PHYSICAL REVIEW D **87,** 114004 (2013)

Direct connection between the different QCD orders for parton distribution and fragmentation functions

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(Received 26 March 2013; published 5 June 2013)

The formulas directly connecting parton distribution functions and fragmentation functions at the next-to-leading-order QCD with the same quantities at the leading order are derived. These formulas are universal, i.e., have the same form for all kinds of parton distribution functions and fragmentation functions, differing only in the respective splitting functions entering there.

DOI: 10.1103/PhysRevD.87.114004 PACS numbers: 12.38.Bx, 13.60.Hb, 13.85.Ni, 13.88.+e

The extraction of parton distribution functions (PDFs) and fragmentation functions (FFs) from the experimental data is one of the important tasks of the modern hadron physics. The most simple and transparent way to do it is the QCD analysis of the data on measured asymmetries and cross sections in leading-order (LO) QCD. One of the main advantages of such an analysis is that the central values and uncertainties of measured asymmetries and cross sections directly propagate to the central values and errors of PDFs and FFs extracted from these data in LO QCD (see, for instance, Fig. 3 in Ref. [1]). At the same time, the situation with the next-to-leading-order (NLO) analysis is much more difficult because, instead of simple algebraic equations (see, for example, Eq. (2) in Ref. [1]), one deals there with complex integral equations (like, for instance, Eqs. (10)–(13) in Ref. [2]) for finding the PDFs (FFs) in which we are interested. The standard way to solve this problem is to apply the QCD analysis based on the fitting procedure (see Ref. [2] and references therein). However, there are unavoidable ambiguities inherent in a fitting procedure that become especially important when the quality of the fitted data is rather bad (a small number of points with large errors). These are arbitrariness in the choice of the functional form (with a lot of varied parameters) of the fitted PDFs and FFs at the initial scale and also ambiguities in the error band calculation (ambiguities related to the deviation of the χ^2 profile from the quadratic parabola and to the choice of $\Delta\chi^2$ determining the uncertainty size—see the discussion on this subject in Ref. [2]). Thus, it seems to be very useful if one could obtain NLO (NNLO, ...) results on PDFs/FFs using the respective LO results as an input, without loosing, thereby, all advantages of LO analysis.

We start with some necessary notation and definitions. For the flavor nonsinglet and singlet quantities, we introduce the notation Q_{NS} and $\mathbf{V} = (\mathrm{Q}_S, \mathrm{G})$, where Q_{NS} can be either q_{NS} (nonsinglet combinations of quark densities); Δq_{NS} (nonsinglet combinations of helicity PDFs); combinations of transversity PDFs $\Delta_T q(\bar{q}) \equiv h_{1q,\bar{q}}, \ldots$; or D^h_{NS} (a "nonsinglet" combination of FFs D^h_q), ..., while Q_S can be $q_S, \Delta q_S, D^h_S, \ldots$; G can be $g, \Delta g, D^h_g, \ldots$. In this notation, the DGLAP evolution equations (see Ref. [3] for review) look like

$$Q^2 d\mathbf{V}(Q^2, x)/dQ^2 = (\alpha_s/2\pi)[\mathbf{P}^{(0)}(x) + (\alpha_s/2\pi)\mathbf{P}^{(1)}(x) + O(\alpha_s^2)] \otimes \mathbf{V}(Q^2, x), \quad (1)$$

and analogously for Q_{NS} with the replacement $\mathbf{P}(x, \alpha_s) \to P(x, \alpha_s) = P^{(0)}(x) + (\alpha_s/2\pi)P^{(1)}(x) + O(\alpha_s^2)$. Here, $\mathbf{P}$ is a 2×2 matrix with the elements $P_{qq}, P_{qg}, P_{gq}, P_{gg}$, and the splitting functions for unpolarized PDFs and helicity PDFs can be found in Ref. [4], for transversity PDFs—in Ref. [5], for FFs—in Ref. [6] and references therein.

Following Ref. [7] it is convenient to define the evolution operators E and $\mathbf{E}$ (2×2 matrix with the elements $E_{qq}, E_{qg}, E_{gq}, E_{gg}$) as

$$\mathrm{Q}_{\mathrm{NS}}(Q^2, x) = \mathrm{E}(Q^2, x) \otimes \mathrm{Q}_{\mathrm{NS}}(Q_0^2, x),$$
$$\mathbf{V}(Q^2, x) = \mathbf{E}(Q^2, x) \otimes \mathbf{V}(Q_0^2, x). \quad (2)$$

Here, we are interested in the initial conditions[1]

$$\mathrm{E}(Q^2 = Q_0^2, x) = \delta(1-x),$$
$$\mathbf{E}(Q^2 = Q_0^2, x) = \mathbf{1}\delta(1-x), \quad (3)$$

which allow us to evolve Q_{NS} and $\mathbf{V}$ from the initial scale Q_0^2 to an arbitrary scale Q^2. It is also convenient to use, following Ref. [7], the evolution variable $t = (2/\beta_0) \times \ln(\alpha_s(Q_0^2)/\alpha_s(Q^2))$ instead of the standard variable $\ln(Q^2/\mu^2)$. Besides, we introduce the notation

$$A|_{\mathrm{LO}} \equiv \hat{A}, \qquad A|_{\mathrm{NLO}} \equiv A, \quad (4)$$

for any quantity A at LO and NLO, respectively.

From now on, we consider only the nontrivial singlet case. The transition to the simple nonsinglet case will be easily done in the end of calculations by making the

[1]We do not consider the asymptotic conditions [7] $\mathbf{E}(\mathrm{E}) \to \hat{\mathbf{E}}(\hat{\mathrm{E}})$ as $Q^2 \to \infty$ (see Eq. (5.57) in Ref. [7]) since we deal only with particular realization (2) of the general conditions given by Eqs. (5.18) in Ref. [7].

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replacement of the matrices with the respective commuting quantities.

In terms of quantities t and $\mathbf{E}$, the DGLAP equations are rewritten in LO as

$$\frac{d}{d\hat{t}}\hat{\mathbf{E}}(\hat{t},x) = \mathbf{P}^{(0)} \otimes \hat{\mathbf{E}}(\hat{t},x), \tag{5}$$

while in NLO they look like

$$\frac{d}{dt}\mathbf{E}(t,x) = \left[\mathbf{P}^{(0)}(x) + \frac{\alpha_s}{2\pi}\mathbf{R}(x) + O(\alpha_s^2)\right] \otimes \mathbf{E}(t,x), \tag{6}$$

where

$$\mathbf{R}(x) \equiv \mathbf{P}^{(1)}(x) - \frac{\beta_1}{2\beta_0}\mathbf{P}^{(0)}(x). \tag{7}$$

The solution of Eq. (5) with the initial condition (3) $\hat{\mathbf{E}}(\hat{t}=0,x) = \mathbf{1}\delta(1-x)$ reads [7]

$$\hat{\mathbf{E}}(\hat{t},x) = \mathrm{Exp}(\mathbf{P}^{(0)}(x)\hat{t}) = \mathbf{1}\delta(1-x) + \hat{t}\mathbf{P}^{(0)}(x) + \frac{\hat{t}^2}{2!}\mathbf{P}^{(0)}(x) \otimes \mathbf{P}^{(0)}(x) + \cdots, \tag{8}$$

while to solve the NLO equation (6), one can apply the elegant method of Ref. [7] based on the analogy with the perturbative quantum mechanics (see Eqs. (5.47)–(5.54) in Ref. [7]). Operating in this way, one obtains the general solution of Eq. (6) in the form [for a moment, we omit x dependence and $\delta(1-x)$]

$$\mathbf{E}(t) = \left\{\hat{\mathbf{E}}(t) \otimes \left[\mathbf{1} + \frac{\alpha_s(Q_0^2)}{2\pi}\int_{t'}^{t} d\tau e^{-\beta_0\tau/2}\hat{\mathbf{E}}(-\tau) \otimes \mathbf{R} \otimes \hat{\mathbf{E}}(\tau)\right] \otimes \hat{\mathbf{E}}(-t')\right\} \otimes \mathbf{E}(t'). \tag{9}$$

Putting $t' \to \infty$ in Eq. (9), one reproduces the solution (Eq. (5.54) in Ref. [7]), satisfying the boundary condition $\mathbf{E} \to \hat{\mathbf{E}}$ as $t \to \infty$. In turn, putting $t' = 0$ in Eq. (9), one gets the solution

$$\mathbf{E}(t) = \left[\mathbf{1} + \frac{\alpha_s(Q^2)}{2\pi}\int_0^t d\tau e^{\beta_0\tau/2}\hat{\mathbf{E}}(\tau) \otimes \mathbf{R} \otimes \hat{\mathbf{E}}(-\tau)\right] \otimes \hat{\mathbf{E}}(t), \tag{10}$$

satisfying the boundary condition (3) with which we deal.

The *key point* to proceed is the condition that all PDFs and FFs should take the same values in LO and NLO (as well as in NNLO, ...) as $Q^2 \to \infty$:

$$\begin{aligned} \mathrm{Q}_{\mathrm{NS}}(Q^2 \to \infty, x) &= \hat{\mathrm{Q}}_{\mathrm{NS}}(Q^2 \to \infty, x), \\ \mathbf{V}(Q^2 \to \infty, x) &= \hat{\mathbf{V}}(Q^2 \to \infty, x). \end{aligned} \tag{11}$$

Although this asymptotic condition seems to be intuitively clear, let us argue it in some detail because of its great importance for what follows.

Imagine that two researchers analyze in LO (the first) and NLO (the second) the same "ideal" data—the data available with tremendous statistics even in the Bjorken "sublimit" (such high Q^2 values are accessible that the Bjorken scaling violation becomes invisible even within extremely small uncertainties on measured asymmetries and cross sections). For determinacy and simplicity, let us suppose that they analyze the imaginary ideal polarized semi-inclusive DIS (SIDIS) data on pion production and extract the valence helicity PDFs Δu_V, Δd_V from the proton and deuteron difference asymmetries (see Ref. [8] and references therein) measured in the Bjorken sublimit. The first uses LO formulas $A_p^{\pi^+-\pi^-} \sim (4\Delta u_V - \Delta d_V)/(4u_V - d_V)$ and $A_d^{\pi^+-\pi^-} \sim (\Delta u_V + \Delta d_V)/(u_V + d_V)$ (i.e., performs the analysis analogous to one of COMPASS [9]), and the second uses their NLO generalization (Eqs. (6–10) in Ref. [8]). Besides, for self-consistency, both imaginary researches do not use the existing parametrizations on u_V, d_V but extract these quantities themselves (as well as the integrated over cut in z difference[2] $D_1 - D_2$ of favored and unfavored pion FFs) using the same SIDIS data on pion production averaged over spin and studying the quantities $F_{2p(d,{}^3\mathrm{He},\ldots)}^{\pi^+} - F_{2p(d,{}^3\mathrm{He},\ldots)}^{\pi^-}$, where in both LO and NLO only, u_V, d_V and $D_1 - D_2$ survive. It is obvious that all terms with convolutions $\otimes$ (see Eqs. (6–10) in Ref. [8]) distinguishing NLO and LO equations for finding Δu_V, Δd_V, and u_V, d_V, $D_1 - D_2$ just disappear as one approaches the Bjorken limit, so that, comparing the results on these quantities obtained in the Bjorken sublimit, both researchers could not discriminate between them.

So, let us pass to limit $Q_0^2 \to \infty$ in Eq. (2) using the asymptotic condition (11). Then, on the one hand (NLO evolution),

$$\begin{aligned} \mathbf{V}(Q^2,x) &= \mathbf{E}(t \to -\infty, x) \otimes \mathbf{V}(Q_0^2 \to \infty, x) \\ &= \mathbf{E}(t \to -\infty, x) \otimes \hat{\mathbf{V}}(Q_0^2 \to \infty, x), \end{aligned} \tag{12}$$

and, on the other hand (inverse LO evolution),

$$\hat{\mathbf{V}}(Q_0^2 \to \infty, x) = \hat{\mathbf{E}}(\hat{t} \to \infty, x) \otimes \hat{\mathbf{V}}(Q^2, x). \tag{13}$$

Combining Eqs. (12) and (13), one obtains

$$\mathbf{V}(Q^2,x) = \left[\lim_{Q_0^2\to\infty}\mathbf{E}(t,x) \otimes \hat{\mathbf{E}}(-\hat{t},x)\right] \otimes \hat{\mathbf{V}}(Q^2,x). \tag{14}$$

Using Eqs. (8) and (10) and the relation $\lim_{Q^2\to\infty}(\alpha_s/\hat{\alpha}_s) = 1$, we arrive at the connection formula between NLO and LO flavor singlet PDFs (FFs) $\mathbf{V}$ and $\hat{\mathbf{V}}$ at the same finite Q^2 value,

[2] On the simultaneous determination of valence PDFs and $D_1 - D_2$ from the SIDIS data see, for example, Ref. [10].

$$\mathbf{V}(Q^2,x) = \left[\mathbf{1}\delta(1-x) - \frac{\alpha_s(Q^2)}{2\pi}\int_{-\infty}^{0} d\tau e^{\beta_0\tau/2}\hat{\mathbf{E}}(\tau,x)\otimes\mathbf{R}(x)\otimes\hat{\mathbf{E}}(-\tau,x)\right]\otimes \mathrm{Exp}\left(-\frac{2}{\beta_0}\ln\frac{\alpha_s(Q^2)}{\hat{\alpha}_s(Q^2)}\mathbf{P}^{(0)}(x)\right)\otimes\hat{\mathbf{V}}(Q^2,x), \tag{15}$$

where all dependence on the unreachable infinite point Q_0^2 just cancels out.

In the nonsinglet case, the relation (15) is significantly simplified. The terms $\hat{E}(\tau,x)\equiv \mathrm{Exp}(\tau P^{(0)}(x))$ and $\hat{E}(-\tau,x)$ cancel out each other in the integrand, and one easily obtains

$$\mathrm{Q}_{\mathrm{NS}}(Q^2,x) = \left[\delta(1-x) + \frac{\alpha_s(Q^2)}{2\pi}\left(\frac{\beta_1}{\beta_0^2}P^{(0)}(x) - \frac{2}{\beta_0}P^{(1)}(x)\right)\right]\otimes\mathrm{Exp}\left(-\frac{2}{\beta_0}\ln\frac{\alpha_s(Q^2)}{\hat{\alpha}_s(Q^2)}P^{(0)}(x)\right)\otimes\hat{\mathrm{Q}}_{\mathrm{NS}}(Q^2,x). \tag{16}$$

Equations (15) and (16) connecting flavor singlet and nonsinglet quantities in NLO with the same quantities in LO is the *main result* of the paper. Let us briefly discuss their practical use.

There are not any problems with the application of Eq. (16), and the task of reconstruction of NLO nonsinglet quantities from LO ones is reduced just to the trivial calculation of the integrals entering the convolutions $\otimes$. Indeed, the parameter $\epsilon \equiv -(2/\beta_0)\ln(\alpha_s/\hat{\alpha}_s)$ is very small even at the minimal (the lower boundary of the experimental cut on Q^2 is usually about 1 GeV^2) really available Q^2 values, so that one can achieve very good accuracy keeping only a few first terms in the expansion $\mathrm{Exp}(\epsilon P^{(0)}(x)) = \delta(1-x) + \epsilon P^{(0)}(x) + (\epsilon^2/2!)P^{(0)}(x)\otimes P^{(0)}(x) + \cdots$ Certainly, the same statement holds for the term $\mathrm{Exp}(\epsilon\mathbf{P}^{(0)}(x))$ in Eq. (15), but there arises an additional problem of how to deal with the integral over τ. As usual, the problem is easily solved in the space of Mellin moments. Notice that the Q^2 independent integral over τ in Eq. (15) just coincides[3] with the quantity $-U(x)$ in Ref. [7] (see Eq. (5.45) in Ref. [7]), which enters the solution of DGLAP with the boundary conditions $\lim_{Q^2\to\infty}\mathbf{E}(\mathrm{E}) = \hat{\mathbf{E}}(\hat{\mathrm{E}})$ (see footnote 1). Then, applying the inverse Mellin transformation, one easily obtains instead of Eq. (15) the formula suitable[4] for numerical calculations:

$$\mathbf{V}(Q^2,x) = \left[\mathbf{1}\delta(1-x) + \frac{\alpha_s(Q^2)}{2\pi}\int_{C-i\infty}^{C+i\infty} dn\frac{x^{-n}}{2\pi i}U(n)\right] \otimes \mathrm{Exp}(\epsilon(Q^2)\mathbf{P}^{(0)}(x))\otimes\hat{\mathbf{V}}(Q^2,x), \tag{17}$$

where a 2×2 matrix $U(n)\equiv\int_0^1 dx x^{n-1}U(x)$ is given by Eq. (5.41) in Ref. [7].

In summary, the formulas allowing one to transform LO parton distribution and fragmentation functions to NLO ones are derived. To obtain these formulas, we use as an input only the DGLAP evolution equations and the asymptotic condition that PDFs (FFs) at different QCD orders become the same in the Bjorken limit. Because of the universality of this input, the connection formulas are also universal, i.e., they are valid for any kind of PDFs (FFs) with which we deal. Besides, it is obvious that, operating in the same way, one can also establish the connection of PDFs (FFs) at LO (as well as at NLO) with these quantities at any higher QCD order (NNLO, NNNLO, ...), and the only restriction here is the knowledge of the respective splitting functions.

[3]Using Eq. (5.28) in Ref. [7] for U and the obvious relation $Q^2 d[\mathrm{Exp}((2/\beta_0)\ln\hat{\alpha}_s\mathbf{P}^{(0)})\otimes\hat{\mathbf{V}}]/dQ^2 = 0$, one can immediately check that the rhs of Eq. (15) indeed satisfies the NLO DGLAP equation (1).

[4]In Ref. [11], one can find the efficient algorithm for the numerical calculation of the integral over n (proper choice of the integration contour, etc.—see the discussion around Eq. (3.2) in Ref. [11]).

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