Pais-Uhlenbeck oscillator with a benign friction force

Matej Pavšič*

Jožef Stefan Institute, Jamova 39, 1000 Ljubljana, Slovenia

(Received 4 April 2013; published 15 May 2013)

It is shown that the Pais-Uhlenbeck oscillator with damping, considered by Nesterenko, is a special case of a more general oscillator that has not only a first order, but also a third order friction term. If the corresponding damping constants, α and β, are both positive and below certain critical values, then the system is stable. In particular, if $\alpha = -\beta$, then we have the unstable Nesterenko's oscillator.

DOI: 10.1103/PhysRevD.87.107502 PACS numbers: 04.60.−m, 03.50.Kk, 04.20.Cv, 11.10.−z

I. INTRODUCTION

The Pais-Uhlenbeck (PU) oscillator [1] is a toy model for higher-derivative theories. The latter theories are very important for quantum gravity, but because of the presence of negative energies, they are generally considered as very problematic, if not completely unsuitable for physics. Negative energies arise from the wrong signs of certain terms in the Ostrogradsky Hamiltonian. In a quantized theory, such wrong signs can manifest themselves in the presence of ghost states [2] that break unitarity. With an alternative quantization procedure, based on a different choice of vacuum [3–5], one has negative energy states, just as in the classical higher-derivative theory, and no ghost states.

Several authors have argued that the presence of negative energies in PU oscillator does not lead to inconsistencies [6] (see also Ref. [7]). Those arguments hold for a free oscillator, and are no longer valid if one includes an interaction term that couples positive and negative energy degrees of freedom. The interacting PU oscillator has to be analyzed afresh. In Refs. [8–11] it has been found that for small initial velocities and coupling constants there exist islands of stability. Moreover, an example of an unconditionally stable interacting system was found [11]. This system, which is a nonlinear extension of the PU oscillator, is a close relative of a supersymmetric higher-derivative system [12]. Further, if to the ordinary, linear, PU oscillator we add a self-interaction term that is bounded from below and from above, such as $\frac{1}{4}\sin^4 x$, then, as shown in Ref. [13], such a system is stable for any value of initial velocity, and is thus an example of a viable higher-derivative theory.

But there remains an important issue that has to be resolved. Every physical system in contact with an environment undergoes dissipative forces. An ordinary oscillator is subjected to a damping force that exponentially diminishes the amplitude of oscillations. For the PU oscillator, this could be different. Indeed, according to Nesterenko [14], the PU oscillator with an external friction force undergoes an exponential instability: the amplitude grows into infinity.

In this paper, it will be shown that the friction force, considered by Nesterenko, is a special case of a more general friction force that, in general, does not cause the exponential instability. The stability of such a system is also preserved in the presence of an external time dependent force.

II. PAIS-UHLENBECK OSCILLATOR WITH DAMPING

Without damping, the Pais-Uhlenbeck oscillator satisfies the following forth order equation of motion:

$$\left(\frac{d^2}{dt^2} + \omega_2^2\right)\left(\frac{d^2}{dt^2} + \omega_1^2\right)x = 0. \tag{1}$$

The latter equation can be generalized to include damping terms:

$$\left(\frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_2^2\right)\left(\frac{d^2}{dt^2} + 2\alpha\frac{d}{dt} + \omega_1^2\right)x = 0. \tag{2}$$

Explicitly we thus have

$$\begin{aligned} x^{(4)} &+ 2(\alpha+\beta)\dddot{x} + (\omega_1^2 + \omega_2^2 + 4\alpha\beta)\ddot{x} \\ &+ 2(\omega_1^2\beta + \omega_2^2\alpha)\dot{x} + \omega_1^2\omega_2^2 x = 0. \end{aligned} \tag{3}$$

If $\alpha = -\beta$ we obtain the equation

$$x^{(4)} + (\omega_1^2 + \omega_2^2 - 4\beta^2)\ddot{x} + 2\beta(\omega_1^2 - \omega_2^2)\dot{x} + \omega_1^2\omega_2^2 x = 0, \tag{4}$$

which can be written in the form

$$x^{(4)} + (\Omega_1^2 + \Omega_2^2)\ddot{x} + 2\gamma\dot{x} + \Omega_1^2\Omega_2^2 x = 0, \tag{5}$$

where $\gamma = \beta(\omega_1^2 - \omega_2^2)$. Here

$$\Omega_1^2 + \Omega_2^2 = \omega_1^2 + \omega_2^2 - 4\beta^2, \tag{6}$$

$$\Omega_1^2\Omega_2^2 = \omega_1^2\omega_2^2, \tag{7}$$

with the solution

*matej.pavsic@ijs.si

$$\Omega_{1,2}^2 = \frac{1}{2}\left[\omega_1^2 + \omega_2^2 - 4\beta^2 \pm \sqrt{(\omega_1^2 + \omega_2^2 - 4\beta^2)^2 - 4\omega_1^2\omega_2^2}\right]. \tag{8}$$

Equation (5) is just the equation for the Pais-Uhlenbeck oscillator in the presence of a friction force, considered by Nesterenko [14].

The general solution of Eq. (2) is

$$x = e^{-\alpha t}\left(C_1 e^{t\sqrt{\alpha^2-\omega_1^2}} + C_2 e^{-t\sqrt{\alpha^2-\omega_1^2}}\right) + e^{-\beta t}\left(C_3 e^{t\sqrt{\beta^2-\omega_2^2}} + C_4 e^{-t\sqrt{\beta^2-\omega_2^2}}\right). \tag{9}$$

If $\alpha^2 < \omega_1^2$, $\beta^2 < \omega_2^2$, this is oscillatory function, and if α and β are both positive, the amplitude of oscillations exponentially decreases.

In particular, if $\alpha = -\beta$, the solution of (2) is

$$x = e^{\beta t}\left(C_1 e^{t\sqrt{\beta^2-\omega_1^2}} + C_2 e^{-t\sqrt{\beta^2-\omega_1^2}}\right) + e^{-\beta t}\left(C_3 e^{t\sqrt{\beta^2-\omega_2^2}} + C_4 e^{-t\sqrt{\beta^2-\omega_2^2}}\right). \tag{10}$$

For $\beta^2 < \omega_1^2, \omega_2^2$, the $x(t)$ is oscillating function consisting of a part with exponential growth and a part with exponential damping. Such behavior was found by Nesterenko using a perturbative solution of Eq. (5). But as we see here, Eq. (5) can be solved exactly through the steps (2)–(8), and by taking $\alpha = -\beta$. Since Eq. (5) is equivalent to the system of two oscillators with the damping constants of opposite signs, it describes an unstable system.

In general, for positive $\alpha \neq \beta$, Eq. (3) has stable solutions, provided that $|\alpha|$, $|\beta|$ are sufficiently small, so that all terms are oscillating and damped by $e^{-\alpha t}$ and $e^{-\beta t}$.

III. PRESENCE OF AN ARBITRARY EXTERNAL FORCE

To the right-hand side of the homogeneous equation (2) we can add an arbitrary time dependent force $f(t)$:

$$\left(\frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_2^2\right)\left(\frac{d^2}{dt^2} + 2\alpha\frac{d}{dt} + \omega_1^2\right)x = f(t). \tag{11}$$

In the absence of damping, $\alpha = \beta = 0$, the general solution to the latter equation can be expressed as [14]

$$x(t) = x_0(t) + \int_{-\infty}^{\infty} G(t-t')f(t')dt'. \tag{12}$$

Here $x_0(t)$ is the general solution of the homogeneous equation (1)

$$x_0(t) = C_1\cos\omega_1 t + C_2\sin\omega_1 t + C_3\cos\omega_2 t + C_4\sin\omega_2 t, \tag{13}$$

and

$$G(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega t}\tilde{G}(\omega)d\omega, \tag{14}$$

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega t}\tilde{f}(\omega)d\omega, \tag{15}$$

where [14]

$$\tilde{G}(\omega) = \frac{1}{\sqrt{2\pi}(\omega_1^2 - \omega_2^2)}\left(\frac{1}{\omega^2-\omega_1^2} - \frac{1}{\omega^2-\omega_2^2}\right). \tag{16}$$

By inserting (16) into (14), we obtain the following for $t > 0$:

$$G(t) = \frac{1}{2(\omega_1^2-\omega_2^2)}\left(\frac{\sin\omega_2 t}{\omega_2} - \frac{\sin\omega_1 t}{\omega_1}\right). \tag{17}$$

As an example let us first consider the force

$$f(t) = a\cos\omega_1 t + b\cos\omega_2 t, \tag{18}$$

to which there corresponds the spectral density

$$\tilde{f}(\omega) = \frac{a\sqrt{2\pi}}{2}[\delta(\omega-\omega_1) + \delta(\omega+\omega_1)] + \frac{b\sqrt{2\pi}}{2}[\delta(\omega-\omega_2) + \delta(\omega+\omega_2)]. \tag{19}$$

Then Eq. (12) gives

$$x(t) = x_0(t) - \frac{1}{2\omega_1\omega_2(\omega_1^2-\omega_2^2)^2} \times [(\omega_1^2-\omega_2^2)(a\omega_2 t\sin\omega_1 t - b\omega_1 t\sin\omega_2 t) + 2(a-b)\omega_1\omega_2(\cos\omega_1 t - \cos\omega_2 t)]. \tag{20}$$

The same solution of Eq. (11) can be obtained also by using the Mathematica command DSolve.

The amplitude in Eq. (20) increases linearly with t. This was the case without damping. If we include damping, we find the following general solution of Eq. (11):

$$x(t) = x_0(t) + \frac{-2\beta\omega_1 a\cos\omega_1 t - a(\omega_1^2-\omega_2^2)\sin\omega_1 t}{2\alpha\omega_1[4\beta^2\omega_1^2 + (\omega_1^2-\omega_2^2)^2]} + \frac{-2\alpha\omega_2 b\cos\omega_2 t + b(\omega_1^2-\omega_2^2)\sin\omega_2 t}{2\beta\omega_2[4\alpha^2\omega_2^2 + (\omega_1^2-\omega_2^2)^2]}, \tag{21}$$

where $x_0(t)$ is now the general solution of the homogeneous equation (2) [see Eq. (9)]. For positive α and β, satisfying $\alpha^2 < \omega_1^2$, $\beta^2 < \omega_2^2$, the function $x(t)$ has the oscillating exponentially decreasing part $x_0(t)$ [Eq. (9)], and the oscillating part due to the external force (18). Notice that now, differently than in Eq. (20), the amplitude does not linearly increase with t.

More generally, if the spectral density is localized around ω_1^2 and ω_2^2 according to

$$\tilde{f}(\omega) = \frac{a\sqrt{2\pi}}{2}\sqrt{\frac{c}{\pi}}(e^{-c(\omega-\omega_1)^2} + e^{-c(\omega+\omega_1)^2}) + \frac{b\sqrt{2\pi}}{2}\sqrt{\frac{c}{\pi}}(e^{-c(\omega-\omega_2)^2} + e^{-c(\omega+\omega_2)^2}), \tag{22}$$

then

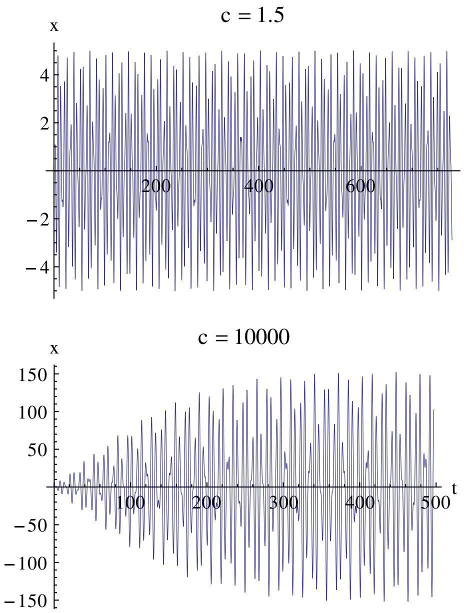

FIG. 1 (color online). Solution of the undamped Pais-Uhlenbeck oscillator ($\alpha = \beta = 0$) in the presence of an external force with the spectral density localized around $\omega_1^2 = 1$ and $\omega_2^2 = 1.5$ according to (22) for two different values of the width parameter c. We took the constants $a = b = 1$, and the initial conditions $x(0) = 1$, $\dot{x}(0) = 0.2$, $\ddot{x}(0) = -0.7$, $\dddot{x}(0) = 0.5$.

$$f(t) = e^{-\frac{t^2}{4c}}(a\cos\omega_1 t + b\cos\omega_2 t), \tag{23}$$

and solutions to Eq. (11) are stable, oscillating functions even in the absence of damping. This can be verified by solving Eq. (11) numerically, using the command NDSolve in Mathematica. Examples of solutions are given in Fig. 1. We see that nothing unphysical happens with the classical displacement $x(t)$. It displays decent oscillatory behavior. Quantum behavior of the propagator has been recently investigated by Ilhan and Kovner [10].

IV. CONCLUSION

We have clarified the important point raised by Nesterenko and found that the Pais-Uhlenbeck oscillator with external friction force is not necessarily unstable. It can be stable because, in general, the Pais-Uhlenbeck oscillator has not only one, but two damping terms: a term with the first and a term with the third derivative of the displacement $x(t)$ with respect to the time t. If the corresponding damping constants, α and β, are both positive and lower than the critical values determined by ω_1^2, ω_2^2, then the system is stable in the sense that it oscillates with exponentially decreasing amplitude. In particular, if $\alpha = -\beta$, then the third order term vanishes and we have the oscillator that has only the first derivative damping term considered by Nesterenko [14]. Such a "damping" term causes the exponential growth of the oscillator's amplitude. Nesterenko's conclusion that the theories with higher derivatives suffer exponential instability holds only in the latter particular case. In general, such theories can be stable because the third order damping term provides the mechanism that prevents the exponential instability.

We have also analyzed the Pais-Uhlenbeck oscillator that experiences an arbitrary external force $f(t)$. For the force whose spectral density is sharply localized around ω_1^2 and ω_2^2, we have found the exact general solution whose amplitude linearly increases with time if $\alpha = \beta = 0$ and decently oscillates if $\alpha > 0$, $\beta > 0$. For a force whose spectral density has a Gaussian (and not the physically unrealistic δ-like) distribution around ω_1^2 and ω_2^2, we obtain stable, oscillating solutions even in the absence of damping.

ACKNOWLEDGMENTS

This work has been supported by the Slovenian Research Agency.

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