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Remarks on correlators of Polyakov loops

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Polyakov loop eigenvalues and their N dependence are studied in two- and four-dimensional $SU(N)$ Yang-Mills (YM) theory. The connected correlation function of the single-eigenvalue distributions of two separated Polyakov loops in two-dinemsional YM is calculated and is found to have a structure differing from the one of corresponding Hermitian random matrix ensembles. No large-N nonanalyticities are found for two-point functions in the confining regime. Suggestions are made for situations in which large-N phase transitions involving Polyakov loops might occur.

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I. INTRODUCTION

This work is concerned with $SU(N)$ Yang-Mills (YM) theory quantum chromodynamics with N colors (QCDN) in four dimensions. QCDN admits a large-N expansion [1]. Lattice work has shown that there is confinement at finite and infinite N [2]. Then, QCDN at $N = \infty$ (QCD∞) is similar to the starting point of the topological expansion (TE) [3]. In TE one constructs iteratively an S matrix from a set of postulated basic general properties. Another starting point for the TE is provided by string theory. In both cases one starts from a system consisting of an infinite set of stable particles interacting weakly at linear order. Upon iteration, other singularities build up. The expansion is organized diagrammatically with an order given by the genus of a Riemann surface.

The QCDN route is better founded than the string one. We can safely assume that there exist Wightman n-point functions of local gauge invariant observables that admit a single valued continuation to the extended tube $\mathcal{T}'_{n-1}$ [4] for any N. These functions determine the leading nontrivial term in $\frac{1}{N}$ of any amplitude entering the S matrix. From this off-shell starting point, one might be able to build a better founded QCDN string theory [5]. Concretely, one would need explicit forms of a least some of the sets of entries of the S matrix.

Despite quite a few papers which achieved high levels of popularity, there is not one nonperturbative physical number that has been analytically calculated, or at least credibly estimated, in QCDN (with or without a finite number of quarks) at leading order in $\frac{1}{N}$ or $\frac{1}{N^2}$. Nevertheless, interest in large N does not seem to die out. Quite a few workers, me included, still are trying to get some new quantitative result in QCDN which rests on the simplification afforded by $N \gg 1$.

My idea has been to find a simple physical single scale observable whose behavior as a function of this scale showed a universal behavior at the crossover separating long from short scales. Large N comes in to provide this universality by a large-N phase transition. The universality then becomes a random-matrix type of universality. The hope is to exploit it in order to match effective string descriptions holding at large distances to perturbation theory holding at short distances. For example, consider a circular Wilson loop of radius r. For r large effective string theory provides some universal information about the r dependence, while at small r perturbation theory applies; the new ingredient is that random-matrix universality would provide the means to connect these two dependencies. The hope is that an approximate connection between the string tension and some standard perturbative scale would then be calculable. The existence of the large N phase transition is believable for the circular loop because it has been established numerically for square loops. However, it would be preferable to consider smooth loops also on the lattice, and this leaves us with only Polyakov loops winding around a compactified direction. The length of this circle has to be bounded from below in order to stay in the confined phase. The single-eigenvalue density, $\rho^{(1)}$, of a Polyakov loop becomes uniform at $N = \infty$ on account of the well-known $Z(N)$ symmetry. This leaves us with $\rho^{(2)}$, the connected correlation function of the $\rho^{(1)}$'s of two separated Polyakov loops, as the simplest smooth observable on the lattice.

In this paper, I focus on Polyakov loops. The outline of the papers is as follows. Sections II and III A provide background material. The concrete new results are in Sec. III B. They consist of an evaluation of the single Polyakov eigenvalue density connected two-point correlation function under the assumption of second rank Casimir dominance. A formula for any N (taken as odd, for simplicity) is provided, the large-N limit is taken, and the validity of the latter is checked numerically. Next, a brief comparison with Monte Carlo data in four-dimensional $SU(N)$ Yang-Mills theory is carried out. There are no large-N phase transitions. Incidentally it is noted that the result does not show universal features known to hold for large Hermitian matrix ensembles. Section IV contains ideas for future work. A short summary concludes the paper.

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II. VOLUME REDUCTION

QCDN is a field theory, but geometrically the fundamental variables are not fields defined over $\mathcal{M} = R^4$ but rather fields defined over loops in R^4. This becomes particularly evident when one introduced a lattice UV cutoff: One can derive an infinite set of equations connecting various loop operators, and the equations reflect the ordinary locality of four-space the field theoretical formulation rests on, without any of the collateral expenditures (gauge fixing, Faddeev-Popov ghosts, Gribov ambiguities) associated with formulating the continuum theory in terms of gauge fields [6]. The loop equations self-truncate at infinite N, feeding the hope that it ought to be easier to handle nonperturbative issues of QCDN at $N = \infty$ [6]. Taking the equations to the continuum is hampered by the nonexistence of anything resembling a decent calculus in loop space. One way to go around this obstacle is to try to guess a well-defined solution directly in the continuum (which obeys general symmetry/unitarity constraints) and show that it satisfies a set of equations that can be viewed as a concrete realization of the formal continuum loop equations [7]. This has led to progress in string theory and even to a connection back to field theory, but not for QCDN [8]. As far as I know, we still do not have even one nontrivial example where the formal loop equations have been credibly defined in the continuum.

One consequence of the loop equation is that at $N = \infty$ the replacement of R^4 by T^4, where the sides of the torus are all larger than the inverse deconfinement temperature, preserves a large subset of observables with no dependence on the actual finite length of these sides [9]. This is of some help in numerical work, but the saving is quite limited [10]. The term describing this phenomenon is "reduction," on account of a reduction in the number of degrees of freedom as far as the four volume size goes. Reduction can be applied to any number of directions, and one assumes that there is a hierarchy of scales associated with the preservation of the associated $Z^k(N)$, $k = 1, 2, 3, 4$ [9].

The restriction on the sides of T^4 ensures that the global $Z^4(N)$ symmetry now present is not broken spontaneously by the $N = \infty$ limit. The preservation of the consequence of this symmetry on expectation values of parallel transport round noncontractible loops is a necessary ingredient for reduction [11]. The equivalence between the R^4 and T^4 loops equations breaks down if $Z^4(N)$ is not obeyed by expectation values of winding loops at $N = \infty$ [12]. If the $Z^4(N)$ is preserved at $N = \infty$, the spacings between momenta induced by the finite volume of T^4 get continuously filled in by the eigenvalue sets of winding loops [9].

Polyakov loops are the natural extra observables one has when considering $\mathcal{M} = S \times R^3$. We assume that the compact direction is large enough that the one $Z(N)$ is preserved at all N. When I slice $\mathcal{M}$ by three spaces parallel, or orthogonal, to the compact direction, I find two different transfer matrices. They provide two Hamiltonian pictures. In one picture $Z(N)$ is just an extra global symmetry of the Hilbert space, and the system is at zero temperature. In the other, the system is at finite temperature. In either case, one has a Hilbert space which can be chosen to transform irreducibly under the symmetries commuting with the Hamiltonian. The spaces and Hamiltonians are different, providing different spectral representations of identical observables.

Reduction applies only to nonwinding loops; it is of interest to see if any reduction-related simplifications hold at $N = \infty$ also for winding loops [13].

III. CORRELATIONS OF TWO POLYAKOV LOOPS

Define the parallel transport winding loop operator by

$$U_P(x) = \mathcal{P} e^{i \oint_{x_4}^{x_4} A_4(\vec{x},\tau) d\tau}. \tag{1}$$

The compact direction is 4. $A_\mu(x)$ are the gauge fields given by traceless Hermitian $N \times N$ matrices and $(\vec{x})_i = x_i$, $i = 1, 2, 3$. $\mathcal{P}$ is path ordering. Polyakov loops are independent of x_4:

$$P_R(\vec{x}) = \frac{1}{d_R} \chi_R(U_P(x)). \tag{2}$$

R labels an irreducible representation of $SU(N)$, χ_R is the character in R, and d_R is the dimension of R. If the number of boxes in the Young pattern corresponding to R is m_R, the N-ality of R is given by $n = \mathrm{mod}(m_R, N)$. Under $Z(N)$ I have $P_R \to e^{2i\pi n/N} P_R$.

Consider the case $n \neq 0$. Then $\langle P_R \rangle = 0$, and $G_R(r) = \langle P_R(0) P_{\bar{R}}(r) \rangle$ is generically nonzero. Here $\bar{R}$ is conjugate to R. r is a positive distance. Denoting the length of the compact direction by l, the two-point function is a function of R and the two length scales l, r. Define (formally) $W_R(l, r) = \log G_R(r)$. We assume that the θ parameter in front of the $\int d^4x \,\mathrm{Tr}[F \wedge F]$ term in the action is set to zero, so $G_R(r)$ is real and positive. The definition is only formal because after renormalization there will be an arbitrary term in $W_R(l, r)$ of the form $\mu_R l$. Thus, $\partial W(l, r)/\partial l$ is well defined up to an additive constant [14].

A. Simplest asymptotic properties

Since $l > 1/T_c$ where T_c is the deconfinement temperature, the first reaction would be that W cannot be computed in perturbation theory. Nevertheless, to some extent, the quantity

$$\lim_{l\to\infty} \partial^2 W(l, r)/\partial l \partial r = dV_R(r)/dr \tag{3}$$

can [15]. Instead of the $l \to \infty$ limit in Eq. (3), one takes an infinite uncompactified x_4 axis and replaces l by T and $W(l, r)$ by the logarithm of a rectangular Wilson loop of shape $T \times r$. Inspection of Feynman diagrams shows that in the $T \to \infty$ limit, one has an expansion for $V_R(r)$ up to two-loop order in the coupling which ought to be useful for r small enough. To go to higher loops, one needs to include nonanalytic terms in the coupling [16]. Potential nonrelativictic quantum chromodynamics provides a prescription for how to do this, but I doubt that it is unique [17].

In this expansion the path integral over A_4 is expanded around $A_4 = 0$, which breaks the $Z(N)$. This may not matter for the two-point Polyakov loop function at infinite l. Whether it can be used at any finite l is an open question, so long as $l > 1/T_c$. If there were a credible perturbative regime in which the l scale can somehow be removed from the problem, one would expect that $\rho^{(2)}$ might show some crossover as r is varied. Then, the ingredients necessary for a large-N phase transition to develop are present. The hope is that, for r small enough, the eigenvalues of one Polyakov loop would restrict the fluctuations of the eigenvalues of the other Polyakov loop to such a degree that the periodicity in the angle differences would barely be felt. For r large this periodicity would get restored to full strength. A separating crossover at finite N would become a phase transition at infinite N. The intuition behind the focus on eigenvalues is that collectively their fluctuations explore the distance of parallel transport operators from unity. Only when the compactness of the group is felt does one expect nonperturbative effects to become important. Compactness is felt only when parallel transport exceeds a certain distance from unity. We shall see that the hope for a transition is not realized.

Beyond perturbation theory, $V_R(r)$ is the ground state energy of the Hamiltonian associated with evolution in the x_4 direction in the sector defined to transform under the local gauge group as R at $\vec{r} = (0, 0, 0)$ and $\bar{R}$ at $\vec{r} = (r, 0, 0)$, $r > 0$. This ground state is d_R^2 degenerate. When one computes the partition function viewing l as the inverse temperature, the degeneracy of the ground state cancels the prefactors normalizing the Polyakov loop operators. There is no overall factor of N^2 in the physical piece of the free energy.

Because of the representation content of the Hilbert space which breaks translation invariance, there is no physical interpretation in this picture for plane waves propagating in the $x_{1,2,3}$ directions superposing ground state states.[1] The dependence of $dV(r)/dr$ on r for $r \to \infty$ starts with a constant (the string tension of open strings with fixed endpoints transforming as R and $\bar{R}$, respectively) and continues to subleading orders; several terms in this expansion are universal. I shall describe below in more detail this aspect in another asymptotic limit.

[1] In potential nonrelativictic quantum chromodynamics one deals with an expansion in inverse quark mass; in this context it is possible to provide a meaningful definition of the spatial Fourier transform of $V(r)$ because the sources can move. At three-loop order, the expansion in powers of the strong-force coupling breaks down, but the nonanalytic term responsible for this (in this framework) can be derived.

One can also consider the large r-separation fixed l limit:

$$\mathcal{F}_R(l) = \lim_{r\to\infty} \partial^2 W_R(l, r)/\partial l \partial r. \tag{4}$$

Confinement in this context means that $\lim_{l\to\infty} \mathcal{F}_R(l) = \sigma_n > 0$, with σ_n depending only on the assumed nonzero N-ality of R, n; $\sigma_n = \sigma_{N-n}$. $\mathcal{F}_R(l)$ gives the l derivative of the r derivative at $r = \infty$ of eigenvalues of the Hamiltonian describing evolution in any one of the directions x_j, $j = 1$, 2, 3 in the n sector of the global $Z(N)$. Now, it does make sense to superpose states and project on zero spatial momentum. This gives the ground state energy in the n-winding sector. One can look at several subleading terms in the large-l expansion. For this define $\mathcal{F}_R(l) = \sigma_n \hat{F}_R(l\sqrt{\sigma_n})$,

$$\hat{F}(x) = 1 + c_1/x^2 + c_2/x^4 + c_3/x^6 + \cdots \tag{5}$$

Assuming an effective string theory description, one derives from symmetry principles alone that the coefficients $c_{1,2,3}$ are universal calculable finite numbers, independent of R (and consequentially of n). They are actually also independent of any other detail regarding the field theory, except the assumption of confinement and applicability of effective string theory [18].

To summarize, at any finite N, $W_R(\infty, r)$ and $W_R(l, \infty)$ with R of nonzero N-ality have some universal coefficients in their asymptotic expansions in r and l, respectively, which follow from the assumption of confinement and applicability of effective string theory. Numerical checks have yielded results consistent with this. The $N \to \infty$ limit provides no further simplification with respect to these properties.[2] There is a hope that the $N \to \infty$ limit could provide a clear demarcation point for the domain in which the asymptotic expansion of long strings can make any sense at all, but the universal predictions of effective string theory are insensitive to this.

B. Looking for large-N phase transitions

Effective string theory can also be applied to the case where l and r are both taken to infinity; a brief overview with references to original work can be found in Ref. [10]. This can be done also for ordinary contractible rectangular Wilson loops in R^4. Reduction applies in these cases. There

[2] One should distinguish between the dream string theory, which is equivalent to QCDN (it is unknown whether such a dream string theory actually exists), and the effective string theory I am talking about; the dream (closed-)string theory would have a coupling constant, which goes as $\frac{1}{N^2}$, and the $N \to \infty$ limit would turn the interactions off. One would need the $N \to \infty$ limit in order to justify the focus on the lowest genus surface relevant to the correlation under investigation. The effective string theory on the other hand has already summed up all contributions from handles of the dream string theory, and the universal results provided by the cylinder are N independent.

is no lower limit on l or r, and therefore there is a perturbative regime. Renormalization to remove perimeter divergences is still required. Again, for individual irreducible representations, large N provides no additional constraints on the universal large l, r results of effective string theory. However, now there is an evident connection to a perturbative regime where both l and r are small. This connection is smooth. Only by looking at many representations simultaneously does one detect a large-N transition: one finds a nonanalyticity in the single-eigenvalue distribution at a point which serves as a boundary between the perturbative and nonperturbative regime [19]. The boundary location depends on an arbitrary dimensional smearing parameter. This parameter is a remnant of the need to eliminate the perimeter divergence. It is defined in a manner independent of R [20]. Only at infinite N is there a well-defined transition point. For any finite N, one has only a crossover. Numerical and analytical considerations indicate that the large-N transition has a certain random-matrix model universality [19].

At any finite N, the Polyakov loop two-point function has a discontinuity as l is varied through $l = \frac{1}{T_c}$ separating the confined and deconfined phases. Since one can compute $V_R(r)$ in perturbation theory at $l = \infty$, one may hope that a specifically *large* N transition as r is varied takes place at any l in the confined phase.

Lattice checks of predictions of effective string theory have overall been successful, at times even surprisingly successful; this has nothing to do with N being large enough. Just like effective Lagrangians for massless pions work already on the lattice since these Lagrangians can parametrize any reasonable UV behavior, effective string theory should also hold directly on the lattice so long as one is in the rough phase. For long strings the lattice violation of $O(4)$ invariance makes an impact determined by symmetries. If I replace the hypercubic lattice by an F_4 lattice [21], the effective string theory predictions may apply even better. So, the success of lattice checks of effective string theory might be "explained" by the relative unimportance of the proximity of the field theoretical continuum limit. For example, QCDN with scalar matter fields in the adjoint has no continuum limit; on the lattice it would confine, and effective string theory would make some universal predictions in the rough phase of the loop under consideration.

In general, strings are less sensitive to high mode cutoffs than field theories are [22]. In four dimensions there is an exception [22] which has to do with the fluctuations in the extrinsic string curvature. Potentially significant deviations from effective string theory were observed in Ref. [10] and independently in Ref. [23]; in Ref. [10] these deviations were tentatively attributed to the corners of lattice loops because of the perturbation theory experience with the corner divergences in Wilson loops.

The technical reason for the universality of the large-N transition in the case of contractible Wilson loops has been second rank Casimir dominance of the dependence[3] on the representation R. One may then map the size parameters that are varied into an appropriate measure of separation in two-dimensional YM. So, I ask whether there is a large-N phase transition to be found in an analysis of Polyakov loop correlators in YM on a two-dimensional cylinder.

1. Eigenvalue-eigenvalue correlation function in two-dimensional YM

In general, it is known that there are large-N nonanalyticities in two-dimensional (2D) YM on a cylinder [24]. Here I wish to see if they show up in the two-point function of the single-eigenvalue distributions associated with two separated Polyakov loops. In the context of large-size Hermitian matrices, two-point single-eigenvalue correlation functions have been shown to have some universal properties [25]. A side result of the calculation below will be to check whether this universality extends to the simplest unitary matrices' ensemble with a global $Z(N)$ symmetry. From experience with Hermitian matrix models, I expect smooth and strongly fluctuating contributions to the two-point eigenvalue function of equal magnitude; the oscillating piece has to be first separated out, and only the remaining smooth piece can exhibit a universal large-N phase transition.

In the subsequent equations, the assumption that N is odd is made implicitly. The partition function of $SU(N)$ YM on a 2D cylinder connecting two loops, 1 and 2, is given by [24]

$$Z_N(U_{P_1}, U_{P_2}|t) = \sum_R \chi_R(U_{P_1}) e^{-\frac{t}{2N}C_2(R)} \chi_{\bar{R}}(U_{P_2}). \qquad (6)$$

t is the area in some area unit. Let $\rho_1^{(1)}(\alpha)$ and $\rho_2^{(1)}(\beta)$ be the single-eigenvalue distributions associated with the $N \times N$ unitary matrices $U_{P_{1,2}}$ [26]:

$$\rho^{(1)}(\theta; U) = \frac{2\pi}{N} \sum_{k=1}^{N} \delta_{2\pi}(\theta - \theta_k). \qquad (7)$$

The θ_k are the eigenvalues of U, and $\delta_{2\pi}$ is the 2π-periodic delta function. The character expansion of ρ is

$$\rho^{(1)}(\theta; U) = 1 + \frac{1}{2N} \lim_{\epsilon \to 0^+} \sum_{p=0}^{N-1} \sum_{q=0}^{\infty} (-1)^p e^{-\epsilon(p+q+1)} \times [e^{i(p+q+1)\theta} \chi_{(p,q)}(U) + e^{-i(p+q+1)\theta} \chi_{\overline{(p,q)}}(U)]. \qquad (8)$$

[3] The expectation value of contractible Wilson loops of all sizes can be written as exponents of expressions in which the dependence on the representation enters predominantly through the second rank Casimir. In order to get eigenvalue distributions, one needs to sum over a set of representations. The factors multiplying the Casimirs are smooth in the geometrical parameters of the loop. This structure can induce large-N phase transitions.

The irreducible representation (p, q) has a Young pattern in the shape of a width-one hook, with $1 + p$ rows and $1 + q$ columns. We wish to calculate the connected two-point function

$$\langle\rho_1^{(1)}(\alpha)\rho_2^{(1)}(\beta)\rangle_c = \int dU_{p_1} dU_{p_2} \rho_1^{(1)}(\alpha)\rho_2^{(1)}(\beta) \times [Z_N(U_{p_1}, U_{p_2}|t) - 1]. \quad (9)$$

dU is the Haar measure on $SU(N)$. In order for a pair $(p, q)_1$ and $(p, q)_2$ to contribute, I need that there be a singlet in their direct product. As N is odd, this will happen only when one pair is the conjugate of the other. For odd N there are no (p, q) self-conjugate pairs. For $t > 0$ I have

$$\langle\rho_1^{(1)}(\alpha)\rho_2^{(1)}(\beta)\rangle_c = \frac{1}{N^2}\sum_{p=0}^{N-1}\sum_{q=0}^{\infty}(-1)^p e^{-\frac{t}{2N}C(p,q)} \times \cos[(p+q+1)(\alpha-\beta)]. \quad (10)$$

Here [26],

$$C(p, q) = (p+q+1)\left(N - \frac{p+q+1}{N} + q - p\right). \quad (11)$$

One can reorganize the sums to get

$$N^2\langle\rho_1^{(1)}(\alpha)\rho_2^{(1)}(\beta)\rangle_c \equiv \Re\mathcal{J} = \mathcal{J}_a + \mathcal{J}_b. \quad (12)$$

The real part is taken using Eq. (8) with fixed small ϵ and subsequently setting $\epsilon = 0$:

$$\mathcal{J}_a = \sum_{n=1}^{N} e^{-\frac{t(N-1)}{2N^2}n(n+N)} \frac{1-(-1)^n e^{\frac{t}{N}n^2}}{1+e^{\frac{t}{n}}} \cos(n(\alpha-\beta)) \quad (13)$$

$$\mathcal{J}_b = \sum_{n=N+1}^{\infty} e^{-\frac{t(N-1)}{2N^2}n(n+N)} \frac{1-(-1)^N e^{tn}}{1+e^{\frac{t}{n}}} \cos(n(\alpha-\beta)). \quad (14)$$

$\mathcal{J}_a$ contains the two-point connected correlations $\langle \mathrm{Tr}(U_{p_1}^k)\,\mathrm{Tr}(U_{p_2}^{\dagger k})\rangle(t)$ for $k = 1, \ldots, N$.

To obtain the large-N limit, I write an integral representation [26], exploiting the quadratic nature of the $C(p, q)$ dependence on p, q:

$$\mathcal{J} = \frac{Nu}{t} e^{-\frac{t}{2}(1-\frac{1}{N^2})} \int \frac{dxdy}{2\pi} e^{-\frac{N}{2t}(x^2+y^2)+\frac{(x+iy)^2}{2t}} \times \frac{1+u^N e^{-N(x+\frac{t}{2})+\frac{t}{2}+\frac{t}{N}}}{1+ue^{-(x+\frac{t}{2})+\frac{t}{2N}+\frac{t}{N^2}}} \frac{1}{1-ue^{iy-\frac{2}{2}+\frac{t}{2N}+\frac{t}{2N^2}}}. \quad (15)$$

Here, $u = \exp[i(\alpha-\beta)]$. The y integral can be done by saddle point with $y_{\rm sp} = 0$. To leading order in N, I get

$$\mathcal{J} \approx \sqrt{\frac{N}{t}} u e^{-\frac{t}{2}+\frac{t}{2N^2}} \int \frac{dx}{\sqrt{2\pi}} e^{-\frac{N}{2t}x^2+\frac{1}{2t}x^2} \frac{1+u^N e^{-N(x+\frac{t}{2})+\frac{t}{2}}}{1+ue^{-x-\frac{t}{2}+\frac{t}{2N}}} \times \frac{1}{1-ue^{-\frac{t}{2}}}. \quad (16)$$

We need to keep the factor $e^{t/2N}$ in the denominator of the first fraction in the integrand in order to ensure its regularity for $x + t/2 < 0$ because then the numerator divides evenly by the denominator. Carrying out the integral by saddle point, I find two contributions, corresponding to the saddle points $x_{\rm sp}^1 = 0$ and $x_{\rm sp}^2 = -t$. The large-N limit is taken at finite nonzero $\Im(u)$ and finite positive t. The limits $\Im(u) \to 0$ and $t \to 0$ do not commute with the limit $N \to \infty$. The final results for the leading large-N behavior comes out to be

$$\Re\mathcal{J} \approx \frac{1}{2}\frac{\sinh\frac{t}{2}\cos\phi + e^{\frac{t}{2}}(\sinh\frac{t}{2}\cos N\phi - \sin\phi\sin N\phi)}{\sinh^2\frac{t}{2}+\sin^2\phi}. \quad (17)$$

Here, $\phi = \alpha - \beta$. The result is the sum of a smooth term and a rapidly oscillating one. There are no large-N phase transitions.

The nonoscillating term is

$$\Re\mathcal{J}_{\text{non-oscillating}} \approx \frac{1}{2}\frac{\sinh\frac{t}{2}\cos\phi}{\sinh^2\frac{t}{2}+\sin^2\phi}. \quad (18)$$

It does not exhibit the universal structure seen in continuous chains of large Hermitian matrix models [25]. Most likely, the main difference is that for the unitary matrix ensemble I solved above, there is no analogue of the nontrivial potential $\int dt\,\mathrm{Tr}V(M(t))$ term in the action for the Hermitian $M(t)$ matrices. While the explicit form of V is irrelevant, its mere presence is relevant.

To get a feel for the goodness of the large-N limit and also check whether its derivation was correct, I present Figs. 1 and 2. One can see that the large-N approximation deteriorates when N decreases, when ϕ is close to $k\pi$, $k \in Z$ and when t is small, but otherwise holds well.

2. Some four-dimensional examples

I now turn to four dimensions and report on some numerical simulation done in order to see qualitatively whether, overall, the data looks similar to the 2D case. I only wish to confirm that also in the four-dimensional case, there are no signs of a large-N transition. I do not aim here for anything quantitative and am content with low numerical precision.

The connected single-eigenvalue distribution for two Polyakov loops in four dimensions will be a function of α and β similarly to the 2D case. For finite N, there is no reason for this function to depend only on the angle difference. The $Z(N)$ symmetry only provides invariance under

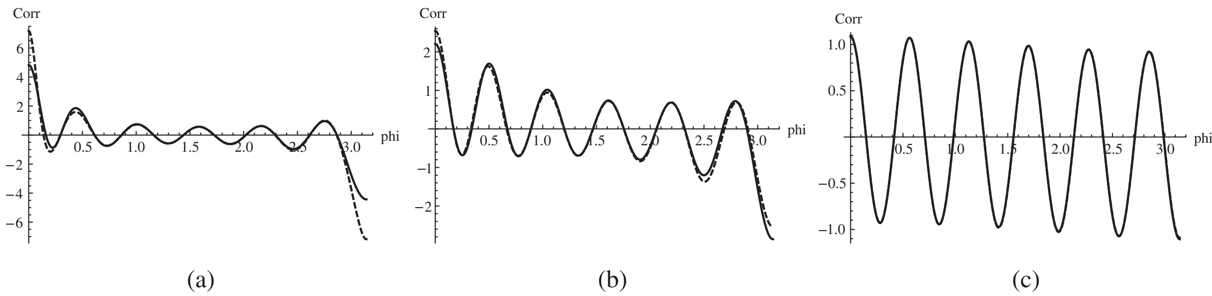

FIG. 1. Solid line: exact result; dashed line: large N. (a) $N = 11$, $t = 0.3$; (b) $N = 11$, $t = 1$; (c) $N = 11$, $t = 5$.

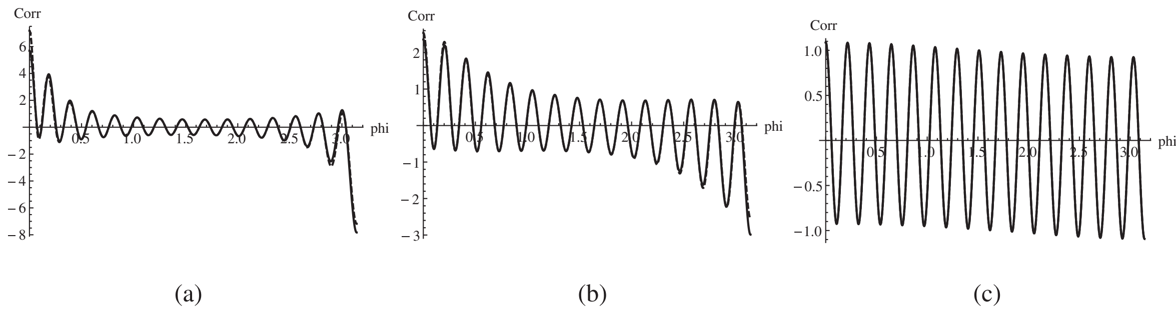

FIG. 2. Solid line: exact result; dashed line: large N. (a) $N = 29$, $t = 0.3$; (b) $N = 29$, $t = 1$; (c) $N = 29$, $t = 5$.

simultaneous shifts of α and β by $2\pi k/N$. Initial simulations were done collecting two-dimensional histograms in the α, β plane. Is was found that within practical numerical accuracy, collapsing the histograms along constant $\alpha - \beta$ lines did not lose any information. This means that I may as well consider the following finite-N definition of $\rho^{(2)}$:

$$\rho^{(2)}(\alpha - \beta) = \frac{N}{2\pi} \int_{-\pi/N}^{\pi/N} d\theta \langle \rho_1^{(1)}(\alpha + \theta) \rho_2^{(1)}(\beta + \theta) \rangle_c. \tag{19}$$

$\rho^{(2)}$ depends only on the angle difference on account of the $Z(N)$ symmetry.

In order to eliminate the UV divergences in four dimensions, the gauge field configurations were smeared. Unlike in previous work [20], gauge fields along the direction separating the Polyakov loops were left unsmeared; smearing was only done for gauge fields tangent to all orthogonal three spaces. Thus, the gauge fields entering the Polyakov loop are smeared. This is enough to remove the perimeter divergence, as can be seen from the formula of a massless propagator smeared in the above manner at infinite volume:

$$G(x, s) = \int \frac{d^4 p}{(2\pi)^4} e^{i p_4 x_4 + i \vec{p} \cdot \vec{x}} \frac{e^{-2s\vec{p}^2}}{p_4^2 + \vec{p}^2}. \tag{20}$$

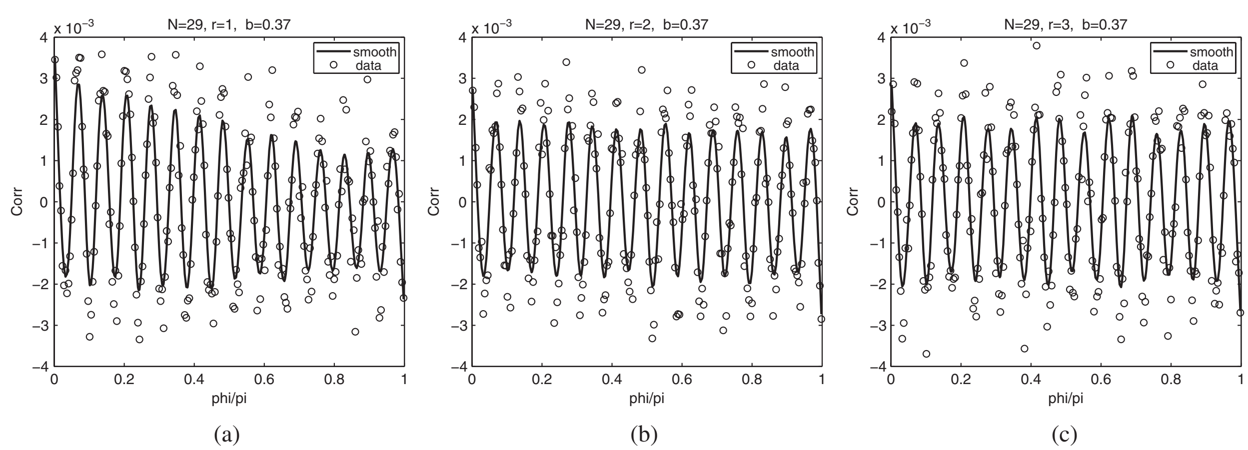

FIG. 3. MC data on 12^4 at $b = 0.36$ and at smearing $s = 0.25$. (a) $N = 29$, $r = 1$, $b = 0.36$; (b) $N = 29$, $r = 2$, $b = 0.36$; (c) $N = 29$, $r = 3$, $b = 0.36$.

One has then $G(0,s)=\frac{1}{16\pi s}$, and the short distance singularity is regulated away. This removes the perimeter divergence at the cost of a dependence on the new scale $\sqrt{s}$. At the level of Feynman diagrams, it is obvious that this eliminates all UV perimeter divergences to any finite order because the extra diagrams the smearing introduces have a tree structure, reflecting the determinism of the smearing equation, which, in continuum notation reads

$$F_{is}=D_j^{\mathrm{adjoint}}F_{ij}. \tag{21}$$

Like in Ref. [20], s is a coordinate along a new direction. i, j label directions orthogonal to the direction of separation between the two Polyakov loops. D and F are the covariant derivative and field strength, respectively. The three-dimensional character of the smearing means that the quantities $\mathcal{F}_R(l)$ are s independent on account of the limit $r\to\infty$, which projects on the ground state of the relevant Hamiltonian. Smearing only affects the (regularized) matrix element of the operator between the singlet $Z(N)$ ground state and the nontrivial $Z(N)$ ground state.

In simulations I employed $s=0.25$ in lattice units, which is a moderate amount of smearing, found adequate in the study of contractible Wilson loops [10].

Figures 3–5 show results from a lattice volume of 12^4 with $N=29$ at inverse 't Hooft couplings $b=0.360$, 0.365, 0.370 and separation $r=1$, 2, 3 in lattice units. We see that at fixed b, the general behavior resembles the analytical results in two dimensions with t increasing with r. One also sees a trend of increase in the difference from two dimensions as the angle difference increases. Only the angle difference range of $(0,\pi)$ is plotted on account of the symmetry under the simultaneous sign switch of α, β. As b increases the effective t decreases, as expected on account of asymptotic freedom. The errors on the Monte Carlo (MC) data are of the order of 10% but cannot be reliably estimated.

Each figure shows, in addition to raw data, a smoothed curve obtained by a cubic spline smoothing method. The method of smoothing consists of a minimization of a weighted combination of some average of curve curvature and deviation from the data. The smoothing procedure

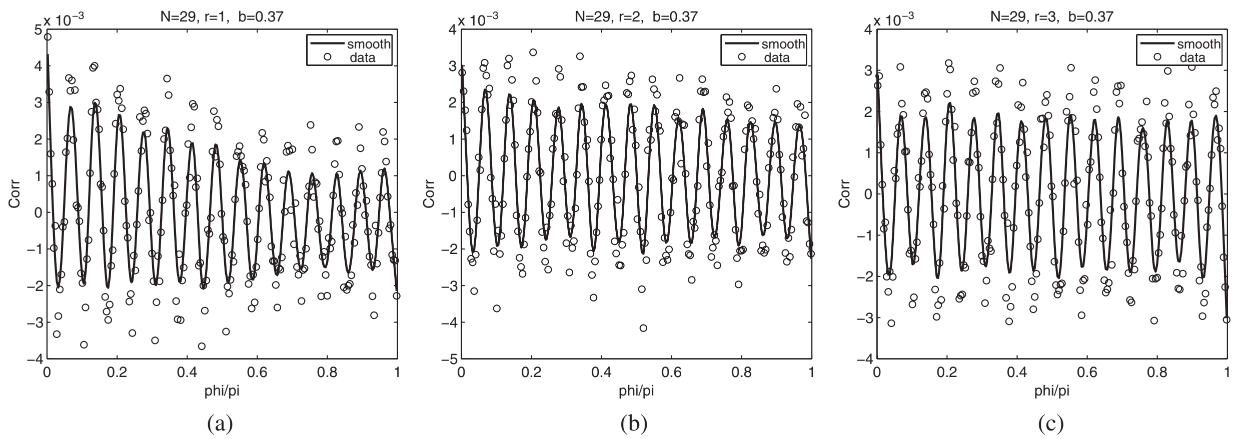

FIG. 4. MC data on 12^4 at $b=0.365$ and at smearing $s=0.25$. (a) $N=29$, $r=1$, $b=0.365$; (b) $N=29$, $r=2$, $b=0.365$; (c) $N=29$, $r=3$, $b=0.365$.

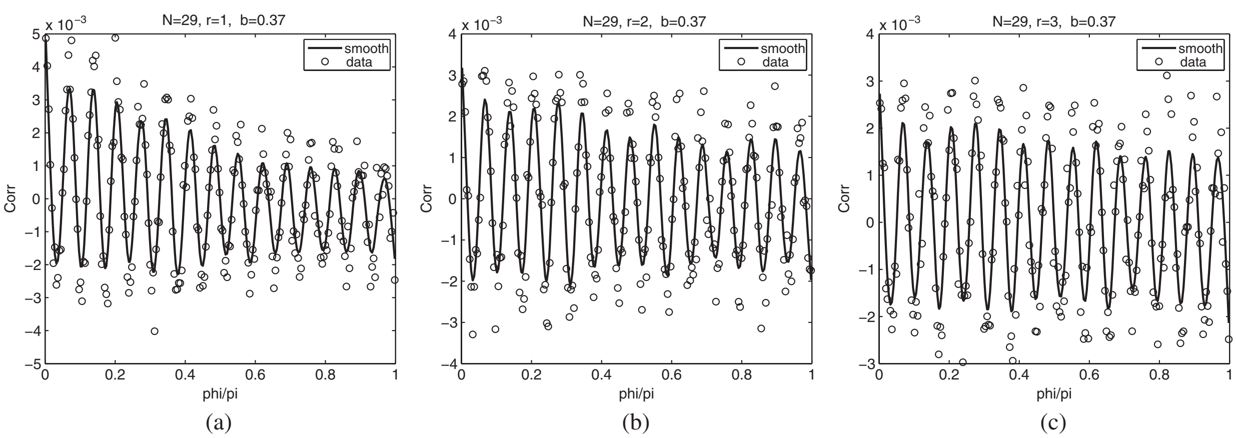

FIG. 5. MC data on 12^4 at $sb=0.37$ and at smearing $s=0.25$. (a) $N=29$, $r=1$, $b=0.37$; (b) $N=29$, $r=2$, $b=0.37$; (c) $N=29$, $r=3$, $b=0.37$.

is quite *ad hoc* and only serves to produce curves to guide the eye.

IV. OTHER POSSIBILITIES

A. Where to look for a large-N phase transition in four dimensions

I discovered no analogues to the large-N phase transition found for contractible Wilson loops in R^4 in this case. The reason must be that one cannot make l small. If I try, I hit a discontinuity already at finite N; this discontinuity becomes quite spectacular at $N = \infty$ [27]. There seems to be no way to qualitatively distinguish between the finite-N discontinuity and the $N = \infty$ one.

So, I would like to work in a metastable phase where $l < \frac{1}{T_c}$ but $Z(N)$ is still a good symmetry. More mathematically, I want to analytically continue in l from the low-temperature to the high-temperature phase. As pointed out in Ref. [28], there is a way to define also a string description there. The instability in the string theory occurs when some winding string states go tachyonic [29].[4]

Numerically, one might try to go into the metastable phase by using quenching. Originally quenching was introduced as a device to maintain the global $Z^4(N)$ of T^4 and reduction to zero volume [12], but the idea was flawed [31]. The flaw was that it still left alive an annealed mechanism for breaking the $Z^4(N)$ to some proper subgroup. Without respecting full $Z^4(N)$ symmetry, reduction fails. However, when a single direction is compactified, the preservation of the single $Z(N)$ would not suffer from this flaw. To be sure, I choose prime N because then, unlike $Z^4(N)$, $Z(N)$ has not proper subgroups.

The point would be to check numerically whether the condensation of winding states, which would occur beyond the Hagedorn temperature, is something that occurs in the analytically continued field theory at finite N as a phase transition (continuous or not). The alternative might be that there is no such ordinary phase transition, since l is small enough to enter a field theoretic perturbative regime where string theory of any traditional sort is inapplicable. Then, on the basis of analogy with the contractible Wilson loops, one would guess that a large-N phase transition would develop in the two-point single-eigenvalue correlation function. The investigation of this is left for the future.

The large-N phase transition for contractible loops is seen only when considering simultaneously many irreducible representations. They may be viewed as coming from multiple windings of the boundary of the loop. Since the loop is contractible, winding numbers are not conserved. In the Polyakov case, they are, at least for windings between 0 and $N - 1$.

The analytic continuation from $lT_c > 1$ to $lT_c < 1$ should provide a way to compute (for small r and l) $W(l, r)$ from the two Polyakov loop correlator directly and for arbitrary irreducible representations. The analytic continuation would amount to expanding around the one-loop unstable saddle point, given by

$$P(x) = \frac{1}{d_R}\chi_R[\mathrm{diag}(e^{i\frac{2\pi j}{N}})]. \tag{22}$$

For odd N, $j = 0, \ldots, N - 1$, where "diag" indicates a diagonal matrix with the listed elements on its diagonal. This configuration is $Z(N)$ invariant but unstable at one-loop order.

B. Correlations of three Polyakov loops

There are two ingredients in 2D YM: One is the "propagator" defining the cylinder with fixed circular boundaries, and the other is the "vertex," which sews together three boundaries [24]. This indicates that it would be of interest to study the connected correlation function of three Polyakov loops.

The simplest example is to take two Polyakov loops in the fundamental and a third in the irreducible representation made by combining two antifundamentals into a symmetric or antisymmetric irreducible representation. Take N as greater than or equal to 5 and odd. The three loops are positioned at distinct locations in R^3. In Euclidean space, using a different slicing, this looks like a finite temperature setting for one among the many possible generalizations to large N of $N = 3$ baryons. Three infinitely heavy quarks are connected by a V-shaped string configuration.

One could go to Minkowski space and endow these locations with zero masses, forcing them to evolve in time at the velocity of light. For open strings in Minkowski space, such a situation was looked at in the context of cosmic strings. For $N = 3$ this was considered in several papers, but the string tensions were taken to be equal [32]. An effective string theory valid at large separations would need to handle a case where one couples two strings of the same tension to one of a different tension. It seems to me that string tension considerations would favor a V-shaped arrangement of "fundamental" strings. It would be interesting to apply the methods of effective string theory to this setup. Assuming the V shape, in the field theory, there would be a coupling associated with the vertex of the V. For large N it would go as $\frac{g}{N^4}$ with a finite g.

[4]A simple way to understand winding states is by bosonizing the two-dimensional scalar field describing the compactified dimension on a cylinder [30]. The fermions one gets are solitons of the original theory, and their integer charges under $U(1)$ and $U(1)_5$ are given by the right and left winding numbers. The constraint on the closed string states that ties the left and right movers ends up leaving one extra integer labeling the string modes. For a small circumference, nontrivial winding modes have positive mass squares, overcoming the negative additive contribution reflecting the ordinary tachyon, but for a larger circumference, some of the winding modes have negative mass square. The smallest nonzero windings cross the tachyonic threshold at the Hagedorn temperature.

Back to fixed sources, the three-point function of Polyakov loops for the antisymmetric case is given by

$$\frac{2}{N^3(N-1)}\langle \mathrm{Tr}(U_{P_1})[\mathrm{Tr}(U_{P_2}^{\dagger 2}) - (\mathrm{Tr}(U_{P_2}^{\dagger}))^2]\mathrm{Tr}(U_{P_3})\rangle \times (r_a, r_b, \theta). \quad (23)$$

Here, P_1 is at (0, 0, 0), P_2 is at $(r_a, 0, 0)$, and P_3 is at $(r_a + r_b\cos\theta, r_b\sin\theta, 0)$ with $r_a \neq r_b$ and $r_a, r_b > 0$. The new ingredient is the presence of corners. In the case of rectangular Wilson loops, corners may change the rules of effective string theory by exhibiting a field theoretical dependence on loop sides, which is not exponentially suppressed even for asymptotically large loops. Here, the same question can be addressed in a different setup. On a hypercubic lattice, only θ values which are multiples of $\frac{\pi}{2}$ are accessible. Numerically there would be high noise problems, but it is worth a try. One could then get back at our main theme and consider the connected three-point function of the eigenvalues of the three loops. To search for large-N phase transitions, one would need to look at three-point connected correlations $\rho^{(3)}$, depending on two eigenvalue-angle differences at infinite N.

It might be of interest to consider the problem of colliding two same-direction wound Polyakov loops in Minkowski space. The three-point vertex would enter twice to produce a two to two particle scattering dominated by the exchange of the symmetric and antisymmetric long strings with masses above and below threshold. The distribution of the excited string modes of the two separate outgoing strings might provide a thought experiment reminiscent to the Bjorken model for high-energy nucleus-nucleus collisions [33].

V. SUMMARY

The correlations among single-eigenvalue distributions associated with various Polyakov loops have been studied for the simplest arrangement and found to provide no large-N generated nonanalyticities. The results might be of some interest in random-matrix theory. One needs the interplay between different windings to get large-N phase transitions and also a perturbative regime. One idea was to somehow analytically continue in l to $lT_c \ll 1$ and follow the evolution of the single-eigenvalue distribution of a Polyakov loop as a function of l. The other was to construct arrangements involving mixtures of Polyakov loops of different winding numbers.

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[1] G. t Hooft, Nucl. Phys. **B72**, 461 (1974).
[2] B. Lucini and M. Panero, Phys. Rep. **526**, 93 (2013).
[3] G. F. Chew and C. Rosenzweig, Phys. Rep. **41**, 263 (1978); G. Veneziano, Nucl. Phys. **B74**, 365 (1974); Phys. Lett. **52B**, 220 (1974).
[4] R. F. Streater, A. S. Wightman, *PCT, Spin and Statistics, and All That* (Benjamin, New York, 1964).
[5] G. Veneziano, Nuovo Cimento A **57**, 190 (1968).
[6] Yu. M. Makeenko and A. A. Migdal, Phys. Lett. **88B**, 135 (1979); D. Foerster, Phys. Lett. **87B**, 87 (1979); T. Eguchi, Phys. Lett. **87B**, 91 (1979); D. Weingarten, Phys. Lett. **87B**, 97 (1979).
[7] A. M. Polyakov, Nucl. Phys. B, Proc. Suppl. **68**, 1 (1998).
[8] J. Maldacena, Int. J. Theor. Phys. **38**, 1113 (1999).
[9] J. Kiskis, R. Narayanan, and H. Neuberger, Phys. Lett. B **574**, 65 (2003).
[10] R. Lohmayer and H. Neuberger, J. High Energy Phys. 08 (2012) 102.
[11] T. Eguchi and H. Kawai, Phys. Rev. Lett. **48**, 1063 (1982).
[12] G. Bhanot, U. Heller, and H. Neuberger, Phys. Lett. **113B**, 47 (1982).
[13] H. Neuberger, Phys. Lett. **119B**, 179 (1982).
[14] A. M. Polyakov, Nucl. Phys. **B164**, 171 (1980).
[15] O. Jahn and O. Philipsen, Phys. Rev. D **70**, 074504 (2004).
[16] T. Appelquist, M. Dine, and I. J. Muzinich, Phys. Rev. D **17**, 2074 (1978).
[17] A. Pineda, Prog. Part. Nucl. Phys. **67**, 735 (2012).
[18] O. Aharaony and Z. Komargodski, J. High Energy Phys. 05 (2013) 118.
[19] R. Lohmayer and H. Neuberger, Phys. Rev. Lett. **108**, 061602 (2012).
[20] R. Narayanan and H. Neuberger, J. High Energy Phys. 03 (2006) 064; R. Lohmayer and H. Neuberger, Proc. Sci., LATTICE2011 (2011) 249.
[21] H. Neuberger, Phys. Lett. B **199**, 536 (1987).
[22] M. Karliner, I. Klebanov, and L. Susskind, Int. J. Mod. Phys. A **03**, 1981 (1988).
[23] A. Gonzalez-Arroyo and M. Okawa, Phys. Lett. B **718**, 1524 (2013).
[24] D. J. Gross and A. Matytsin, Nucl. Phys. **B437**, 541 (1995).
[25] C. W. J. Beenakker, Rev. Mod. Phys. **69**, 731 (1997); B. Eynard, J. Phys. A: Math. Gen. **31**, 8081 (1998).
[26] R. Lohmayer, H. Neuberger, and T. Wettig, J. High Energy Phys. 05 (2009) 107; H. Neuberger, Phys. Lett. B **666**, 106 (2008).

[27] C. B. Thorn, Phys. Lett. **99B**, 458 (1981).

[28] J. Polchinski, Phys. Rev. Lett. **68**, 1267 (1992).

[29] J. J. Atick and E. Witten, Nucl. Phys. **B310**, 291 (1988).

[30] H. Neuberger, Master's Thesis, Tel Aviv University, RX-744, 1976 (unpublished).

[31] B. Bringoltz and S. Sharpe, Phys. Rev. D **78**, 034507 (2008).

[32] E. J. Copeland, T. W. B. Kibble, and D. A. Steer, Phys. Rev. Lett. **97**, 021602 (2006); G. S. Sharov, Theor. Math. Phys. **140**, 1109 (2004); **140**, 1109 (2004); Phys. Rev. D **79**, 114025 (2009); G. 't Hooft, arXiv: hep-th/0408148.

[33] J. D. Bjorken, Phys. Rev. D **27**, 140 (1983).