χ and q_{ab}—with tangent vector $N^a = \partial_r^a + (2\kappa_{\rm IH}(r-r_0))^{-1}\partial_v^a$—is given by $\ell = \sqrt{2(r-r_0)/\kappa_{\rm IH}}$, while $\chi\cdot\chi = 2\kappa_{\rm IH}(r-r_0)$. Therefore,

$$\bar{\kappa} = \frac{\kappa_{\rm IH}}{\|\chi\|} = \frac{1}{\ell}. \tag{23}$$

Notice that $\bar{\kappa}$ is well defined in $[\chi]$ in contrast with $\kappa_{\rm IH}$. As the form of the perturbed Raychaudhuri equation (17) is the same for the generators of IH (as their expansion, shear, and twist vanish by definition), the same arguments given below Eq. (14) yield the quasilocal first law,

$$\delta E = \frac{\bar{\kappa}}{8\pi} \delta A, \tag{24}$$

where the energy notion $E = \frac{A}{8\pi\ell}$, and we have used that $\ell^2 \ll A$, provide the right quasilocal framework for the statistical mechanics study of quantum IHs.

Summarizing, even though we have first justified the quasilocal first law (24) starting from the analysis of the first law for stationary spacetimes and its translation in terms of the quasilocal observers $\mathcal{O}$, the final analysis for IHs implies that the result can be recovered entirely from local considerations that know nothing about the global structure. In this paper we are proposing to use this remarkable fact in order to reverse the perspective, and thus take the local definition of IHs with its null normals $[\chi]$, the quasilocal first law (24), the energy (14), and the intrinsic notion of surface gravity (23) [both associated to the quasilocal observers (5)] as the fundamental structure behind BH thermodynamics and the statistical mechanical treatments in the framework of loop quantum gravity [8].

Notice also that the quasilocal first law and the universality of $\bar{\kappa}$ implies the Gibbs relation $E = TS$ where $T = \ell_p^2\bar{\kappa}/(2\pi)$, and $S = A/4\ell_p^2$. This simple property of usual thermodynamic systems is not realized by the quantities taking part in the standard first law (6). This is an extra bonus of our quasilocal description.

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