

# Greybody factors for nonminimally coupled scalar fields in Schwarzschild–de Sitter spacetime

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We compute the greybody factors for nonminimally coupled scalar fields in four-dimensional Schwarzschild–de Sitter spacetime. In particular, we demonstrate that the zero-angular-momentum greybody factor generically tends to zero in the zero-frequency limit like frequency squared if there is nonvanishing coupling to the scalar curvature. This is in contrast with the minimally coupled case, where this greybody factor is known to tend to a finite constant. We also study the Hawking radiation for nonminimally coupled massless scalar fields in Schwarzschild–de Sitter spacetime, formulate a sensible notion of a generalized absorption cross section and investigate its properties.

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## I. INTRODUCTION

The study of phenomena in de Sitter spacetime is a subject of importance beyond pure academic interest, given that our cosmological neighborhood is presently undergoing accelerated expansion [1]. Moreover, de Sitter (dS) spacetime is a very good approximation to the exponentially expanding phase postulated by the inflationary paradigm [2]. Finally, possible connections with conformal field theories provided by the dS/CFT correspondence [3] add value to the study of asymptotically dS spacetimes. On the other hand, black holes are among the most relevant objects in any gravitational theory, besides there being clear indications of their presence at the center of galaxies [4] and the possibility of black hole (BH) formation in particle colliders [5]. These observations motivate the study of black holes in asymptotically dS spacetimes.

Most of the literature investigating the scattering and absorption properties of waves in BH spacetimes focus on the asymptotically flat case. Recently, such problems have been studied extensively [6–12]. In asymptotically flat spacetimes it is common to express the outcome of scattering a wave off a black hole in terms of an absorption cross section. The absorption cross section is directly connected to the *greybody factors*, i.e., the probability for a given wave coming in from infinity to be absorbed by the black hole [13,14]. This has been shown to be equal to the transmission probability for an outgoing wave “emitted” from the black hole event horizon to reach the asymptotic region (see, e.g., Ref. [14]). It is the nontriviality of greybody factors that cause the semiclassical spectrum of

emission of black holes to depart from that of a pure blackbody.

For asymptotically flat BH spacetimes there are indications that the greybody factor  $\gamma_l(\omega)$  for waves of arbitrary spin  $s$  and angular quantum number  $l$  in any number of dimensions  $d$  vanishes in the zero-frequency limit  $\omega \rightarrow 0$ . This general statement is confirmed by all cases studied so far, and has actually been proven for massless minimally coupled scalar fields in stationary black hole backgrounds [15,16] in general spacetime dimensions. In  $d = 4$  this phenomenon also occurs for massless spin-1/2 fermions, gauge bosons and gravitons [17] and persists even in the presence of nonminimal coupling of the scalar field with the curvature scalar [18].

However, the  $l = 0$  greybody factor tends to a positive constant in the infrared limit for a minimally coupled massless scalar field in Schwarzschild–de Sitter spacetime (SdS). The expression obtained in the zero-frequency limit in four dimensions was first reported in Ref. [19], and the result for arbitrary dimensions is [20]

$$\gamma_0(\omega = 0) = \frac{4(r_C r_H)^{d-2}}{(r_C^{d-2} + r_H^{d-2})^2}, \quad (1)$$

where  $r_C$  and  $r_H$  stand for the radial location of the cosmological horizon and the black hole event horizon, respectively. Several other authors have confirmed this result, e.g., Refs. [14,21,22]. This means that at low energies the cosmological constant has an important effect on the greybody factor of massless minimally coupled scalar particles. The explanation put forward in Ref. [20] for this phenomenon is that zero-energy particles are fully delocalized, and have therefore a finite probability to traverse the distance between the two horizons [23]. This argument then suggests that the infrared enhancement of transmitted flux in SdS is not present when fields are massive, or “effectively massive” due to nonzero coupling to the

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scalar curvature, since the dispersion relation gets modified by the addition of the mass. In this paper we will indeed find that this phenomenon is specific to the massless case with minimal coupling.

To the best of our knowledge, the greybody factors for massless scalar fields propagating in the SdS geometry have been computed only for minimal coupling. In this paper we consider a massless scalar field with nonzero coupling to the scalar curvature,  $\xi R\phi^2$ , propagating in four-dimensional SdS spacetime and compute the corresponding greybody factors. Such a coupling can also be interpreted as a mass term since the scalar curvature of the background is constant in SdS spacetime. However, we note that this masslike term is tied to the cosmological constant so that it vanishes in the asymptotically flat limit. (For a study of the effect of a scalar mass on the absorption and emission spectra of a Schwarzschild black hole see Ref. [24].) We show that, if  $\xi \neq 0$ , the greybody factor for the  $l = 0$  mode tends to zero in general like  $\omega^2$ . We also show that the rate of Hawking radiation tends to zero as  $\omega \rightarrow 0$  if  $\xi \neq 0$ , unlike the case with  $\xi = 0$ , in which this rate remains finite in the low-frequency limit.

The definition of an absorption cross section in dS spacetimes is an issue that has generated some controversy [14]. We elucidate that this concept, as it is defined in Ref. [20], is generally not meaningful in dS spacetimes. However, for small black holes ( $r_C \gg r_H$ ) it is possible to define a *generalized* absorption cross section, albeit only approximately.

The remainder of this paper is organized as follows. We briefly describe Schwarzschild–de Sitter spacetime in Sec. II and present the framework for finding the behavior of a massless scalar field in this spacetime for general coupling  $\xi$ . In Sec. III we explain how the greybody factor is computed and find its low-frequency limit for  $\xi \neq 0$  to second order in  $\xi$ . We also derive a complementary low-frequency approximation, valid only for small SdS black holes but for arbitrary  $\xi$ . To conclude this section we present numerical results for the greybody factors with several values of  $\xi$ . Hawking radiation is analyzed in Sec. IV and some properties of the generalized absorption cross section are studied in Sec. V. We summarize our results in Sec. VI. In the Appendix we briefly discuss the scattering problem in a negative square-well potential, which may be helpful in understanding the special nature of minimal coupling  $\xi = 0$ .

We use natural units such that  $c = G = \hbar = k_B = 1$  and metric signature  $(-, +, +, +)$ .

## II. SCHWARZSCHILD–DE SITTER SPACETIME

The four-dimensional Schwarzschild–de Sitter spacetime has line element given by

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (2)$$

where

$$f(r) = 1 - \frac{2M}{r} - \frac{\Lambda}{3}r^2, \quad (3)$$

with  $M$  being the black hole mass and  $\Lambda > 0$ . The metric  $g_{\mu\nu}$  defined by  $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$  is a solution of the vacuum Einstein field equations with positive cosmological constant  $\Lambda$ ,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0, \quad (4)$$

where  $R_{\mu\nu}$  and  $R$  denote the Ricci tensor and scalar curvature, respectively.

The roots of  $f$  occur at  $r = r_H$ ,  $r = r_C$  and  $r = -r_H - r_C$ , where  $r_H$  and  $r_C$  are the event and cosmological horizons of the SdS spacetime, respectively (with  $r_H < r_C$ ). It is easy to see that the metric (2) depends only on a single dimensionless parameter, which is commonly taken to be  $\lambda \equiv \Lambda M^2/3$ , up to an overall constant scale factor. The spacetime features two horizons only for  $0 < \lambda < 1/27$  [25,26] (see [27] for the higher-dimensional case). For  $\lambda = 1/27$  the two horizons merge and one obtains the extreme SdS geometry, whereas if  $\lambda > 1/27$  there are no horizons. In the present study we shall restrict ourselves to the case  $0 < \lambda < 1/27$ .

The temperatures associated with the black hole event horizon and the cosmological horizon of the four-dimensional SdS solution are given by [20,25]

$$T_H = \frac{1}{4\pi r_H}(1 - \Lambda r_H^2), \quad (5)$$

$$T_C = \frac{1}{4\pi r_C}(\Lambda r_C^2 - 1). \quad (6)$$

We note that the temperature of the black hole is always larger than the temperature of the universe ( $T_H > T_C$ ), since  $r_H < r_C$ , implying a net energy flow from the black hole horizon towards the cosmological horizon [20].

### A. Nonminimally coupled massless scalar field

We consider a massless Klein-Gordon field  $\Phi(x^\mu)$  in the background geometry (2), coupled to the gravitational field, described by the action

$$S = -\frac{1}{2} \int d^4x \sqrt{-g} [(\nabla_\mu \Phi)(\nabla^\mu \Phi) + \xi R \Phi^2], \quad (7)$$

where  $g$  is the determinant of the spacetime metric and  $\xi$  is the coupling between the scalar and the gravitational fields. Some particular values of  $\xi$  are of specific interest:  $\xi = 0$  is the so-called minimal coupling, whereas  $\xi = 1/6$  is the so-called conformal coupling for which the scalar field theory becomes conformally invariant [28].

The massless scalar field satisfies the Klein-Gordon equation,

$$[\nabla_\mu \nabla^\mu - \xi R]\Phi(t, r, \theta, \phi) = 0. \quad (8)$$

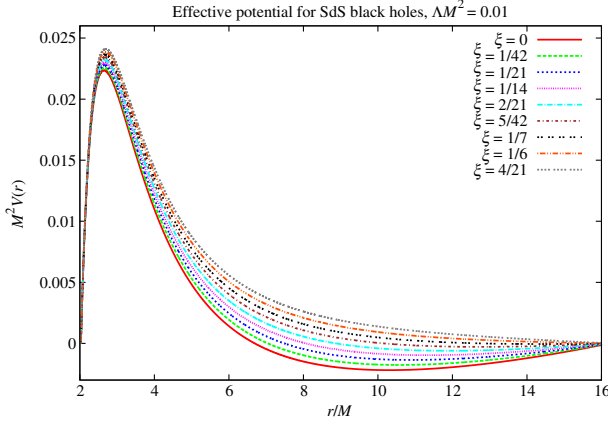


FIG. 1 (color online). Effective potential plotted for  $\Lambda M^2 = 0.01$  and  $l = 0$ , for different choices of the coupling parameter  $\xi$ .

Taking advantage of the spherical symmetry of the problem and of the existence of the Killing vector field  $\partial_t$ , we write the solutions to Eq. (8) in the form

$$\Phi(t, r, \theta, \varphi) = \frac{\psi_{\omega l}(r)}{r} Y_{lm}(\theta, \varphi) e^{-i\omega t}, \quad \omega > 0, \quad (9)$$

where  $Y_{lm}(\theta, \varphi)$  are the scalar spherical harmonics. The radial part of the waves obeys the following equation:

$$-f \frac{d}{dr} \left[ f \frac{d}{dr} \psi_{\omega l}(r) \right] + V(r) \psi_{\omega l} = \omega^2 \psi_{\omega l}, \quad (10)$$

where the effective potential  $V(r)$  is given by

$$V(r) = f(r) \left[ \frac{f'(r)}{r} + \frac{l(l+1)}{r^2} + 4\xi\Lambda \right], \quad (11)$$

with the “prime” standing for the derivative with respect to  $r$ .

In Fig. 1 we plot the effective potential (11) for  $\Lambda M^2 = 0.01$ ,  $l = 0$  and for several values of  $\xi$ . This effective potential vanishes at the SdS black hole horizon  $r_H$  and at the cosmological horizon  $r_C$ . It is apparent that the effective potential presents different behaviors for different values of the coupling parameter  $\xi$ .

### III. GREYBODY FACTOR

#### A. Asymptotic solution and greybody factor

In order to determine the greybody factors, we need to know the asymptotic behavior of the radial function  $\psi_{\omega l}(r)$ .

Since  $V(r)$  goes to zero at the black hole event horizon, near  $r = r_H$  we can write

$$\psi_{\omega l}(r) \approx A_{\omega l}^{\text{tr}} e^{-i\omega r_*}, \quad (12)$$

with  $r_*$  being the tortoise coordinate defined by  $dr_*/dr \equiv f^{-1}$ . We imposed purely ingoing boundary conditions at the black hole horizon, thus eliminating a term proportional to  $e^{+i\omega r_*}$  from the general solution. We may write  $r_*$  as

$$r_* = \sum_{n=1}^3 \frac{1}{f'(r_n)} \log |1 - r_n/r|, \quad (13)$$

where  $r_n$  denotes the roots of  $f$ .

As we approach the cosmological horizon, again  $V(r)$  goes to zero and therefore

$$\psi_{\omega l} \approx A_{\omega l}^{\text{in}} e^{-i\omega r_*} + A_{\omega l}^{\text{out}} e^{+i\omega r_*}, \quad (14)$$

where  $A_{\omega l}^{\text{in}}$  represents the amplitude of the incoming wave, while  $A_{\omega l}^{\text{out}}$  stands for the outgoing wave. Hence, the greybody factors can be expressed as

$$\gamma_l(\omega) = \left| \frac{A_{\omega l}^{\text{tr}}}{A_{\omega l}^{\text{in}}} \right|^2. \quad (15)$$

Since  $|A_{\omega l}^{\text{in}}|^2 = |A_{\omega l}^{\text{tr}}|^2 + |A_{\omega l}^{\text{out}}|^2$  (from flux conservation), the greybody factors can also be written as

$$\gamma_l(\omega) = 1 - \left| \frac{A_{\omega l}^{\text{out}}}{A_{\omega l}^{\text{in}}} \right|^2. \quad (16)$$

We will also refer to the greybody factor as transmission coefficient.

#### B. Low-frequency approximation with small $\xi$

It was shown in Ref. [19] that for  $\xi = 0$  the transmission coefficient for  $l = 0$  is finite and given by

$$\lim_{\omega \rightarrow 0} \gamma_0(\omega) = \frac{4r_H^2 r_C^2}{(r_H^2 + r_C^2)^2}. \quad (17)$$

However, as we stated in the Introduction, if  $\xi \neq 0$  the transmission coefficient  $\gamma_0(\omega)$  generally behaves like  $\omega^2$  in the low-frequency limit. This can be seen as follows.

Consider the limit  $\omega \rightarrow 0$  of  $\Psi_{\omega 0} \equiv \psi_{\omega 0}/A_{\omega 0}^{\text{tr}}$ , where  $A_{\omega l}^{\text{tr}}$  is defined by Eq. (12). We have  $\Psi_{\omega 0} \rightarrow 1$  near the black hole event horizon. Near the cosmological horizon we have instead

$$\lim_{\omega \rightarrow 0} \Psi_{\omega 0} = \lim_{\omega \rightarrow 0} \left( \frac{A_{\omega 0}^{\text{in}}}{A_{\omega 0}^{\text{tr}}} e^{-i\omega r_*} + \frac{A_{\omega 0}^{\text{out}}}{A_{\omega 0}^{\text{tr}}} e^{+i\omega r_*} \right). \quad (18)$$

Now, if there is a solution to Eq. (10) for  $l = 0$  and  $\omega = 0$  that tends to 1 as  $r_* \rightarrow -\infty$  and to a constant  $a$  as  $r_* \rightarrow +\infty$ , then we must have

$$a = \lim_{\omega \rightarrow 0} \left( \frac{A_{\omega 0}^{\text{in}}}{A_{\omega 0}^{\text{tr}}} + \frac{A_{\omega 0}^{\text{out}}}{A_{\omega 0}^{\text{tr}}} \right). \quad (19)$$

Then the  $\omega \rightarrow 0$  limit of  $\gamma_0(\omega) = |A_{\omega 0}^{\text{tr}}/A_{\omega 0}^{\text{in}}|^2$  must be nonzero. This is indeed what happens for  $\xi = 0$  [19]. However, generically the  $\omega = 0$  solution which tends to 1 as  $r_* \rightarrow -\infty$  behaves as  $\Psi_{00} \approx s r_*$  in the limit  $r_* \rightarrow +\infty$ , where  $s$  is a constant. In such cases  $A_{\omega 0}^{\text{out}} \rightarrow -A_{\omega 0}^{\text{in}}$  in the limit  $\omega \rightarrow 0$  and we have

$$s = -2i \lim_{\omega \rightarrow 0} \omega \frac{A_{\omega 0}^{\text{in}}}{A_{\omega 0}^{\text{tr}}}. \quad (20)$$

This leads to the following small  $\omega$  approximation of the transmission coefficient:

$$\gamma_0(\omega) = \left| \frac{A_{\omega 0}^{\text{tr}}}{A_{\omega 0}^{\text{in}}} \right|^2 \approx \frac{4}{s^2} \omega^2 \quad (21)$$

as in the asymptotically flat case. In the rest of this subsection we calculate the constant  $s$  for small  $\xi$  to second order in  $\xi$  and find the low-frequency behavior of  $\gamma_0(\omega)$  to this order.

We first write Eq. (10) with  $\omega = 0$  in terms of  $R(r) \equiv \Psi_{00}(r)/r$  as

$$\frac{d}{dr} \left[ f(r) r^2 \frac{dR}{dr} \right] = 4\xi \Lambda r^2 R(r). \quad (22)$$

We take  $R^{(0)}(r) = 1/r_H$ , i.e.,  $\Psi_{00} = r/r_H$ , as the lowest-order solution. Then by writing

$$R(r) = 1/r_H + \xi R^{(1)}(r) + O(\xi^2) \quad (23)$$

we have

$$\frac{d}{dr} \left[ f(r) r^2 \frac{dR^{(1)}}{dr} \right] = \frac{4\Lambda}{r_H} r^2. \quad (24)$$

This equation can readily be solved by integration. We find

$$\begin{aligned} R^{(1)}(r) = & -\frac{4(r_C^2 + r_C r_H + r_H^2)}{r_C r_H (2r_C + r_H)} \log \left( \frac{r_C - r}{r_C - r_H} \right) \\ & -\frac{4(r_C^2 + r_C r_H + r_H^2)}{r_H (r_C + r_H) (2r_C + r_H)} \log \left( \frac{r + r_C + r_H}{r_C + 2r_H} \right) \\ & + \frac{4r_H}{r_C (r_C + r_H)} \log \frac{r}{r_H}. \end{aligned} \quad (25)$$

Next we note that Eq. (22) can be written as

$$\frac{d}{dr} \left[ r \frac{d\Psi_{00}}{dr_*} - f(r) \Psi_{00} \right] = 4\xi \Lambda r^2 R(r). \quad (26)$$

Since  $f(r) \rightarrow 0$  like  $r_C - r$  as  $r \rightarrow r_C$  whereas from Eq. (13) we see that  $\Psi_{00}$  diverges like  $\ln(r_C - r)$  in this limit, we conclude that  $f(r) \Psi_{00} \rightarrow 0$  as  $r \rightarrow r_C$ . Recalling that we have required  $d\Psi_{00}/dr_* \rightarrow 0$  as  $r \rightarrow r_H$  and that we have defined  $\lim_{r \rightarrow r_C} d\Psi_{00}/dr_* = s$ , we find

$$s = \frac{4\xi \Lambda}{r_C} \int_{r_H}^{r_C} r^2 R(r) dr. \quad (27)$$

By substituting  $R(r) = 1/r_H + \xi R^{(1)}(r)$ , where  $\xi R^{(1)}(r)$  is given by Eq. (25) into this equation we find the constant  $s$  to order  $\xi^2$  as

$$s = \frac{4\xi(r_C - r_H)}{r_C r_H} [1 + 4\xi G(r_C, r_H)], \quad (28)$$

where

$$\begin{aligned} G(r_C, r_H) = & \frac{4}{3} + \frac{r_C^2 r_H^2}{(r_C + r_H)(r_C^3 - r_H^3)} \log \frac{r_C}{r_H} \\ & - \frac{r_C^2 + r_C r_H + r_H^2}{r_C^2 - r_H^2} \log \frac{2r_C + r_H}{r_C + 2r_H}. \end{aligned} \quad (29)$$

Hence by Eq. (21)

$$\gamma_0(\omega) \approx \frac{r_C^2 r_H^2}{4\xi^2 (r_C - r_H)^2} \omega^2 [1 + 4\xi G(r_C, r_H)]^{-2} \quad (30)$$

to second lowest order in  $\xi$ .

### C. Low-frequency approximation with small $\lambda$

In this section we present an analytic computation that provides an approximation to the greybody factor of small dS black holes in the low-frequency regime. The calculation is performed using matched asymptotic expansions, a technique first employed in this context in Ref. [29]. Our calculation follows along the same lines as Refs. [30,31].

The main interest of this result is that it is valid for arbitrary angular quantum number  $l$  and coupling  $\xi$ . On the other hand, the accuracy of the result is guaranteed only if the two asymptotic regions significantly overlap, which implies that it is only valid for small frequencies. In addition, the procedure is justified, as we shall see, only for the class of “small” black holes (compared with the characteristic dS scale), i.e.,  $\lambda \ll 1$ . Thus the approximation for the greybody factor we shall obtain below, Eq. (43), and the result of the previous section, Eq. (30), are valid in complementary regions of parameter space.

The starting point is the radial wave equation (10), written in terms of  $X_{\omega l}(r) \equiv \psi_{\omega l}(r)/r$ , namely

$$\begin{aligned} r^2 f \frac{d}{dr} \left[ r^2 f \frac{d}{dr} X_{\omega l} \right] - r^2 f [l(l+1) + 4\xi \Lambda r^2] X_{\omega l} \\ + \omega^2 r^4 X_{\omega l} = 0. \end{aligned} \quad (31)$$

We then analyze this wave equation in two distinct, but overlapping, regions. The *near* region is defined by  $r - r_H \ll 1/\omega$  whereas the *far* region is such that  $r - r_H \gg 2M$ . The two regions overlap if  $\omega M \ll 1$ .

The essential point of considering small SdS black holes is that in the near region  $r \sim r_H \sim 2M$  we can discard the effects of the cosmological constant. Hence,  $r^2 f(r) \approx r^2 - 2Mr$  and the near region wave equation becomes

$$\begin{aligned} (r^2 - 2Mr) \frac{d}{dr} \left[ (r^2 - 2Mr) \frac{d}{dr} X_{\omega l}^{(\text{near})} \right] \\ - [(r^2 - 2Mr)l(l+1) - (2M)^4 \omega^2] X_{\omega l}^{(\text{near})} = 0. \end{aligned} \quad (32)$$

The (purely ingoing) solution of this equation can be written in terms of a hypergeometric function [31]:

$$X_{\omega l}^{(\text{near})}(r) = A \left(1 - \frac{2M}{r}\right)^{-2iM\omega} \left(\frac{2M}{r}\right)^{l+1} \times F\left(l+1, l+1-4iM\omega, 1-4iM\omega; 1-\frac{2M}{r}\right), \quad (33)$$

with  $A$  being the constant amplitude. It can be shown that

$$X_{0l}^{(\text{near})}(r) = (-1)^l A P_l\left(1 - \frac{r}{M}\right), \quad (34)$$

where  $P_l(x)$  are the Legendre polynomials [32].

For the far region solution we can neglect the effects of the black hole, so  $M \sim 0$  and  $f(r) \approx 1 - \Lambda r^2/3$ . By defining a new variable  $x \equiv 1 - \Lambda r^2/3$  the radial wave equation then becomes

$$x(1-x) \frac{d^2 X_{\omega l}^{(\text{far})}}{dx^2} + \left(1 - \frac{5}{2}x\right) \frac{dX_{\omega l}^{(\text{far})}}{dx} - \left[\frac{l(l+1)}{4(1-x)} - \frac{3\omega^2}{4\Lambda x} + 3\xi\right] X_{\omega l}^{(\text{far})} = 0, \quad (35)$$

whose general solution can again be expressed in terms of hypergeometric functions as

$$X_{\omega l}^{(\text{far})}(x) = C x^{i\frac{\omega}{2}\sqrt{\frac{3}{\Lambda}}} (1-x)^{l/2} F(\nu_+, \nu_-, \mu; x) + D x^{-i\frac{\omega}{2}\sqrt{\frac{3}{\Lambda}}} \times (1-x)^{l/2} F(1+\nu_+, \mu, 1+\nu_- - \mu, 2-\mu; x), \quad (36)$$

where

$$\nu_{\pm} = \frac{1}{4} \left(3 + 2l \pm \sqrt{9 - 48\xi} + 2i\omega\sqrt{\frac{3}{\Lambda}}\right), \quad (37)$$

$$\mu = 1 + i\omega\sqrt{\frac{3}{\Lambda}}. \quad (38)$$

(See also Ref. [33].)

Having the two asymptotic solutions at hand, one can now match them in the overlapping region  $2M \ll r \ll 1/\omega$ . This requires finding the large- $r$  behavior of Eq. (33) and the small- $r$  limit of Eq. (36), which corresponds to  $x \rightarrow 1$ . In the overlapping region both solutions are expressed as a superposition of two terms, with corresponding behaviors  $\sim r^l$  and  $\sim r^{-(l+1)}$ . By matching the respective coefficients we determine the constants  $C$  and  $D$  as functions of all the parameters in our problem, namely  $\omega$ ,  $\Lambda$ ,  $M$ ,  $l$  and  $\xi$ . Obviously, both coefficients  $C$  and  $D$  are proportional to the amplitude  $A$ , so when one computes their ratio,  $A$  cancels out.

The greybody factor is computed from Eq. (16) so we must relate coefficients  $C$  and  $D$  with  $A_{\omega l}^{\text{in}}$  and  $A_{\omega l}^{\text{out}}$ . The solution near the cosmological horizon,  $x \sim 0$ , behaves like

$$X_{\omega l} \approx C x^{i\frac{\omega}{2}\sqrt{\frac{3}{\Lambda}}} + D x^{-i\frac{\omega}{2}\sqrt{\frac{3}{\Lambda}}}. \quad (39)$$

Also, in this spacetime region ( $x \sim 0$ ) the coordinate  $x$  is related with the tortoise coordinate  $r_*$  through

$$r_* \approx -\frac{1}{2} \sqrt{\frac{3}{\Lambda}} \log \frac{x}{2}. \quad (40)$$

Thus, comparing Eq. (39) with Eq. (14) we conclude that

$$C = \frac{A_{\omega l}^{\text{in}}}{r_C} 2^{-i\frac{\omega}{2}\sqrt{\frac{3}{\Lambda}}}, \quad D = \frac{A_{\omega l}^{\text{out}}}{r_C} 2^{i\frac{\omega}{2}\sqrt{\frac{3}{\Lambda}}}, \quad (41)$$

and therefore the greybody factor may be expressed as

$$\gamma_l(\omega) = 1 - \frac{|D|^2}{|C|^2}. \quad (42)$$

Substituting into the above expression the coefficients  $C$  and  $D$  obtained from the matching procedure, we arrive at the following analytic small-frequency approximation for the greybody factor of small dS black holes:

$$\gamma_l(\omega) = \frac{16\pi^{7/2}(-1)^l l! ((2l)!)^2 \Gamma(2l+2) \Gamma(-l-1/2)}{-\cos(\pi\sqrt{9-48\xi}) - \cosh(2\pi\frac{\omega M}{\sqrt{\Lambda}})} \times \frac{\sinh(\pi\frac{\omega M}{\sqrt{\Lambda}}) \lambda^{l+1/2}}{\sinh(4\pi\omega M) |H(\omega)|^2}, \quad (43)$$

where the function  $H(\omega)$  has been defined as

$$H(\omega) = (2l)! \Gamma(2l+2) \Gamma\left(l + \frac{1}{2}\right) \Gamma(-l-4i\omega M) \Gamma(\alpha_+) \Gamma(\alpha_-) + (-4)^l (l!)^2 \lambda^{l+1/2} \Gamma\left(-l - \frac{1}{2}\right) \Gamma(1+l-4i\omega M) \times \Gamma(\beta_+) \Gamma(\beta_-), \quad (44)$$

and

$$\alpha_{\pm} = \frac{1}{4} \left(1 - 2l \pm \sqrt{9 - 48\xi} - 2i\frac{\omega M}{\sqrt{\Lambda}}\right), \quad (45)$$

$$\beta_{\pm} = \frac{1}{4} \left(3 + 2l \pm \sqrt{9 - 48\xi} - 2i\frac{\omega M}{\sqrt{\Lambda}}\right). \quad (46)$$

This result is fully consistent with Refs. [14,19,20], meaning that the zero-frequency limit of the greybody factor for  $l = \xi = 0$  reproduces Eq. (17) in the small SdS black hole regime. For  $\xi \neq 0$  Eq. (43) becomes in the low-frequency limit

$$\gamma_l(\omega) = \frac{\pi(l!)^2 \lambda^l |\Gamma(\beta_+^0) \Gamma(\beta_-^0)|^2}{4^l |\Gamma(l + \frac{1}{2}) \Gamma(l + \frac{3}{2})|^2} (r_H \omega)^2, \quad (47)$$

where  $\beta_{\pm}^0 = \beta_{\pm}|_{\omega=0}$ . This result reproduces Eq. (30) for  $l = 0$  with small  $\xi$  at zeroth order in  $\lambda$ . Moreover, it is applicable to more general cases, with  $\xi \neq 0$  (not necessarily small) and general  $l$ .



### D. Computational methods and numerical results

In this section we present results valid for arbitrary frequencies that were obtained by solving numerically Eq. (10). We develop this solution from  $r = r_H(1 + \varepsilon_1)$  to  $r = r_C(1 - \varepsilon_2)$ , with  $\varepsilon_1, \varepsilon_2 \ll 1$ . The reflection

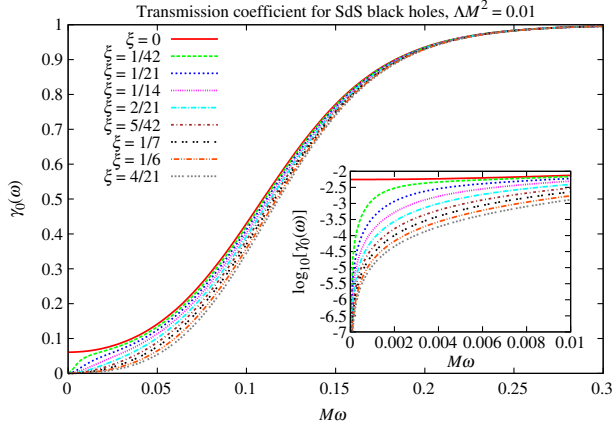


FIG. 2 (color online). Greybody factor plotted as a function of the frequency for  $\Lambda M^2 = 0.01$  and for different choices of the coupling  $\xi$ .

coefficient  $|A_{\omega l}^{\text{out}}/A_{\omega l}^{\text{in}}|^2$  may be obtained by comparing the numerical solution with the asymptotic form (14).

In Fig. 2 we plot our numerical results for the greybody factor. We see that the results go to zero as  $\omega \rightarrow 0$ , except for the minimally coupled case. The behavior of the transmission coefficient in the SdS spacetime (except for the minimally coupled case) keeps the same structure as in asymptotically flat spacetimes: it is zero in the low-frequency limit and tends to unity in the high-frequency limit. The explanation for this behavior in asymptotically flat spacetimes relies on the semiclassical approximation, in which we can relate the apparent impact parameter with the wave frequency and angular momentum as  $b \approx (l + 1/2)/\omega$ . Roughly speaking, for a fixed  $l$ , if  $b > b_c$ , where  $b_c$  is the critical impact parameter [34], the transmission coefficient is zero. As  $\omega$  increases,  $b$  approaches the value of  $b_c$ , and when  $b < b_c$  the particle is absorbed by the black hole. In Sec. VA we develop the geodesic analysis in the SdS spacetime, and, although the concepts of impact parameter are physically different in asymptotically flat and dS spacetimes, this analysis may be helpful in understanding the absorption process in asymptotically dS spacetimes.

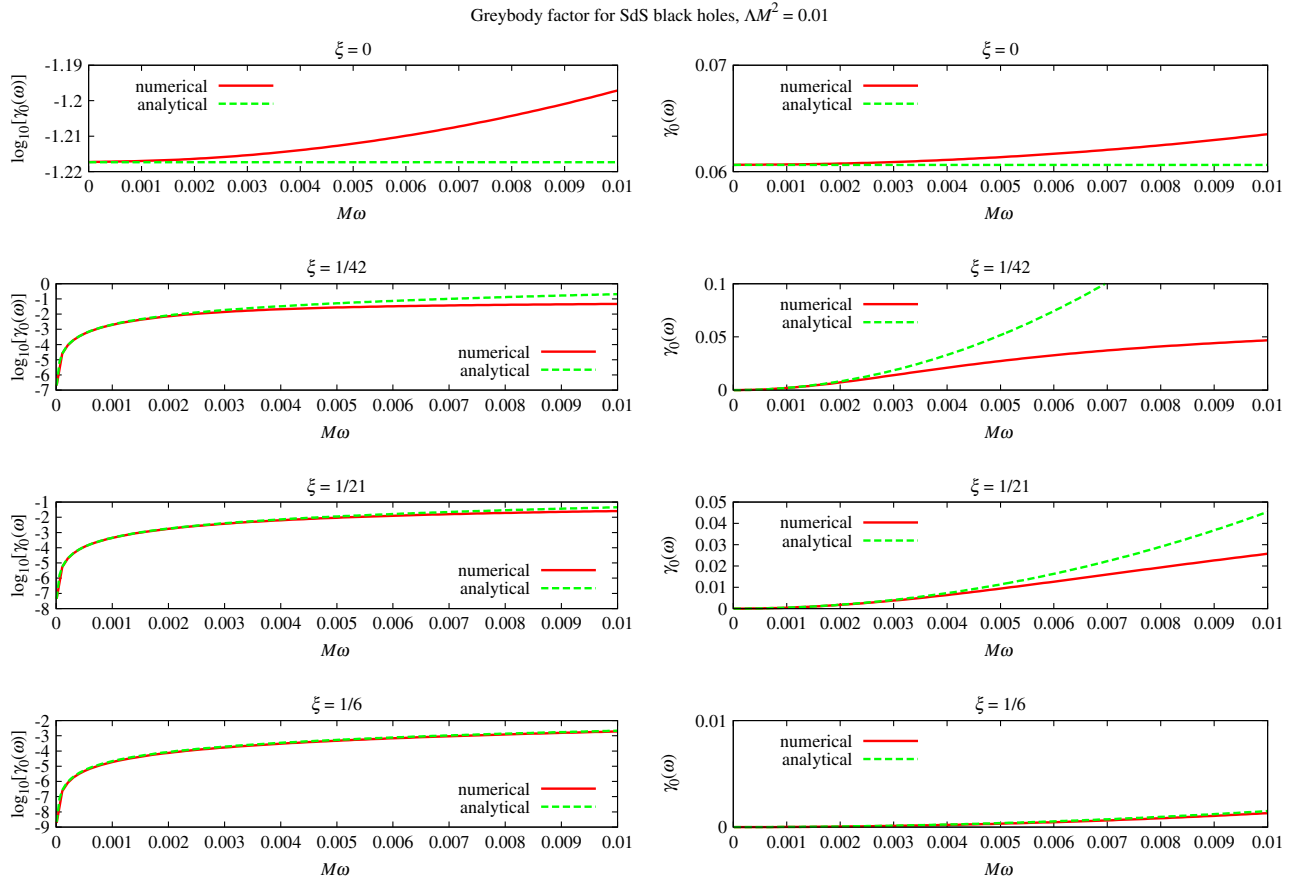


FIG. 3 (color online). Greybody factor plotted as a function of the frequency for  $\Lambda M^2 = 0.01$  compared with the analytic approximation given by Eq. (30), for different choices of the coupling  $\xi$ . Note the logarithmic scale on the vertical axis of the left plots.

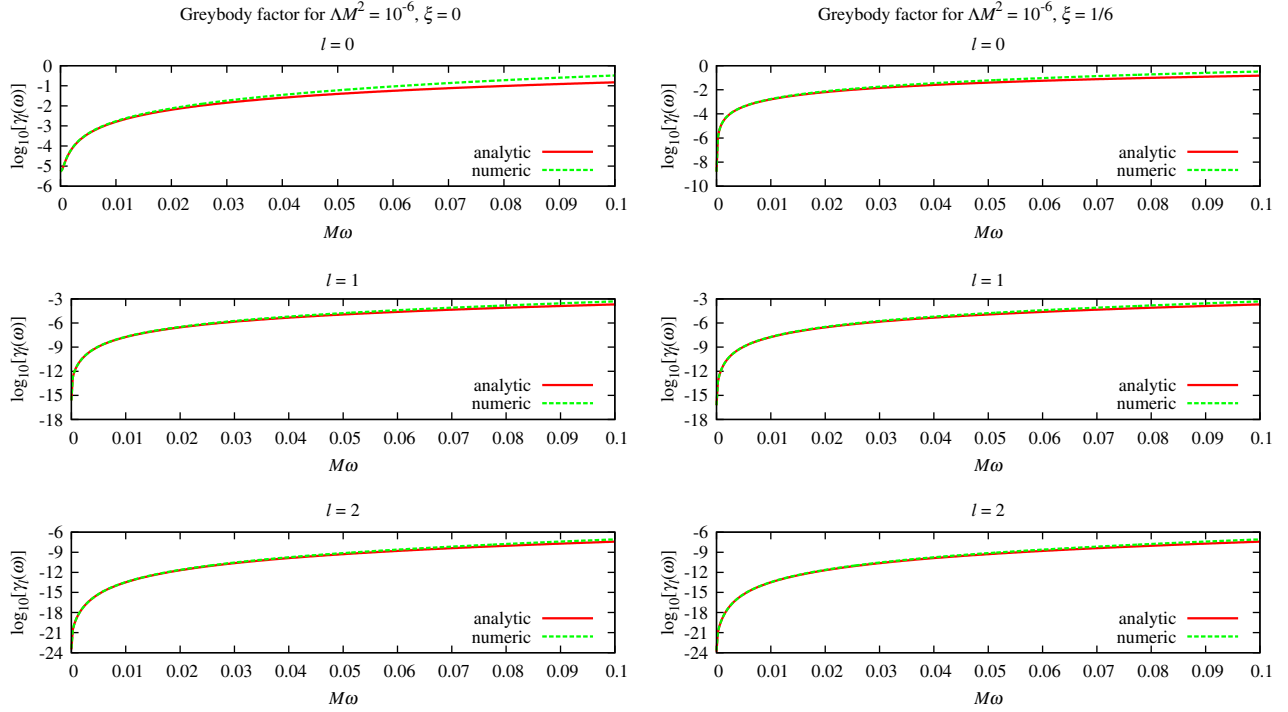


FIG. 4 (color online). Left plots: Greybody factor for small black holes in which  $\Lambda M^2 = 10^{-6}$ ,  $\xi = 0$ , and  $l = 0, 1, 2$ . Note the logarithmic scale on the vertical axes. Right plots: Analogous plots for  $\xi = 1/6$ . Excellent agreement is found between the numerical results and the analytic results given by Eq. (43).

In Fig. 3 we compare the numerical results for the greybody factor with the ones obtained using the analytical approximation (30). We find good agreement for low  $\omega$  even in the conformally coupled case ( $\xi = 1/6$ ). [We recall that Eq. (30) is valid in the small- $\xi$  regime.]

In Fig. 4 numerical results are compared with the analytic results for small black holes, given by Eq. (43), for  $\Lambda M^2 = 10^{-6}$ ,  $l = 0, 1, 2$ , with  $\xi = 0$  (left plots), and  $\xi = 1/6$  (right plots). We note an excellent agreement in the low-frequency limit, which is the regime of validity of approximation (43). This can be regarded as a simple consistency check of our results.

#### IV. ENERGY EMISSION

The number of massless scalar particles emitted by the black hole per unit time, also called flux spectrum, is given by

$$\frac{dN(\omega)}{dt} = \frac{d\omega}{2\pi} \frac{1}{e^{\omega/T_H} - 1} \sum_{l=0}^{\infty} (2l+1) \gamma_l(\omega). \quad (48)$$

The differential energy emission rate reads

$$\frac{d^2 E(\omega)}{dt d\omega} = \frac{1}{2\pi} \frac{\omega}{e^{\omega/T_H} - 1} \sum_{l=0}^{\infty} (2l+1) \gamma_l(\omega). \quad (49)$$

In Fig. 5 we plot  $d^2 E(\omega)/(dt d\omega)$ . As for the transmission coefficient, the emission rate for zero frequency is nonzero only in the case of a minimally coupled scalar

field. Therefore, we conclude that the conjectured distinctive feature caused by the presence of the nonvanishing cosmological constant, namely the emission of a significant number of ultrasoft quanta, pointed out in Ref. [20] (even in the four-dimensional case—cf. Sec. V of that paper), will not occur for the case of a nonminimally coupled scalar field. Note, however, that there is still an enhancement in the emission of soft quanta for low  $\xi$  with the rate dropping to zero rapidly as  $\omega$  becomes very close to zero.

It is apparent in Fig. 5 that the difference in the emission rates for distinct couplings is only considerable in the low-frequency regime. In other words, the coupling to the scalar curvature becomes irrelevant above intermediate values of the frequency ( $\omega \sim M^{-1}$ ). We also note that these differences for low frequencies become more and more pronounced as the value of  $\lambda$  increases. On the other hand, the enhancement in the emission rate occurring at low frequencies becomes less significant as the coupling parameter  $\xi$  grows.

#### V. GENERALIZED ABSORPTION CROSS SECTION

The definition of absorption cross section in nonasymptotically flat spacetimes cannot be formulated as it is done in asymptotically flat geometries. In the latter case, one may consider an impinging plane wave,  $\Phi_{\text{inc}} = e^{i\omega(z-t)}$ , which is a solution of the field equation in the asymptotic region since the metric there approaches that of Minkowski

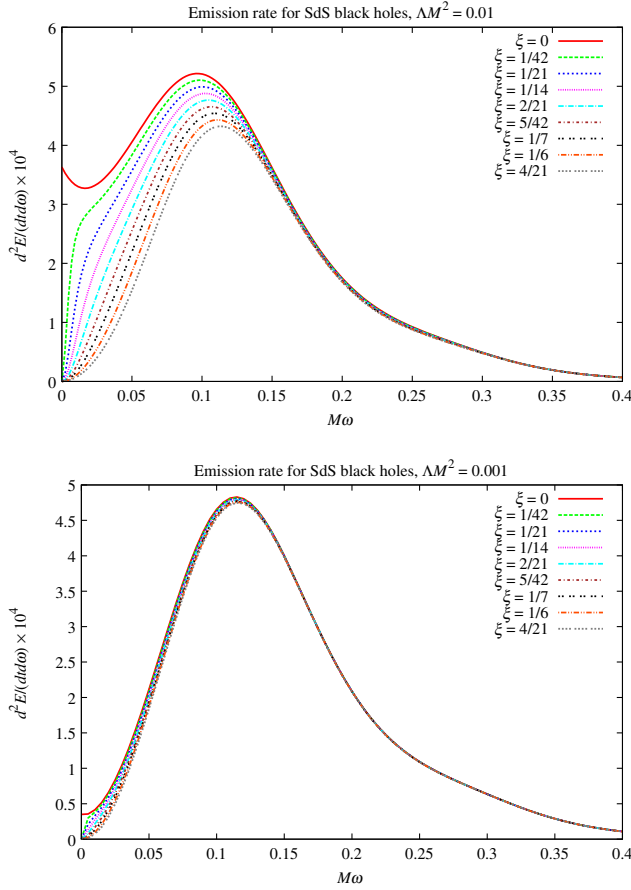


FIG. 5 (color online). Emission rate for Schwarzschild-de Sitter black holes for  $\Lambda M^2 = 0.01$  (top panel) and  $\Lambda M^2 = 0.001$  (bottom panel), and different values of  $\xi$ . A nonvanishing emission rate at zero frequency occurs only in the minimally coupled case ( $\xi = 0$ ).

spacetime. After being scattered off the spherically symmetric black hole, the scalar field in the asymptotic region  $r \rightarrow \infty$  can be decomposed into spherical harmonics [35]

$$\Phi_{\text{sca}} \approx i \sum_{l=0}^{\infty} \frac{(-1)^l (2l+1)}{2\omega} \times \frac{[e^{-i\omega r} + R_{\omega l} e^{i\omega r}]}{r} P_l(\cos \theta) e^{-i\omega t}, \quad (50)$$

with  $|R_{\omega l}|^2$  being the reflection coefficient. Then, using the definition of the absorption cross section [36]

$$\sigma \equiv \frac{\text{absorbed flux}}{\text{incident wave current}}, \quad (51)$$

one concludes that the total absorption cross section for an asymptotic plane wave incident on an asymptotically flat spherically symmetric spacetime is

$$\sigma = \sum_{l=0}^{\infty} \sigma_l = \frac{\pi}{\omega^2} \sum_{l=0}^{\infty} (2l+1) \gamma_l(\omega), \quad (52)$$

where  $\sigma_l$  represents the absorption cross section of each partial wave [37], usually referred to as partial absorption cross section.

In the case of asymptotically dS spacetimes it is easy to see that a plane wave is no longer an asymptotic solution of the field equation. Indeed,  $\psi_{\omega l}(r) = e^{\pm i\omega r}$  does not satisfy Eq. (10) near the cosmological horizon  $r_C$ , and therefore a sum over angular modes like the one displayed on the right-hand side of Eq. (50) does not yield an asymptotic solution. However,  $\psi_{\omega l}(r) = e^{\pm i\omega r_*}$  is an asymptotic solution of the Regge-Wheeler equation (10) and one might try to construct an analogue of a plane wave out of such solutions.

Let us consider then an incident wave of the form

$$\Phi_{\text{inc}}^{\text{dS}} = \frac{r_*}{r} e^{i(\omega r_* \cos \theta - t)}, \quad (53)$$

which can also be decomposed into spherical waves. This is understood by first recalling the identity [38]

$$e^{i\omega r_* \cos \theta} = \sum_{l=0}^{\infty} i^l (2l+1) j_l(\omega r_*) P_l(\cos \theta), \quad (54)$$

and then using the asymptotic form of the spherical Bessel functions of the first kind,  $j_l(x)$ , thus obtaining

$$\frac{r_*}{r} e^{i\omega r_* \cos \theta} \approx i \sum_{l=0}^{\infty} \frac{(-1)^l (2l+1)}{2\omega} \times \frac{[e^{-i\omega r_*} - (-1)^l e^{i\omega r_*}]}{r} P_l(\cos \theta). \quad (55)$$

This approximation is valid as an expansion for  $\omega r_* \gg 1$ .

It turns out that the wave (53) is *not* an asymptotic solution of the Klein-Gordon equation (8). Accordingly,  $\Phi = r^{-1} e^{-i\omega(t \mp r_*)} Y_{lm}(\theta, \varphi)$  is also not an asymptotic solution of Eq. (8), even though it is an asymptotic solution of the Regge-Wheeler equation (10) [39].

Nevertheless, for small black holes there exists an intermediate region  $r_H \ll r \ll r_C$  where  $f \approx 1$  and  $r_* \approx r$ . In this region the wave (53) is an *approximate* solution of the field equation, meaning that the Klein-Gordon operator defined by the left-hand side of Eq. (8) does not annihilate (53) but instead yields terms that are suppressed by powers of  $r_H/r$  and of  $r/r_C$ .

Let us now return to expression (55), which represents a scattered wave in pure dS spacetime at any radial coordinate  $r$  that is large compared to the wavelength  $1/\omega$  but small relative to the cosmological horizon  $r_C$ . In the presence of a black hole, the scattered wave will be modified by the inclusion of a nontrivial reflection coefficient,

$$\Phi_{\text{sca}}^{\text{dS}} \approx i \sum_{l=0}^{\infty} \frac{(-1)^l (2l+1)}{2\omega} \times \frac{[e^{-i\omega r_*} + R_{\omega l} e^{i\omega r_*}]}{r} P_l(\cos \theta) e^{-i\omega t}, \quad (56)$$



with  $R_{\omega l} = A_{\omega l}^{\text{out}}/A_{\omega l}^{\text{in}}$  and  $A_{\omega l}^{\text{in}} = i(-1)^l \sqrt{\pi(2l+1)}/\omega$ . Despite the similarity between Eqs. (56) and (50), in order to define the absorption cross section one must still determine the flux of the wave (56) at some intermediate radius  $r \in (r_H, r_C)$  and the current of the incident wave.

The wave flux is given by

$$\mathcal{F} = - \int J^r r^2 d\Omega, \quad (57)$$

where  $d\Omega$  is the solid angle element and  $J^r$  is the radial (contravariant) component of the current, which is defined for a wave  $\Phi$  as

$$J_\mu \equiv \frac{1}{2i} (\Phi^* \partial_\mu \Phi - \Phi \partial_\mu \Phi^*). \quad (58)$$

Using Eqs. (56)–(58), it can be shown that the flux computed using the scattered wave is

$$\mathcal{F} = \frac{\pi}{\omega} \sum_{l=0}^{\infty} (2l+1) \gamma_l(\omega), \quad (59)$$

which is independent of the radial coordinate  $r$ . To find Eq. (59) we have used  $\gamma_l(\omega) = 1 - |R_{\omega l}|^2$ .

The incident current corresponding to wave (53) is found to be

$$|\vec{J}_{\text{inc}}| = \omega \frac{r_*^2}{r^2} \left( f^{-1} \cos^2 \theta + \frac{r_*^2}{r^2} \sin^2 \theta \right)^{1/2}. \quad (60)$$

This result is general and could be applied for both asymptotically dS and flat spacetimes. In the latter case, we may take the limit  $r \rightarrow \infty$ , in which case  $r_* \rightarrow r$  and  $f \rightarrow 1$ , so that  $|\vec{J}_{\text{inc}}| = \omega$ . This, together with Eq. (59) and the definition of the absorption cross section (51), results in Eq. (52).

On the other hand, for asymptotically dS spacetimes if we take the limit  $r \rightarrow r_C$  then  $r_* \rightarrow \infty$  and  $f \rightarrow 0$ , which makes the incident current diverge. This would lead to an ill-defined absorption cross section and it is a consequence of the wave (53) not being a true asymptotic state.

However, for the case of small SdS black holes there exists an intermediate region where  $f \approx 1$  and  $r_* \approx r$ , and therefore  $|\vec{J}_{\text{inc}}| \approx \omega$ . Hence, it is physically sensible to define an absorption cross section only for small SdS black holes, i.e., for  $\lambda \ll 1$ . For this reason we refer to the quantity  $\sigma$  in Eq. (52) when used in asymptotically dS spacetimes as the “generalized” absorption cross section, and to the quantity  $\sigma_l$  as the partial “generalized” absorption cross section. [We note that  $\sigma$  in Eq. (52) was referred to as absorption cross section in Ref. [20] for asymptotically dS spacetimes without explaining when this notion is physically meaningful].

### A. Analytic results

Our numerical results show that, in the low-frequency regime, the partial generalized absorption cross section is

nonzero for all  $l$  modes. This contrasts with the case of the scalar field in Schwarzschild spacetimes, in which the greybody factor in the low-frequency regime behaves generally as  $\omega^{2l+2}$ , and the partial absorption cross section as  $\omega^{2l}$  (see, for instance, Ref. [40]). Here, as shown in Eq. (47), the greybody factors behave generally as  $\omega^2$ , except for the minimally coupled case, where the greybody factor is constant in the low-frequency regime, as expressed in Eq. (17).

Although the concept of absorption cross section cannot be straightforwardly adopted in asymptotically dS spacetimes, it is interesting that numerical results for expression (52) are consistent with what one could call the “high-frequency generalized absorption cross section,” as we show here.

Let us present a geodesic analysis to illustrate how this quantity can be found. We start by recalling that the motion of a massless particle in the spacetime defined by the line element (2) is governed by

$$-f\dot{t}^2 + f^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = 0, \quad (61)$$

where we have assumed  $\theta = \pi/2$ , without loss of generality (taking advantage of the spherical symmetry), and the overdot represents the derivative with respect to an affine parameter. The constants of motion are

$$E \equiv f\dot{t}, \quad (62)$$

$$L \equiv r^2\dot{\phi}. \quad (63)$$

Writing Eq. (61) in terms of these constants, we get

$$\dot{r}^2 + \frac{fL^2}{r^2} = E^2, \quad (64)$$

which is similar to the equation for the total mechanical energy. In this analogy, the second term on the left-hand side of Eq. (64) plays the role of an “effective potential” for the particle’s motion of total energy  $E^2$ . This effective potential possesses a maximum at  $r = 3M$ , which implies that critical orbits exist in SdS spacetime at radius  $r_c = 3M$ . This result does not depend on the value of the cosmological constant and, therefore, is the same for Schwarzschild and SdS spacetimes.

Making use of the analogy with asymptotically flat spacetimes once again, we can define the quantity  $b = L/E$ , which is an analog of the apparent impact parameter [26,34]. This quantity  $b$ , which is related to the initial conditions of the particle motion, has a critical value  $b_c$  when  $E^2$  equals the maximum of the effective potential, given by

$$b_c = 3M \left( \frac{1}{3} - 3\Lambda M^2 \right)^{-1/2}. \quad (65)$$

Classically, if  $b < b_c$ , then the particle is absorbed by the black hole; if  $b > b_c$ , the particle is scattered away from the black hole; if  $b = b_c$ , the particle ends on the unstable

orbit at  $r = r_c$ , circling the black hole an infinite number of times.

We may push this analogy further and define what would be the “high-frequency generalized absorption cross section” as

$$\sigma_{\text{hf}} \equiv \pi b_c^2 = 9\pi M^2 \left( \frac{1}{3} - 3\Lambda M^2 \right)^{-1}. \quad (66)$$

We note that  $\sigma_{\text{hf}} \rightarrow 27\pi M^2$  as  $\Lambda \rightarrow 0$ , which is a well-known result for Schwarzschild black holes. In the next subsection we compare  $\sigma_{\text{hf}}$  with the numerical results for the quantity (52) in the high-frequency limit and obtain an excellent agreement.

### B. Numerical results

In Fig. 6 we plot our numerical results for the generalized absorption cross section  $\sigma$ , defined by (52). It diverges when  $\omega \rightarrow 0$  in the minimally coupled case, but remains finite for  $\xi \neq 0$ , as we can see from the log plots in the same figure. The results presented in Fig. 6 agree very well with the high-frequency limit given in Eq. (66), which is represented by the straight line, despite the absence of a clear physical meaning for the quantity  $\sigma$  in generic asymptotically dS spacetimes.

In Fig. 7, we present the total and partial generalized absorption cross sections for  $\Lambda M^2 = 0.01$  and  $\xi = 1/6$  (conformal coupling). The sum in Eq. (52) is performed up to  $l = 20$ . It can be seen that the partial generalized absorption cross sections are nonzero for  $\omega = 0$ , and that they get smaller for bigger values of  $l$ , presenting a value  $\sim 10^{-19}M^2$  already for  $l = 6$ . This is in agreement with Eq. (47). As in Fig. 6, the total generalized absorption cross section exhibited in Fig. 7 presents an excellent agreement with the high-frequency result given by Eq. (66). We note

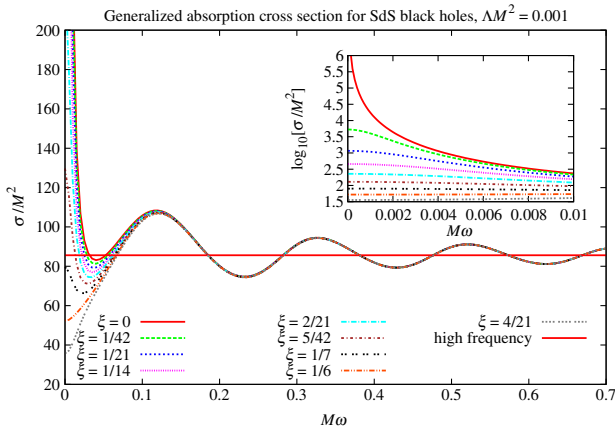


FIG. 6 (color online). Generalized absorption cross section plotted as a function of the frequency for  $\Lambda M^2 = 0.001$  and for different choices of the coupling  $\xi$ . The summation in  $l$  has been performed up to  $l = 3$ . The straight line is the high-frequency limit, given by Eq. (66).

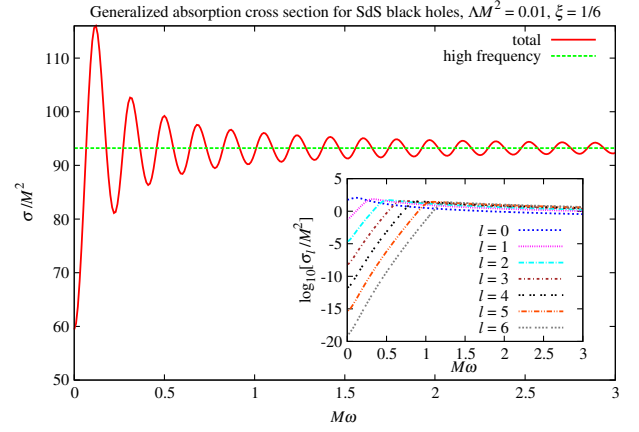


FIG. 7 (color online). Total and partial generalized absorption cross sections for  $\Lambda M^2 = 0.01$  and conformally coupled scalar field,  $\xi = 1/6$ , considering contributions of the angular momentum up to  $l = 20$ . The total generalized absorption cross section is in excellent agreement with the high-frequency limit. The partial contributions are shown in the log plot. One can see that, although finite, the partial generalized absorption cross sections at  $\omega = 0$  are very small for  $l > 0$ , and get smaller for larger values of  $l$ .

that the  $\Lambda$  dependence in  $\sigma_{\text{hf}}$  is not significant for this agreement.

### VI. CONCLUSION

In this paper we have analyzed the greybody factor of Schwarzschild–de Sitter black holes for nonminimally coupled scalar fields. In particular, it was shown that the zero-angular-momentum greybody factor in the zero-frequency regime is nonzero only for the minimally coupled (and massless) scalar field. For all other couplings to the scalar curvature, which may equivalently be regarded as mass terms, the greybody factors tend to zero like  $\omega^2$ , irrespective of the value of the coupling parameter  $\xi$ . In this sense the minimally coupled case is special.

A nonvanishing greybody factor in the low-frequency regime implies a nonzero Hawking emission in the same limit. For  $\xi \neq 0$ , however, the emission rate in this limit is always zero. There is nevertheless an enhancement in the emission rate but it only occurs at finite, albeit small, frequencies.

We have obtained numerical results that are in good agreement with the analytical low- $\xi$  approximations derived in the low-frequency regime. The numerical results also match the analytical small- $\lambda$  approximation in the low-frequency regime for the case of small Schwarzschild–de Sitter black holes. We observed that the coupling to the scalar curvature only has a significant effect on the emission rate at low frequencies,  $\omega \lesssim M^{-1}$ , and that the above-mentioned enhancement is more pronounced for larger black holes (larger  $\lambda$ ) but becomes less significant as the coupling parameter  $\xi$  grows.

Finally, we developed a sensible notion of a generalized absorption cross section in asymptotically de Sitter spacetimes and investigated its properties. In this respect we found an excellent agreement between its high-frequency behavior and a geometric-optics description.

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### APPENDIX: SQUARE-WELL POTENTIAL

In order to understand the change in the low-frequency behavior of the transmission coefficient at certain values of

parameters in the potential, such as  $\xi$  for the potential (11), the scattering problem in a negative square-well potential is instructive. Thus, we consider the scattering problem

$$-\frac{d^2\psi}{dx^2} + V(x)\psi(x) = \omega^2\psi(x), \quad (\text{A1})$$

where

$$V(x) = \begin{cases} -\omega_0^2, & \text{if } 0 < x < a, \\ 0, & \text{otherwise.} \end{cases} \quad (\text{A2})$$

The transmission coefficient  $\gamma(\omega)$  can readily be found as

$$\gamma(\omega) = \frac{4\omega^2\Omega^2}{4\omega^2\Omega^2\cos^2\Omega a + (\omega^2 + \Omega^2)\sin^2\Omega a}, \quad (\text{A3})$$

where we have defined  $\Omega = \sqrt{\omega^2 + \omega_0^2}$ .

If  $\omega_0 a \neq n\pi$  for any integer  $n$ , then for small  $\omega$  we have

$$\gamma(\omega) \approx \frac{4\omega^2}{\omega_0^2\sin^2\omega_0 a}. \quad (\text{A4})$$

We can readily see that the  $\omega = 0$  solution which equals 1 for  $x < 0$ , takes the value  $\psi(x) = \cos\omega_0 a - \omega_0(x - a) \times \sin\omega_0 a$  for  $x > a$ . With  $d\psi/dx = s = -\omega_0 \sin\omega_0 a$  we can write Eq. (A4) as  $\gamma(\omega) \approx 4\omega^2/s^2$  [see Eq. (21)]. If  $\omega_0 a = n\pi$  for some integer  $n$ , then the  $\omega = 0$  solution which equals 1 for  $x < 0$  takes the constant value  $(-1)^n$  for  $x > a$ . For these cases we indeed find  $\gamma(\omega) \rightarrow 1$ , which is a constant, in the limit  $\omega \rightarrow 0$ .

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In this case the distance between the two horizons is infinite.

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