PHYSICAL REVIEW D **87,** 104024 (2013)

Continuous spontaneous localization wave function collapse model as a mechanism for the emergence of cosmological asymmetries in inflation

Pedro Cañate,[1,*] Philip Pearle,[2,†] and Daniel Sudarsky[1,‡]

[1] *Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de Mexico, México D.F. 04510, Mexico*
[2] *Department of Physics, Hamilton College, Clinton, New York 13323, USA*

(Received 3 December 2012; published 21 May 2013)

The inflationary account for the emergence of the seeds of cosmic structure falls short of actually explaining the generation of primordial anisotropies and inhomogeneities. This description starts from a symmetric background, and invokes symmetric dynamics, so it cannot explain asymmetries. To generate asymmetries, we present an application of the continuous spontaneous localization model of wave function collapse in the context of inflation. This modification of quantum dynamics introduces a stochastic nonunitary component to the evolution of the inflaton field perturbations. This leads to passage from a homogeneous and isotropic stage to another, where the quantum uncertainties in the initial state of inflation transmute into the primordial inhomogeneities and anisotropies. We show, by proper choice of the collapse-generating operator, that it is possible to achieve compatibility with the precise observations of the cosmic microwave background radiation.

DOI: 10.1103/PhysRevD.87.104024 PACS numbers: 04.62.+v, 03.65.Ta, 42.50.Lc, 98.80.−k

I. INTRODUCTION

The measurement problem in quantum mechanics remains, almost a century after the theory's formulation, the major obstacle to considering the theory as truly fundamental. Despite heroic efforts by many insightful physicists, the difficulties involved have not yielded and we are still lacking any fully satisfactory option. The basic issue, as described for instance in Ref. [1], is the fact that the theory relies on two different and incompatible evolution processes. Using Penrose's characterization [2], there is the U (unitary) process, where the state changes smoothly according to Schrödinger's deterministic differential equation, and the R (reduction) process, in which the state of the system changes instantaneously, in an indeterministic fashion. The U process is supposed to control a system's dynamics all the time that the system is left alone, while the R process is called upon whenever a measurement has been carried out.

The problem is that no one has been able to characterize in general when a physical process should be considered a *measurement*. This issue has been studied and debated extensively in the scientific and philosophical literature [3]. Of course, in laboratory situations, one clearly knows when a measurement has been carried out. Nonetheless, as characterized by Bell [4], a FAPP (for all practical purposes) approach is not satisfactory at the foundational level, as it involves treating the system differently from the measuring device or the observer, and this division is one for which the theory offers no specific internal rules. Its most conspicuous inadequacy occurs in cosmological applications, where one cannot use any interpretation that relies on an observer, or on a measurement device [5].

One approach [6] to resolving this problem of standard quantum theory is to modify it by incorporating novel dynamical features responsible for "the collapse of the wave function." It may be characterized as the promotion of quantum theory from a theory of measurement to a theory of reality, in which some of the physical properties of a given system take values, regardless of whether they are observed or not. Such an approach *can* be applied to cosmology, and we shall do so by focusing on the problem of emergence of the seeds of structure in inflationary cosmology.

According to the standard account of inflation, in its early stages, a relatively generic state of the universe[1] is driven towards a homogeneous and isotropic Robertson Walker (RW) space-time. This universe expands almost exponentially, driven by the potential of the inflaton field, which acts as a large effective cosmological constant. The inflaton field fluctuation is a quantum field[2] which is taken to initially be in the so-called Bunch-Davies vacuum state.[3] This state is completely homogeneous and isotropic, and the dynamics preserves these properties. Therefore, it cannot be used to explain the observed inhomogeneous and

[1] There are several conditions that are required for this, but we shall not elaborate on that here.

[2] There are scalar, vector, and tensor fields, but we will focus here on just the scalar field which is responsible for the anisotropies that have been observed in the cosmic microwave background (CMB), so far.

[3] The fact that, due to the kinetic term in the energy momentum tensor of the scalar field, the space-time cannot exactly correspond to the de Sitter metric, implies that the state is not exactly the Bunch-Davies state. However, this difference is negligible for our treatment.

*pedro.canate@nucleares.unam.mx
†ppearle@hamilton.edu
‡sudarsky@nucleares.unam.mx

anisotropic distribution of the primordial energy density in our universe. There are various proposals to address this issue. One proposal postulates that the state vector can be expanded in some "natural" basis and, somehow, it is one of these basis states which describes our inhomogeneous and anisotropic universe. As discussed in detail in Ref. [7], none of these proposals can be regarded as satisfactory.

There are analogous instances where quantum theory presents us with a symmetric quantum state whereas nature exhibits asymmetric behavior. Perhaps the best known example is the α decay of a $J = 0$ nucleus. Although the wave function is rotationally invariant, the alpha particles are seen to move on linear trajectories. The problem was studied by Mott [8], and was resolved by heavy usage of the collapse postulate appropriate to a measurement situation [7]. However, in the cosmological problem at hand, even if we wanted to, we could not achieve a similar explanation, for we cannot call upon any external entity making a measurement.

The inflaton field fluctuation, according to the standard accounts, is supposed to describe the seeds of the growth of structure and lead to the formation of galaxy clusters, galaxies, stars, etc. The prediction, which involves the expectation value of the product of two inflaton field operators at the end of the inflationary period, is phenomenologically quite successful. But, as we have argued, in order to be truly successful those accounts should also explain the actual emergence of inhomogeneities and anisotropies from the quantum uncertainties of a quantum state that is completely homogeneous and isotropic. The issue is sometimes referred as that of the "classicalization of the fluctuations." There have been important efforts in this direction, among which are works like [9,10] which focus on the squeezing of the quantum state of the inflaton field fluctuations as a result of the cosmological expansion as well as others that focus on the role of decoherence [11], as well as works where both aspects are emphasized [12]. As explained in detail in Ref. [7], those approaches are not fully satisfactory. Basically, there is no way we can invoke anything like a measurement postulate as part of the explanation of the emergence of those primordial seeds of structure. Not only were there no observers or anything to play the role of a measuring apparatus at the time but, because we and our measuring devices (and indeed any conceivable kind of astronomers of alien civilizations) owe their existence to the process generating those seeds, they cannot be any part of their cause! A careful reading of, say Ref. [9], uncovers the important role that measurements would have to play in any such account.

To summarize, the problem is that one cannot explain the emergence of the observed asymmetries in homogeneity and isotropy when one has a theory with an initial state, the Bunch-Davies vacuum on the RW space-time, which is 100% homogeneous and isotropic and a dynamics that does not break such symmetries. Nonetheless, we wish to recover the phenomenological success of the standard account, where quantum fluctuations in the vacuum state are the source of the first inhomogeneities and anisotropies (we emphasize that the so-called quantum fluctuations are quantum uncertainties and should not be confused with thermal or statistical fluctuations).

In the case of alpha decay, dynamical collapse theories, either the Ghirardi-Rimini-Weber (GRW) [13] model or the continuous spontaneous localization (CSL) [14] model, satisfactorily resolves the problem. In CSL, which is usually considered to have superseded GRW, the Schrödinger equation is modified by adding to the usual Hamiltonian a nonunitary term iH_C, where H_C is a unitary "collapse Hamiltonian." H_C depends upon a randomly fluctuating classical field $w(\mathbf{x}, t)$. The eigenstates of H_C are essentially mass density eigenstates. This modified Schrödinger equation is supplemented by a second equation, the probability rule, which gives the probability that nature chooses a particular $w(\mathbf{x}, t)$. The dynamics is such that, for each $w(\mathbf{x}, t)$ of high probability, the collapse Hamiltonian evolves the state vector toward one or another eigenstate of mass density, and this occurs according to the Born probability rule.

In the description of alpha decay, under the usual Hamiltonian, the state vector describes the alpha particle interacting with the gas atoms in its path. With the added CSL dynamics, as more and more gas atoms get involved, for each high probability $w(\mathbf{x}, t)$, the collapse Hamiltonian more and more rapidly drives the state vector toward an approximate mass-density eigenstate describing the alpha particle moving in a straight-line path among its associated atoms.

Returning to the standard inflationary scenario, since there is no physical mechanism in standard quantum theory that could account for the emergence of the inhomogeneities and anisotropies, it is natural to consider addition of a physical mechanism that can do so. The mechanism of wave function collapse in the cosmological setting was suggested and modeled in an *ad hoc* and phenomenological way, in previous work along this line [15]. This led to phenomenological constraints on the parameters characterizing the suggested collapse mechanism, in order to ensure the theory is able to reproduce the observational results. In this manuscript, we consider this idea anew, employing however the rather well developed CSL formalism for the description of wave function collapse.

The article is organized as follows. Sections II, III, IV, and V present the noncollapse setting, and the remaining sections invoke the collapse.

Section II discusses the inflationary paradigm and gives estimates of important numerical quantities. Section III obtains the Hamiltonian for the quantum perturbations on the inflaton field and presents the dynamical solutions without collapse. Section IV discusses the the metric perturbation on the Robertson Walker metric known as the

Newtonian potential. The connection between classical gravity and quantum variables is obtained by invoking *semiclassical gravity*, whereby the Newtonian potential is related to a certain quantum expectation value, and this important assumption is discussed. Section V reviews the relation of experimentally observed quantities to the Newtonian potential, and thereby to those quantum expectation values. Section VI then reviews the CSL formalism. Section VII adds the CSL modification to the Hamiltonian evolution of the inflaton perturbations. Sections VIII and IX show how it is possible to obtain agreement with the observations. Section X uses this result to obtain expressions for some physical quantities, and their probabilities of realization. Sections XI and XII discuss our conclusions.

Regarding notation, we will use signature $(-+++)$ for the metric and Wald's convention for the Riemann tensor.

II. INFLATION

The starting point for the discussion of inflation is the action of a scalar field coupled to gravity:

$$S[\phi] = \int d^4x\sqrt{-g}\bigg[\frac{1}{16\pi G}R[g] - \frac{1}{2}\nabla_a\phi\nabla_b\phi g^{ab} - V[\phi]\bigg]. \tag{1}$$

The field equations derived from the action [Eq. (1)] are

$$G^a_b = 8\pi G T^a_b, \tag{2}$$

where T^a_b is give by

$$T^a_b = g^{ac}\partial_c\phi\partial_b\phi + \delta^a_b\bigg(-\frac{1}{2}g^{cd}\partial_c\phi\partial_d\phi - V(\phi)\bigg). \tag{3}$$

Now we address a fundamental issue, how to combine quantum theory and gravitational physics. Despite important advances, it is well known that we are still lacking a fully workable and satisfactory theory of quantum gravity. It is also well known that a quantum gravity approach based on canonical quantization leads to what is, in effect, a timeless theory [16].

On the other hand, time seems not only an important aspect of any discussion of cosmology but, also, time is needed for the CSL dynamical reduction theory which we wish to incorporate. These considerations lead us to use the approach based on semiclassical gravity, where matter fields are treated at the quantum level whereas gravitation (although quantum at the fundamental level) is assumed to be in a regime where it can be treated in a classical manner to a very good approximation [7]. The fact that inflation is thought to occur at energy scales well below the Planck mass lends support to this assumption. (For further discussion on the issue and on the kind of treatment capable of incorporating dynamical reduction in such a context, see Ref. [17].[4]) This approach differs from that followed in standard treatments. There (as here), the starting point is a classical background for both the gravitation and matter fields, but a quantum treatment is used for the perturbations of *both* the metric and the inflaton. In the present work we adopt in principle the strict semiclassical approach described above, limiting the quantum treatment to the perturbation of the inflaton field. This difference should be kept in mind when comparing the standard approach to our treatment using the CSL theory.

We now proceed to describe the basic setting of the problem, starting with characterization of the background metric and inflaton field, followed by the treatment of the corresponding perturbed quantities.

A. The background

The analysis here will be based, as is usual, on separating the metric and scalar field into a spatially homogeneous-isotropic *background* part and a *fluctuation* part. That is, the scalar field is written $\phi = \phi_0 + \delta\phi$, while the metric is written as $g = g_0 + \delta g$, where ϕ_0 is a function of the conformal time[5] η only, and g_0 characterizes the spatially flat Robertson Walker cosmology, i.e.,

$$ds_0^2 = a^2(\eta)[-d\eta^2 + \delta_{ij}dx^i dx^j], \qquad \phi_0(\eta). \tag{4}$$

The background metric and scalar field are treated classically. The scalar field and Friedman evolution equations are

$$\begin{aligned} &\phi_0'' + 2\frac{a'}{a}\phi_0' + a^2\partial_{\phi_0}V[\phi_0] = 0,\\ &3\Big(\frac{a'}{a}\Big)^2 = 4\pi G[\phi_0'^2 + 2a^2V[\phi_0]], \end{aligned} \tag{5}$$

where the prime denotes derivative with respect to the conformal time η, whose range during the inflationary era is *negative*, i.e., $\eta \in (-\infty, -\delta)$ with $\delta > 0$.

The scale factor corresponding to the inflationary era of standard inflationary cosmology which follows from Eq. (5) is, to a good approximation,

$$a(\eta) = \Big(\frac{-1}{H_I\eta}\Big)^{1+\epsilon}, \tag{6}$$

where the Hubble constant (the expansion rate in time t) is

$$H_I \equiv [(8\pi/3)GV[\phi_0]]^{1/2}. \tag{7}$$

[4]In contrast with the standard approach where the quantum-classical cut is tied to the background-perturbation separation, in the treatment developed in Ref. [17], the quantum-classical cut is tied to the separation of gravitation and matter fields. In particular, the so-called zero mode of the inflaton field is treated at the quantum level.

[5]Note that $a(\eta)d\eta = dt$, where t is ordinary time, so if $a(\eta) \sim -1/\eta$, then $a(\eta)$ grows exponentially with t.

Written in terms of $\mathcal{H} \equiv a'(\eta)/a(\eta)$, the so-called slow-roll parameter $\epsilon \equiv 1 - \mathcal{H}'/\mathcal{H}^2$ is considered to be very small: $\epsilon \ll 1$ during the inflationary stage. As for the scalar field ϕ_0, the slow-roll regime corresponds to $\phi_0' = -(a^3/3a')\partial_\phi V$.[6] Also, it is assumed that the change in the scalar potential is so small that the Hubble parameter H_I is essentially constant.

According to the standard scenario, this inflationary era is followed by a brief reheating period in which the inflaton field (including the fluctuation field) decays, populating the universe with ordinary matter (a process that, for simplicity, we take to occur instantaneously). This is immediately followed by the radiation-dominated era, which begins the evolution of standard hot big bang cosmology, leading up to the present cosmological time. While the functional form of $a(\eta)$ changes during these later periods, that is irrelevant for our calculation, which deals solely with the inflationary era at whose end, it is assumed, the fluctuation field provides seeds of structure which are transformed into anisotropies and inhomogeneities in the ordinary matter distribution by the reheating conversion process.

We shall take inflation to start at $\eta = -\mathcal{T}$, assume that the inflationary regime ends at the start of the radiation-dominated era at $\eta = -\tau$, and set $a = 1$ at the "present cosmological time."

The effects of the late time physics, comprising the physical processes occurring between the time of reheating and the time of decoupling (where the matter becomes transparent to the radiation), to the extent that we shall refer to them, will be taken as fully codified in what are called transfer functions (which relate the primordial fluctuations to those fluctuations that are directly observable in the CMB) and which will be, for the most part, ignored in the present work.

The only additional information (beyond the behavior of quantities during the inflationary regime) we need, in order to make the estimates below, is the relationship between the scale factor at the end of inflation, the scale factor at the time of decoupling and the scale factor today.

Of course, the quantities directly observed are those that were present at the time of the decoupling (whose conformal time is η_D) which lies in the matter-dominated era. However, we shall evaluate such quantities at the end of inflation, at $\eta = -\tau$. We can do this because the transfer functions allow us to go backward from $\eta = \eta_D$ to $\eta = -\tau$. For example, the famous acoustic peaks, the most noteworthy feature of the CMB power spectrum, which are related to aspects of plasma physics, can be effectively subtracted out using the transfer functions. The results lead back (as we shall discuss later in detail) to the scale-free Harrison-Zeldovich (H-Z) spectrum at the end of inflation. Therefore, it is the H-Z spectrum for which our calculation must account. Alternatively said, if the transfer functions were constants, we would be directly observing the H-Z spectrum today.

B. Estimates

Now, we need to estimate the values of the conformal time $\eta = -\tau$ at the end of inflation and the conformal time $\eta = -\mathcal{T}$ at which the inflation starts. This may be done as follows.

Recall that the temperature of radiation, regardless of era, scales[7] like $1/a$. The radiation temperature today corresponds to $2.7\,\mathrm{K} = 2.4 \times 10^{-13}$ GeV. We adopt the widespread assumption that the inflation scale, i.e., the effective temperature at the end of inflation, corresponds to the grand unification theory (GUT) scale of about 10^{15} GeV. Therefore, we estimate $a(-\tau) = 2.4 \times 10^{-28}$. Next, we must find the value $-\tau$ corresponding to this scale factor.

Using the GUT inflationary scale for $V \sim (\mathrm{GUT\ scale})^4$, and as we use $c = \hbar = 1$ (so $1\,\mathrm{GeV} \approx 10^{37}\,\mathrm{Mpc}^{-1}$ and $G = M_{\mathrm{Pl}}^{-2}$), we employ Eq. (7) to determine $H_I \approx 3(M_{\mathrm{GUT}}^2/M_{\mathrm{Pl}}) = 3(M_{\mathrm{GUT}}/M_{\mathrm{Pl}})^2 \times 10^{19}$ GeV $= 3 \times 10^{11}$ GeV. Knowing $a(-\tau)$ and H_I, we find $\tau \simeq 10^{-22}$ Mpc from Eq. (6). (The result can be written $\tau \simeq (M_{\mathrm{Pl}}/M_{\mathrm{GUT}})10^{-26}$ Mpc, where M_{Pl} is the Planck mass).

Next, in order to estimate the value of the conformal time at the start of inflation, $\eta = -\mathcal{T}$, we assume that inflation lasts, say, 70 e-folds (usually considered a lower bound for inflation to solve the naturalness problems). Thus $\mathcal{T}/\tau = a(-\tau)/a(-\mathcal{T}) = e^{70} \approx 10^{30}$. Combining this with the previous result $\tau \simeq 10^{-22}$ Mpc. gives $\mathcal{T} = 10^8$ Mpc. Note that this time becomes larger if we require inflation to last more than that number of e-folds.

Also, we shall need an estimate of the comoving wave number $k = |\mathbf{k}|$ for the modes $\mathbf{k}$ which are relevant for the observed CMB data.

These are obtained by noting that the surface of last scattering/decoupling is at $a_{\mathrm{LS}} = 2.7K^o/3000K^o = 1/1100$ and that the comoving radius R of the last scattering sphere is determined by the requirement that the photons that started from a point on that sphere at the time of decoupling are just reaching us today. This gives $R \approx 2/H_0 \approx 6 \times 10^3$ Mpc. The physical radius of that sphere is then $R_{\mathrm{Phys}} = a_{\mathrm{LS}}R \approx 5.5$ Mpc.

[6]This slow-roll stage corresponds mathematically to the "terminal velocity" of a body subject to a constant force in addition to a friction term. Thus, the condition is $\frac{\partial^2\phi}{\partial t^2} = 0$ which corresponds in conformal time to $\phi_0'' = \mathcal{H}\phi_0'$. Using this in the first equation in (5) leads to $\phi_0' = -(a^3/3a')\partial_\phi V$.

[7]For a brief period prior to the end of reheating (which we take here to occur essentially instantaneously at the end of inflation), this relationship between the temperature and the scale factor does not hold, because the radiation and particle content of the universe are replenished at that time as a result of the process of inflaton field decay.

Now, the scales that are relevant range from those corresponding to this radius to, say, 10^{-5} of this value. This corresponds to angular scales of $2\pi \times 10^{-5}$ which is the smallest scale that can be seen with current technology in the CMB. Thus, the relevant modes are those whose physical wavelength $a_{\rm LS} 2\pi/k$ at last scattering roughly lies between $R_{\rm Phys}$ and $10^{-5} R_{\rm Phys}$. This gives $10^{-3}\ {\rm Mpc}^{-1} \lesssim k \lesssim 10^2\ {\rm Mpc}^{-1}$.

We emphasize that this discussion about the values of k is completely independent of the precise functional form of $a(\eta)$ between the end of inflation and the decoupling time: the only relevant data is the relationship between the values of the scale factor at the end of inflation and the scale factor at the decoupling time.

It is clear that the conditions $k\mathcal{T} \gg 1$ and $k\tau \ll 1$, which we shall use to make approximations, hold by very wide margins for the relevant modes.

III. THE FLUCTUATION OF THE INFLATON FIELD

Now we consider the fluctuation of the inflaton field, $\delta\phi(\mathbf{x}, \eta)$. We start with the perturbed action up to second order in the scalar field fluctuation, written in term of the auxiliary field $y \equiv a\delta\phi$,

$$\delta S^{(2)} = \frac{1}{2}\int d\eta d^3x (y'^2 - (\nabla y)^2 + \mathcal{H}^2 y^2 - 2\mathcal{H} y y'). \tag{8}$$

The Lagrangian density is then

$$\delta \mathcal{L}^{(2)} = \frac{1}{2}(y'^2 - (\nabla y)^2 + \mathcal{H}^2 y^2 - 2\mathcal{H} y y'). \tag{9}$$

The canonical momentum π conjugate to y is $\pi \equiv \partial\delta\mathcal{L}^{(2)}/\partial y' = y' - \mathcal{H} y$. Note that $\delta\phi' = (y/a)' = \pi/a$. With $\mathcal{H} \equiv a'(\eta)/a(\eta)$ and $a(\eta) = -1/H_I \eta$, the nonvanishing equal-time Poisson bracket and the Hamiltonian are

$$[y(\mathbf{x}), \pi(\mathbf{x}')] = \delta(\mathbf{x} - \mathbf{x}'),$$
$$\delta\mathcal{H}^{(2)} = \frac{1}{2}\int d\mathbf{x}\left[\pi^2(\mathbf{x}) - \frac{2}{\eta}\pi(\mathbf{x})y(\mathbf{x}) + (\boldsymbol{\nabla} y(\mathbf{x}))^2\right], \tag{10}$$

where here, and in what follows, we suppress the dependence of all variables on η.

We next focus on the individual modes of the field:

$$y(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}}\int d\mathbf{k}\, y(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}},$$
$$\pi(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}}\int d\mathbf{k}\, \pi(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{11}$$

In terms of $y(\mathbf{k})$, $\pi(\mathbf{k})$, which are no longer real, the nonvanishing equal-time Poisson bracket and the Hamiltonian then become

$$[y(\mathbf{k}), \pi(\mathbf{k}')] = \delta(\mathbf{k} - \mathbf{k}'),$$
$$\delta H^{(2)} = \frac{1}{2}\int d\mathbf{k}\Big[\pi(\mathbf{k})\pi^*(\mathbf{k}) - \frac{1}{\eta}[\pi^*(\mathbf{k})y(\mathbf{k}) + \pi(\mathbf{k})y^*(\mathbf{k})] + k^2 y(\mathbf{k})y^*(\mathbf{k})\Big]. \tag{12}$$

In making the transition from this classical description to the quantum description, we replace a c-number real variable α by a Hermitian operator $\hat{\alpha}$, and replace the Poisson bracket by $i^{-1} \times$ (commutator bracket). (We shall work in the Schrödinger picture, so the operators will be time independent, and the state vector, evolving according to Schrödinger's equation, describes the time evolution.) Thus, $\hat{y}(\mathbf{x})^\dagger = \hat{y}(\mathbf{x})$, and therefore $\hat{y}(\mathbf{k})^\dagger = \hat{y}(-\mathbf{k})$. Similarly, $\hat{\pi}(\mathbf{x})^\dagger = \hat{\pi}(\mathbf{x})$, and therefore $\hat{\pi}(\mathbf{k})^\dagger = \hat{\pi}(-\mathbf{k})$. Therefore, because the classical variables describing the modes are not real, the operators $\hat{y}(\mathbf{k})$ and $\hat{\pi}(\mathbf{k})$ are not Hermitian. In order to work with Hermitian operators, we turn to a description in terms of symmetric and antisymmetric fields.

A. Symmetric and antisymmetric fields

Still in the classical theory, we write each field as the sum of symmetric and antisymmetric parts:

$$y(\mathbf{x}) = \frac{1}{2}[y(\mathbf{x}) + y(-\mathbf{x})] + \frac{1}{2}[y(\mathbf{x}) - y(-\mathbf{x})] \equiv y_S(\mathbf{x}) + y_A(\mathbf{x}), \tag{13}$$

and similarly for $\pi(\mathbf{x})$. Putting (13) into (10), and using $\int d\mathbf{x} f_S(\mathbf{x}) g_A(\mathbf{x}) = 0$, Eq. (10) becomes

$$\delta H^{(2)} = \frac{1}{2}\int d\mathbf{x}[\pi_S^2(\mathbf{x}) - \frac{2}{\eta}\pi_S(\mathbf{x})y_S(\mathbf{x}) + (\boldsymbol{\nabla} y_S(\mathbf{x}))^2] + \frac{1}{2}\int d\mathbf{x}[\pi_A^2(\mathbf{x}) - \frac{2}{\eta}\pi_A(\mathbf{x})y_A(\mathbf{x}) + (\boldsymbol{\nabla} y_A(\mathbf{x}))^2], \tag{14}$$

i.e., $\delta H^{(2)} = \delta H_S^{(2)} + \delta H_A^{(2)}$.

The equal-time Poisson brackets of the symmetric and antisymmetric fields follow from (10):

$$[y_S(\mathbf{x}), \pi_S(\mathbf{x}')] = \frac{1}{2}[\delta(\mathbf{x} - \mathbf{x}') + \delta(\mathbf{x} + \mathbf{x}')],$$
$$[y_A(\mathbf{x}), \pi_A(\mathbf{x}')] = \frac{1}{2}[\delta(\mathbf{x} - \mathbf{x}') - \delta(\mathbf{x} + \mathbf{x}')], \tag{15}$$

and $[y_S(\mathbf{x}), \pi_A(\mathbf{x}')] = [y_A(\mathbf{x}), \pi_S(\mathbf{x}')] = 0$.

B. Modes, symmetric case

Now, we consider the individual modes of the field,

$$y_S(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} y_S(\mathbf{k}),$$
$$\pi_S(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} \pi_S(\mathbf{k}). \tag{16}$$

Because $y_S(\mathbf{x})$, $\pi_S(\mathbf{x})$ are symmetric, we get $y_S(-\mathbf{k}) = y_S(\mathbf{k})$, $\pi_S(-\mathbf{k}) = \pi_S(\mathbf{k})$. Because $y_S(\mathbf{x})$, $\pi_S(\mathbf{x})$ are real, we get $y_S^*(\mathbf{k}) = y_S(-\mathbf{k})$, $\pi_S^*(\mathbf{k}) = \pi_S(-\mathbf{k})$. So, $y_S^*(\mathbf{k}) = y_S(\mathbf{k})$, $\pi_S^*(\mathbf{k}) = \pi_S(\mathbf{k})$: these classical variables are real. Therefore, the nonvanishing Poisson bracket is, using (15),

$$\begin{aligned}[y_S(\mathbf{k}), \pi_S(\mathbf{k}')] &= [y_S(\mathbf{k}), \pi_S^*(\mathbf{k}')] \\ &= \frac{1}{(2\pi)^3} \int d\mathbf{x} \int d\mathbf{x}' e^{-i\mathbf{k}\cdot\mathbf{x}} e^{i\mathbf{k}'\cdot\mathbf{x}'} [y_S(\mathbf{x}), \pi_S(\mathbf{x}')] \\ &= \frac{1}{2}[\delta(\mathbf{k}-\mathbf{k}') + \delta(\mathbf{k}+\mathbf{k}')]. \end{aligned} \tag{17}$$

In evaluating the Hamiltonian, the first term is

$$\begin{aligned}\frac{1}{2}\int d\mathbf{x} \pi_S^2(\mathbf{x}) &= \frac{1}{2}\int d\mathbf{x} \pi_S(\mathbf{x})\pi_S^*(\mathbf{x}) \\ &= \frac{1}{2(2\pi)^3}\int d\mathbf{x}\int d\mathbf{k}\int d\mathbf{k}' e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k}'\cdot\mathbf{x}} \pi_S(\mathbf{k})\pi_S(\mathbf{k}') \\ &= \frac{1}{2}\int d\mathbf{k} \pi_S^2(\mathbf{k}) \\ &= \int_+ d\mathbf{k} \pi_S^2(\mathbf{k}). \end{aligned} \tag{18}$$

In the last step, the integral over all $\mathbf{k}$ is converted to the integral over the upper half $\mathbf{k}$-plane, since the lower half $\mathbf{k}$-plane makes the identical contribution. Similar steps are to be taken for the other terms in the Hamiltonian, which then becomes

$$\delta H_S^{(2)} = \int_+ d\mathbf{k}\left[\pi_S^2(\mathbf{k}) - \frac{2}{\eta}\pi_S(\mathbf{k})y_S(\mathbf{k}) + k^2 y_S^2(\mathbf{k})\right]. \tag{19}$$

Finally, in limiting to the upper half $\mathbf{k}$-plane, we note that the $\delta(\mathbf{k}+\mathbf{k}')$ term in the Poisson bracket is not used in calculating the equations of motion, so the Poisson bracket is effectively just the first term in (17). However, there is a factor of $1/2$ in the Poisson bracket which makes this not quite canonical. So, we shall define new variables:

$$X_S(\mathbf{k}) \equiv \sqrt{2} y_S(\mathbf{k}), \qquad P_S(\mathbf{k}) \equiv \sqrt{2}\pi_S(\mathbf{k}). \tag{20}$$

In terms of these variables, the equal time Poisson bracket and Hamiltonian are

$$[X_S(\mathbf{k}), P_S(\mathbf{k}')] = \delta(\mathbf{k}-\mathbf{k}'),$$
$$\delta H_S^{(2)} = \frac{1}{2}\int_+ d\mathbf{k}\left[P_S^2(\mathbf{k}) - \frac{2}{\eta}P_S(\mathbf{k})X_S(\mathbf{k}) + k^2 X_S^2(\mathbf{k})\right]. \tag{21}$$

Lastly, we proceed to quantize, so Eq. (21) becomes

$$[\hat{X}_S(\mathbf{k}), \hat{P}_S(\mathbf{k}')] = i\delta(\mathbf{k}-\mathbf{k}'),$$
$$\delta H_S^{(2)} = \frac{1}{2}\int_+ d\mathbf{k}\left[\hat{P}_S^2(\mathbf{k}) - \frac{1}{\eta}[\hat{P}_S(\mathbf{k})\hat{X}_S(\mathbf{k}) + \hat{X}_S(\mathbf{k})\hat{P}_S(\mathbf{k})] + k^2\hat{X}_S^2(\mathbf{k})\right]. \tag{22}$$

Again, we emphasize that, while the classical variables are functions of conformal time η, we are choosing to work in the Schrödinger picture. Thus, the operators are η independent (except for the Hamiltonian where the η dependence is explicit): the η dependence is relegated to the behavior of the state vector.

C. Modes, antisymmetric case

Here we proceed in a manner exactly parallel to that used in the previous case.

We start with

$$y_A(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}}\int d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} i y_A(\mathbf{k}),$$
$$\pi_A(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}}\int d\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}} i\pi_A(\mathbf{k}). \tag{23}$$

However, as $y_A(\mathbf{x})$, $\pi_A(\mathbf{x})$ are antisymmetric, we get $y_A(-\mathbf{k}) = -y_A(\mathbf{k})$, $\pi_A(-\mathbf{k}) = -\pi_A(\mathbf{k})$. Because $y_A(\mathbf{x})$, $\pi_A(\mathbf{x})$ are real, we get $y_A^*(\mathbf{k}) = -y_A(-\mathbf{k})$, $\pi_A^*(\mathbf{k}) = -\pi_A(-\mathbf{k})$. So, $y_A^*(\mathbf{k}) = y_A(\mathbf{k})$, $\pi_A^*(\mathbf{k}) = \pi_A(\mathbf{k})$: the i factors in the definitions of $y_A(\mathbf{k})$, $\pi_A(\mathbf{k})$ were chosen to make these real so that they become Hermitian operators in the transition to quantum theory. So, although $\pi(\mathbf{k})$ is not Hermitian, it has been expressed in terms of Hermitian operators:

$$\pi(\mathbf{k}) = \pi_S(\mathbf{k}) + i\pi_A(\mathbf{k}). \tag{24}$$

The nonvanishing equal time Poisson bracket is, using (15),

$$\begin{aligned}[y_A(\mathbf{k}), \pi_A(\mathbf{k}')] &= [y_A(\mathbf{k}), \pi_A^*(\mathbf{k}')] \\ &= \frac{1}{(2\pi)^3}\int d\mathbf{x}\int d\mathbf{x}' e^{-i\mathbf{k}\cdot\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{x}'}[y_A(\mathbf{x}), \pi_A(\mathbf{x}')] \\ &= \frac{1}{2}[\delta(\mathbf{k}-\mathbf{k}') - \delta(\mathbf{k}+\mathbf{k}')]. \end{aligned} \tag{25}$$

The rest of the argument goes through just as for the symmetric case. The only difference is that the sign of $\delta(\mathbf{k}+\mathbf{k}')$ is positive for the symmetric case, Eq. (17), but negative for the antisymmetric case, Eq. (25). However, when we limit to the upper half $\mathbf{k}$-plane, $\delta(\mathbf{k}+\mathbf{k}')$ plays no role. Therefore, after quantization, we obtain the antisymmetric version of Eq. (22):

$$[\hat{X}_A(\mathbf{k}), \hat{P}_A(\mathbf{k}')] = i\delta(\mathbf{k}-\mathbf{k}'),$$
$$\delta H_A^{(2)} = \frac{1}{2}\int_+ d\mathbf{k}\Big[\hat{P}_A^2(\mathbf{k}) - \frac{1}{\eta}[\hat{P}_A(\mathbf{k})\hat{X}_A(\mathbf{k}) + \hat{X}_S(\mathbf{k})\hat{P}_A(\mathbf{k})] + k^2\hat{X}_A^2(\mathbf{k})\Big]. \tag{26}$$

In this way, the field has become a simple collection of independent modified-harmonic oscillators, with each mode evolving independently. If we define

$$\hat{X} \equiv \sqrt{d\mathbf{k}}\hat{X}_\alpha(\mathbf{k}), \qquad \hat{P} \equiv \sqrt{d\mathbf{k}}\hat{P}_\alpha(\mathbf{k}), \tag{27}$$

with indices $\alpha = S, A, \mathbf{k}$ suppressed, the commutation relations and Hamiltonian for a mode characterized by $k \equiv |\mathbf{k}|$ is

$$[\hat{X}, \hat{P}] = i,$$
$$\hat{H}_k = \frac{1}{2}\Big[\hat{P}^2 - \frac{1}{\eta}[\hat{P}\hat{X} + \hat{X}\hat{P}] + k^2\hat{X}^2\Big]. \tag{28}$$

D. Noncollapse theory: Expectation values

We emphasize that, in our treatment, the Hamiltonian $\hat{H}_k$ is not the sole cause of the dynamics. We have yet to incorporate the CSL modification of quantum theory (Sec. VI and beyond). Nonetheless, it is interesting, and shall prove useful, to calculate expectation values due to the collapse-free dynamics characterized by H_k alone.

This is easy to do. For any operator $\hat{A}$, $\langle\psi, t|\hat{A}|\psi, t\rangle$ satisfies

$$\frac{d}{d\eta}\langle\psi, \eta|\hat{A}|\psi, \eta\rangle = -i\langle\psi, \eta|[\hat{A}, \hat{H}_k]|\psi, \eta\rangle. \tag{29}$$

Because $\hat{H}_k$ is quadratic, the set of expectation values of any power of operators forms a closed set which can be solved. The initial condition is that the mode wave function is in the Bunch-Davies vacuum, which is just the harmonic oscillator ground state, at the initial time $\eta = -\mathcal{T}$:

$$\langle p|\psi, -\mathcal{T}\rangle = \frac{1}{(\pi k)^{1/4}}e^{-p^2/2k},$$
$$\langle x|\psi, -\mathcal{T}\rangle = (\pi/k)^{1/4}e^{-x^2k/2}. \tag{30}$$

(We emphasize that this initial condition is specified at a fixed time and not, as often done, in the limit as $\eta\to-\infty$.)

Writing $\langle\hat{A}\rangle \equiv \langle\psi, \eta|\hat{A}|\psi, \eta\rangle$, the first order equations, the consequent equations of motion and their solutions are

$$\frac{d}{d\eta}\langle\hat{X}\rangle = \langle\hat{P}\rangle - \frac{\langle\hat{X}\rangle}{\eta}, \qquad \frac{d}{d\eta}\langle\hat{P}\rangle = -k^2\langle\hat{X}\rangle + \frac{\langle\hat{P}\rangle}{\eta}, \tag{31a}$$
$$\frac{d^2}{d\eta^2}\langle\hat{X}\rangle = -\Big[k^2 - \frac{2}{\eta^2}\Big]\langle\hat{X}\rangle, \qquad \frac{d^2}{d\eta^2}\langle\hat{P}\rangle = -k^2\langle\hat{P}\rangle, \tag{31b}$$
$$\langle\hat{X}\rangle = C_1\frac{-i}{k}e^{ik\eta}\Big[1+\frac{i}{k\eta}\Big] + C_2\frac{i}{k}e^{-ik\eta}\Big[1-\frac{i}{k\eta}\Big], \qquad \langle\hat{P}\rangle = C_1e^{ik\eta} + C_2e^{-ik\eta}. \tag{31c}$$

(It is notable that $\langle\hat{P}\rangle$ has the usual harmonic oscillator solution, even though the Hamiltonian is not the usual harmonic oscillator Hamiltonian.) From the initial conditions [Eq. (30)], we see that $\langle\hat{X}\rangle$ and $\langle\hat{P}\rangle$ vanish initially, $C_1 = C_2 = 0$, so for all η,

$$\langle\hat{X}\rangle = \langle\hat{P}\rangle = 0. \tag{32}$$

The second order equations, with $Q \equiv \langle\hat{X}^2\rangle$, $R \equiv \langle\hat{P}^2\rangle$, $S \equiv \langle[\widehat{XP} + \widehat{PX}]\rangle$, are

$$\frac{d}{d\eta}Q = S - \frac{2Q}{\eta}, \qquad \frac{d}{d\eta}R = -k^2S + \frac{2R}{\eta}, \qquad \frac{d}{d\eta}S = 2[R - k^2Q]. \tag{33}$$

Because the algebra of the commutator brackets is identical to that of the Poisson brackets, and for the classical variables the product of two solutions is the solution for the product, the same is true for the solutions of Eq. (33). They are the product of the solutions [Eq. (31c)]:

$$Q = -C_1\frac{1}{k^2}e^{2ik\eta}\Big[1+\frac{i}{k\eta}\Big]^2 - C_2\frac{1}{k^2}e^{-2ik\eta}\Big[1-\frac{i}{k\eta}\Big]^2 + C_3\frac{1}{k^2}\Big[1+\frac{1}{(k\eta)^2},\Big], \tag{34a}$$
$$R = C_1e^{2ik\eta} + C_2e^{-2ik\eta} + C_3, \tag{34b}$$
$$S = -2iC_1\frac{1}{k}e^{2ik\eta}\Big[1+\frac{i}{k\eta}\Big] + 2iC_2\frac{1}{k}e^{-2ik\eta}\Big[1-\frac{i}{k\eta}\Big] + C_3\frac{2}{k^2\eta}. \tag{34c}$$

The initial conditions are

$$Q(-\mathcal{T}) = 1/2k, \qquad R(-\mathcal{T}) = k/2, \qquad S(-\mathcal{T}) = 0. \tag{35}$$

Assuming $k\mathcal{T} \ll 1$, we obtain $C_1 = -C_2 = 0$, $C_3 = k/2$ and so

$$Q = \langle \hat{X}^2 \rangle = \frac{1}{2k}\left[1 + \frac{1}{(k\eta)^2},\right], \qquad R = \langle \hat{P}^2 \rangle = \frac{k}{2},$$
$$S = \langle [\widehat{XP} + \widehat{PX}] \rangle = \frac{1}{k\eta}. \tag{36}$$

It may be noted in passing, as a consequence of Eq. (36), that an alternative choice of canonically conjugate variables exhibits squeezing:

$$\left\langle \left[\sqrt{\frac{k}{2}}\hat{X} \pm \sqrt{\frac{1}{2k}}\hat{P}\right]^2 \right\rangle = \frac{1}{4}\left[1 \pm \frac{1}{k\eta}\right]^2 + \frac{1}{4},$$

a behavior noted [9,10,12] as characteristic of the evolution of the cosmological quantum fluctuations. Here, both variables initially [$\eta = -\mathcal{T}$, with $(k\mathcal{T})^{-1}$ considered negligibly small] have the usual harmonic oscillator ground state minimum uncertainty in position and momentum but, for a range of η, the uncertainty of one of them decreases below that value, achieving a minimum at $k\eta = -1$.[8]

It shall be seen that the result $\langle \hat{P}^2 \rangle = k/2$ which, together with Eqs. (24), (20), (27), and (32) *et. seq.* implies

$$\begin{aligned}
\langle \hat{\pi}(\mathbf{k})\hat{\pi}(\mathbf{k}')^* \rangle &= \langle (\hat{\pi}_S(\mathbf{k}) + i\hat{\pi}_A(\mathbf{k}))(\hat{\pi}_S(\mathbf{k}') - i\hat{\pi}_A(\mathbf{k}')) \rangle \\
&= \langle \hat{\pi}_S(\mathbf{k})\hat{\pi}_S(\mathbf{k}') \rangle + \langle \hat{\pi}_A(\mathbf{k})\hat{\pi}_A(\mathbf{k}') \rangle \\
&= 2\left\langle \left[\frac{\hat{P}}{\sqrt{2d\mathbf{k}}}\right]^2 \right\rangle \delta_{\mathbf{k}\mathbf{k}'} \\
&= \langle \hat{P}^2 \rangle \delta(\mathbf{k} - \mathbf{k}') \\
&= \frac{k}{2}\delta(\mathbf{k} - \mathbf{k}'),
\end{aligned} \tag{37}$$

in the usual treatment is what is cited as causing agreement between the theory of inflaton perturbations and the effectively observed H-Z spectrum of temperature fluctuations.

IV. THE FLUCTUATION OF THE METRIC: NEWTONIAN POTENTIAL

In order to connect with observations, we need to study the metric perturbation Ψ known as the Newtonian potential at the end of the inflationary period. As we have mentioned, immediately following that time, the universe entered into the brief so-called "reheating period," when the inflaton field is supposed to have decayed, in a not completely understood manner, into the matter content of the present universe: dark matter, baryons, electrons, photons, etc. As we shall see, the Newtonian potential is related to the fluctuation (inhomogeneity) in the inflaton field, so that it ends up being tied to the inhomogeneities of the matter created during reheating. During reheating, it is supposed that the Newtonian potential did not change in any appreciable manner from its value at the end of inflation. Thus, this primordial Newtonian potential can be used to determine the ensuing evolution of the matter inhomogeneities, entailing well-known physics such as baryon acoustic oscillations, and other processes described by the so-called transfer functions.

Eventually, the universe evolved to what is variously called the *time of last scattering* or the *time of decoupling* of the photons from the plasma, or the *time of recombination* of electrons and protons, when the atoms formed and universe suddenly became transparent to the radiation which we now detect. At that time, the universe is considered to have been in local thermal equilibrium. The quantity that we presently measure is the temperature variation as a function of coordinates on the celestial sphere, $\Delta T(\theta, \varphi)/\bar{T}$ [$\Delta T(\theta, \varphi) \equiv T(\theta, \varphi) - \bar{T}$, where $\bar{T}$ is the mean temperature over the sky]. This temperature variation is due to the inhomogeneities in the matter density distribution at the time of last scattering.

The satellites measure the temperature by detecting the blackbody microwave radiation at a number of frequencies. If ν is the frequency at the peak of the spectrum, we have the relations

$$\frac{\Delta T}{\bar{T}} = \frac{\delta\nu}{\nu} \approx \frac{1}{3}\Psi, \tag{38}$$

where the last step uses the transfer functions to strip away the physics ensuing between the end of reheating and the time of decoupling, effectively considering that we are directly observing Ψ. The effect of Ψ is, first, a gravitational red- or blueshift. Second there is an effect on the rate of expansion of the universe whose combined (Sachs-Wolfe) effect gives the $1/3$ factor.[9]

The Newtonian potential is related to the fluctuation of the inflaton field as follows. When the perturbation of the Robertson Walker space-time is taken into account, with the appropriate choice of gauge (conformal Newton gauge), and ignoring the vector and tensor part of the metric perturbations, Eq. (4) is replaced by

$$ds^2 = a(\eta)^2[-(1 + 2\Phi)d\eta^2 + (1 - 2\Psi)\delta_{ij}dx^i dx^j],$$

where Φ and Ψ are functions of the space-time coordinates η, x^i.

Next, consider Einstein's equations to first order in the perturbations. The expression for the energy-momentum tensor T^a_b for the inflaton field is Eq. (3). Its linear perturbation components are

[8]The minima at $k\eta = +1$ would correspond to $\eta > 0$ which is not physical.

[9]The red-/blueshift contribution is $\frac{\delta\nu}{\nu} = \frac{\delta\sqrt{g_{00}^{(e)}}}{\sqrt{g[(o)_{00}}} \approx \Psi$ where $g_{00}^{(e,o)}$ represent the "time-time" metric components at the events of emission and observation, respectively. For details about the $-2/3$ contribution, see Ref. [18], p. 139, or Ref. [19].

$$\delta T^0_0 = a^{-2}[\phi_0'^2\Phi - \phi_0'\delta\phi' - \partial_\phi V a^2 \delta\phi],$$
$$\delta T^0_i = \partial_i(-a^{-2}\phi_0'\delta\phi), \tag{39}$$
$$\delta T^i_j = a^{-2}[\phi_0'\delta\phi' - \phi_0'^2\Phi - \partial_\phi V a^2\delta\phi]\delta^i_j.$$

Then, Einstein's equations to first order, $\delta G_{\mu\nu} = 8\pi G\delta T_{\mu\nu}$, lead to $\Psi = \Phi$ and

$$\nabla^2\Psi + \mu\Psi = 4\pi G(\omega\delta\phi + \phi_0'\delta\phi'), \tag{40}$$

where $\mu \equiv \mathcal{H}^2 - \mathcal{H}'$ and $\omega \equiv 3\mathcal{H}\phi_0' + a^2\partial_\phi V$.

As discussed following Eq. (6), the slow-roll approximation corresponds to $\omega = 0$. We are ignoring here terms of order ϵ, which implies $\mu = 0$. Thus Eq. (40) and its Fourier transform become

$$\begin{aligned}\nabla^2\Psi(\eta, \mathbf{x}) &= 4\pi G\phi_0'(\eta)\delta\phi'(\eta, \mathbf{x})\\ &= \frac{4\pi G\phi_0'(\eta)}{a}\pi(\eta, \mathbf{x}),\\ -k^2\Psi(\eta, \mathbf{k}) &= 4\pi G\phi_0'(\eta)\delta\phi'(\eta, \mathbf{k})\\ &= \frac{4\pi G\phi_0'(\eta)}{a}\pi(\eta, \mathbf{k}).\end{aligned} \tag{41}$$

Now we make the transition to quantum theory. The usual procedure is to quantize both sides of Eq. (41), so that $\hat\Psi(\mathbf{k}) \sim \hat\pi(\mathbf{k})$. But then, as a consequence of Eqs. (41) and (32), $\langle\hat\Psi(\mathbf{k})\rangle \sim \langle\hat\pi(\mathbf{k})\rangle = 0$. Therefore, $\langle\psi, \eta|\hat\Psi(\mathbf{x})|\psi, \eta\rangle = 0$.

How is this to be interpreted? One would like to identify the expectation value of the Newtonian potential operator with the value of the Newtonian potential in nature. However, for a state $|\psi, \eta\rangle$ representing nature, the expectation value should vary with position. Therefore, the state $|\psi, \eta\rangle$ does not represent nature. It may then be considered as a superposition of possible states of nature, but there is no guideline how to determine the states in the superposition.

Moreover, in the usual approach, $\langle\psi, \eta_D|\hat\Psi(\mathbf{x})\times\hat\Psi(\mathbf{x}')|\psi, \eta_D\rangle$ (η_D represent the conformal time of decoupling) can be readily shown to be rotationally invariant, a function of $\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'$. Therefore, again, $|\psi, \eta\rangle$ does not represent the state of our universe, but at best is some superposition of possible states.

So, as we have emphasized, the homogeneous and isotropic initial state and dynamics does not explain the observed inhomogeneity and anisotropy.

Therefore, as previously mentioned, following Refs. [7,15] we take a different approach, utilizing the semiclassical description of gravitation [20,21], where gravity is treated classically while other fields are treated in the standard quantum field theory (in curved space-time) fashion. The classical gravity and the quantum fields are thus related by

$$G_{ab} = 8\pi G\langle\hat T_{ab}\rangle. \tag{42}$$

There is an immediate objection to semiclassical gravity. Suppose a quantum experiment is performed with two possible macroscopic outcomes, a large object being put in one or another place. Using the Schrödinger equation to describe this, including the apparatus, the resulting state vector describes a superposition of these two outcomes. Then, $\langle\hat T_{ab}\rangle$ is large in two places and so, according to semiclassical gravity, the gravitational field acts as if there were sources in two places. Such an experiment was actually performed [22] and, as expected, the gravitational field (as measured by a Cavendish balance) was that due to only a source in one place. However, this objection no longer obtains if the Schrödinger equation is modified, *à la* CSL, to include collapse, since then the state vector rapidly ends up describing the object in one place only.[10]

However, the resolution of this problem by invoking collapse brings on another problem. As is well known, introducing CSL dynamical collapse violates the conservation of energy, so the divergence of the energy-momentum tensor does not vanish. In the current epoch this energy nonconservation is quite small, and in the present application it is also small compared to the retained terms, so it may practically be neglected. However, from a fundamental point of view, if the divergence of the energy-momentum tensor does not vanish, and it is equated to the Einstein tensor, then of course this conflicts with the vanishing divergence of the latter.

Here we shall take the view that Einstein's equations are an emergent, approximate description of the collective behavior of the fundamental degrees of freedom of quantum space-time, and as such those equations will not hold under all circumstances. It has been argued in Ref. [17] that this should be considered in analogy with the breakdown of the Navier-Stokes characterization of a fluid. This can be expected to occur, not only for phenomena at scales smaller than the mean intermolecular distance of the fluid constituents, but also when there are important energy fluxes between the micro- and macroscopic degrees of freedom, such as when a part of the fluid undergoes a phase transition.

In the same manner, the collapse should be thought of as accompanied by a backreaction in the fundamental quantum gravity degrees of freedom, which are not fully represented in the metric characterization. Then, a more precise description would include a compensating term appearing in the Einstein's equation. Another approach, discussed for instance in Ref. [23], involves assigning energy and momentum to the stochastic field driving the CSL dynamics such that the stress tensor then does have vanishing divergence. We shall not explore this issue any further in the present work.

[10]For more discussion about the applicability of semiclassical gravity to the problem at hand, see Ref. [17] where the first steps of a formalism capable of incorporating collapse of the wave function at the semiclassical level was developed.

It follows from Eqs. (41) and (42) that

$$-k^2\Psi(\eta,\mathbf{k}) = 4\pi G\phi_0'(\eta)\langle\hat{\delta}\phi'(\mathbf{k},\eta)\rangle = \frac{4\pi G\phi_0'(\eta)}{a}\langle\hat{\pi}(\mathbf{k},\eta)\rangle \tag{43}$$

($\langle\hat{\pi}(\mathbf{k},\eta)\rangle \equiv \langle\psi,\eta|\hat{\pi}(\mathbf{k})|\psi,\eta\rangle$). When inflation starts at time $\eta = -\mathcal{T}$, it is supposed that the state is described by the Bunch-Davies vacuum, so $\langle\psi,-\mathcal{T}|\hat{\pi}(\mathbf{k})|\psi,-\mathcal{T}\rangle = 0$, and the space-time is homogeneous and isotropic. While this would remain the case were there only Hamiltonian dynamics, the addition of CSL dynamics causes $\langle\hat{\pi}(\mathbf{k},\eta)\rangle \neq 0$ thereafter. The CSL Hamiltonian depends upon a classical random function of time $w(t)$ [more precisely, one $w(t)$ for each momentum mode]. Each set of such $w(t)$'s gives rise to a different possible inhomogeneous and anisotropic universe.

V. THE OBSERVATIONAL QUANTITIES

The quantity that is measured is $\Delta T(\theta,\varphi)/\bar{T}$, which is a function of the coordinates on the celestial two-sphere. This data is expressed in terms of spherical harmonics as

$$\frac{\Delta T(\theta,\varphi)}{\bar{T}} = \sum_{lm}\alpha_{lm}Y_{lm}(\theta,\varphi), \qquad \alpha_{lm} = \int d^2\Omega\,\frac{\Delta T(\theta,\varphi)}{\bar{T}}Y^*_{lm}(\theta,\varphi). \tag{44}$$

We emphasize, as already discussed, that we are factoring out the late time physics, so "observations" means what would be observed if the transfer functions were constants.

The experimental results are usually expressed in terms of the quantity

$$C_l = \frac{1}{2l+1}\sum_m|\alpha_{lm}|^2. \tag{45}$$

Then, the "observation" is that the quantity

$$OB_l \equiv l(l+1)(2l+1)^{-1}\sum_m|\alpha_{lm}|^2 = l(l+1)C_l \tag{46}$$

is essentially independent of l. This "scale invariant" (the reason for the name and the arcane dependence on l shall be given subsequently), or "Harrison-Zel'dovich" spectrum, is what must be accounted for by the theory.

Our approach produces explicit expressions for the quantities that are most directly extracted from the data. Using Eqs. (38) and (43), we obtain

$$\frac{\Delta T(\theta,\varphi)}{\bar{T}} = c\int d^3k e^{i\mathbf{k}\cdot\mathbf{x}}\frac{1}{k^2}\langle\hat{\pi}(\mathbf{k},\eta)\rangle, \quad \text{where } c \equiv -\frac{4\pi G\phi_0'(\eta)}{3a} \text{ and } \eta = -\tau. \tag{47}$$

Here, $\mathbf{x}$ is the coordinate of the point on the intersection of our past light cone with what will eventually become the last scattering surface in the direction on the sky specified by θ, φ. Then, according to Eq. (44),

$$\alpha_{lm} = c\int d^2\Omega Y^*_{lm}(\theta,\varphi)\int d^3k e^{i\mathbf{k}\cdot\mathbf{x}}\frac{1}{k^2}\langle\hat{\pi}(\mathbf{k},\eta)\rangle. \tag{48}$$

Thus, α_{lm} depends upon $\langle\hat{\pi}(\mathbf{k},\eta)\rangle$, a well-defined quantity in our treatment, which has a stochastic dependence, i.e., it depends upon a random function. (This differs from the standard treatment, where no comparable expression can be given.) The stochasticity occurs because the collapse theory gives an ensemble of possible universes (one for each possible random function, only one of which is actually realized) and the associated probabilities of realization. We shall see that there is not a large deviation from the mean, so we may consider, according to this theory, that our universe is typical.

Continuing, it follows from Eq. (48) and the well-known expansion $e^{i\mathbf{k}\cdot\mathbf{x}} = 4\pi\sum_{l,m}i^l j_l(kr)Y_{lm}(\theta,\varphi)Y^*_{lm}(\hat{k})$ that

$$\alpha_{lm} = i^l 4\pi c\int d^3k j_l(kR_D)Y^*_{lm}(\hat{k})\frac{1}{k^2}\langle\hat{\pi}(\mathbf{k},\eta)\rangle. \tag{49}$$

Here, R_D is the comoving radius of the last scattering sphere, so $\mathbf{x} = \mathbf{R_D}(\sin(\theta)\sin(\varphi), \sin(\theta)\cos(\varphi), \cos(\theta))$, and $\hat{k}$ is the unit vector in the direction $\mathbf{k} = k\hat{k}$. Therefore,

$$|\alpha_{lm}|^2 = (4\pi c)^2\int d^3k d^3k' j_l(kR_D)j_l(k'R_D)Y_{lm}(\hat{k})Y^*_{lm}(\hat{k}') \times\frac{1}{k^2k'^2}(\langle\hat{\pi}(\mathbf{k},\eta)\rangle\langle\hat{\pi}(\mathbf{k}',\eta)\rangle^*). \tag{50}$$

We may consider that the average over the ensemble of possible universes fairly reflects the value expected to be obtained in our own universe. As shall be seen, the form of our expression for the ensemble average at the end of the inflationary period is $\overline{(\langle\hat{\pi}(\mathbf{k},\eta)\rangle\langle\hat{\pi}(\mathbf{k}',\eta)\rangle^*)} = f(k)\delta(\mathbf{k}-\mathbf{k}')$. Then,

$$\overline{|\alpha_{lm}|^2} = (4\pi c)^2\int d^3k j_l^2(kR_D)|Y_{lm}(\hat{k})|^2\frac{1}{k^4}f(k) = (4\pi c)^2\int_0^\infty dk j_l(kR_D)^2\frac{1}{k^2}f(k). \tag{51}$$

Now, we note that if $f(k) = \alpha k$ where α is a constant, the result becomes independent of R_D. That is what is referred to as a scale invariant spectrum. In that case,

$$\overline{|\alpha_{lm}|^2} = (4\pi c)^2\alpha\int_0^\infty dx j_l^2(x)\frac{1}{x} = (4\pi c)^2\alpha\frac{1}{2l(l+1)}. \tag{52}$$

Therefore, our estimate for the quantity that is usually the focus of the analysis is

$$C_l^{\rm th} = \frac{1}{2l+1}\sum_{m=-l}^{l}\overline{|\alpha_{lm}|^2} = (4\pi c)^2\alpha\frac{1}{2l(l+1)}. \tag{53}$$

Thus we have obtained the result that $l(l+1)C_l$ is constant, independent of l, in agreement with "observation," as mentioned following Eq. (46). However, as we have seen, that only occurs if

$$\overline{\langle\hat{\pi}(k)\rangle^2} \sim k. \tag{54}$$

So, we now turn to add CSL dynamics to the dynamics discussed in Sec. III D [which is governed by the Hamiltonian $\hat{H}_k$, Eq. (28)], to see whether, under the combined dynamics, the ensemble of possible universes can satisfy [Eq. (54)]. That requires $\overline{\langle\hat{P}\rangle^2} \sim k$.

VI. CSL

We shall be using just the simplest form of CSL, which describes collapse toward one or another eigenstates of an operator $\hat{A}$ with rate $\sim\lambda$.

As we have seen, the relevant operators on which we should focus our attention are the $\hat{\pi}(\mathbf{k}, \eta) \sim \hat{P}$, and their expectation values $\langle\hat{\pi}(\mathbf{k}, \eta)\rangle \sim \langle\hat{P}\rangle \neq 0$. We may call $\hat{P}$ our "focus" operator to differentiate it from the "collapse generating" operator $\hat{A}$. (Note that we could choose $\hat{A} = \hat{P}$, or make another choice for $\hat{A}$.)

There are two equations we must consider.

The first is a modified Schrödinger equation, whose solution is

$$|\psi, t\rangle = \mathcal{T} e^{-\int_0^t dt'[i\hat{H} + \frac{1}{4\lambda}[w(t') - 2\lambda\hat{A}]^2]}|\psi, 0\rangle. \tag{55}$$

($\mathcal{T}$ is the time-ordering operator.) $w(t)$ is a random classical function of time, of white noise type, whose probability is given by the second equation, the probability rule:

$$PDw(t) \equiv \langle\psi, t|\psi, t\rangle \prod_{t_i=0}^{t} \frac{dw(t_i)}{\sqrt{2\pi\lambda/dt}}. \tag{56}$$

The state vector norm evolves dynamically (does *not* equal 1), so Eq. (56) says that the state vectors with largest norm are most probable. That the total probability is 1 can be seen from

$$\begin{aligned}
\int PDw(t) &= \int Dw(t-dt) \int_{-\infty}^{\infty} \frac{dw(t)}{\sqrt{2\pi\lambda/dt}} \langle\psi, t-dt|e^{-dt'[-i\hat{H} + \frac{1}{4\lambda}[w(t') - 2\lambda\hat{A}]^2]} e^{-dt'[i\hat{H} + \frac{1}{4\lambda}[w(t') - 2\lambda\hat{A}]^2]}|\psi, t-dt\rangle \\
&= \int Dw(t-dt) \int_{-\infty}^{\infty} \frac{dw(t)}{\sqrt{2\pi\lambda/dt}} \langle\psi, t-dt|e^{-\frac{dt'}{2\lambda}[w(t') - 2\lambda\hat{A}]^2}|\psi, t-dt\rangle \\
&= \int Dw(t-dt)\langle\psi, t-dt|\psi, t-dt\rangle = \cdots = \langle\psi, 0|\psi, 0\rangle = 1.
\end{aligned} \tag{57}$$

To see how the dynamics collapses to eigenstates $|a_n\rangle$ of $\hat{A}$ (assuming $\hat{H} = 0$), write $|\psi, 0\rangle = \sum_{n=1}^{N} c_n|a_n\rangle$ so, according to Eqs. (55) and (56),

$$|\psi, t\rangle = e^{-\frac{1}{4\lambda}\int_0^t w^2(t')} \sum_{n=1}^{N} c_n|a_n\rangle e^{B(t)a_n - \lambda t a_n^2}, \qquad P = e^{-\frac{1}{2\lambda}\int_0^t w^2(t')} \sum_{n=1}^{N} |c_n^2| e^{2B(t)a_n - 2\lambda t a_n^2}, \tag{58}$$

where $B(t) \equiv \int_0^t dt' w(t')$. Writing $w(t_i) = B(t_i + dt) - B(t_i)$, so $\prod dw(t_i) = \prod dB(t_i)$, we can integrate P over all $B(t_i)$ except $B(t)$, obtaining the result

$$P'(B(t))dB(t) = \sum_{n=1}^{N} |c_n|^2 \frac{dB(t)}{\sqrt{2\pi\lambda t}} e^{-\frac{1}{2\lambda t}[B(t) - 2\lambda t a_n]^2}. \tag{59}$$

According to Eq. (59), the probability is the sum of Gaussians, each drifting by an amount $\sim a_n t$, but of width $\sim\sqrt{\lambda t}$. Therefore, after a while, they evolve into essentially separate Gaussians. Then, there are ranges of $B(t)$ which correspond to each possible outcome. If $-K\sqrt{\lambda t} \leq B(t) - 2\lambda t a_n \leq K\sqrt{\lambda t}$, ($K > 1$ is some suitably large number), the associated probability integrated over this range of $B(t)$ is essentially $|c_n|^2$, and the state vector given by Eq. (58) is essentially $|\psi, t\rangle \sim |a_n\rangle$.

It should be emphasized that, when $\hat{H} \neq 0$, the Hamiltonian dynamics interferes with the collapse dynamics, and various behaviors may ensue. In some cases, collapse nonetheless takes place. In some cases, a kind of stasis or equilibrium between the two competing dynamics is reached. In other cases, the unitary and nonunitary dynamics interfere with each other in interesting ways.

It is useful to have an expression for the density matrix which describes the ensemble of evolutions. This is obtained from Eq. (55):

$$\begin{aligned}
\rho(t) &= \int PDw(t)\frac{|\psi, t\rangle\langle\psi, t|}{\langle\psi, t|\psi, t\rangle} = \int Dw(t)|\psi, t\rangle\langle\psi, t| \\
&= \int Dw(t)\mathcal{T} e^{-\int_0^t dt'[i\hat{H} + \frac{1}{4\lambda}[w(t') - 2\lambda\hat{A}]^2]}|\psi, 0\rangle \\
&\quad \times \langle\psi, 0|e^{-\int_0^t dt'[-i\hat{H} + \frac{1}{4\lambda}[w(t') - 2\lambda\hat{A}]^2]} \\
&= \mathcal{T} e^{-\int_0^t dt'[i(\hat{H}_L - \hat{H}_R] + \frac{\lambda}{2}[\hat{A}_L - \hat{A}_R]^2]}\rho(0),
\end{aligned} \tag{60}$$

where the subscripts L and R mean that the associated operators are to be put to the left or right of $\rho(0)$, and the $\mathcal{T}$ reverse-time-orders operators to the right of $\rho(0)$. The evolution equation for the density matrix is therefore the simplest of Lindblad equations,

$$\frac{d}{dt}\rho(t) = -i[\hat{H}, \rho(t)] - \frac{\lambda}{2}[\hat{A}, [\hat{A}, \rho(t)]]. \tag{61}$$

It follows that the ensemble expectation value of an operator $\overline{\langle\hat{O}\rangle} = \mathrm{Tr}\hat{O}\rho(t)$ satisfies

$$\frac{d}{dt}\overline{\langle \hat{O}\rangle} = -i\overline{[\hat{O},\hat{H}]} - \frac{\lambda}{2}\overline{[\hat{A},[\hat{A},\hat{O}]]}. \tag{62}$$

VII. APPLICATION OF CSL

CSL dynamics tries to collapse state vectors toward eigenstates of $\hat{A}$. It gives an ensemble of different evolutions of the state vector, each characterized by a different $w(t)$. It is to be applied to the modes described by the focus operator $\hat{P}$, the operator $\hat{X}$, and Hamiltonian $\hat{H}_k$ given in Eq. (28). According to Eq. (54), using the relations which gave us Eq. (37),

$$\begin{aligned}\overline{\langle\hat{\pi}(\mathbf{k})\rangle\langle\hat{\pi}(\mathbf{k}')\rangle^*} &= \overline{\langle(\hat{\pi}_S(\mathbf{k}) + i\hat{\pi}_A(\mathbf{k}))\rangle\langle(\hat{\pi}_S(\mathbf{k}') - i\hat{\pi}_A(\mathbf{k}'))\rangle} = \overline{\langle\hat{\pi}_S(\mathbf{k})\rangle\langle\hat{\pi}_S(\mathbf{k}')\rangle} + \overline{\langle\hat{\pi}_A(\mathbf{k})\rangle\langle\hat{\pi}_A(\mathbf{k}')\rangle}\\ &= 2\overline{\left\langle\frac{\hat{P}}{\sqrt{2d\mathbf{k}}}\right\rangle^2}\delta_{\mathbf{k}\mathbf{k}'} = \overline{\langle\hat{P}\rangle^2}\delta(\mathbf{k}-\mathbf{k}'),\end{aligned} \tag{63}$$

what we need to find is the ensemble average $\overline{\langle\hat{P}\rangle^2}$, and determine under what circumstances, if any, this is $\sim k$. We must therefore choose a collapse-generating operator $\hat{A}$. We shall consider the simplest possibilities in this paper, $\hat{A} = \hat{X}$ and $\hat{A} = \hat{P}$. These correspond to the basic inflaton perturbation operators $\delta\varphi(\mathbf{x})$, $\delta\varphi'(\mathbf{x})$.

Using Eq. (62), one can readily obtain a set of coupled equations for the ensemble average of the expectation value of any power of operators, just as was done in Sec. III D. However, the ensemble average of a *product* of expectation values, in particular,

$$\overline{\langle\hat{P}\rangle^2} \equiv \int PDw(\eta)\frac{\langle\psi,\eta|\hat{P}|\psi,\eta\rangle^2}{\langle\psi,\eta|\psi,\eta\rangle^2} = \int Dw(\eta)\frac{\langle\psi,\eta|\hat{P}|\psi,\eta\rangle^2}{\langle\psi,\eta|\psi,\eta\rangle}, \tag{64}$$

cannot be obtained in this way, and is not so easy to calculate directly. However, for this problem, there is a relationship between $\overline{\langle\hat{P}\rangle^2}$ and $\overline{\langle\hat{P}^2\rangle}$ which allows us to obtain the former more easily.

Because the initial state is a Gaussian and the Hamiltonian and collapse Hamiltonian are quadratic in $\hat{X}$, $\hat{P}$, the form of the state vector in the momentum basis at any time is

$$\langle p|\psi,\eta\rangle = e^{-A(\eta)p^2 + B(\eta)p + C(\eta)}. \tag{65}$$

The initial conditions are $A(-\mathcal{T}) = 1/2k$, $B(-\mathcal{T}) = C(-\mathcal{T}) = 0$. [By solving the Schrodinger equation with this ansatz, which we shall eventually do, one finds that A is independent of $w(t)$, B is linear in $w(t)$ and C is quadratic in $w(t)$.] Therefore, the momentum matrix element is

$$\langle\psi,\eta|\hat{P}|\psi,\eta\rangle = \int dp\, p e^{-(A+A^*)p^2+(B+B^*)p+(C+C^*)} = \frac{1}{2\sqrt{(A+A^*)}}\frac{(B+B^*)}{(A+A^*)}e^{\frac{(B+B^*)^2}{4(A+A^*)}}e^{(C+C^*)}. \tag{66}$$

We also note that the state vector norm is

$$\langle\psi,\eta|\psi,\eta\rangle = \int dp\, e^{-(A+A^*)p^2+(B+B^*)p+(C+C^*)} = \frac{1}{\sqrt{(A+A^*)}}e^{\frac{(B+B^*)^2}{4(A+A^*)}}e^{(C+C^*)}. \tag{67}$$

Therefore, by Eq. (64),

$$\overline{\langle\hat{P}\rangle^2} = \int Dw(\eta)\frac{1}{\sqrt{(A+A^*)}}\left[\frac{(B+B^*)}{2(A+A^*)}\right]^2 e^{\frac{(B+B^*)^2}{4(A+A^*)}}e^{(C+C^*)}. \tag{68}$$

Now, $\overline{\langle\hat{P}^2\rangle}$ can be written as

$$\begin{aligned}\overline{\langle\hat{P}^2\rangle} &= \int Dw\langle\psi,\eta|\hat{P}^2|\psi,\eta\rangle = \int Dw\int dp\, p^2 e^{-(A+A^*)p^2+(B+B^*)p+(C+C^*)}\\ &= \int Dw\frac{1}{\sqrt{(A+A^*)}}\left[\frac{1}{2(A+A^*)} + \left[\frac{(B+B^*)}{2(A+A^*)}\right]^2\right]e^{\frac{(B+B^*)^2}{4(A+A^*)}}e^{(C+C^*)} = \frac{1}{2(A+A^*)} + \overline{\langle\hat{P}\rangle^2},\end{aligned} \tag{69}$$

where the last step follows from $(A + A^*)$ being independent of $w(t)$, from the integral of Eq. (67) being 1 as it is the probability of all possible $w(t)$'s [Eq. (57)], and from Eq. (68).

To summarize,

$$\overline{\langle\hat{P}\rangle^2} = \overline{\langle\hat{P}^2\rangle} - \frac{1}{2(A+A^*)}, \tag{70}$$

i.e., $[2(A+A^*)]^{-1}$ is the standard deviation of the squared momentum. It is also the width of every packet in momentum space.

Thus, to calculate $\overline{\langle\hat{P}\rangle^2}$, we shall find the second term on the right-hand side of Eq. (70) from the Schrödinger equation and we shall find the first term by using the density matrix.

VIII. $\hat{P}$ AS GENERATOR OF COLLAPSE

A. Use of Schrödinger equation

The Schrödinger equation is the time derivative of Eq. (55). In the momentum representation, with $\hat{A} = \hat{P}$, it is

$$\frac{\partial}{\partial\eta}\langle p|\psi,\eta\rangle = \frac{-i}{2}\left[p^2 - \frac{i}{\eta}\left(p\frac{\partial}{\partial p} + \frac{\partial}{\partial p}p\right) - k^2\frac{\partial^2}{\partial p^2}\right] \times\langle p|\psi,\eta\rangle - \left[\frac{1}{4\lambda}w^2(\eta) - w(\eta)p + \lambda p^2\right] \times\langle p|\psi,\eta\rangle. \tag{71}$$

We note that λ is a dimensionless number. Inserting Eq. (65) into Eq. (71), we find the equation A satisfies is

$$\frac{d}{d\eta}A = \left[\frac{i}{2} + \lambda\right] - \frac{2}{\eta}A - 2ik^2A^2. \tag{72}$$

This Ricatti equation is solved by writing $A \equiv \dot{Z}/[2ik^2Z]$. Putting this into Eq. (72) we get

$$\eta\ddot{Z} = 2ik^2\left[\frac{i}{2} + \lambda\right]\eta Z - 2\dot{Z}. \tag{73}$$

Defining $\alpha \equiv k\sqrt{1-2i\lambda}$, the two solutions are

$$\frac{1}{\eta}\cos\alpha\eta \quad \text{and} \quad \frac{1}{\eta}\sin\alpha\eta. \tag{74}$$

Therefore,

$$A(\eta) = \frac{1}{2ik^2\eta}\left[\frac{-\cos\alpha\eta - \alpha\eta\sin\alpha\eta + C(\alpha\eta\cos\alpha\eta - \sin\alpha\eta)}{\cos\alpha\eta + C\sin\alpha\eta}\right]. \tag{75}$$

The constant C is determined by the initial condition $A(-\mathcal{T}) = 1/2k$:

$$C = \frac{(1-ik\mathcal{T})\cos\alpha\mathcal{T} + \alpha\mathcal{T}\sin\alpha\mathcal{T}}{(1-ik\mathcal{T})\sin\alpha\mathcal{T} - \alpha\mathcal{T}\cos\alpha\mathcal{T}}. \tag{76}$$

Putting C into Eq. (75) yields

$$A(\eta) = \frac{i}{2k^2\eta} + \frac{\alpha}{2ik^2}\left[\frac{(1-ik\mathcal{T})\cos\alpha(\eta+\mathcal{T}) + \alpha\mathcal{T}\sin\alpha(\eta+\mathcal{T})}{(1-ik\mathcal{T})\sin\alpha(\eta+\mathcal{T}) - \alpha\mathcal{T}\cos\alpha(\eta+\mathcal{T})}\right] \approx \frac{i}{2k^2\eta} + \frac{\alpha}{2k^2}. \tag{77}$$

The last step has utilized the approximations $k\mathcal{T} \gg 1$, $|\eta| \ll \mathcal{T}$, $\alpha \approx k - i\lambda k$ for $\lambda \ll 1$, yet $\lambda k\mathcal{T} \gg 1$ (to be justified in Sec. X D) so $\cos\alpha(\eta+\mathcal{T}) \approx i\sin\alpha(\eta+\mathcal{T})$. Therefore, we have the result, at $\eta = -\tau$, that

$$\frac{1}{2(A+A^*)} = \frac{k^2}{\alpha+\alpha^*} = \frac{k}{\sqrt{1-2i\lambda}+\sqrt{1+2i\lambda}} = \frac{k}{\sqrt{2}\sqrt{1+\sqrt{1+4\lambda^2}}}. \tag{78}$$

B. Calculation of $\overline{\langle P^2\rangle}$

For this case, Eq. (62) becomes

$$\frac{d}{d\eta}\overline{\langle\hat{O}\rangle} = -i\overline{[\hat{O},\hat{H}]} - \frac{\lambda}{2}\overline{[\hat{P},[\hat{P},\hat{O}]}. \tag{79}$$

Referring to Sec. III D, where we considered the Hamiltonian dynamics alone, the first order equations (31) are unchanged, so here too $\overline{\langle\hat{X}\rangle} = \overline{\langle\hat{P}\rangle} = 0$.

The second order equations (33) for $Q \equiv \overline{\langle\hat{X}^2\rangle}$, $R \equiv \overline{\langle\hat{P}^2\rangle}$, $S \equiv \overline{\langle\hat{X}\hat{P}+\hat{P}\hat{X}\rangle}$ are unchanged except for the first:

$$\dot{Q} = S - \frac{2Q}{\eta} + \lambda, \qquad \dot{R} = -k^2S + \frac{2R}{\eta}, \qquad \dot{S} = 2[R - k^2Q]. \tag{80}$$

The general solution is therefore the sum of the three solutions [Eq. (34)] to the homogeneous equations added to an inhomogeneous solution:

$$Q = \lambda\eta/2, \qquad R = \lambda\eta k^2/2, \qquad S = \lambda/2. \tag{81}$$

For example, the equation which replaces Eq. (34b) is

$$R = C_1 e^{2ik\eta} + C_2 e^{-2ik\eta} + C_3 + \frac{\lambda k^2 \eta}{2}. \quad (82)$$

The constants are determined by the conditions at $\eta = -\mathcal{T}$, which are $Q = 1/2k$, $R = k/2$, $S = 0$. Assuming $k\mathcal{T} \gg 1$, it follows from the modified Eq. (34):

$$\begin{aligned} \frac{k}{2} &= C_1 + C_2 + C_3 - \frac{\lambda k^2 \mathcal{T}}{2}, \\ \frac{1}{2k} &= -C_1 \frac{1}{k^2} - C_2 \frac{1}{k^2} + C_3 \frac{1}{k^2} - \frac{\lambda \mathcal{T}}{2}, \\ 0 &= -2iC_1 \frac{1}{k} + 2iC_2 \frac{1}{k} + \frac{\lambda}{2}. \end{aligned} \quad (83)$$

The solution is

$$C_1 = -\frac{i\lambda}{8k} = -C_2, \qquad C_3 = \frac{k}{2} + \frac{\lambda k^2 \mathcal{T}}{2}. \quad (84)$$

Putting Eq. (84) into Eq. (82), setting $\eta = -\tau$, and using $k\tau \ll 1$, $k\mathcal{T} \gg 1$, we get

$$R = \frac{\lambda k^2 \mathcal{T}}{2} + \frac{k}{2}. \quad (85)$$

When $\lambda k\mathcal{T} \gg 1$ it is clear that the first is the dominant term, but we include the second term to enable consideration of the case $\lambda \approx 0$.

Therefore, by Eq. (70), using the results (78) and (85), we obtain

$$\overline{\langle \hat{P} \rangle^2} = \frac{\lambda k^2 \mathcal{T}}{2} + \frac{k}{2} - \frac{k}{\sqrt{2}\sqrt{1 + \sqrt{1 + 4\lambda^2}}}. \quad (86)$$

If we set $\lambda = 0$ (turn off CSL), we have the standard quantum mechanics result $\overline{\langle \hat{P} \rangle^2} = 0$ since $\langle \hat{P} \rangle = 0$.

We see that agreement with the observed scale-invariant spectrum, $\overline{\langle \hat{P} \rangle^2} \sim k$, can be achieved if we assume the first term is dominant and we also set

$$\lambda = \tilde{\lambda}/k. \quad (87)$$

We note that this replaces the dimensionless collapse rate parameter λ with parameter $\tilde{\lambda}$ of dimension time^{-1}.

In that case we obtain:

$$\overline{\langle \hat{P} \rangle^2} = \frac{\tilde{\lambda} k \mathcal{T}}{2} + \frac{k}{2} - \frac{k}{\sqrt{2}\sqrt{1 + \sqrt{1 + 4(\tilde{\lambda}/k)^2}}}. \quad (88)$$

It is also worth noting that, if $\lambda \ll 1$ but still $\tilde{\lambda}\mathcal{T} \gg 1$, then

$$\frac{\overline{\langle [\hat{P} - \langle \hat{P} \rangle] \rangle^2}}{\overline{\langle \hat{P}^2 \rangle}} \sim \frac{1}{\tilde{\lambda}\mathcal{T}} \ll 1,$$

and the universes in the ensemble do not deviate much from each other.

IX. $\hat{X}$ AS GENERATOR OF COLLAPSE

This proceeds in a manner parallel to the previous section.

A. Use of Schrödinger equation

In the position representation, with $\hat{A} = \hat{X}$, the Schrödinger equation is

$$\begin{aligned} \frac{\partial}{\partial \eta} \langle x | \psi, \eta \rangle = \frac{-i}{2} \Big[-\frac{\partial^2}{\partial x^2} + \frac{i}{\eta} \Big(x \frac{\partial}{\partial x} + \frac{\partial}{\partial x} x \Big) + k^2 x^2 \Big] \\ \times \langle x | \psi, \eta \rangle - \Big[\frac{1}{4\lambda} w^2(\eta) - w(\eta) x + \lambda x^2 \Big] \\ \times \langle x | \psi, \eta \rangle. \end{aligned} \quad (89)$$

We note that λ has dimensions time^{-2}. The wave function in the position representation is $\langle x | \psi, \eta \rangle = \exp[-A' x^2 + B' x + C']$, where $A' = 1/4A$ (A is the coefficient of p^2 in the exponent of the Fourier transform of $\langle x | \psi, \eta \rangle$) satisfies

$$\frac{d}{d\eta} A' = \Big[\frac{ik^2}{2} + \lambda \Big] + \frac{2}{\eta} A' - 2iA'^2. \quad (90)$$

This Ricatti equation is solved by writing $A' \equiv \dot{Z}/[2iZ]$. Putting this into Eq. (90), we get

$$\eta \ddot{Z} = -\beta^2 \eta Z + 2\dot{Z}. \quad (91)$$

Defining $\beta \equiv \sqrt{k^2 - 2i\lambda}$, the two solutions are

$$e^{\pm i\beta\eta}[1 \mp ib\eta]. \quad (92)$$

Using $\dot{Z} = \beta^2 \eta \exp i\beta\eta$,

$$A'(\eta) = \frac{\beta^2 \eta}{2i} \Big[\frac{e^{2i\beta\eta} + C}{e^{2i\beta\eta}(1 - i\beta\eta) + C(1 + i\beta\eta)} \Big]. \quad (93)$$

The constant C is determined by the initial condition $A'(-\mathcal{T}) = k/2$:

$$C = -e^{-2i\beta T} \frac{\beta^2 \mathcal{T} - k\beta \mathcal{T} + ik}{\beta^2 \mathcal{T} + k\beta \mathcal{T} + ik}. \quad (94)$$

Putting C into Eq. (93) yields

$$A'(\eta) = \frac{\beta^2 \eta}{2i} \Big[\frac{e^{2i\beta(\mathcal{T}+\eta)}[\beta^2 \mathcal{T} + k\beta \mathcal{T} + ik] - [\beta^2 \mathcal{T} - k\beta \mathcal{T} + ik]}{e^{2i\beta(\mathcal{T}+\eta)}[\beta^2 \mathcal{T} + k\beta \mathcal{T} + ik](1 - i\beta\eta) - [\beta^2 \mathcal{T} - k\beta \mathcal{T} + ik](1 + i\beta\eta)} \Big] \approx \frac{\beta^2 \eta}{2(i + \beta\eta)}, \quad (95)$$

where the last step has utilized $k\mathcal{T} \gg 1$, and $\exp 2(-\mathrm{Im}\beta)\mathcal{T} \ll 1$.

Using $A = 1/4A'$, we find

$$\frac{1}{2(A+A^*)}=\frac{|A'|^2}{\mathrm{Re}(A')}=(k/2)\frac{(1+4(\lambda/k^2)^2)}{F(\lambda/k^2)+2(\lambda/k^2)^2F^{-1}(\lambda/k^2)-2(\lambda/k^2)(k\eta)^{-1}}, \tag{96}$$

where $F(x)=\frac{1}{\sqrt{2}}\sqrt{1+\sqrt{1+4x^2}}$ $(F(0)=1)$.

B. Calculation of $\overline{\langle P^2\rangle}$

For this case, Eq. (62) becomes

$$\frac{d}{d\eta}\overline{\langle\hat{O}\rangle}=-i\overline{[\hat{O},\hat{H}]}-\frac{\lambda}{2}\overline{[\hat{X},[\hat{X},\hat{O}]]}. \tag{97}$$

Referring to Sec. III D, where we considered the Hamiltonian dynamics alone, the first order equations (31) are unchanged, so here too $\langle\hat{X}\rangle=\langle\hat{P}\rangle=0$.

The second order equations (33) for $Q\equiv\overline{\langle\hat{X}^2\rangle}$, $R\equiv\overline{\langle\hat{P}^2\rangle}$, $S\equiv\overline{\langle\hat{X}\hat{P}+\hat{P}\hat{X}\rangle}$ are unchanged except for the second:

$$\dot{Q}=S-\frac{2Q}{\eta},\qquad \dot{R}=-k^2S+\frac{2R}{\eta}+\lambda,$$
$$\dot{S}=2[R-k^2Q]. \tag{98}$$

The general solution is therefore the sum of the three solutions [Eq. (34)] to the homogeneous equations added to an inhomogeneous solution:

$$Q=\lambda\eta/2k^2,\qquad R=\lambda\eta/2,\qquad S=3\lambda/2k^2. \tag{99}$$

For example, the equation which replaces Eq. (34b) is

$$R=C_1e^{2ik\eta}+C_2e^{-2ik\eta}+C_3+\frac{\lambda\eta}{2}. \tag{100}$$

The constants are determined by the conditions at $t=-\mathcal{T}$, which are $Q=1/2k$, $R=k/2$, $S=0$. Assuming $k\mathcal{T}\gg 1$, it follows from the modified Eq. (34):

$$\frac{k}{2}=C_1+C_2+C_3-\frac{\lambda\mathcal{T}}{2},$$
$$\frac{1}{2k}=-C_1\frac{1}{k^2}-C_2\frac{1}{k^2}+C_3\frac{1}{k^2}-\frac{\lambda\mathcal{T}}{2k^2}, \tag{101}$$
$$0=-2iC_1\frac{1}{k}+2iC_2\frac{1}{k}+3\frac{\lambda}{2k^2}.$$

The solution is

$$C_1=-\frac{3i\lambda}{8k}=-C_2,\qquad C_3=\frac{k}{2}+\frac{\lambda\mathcal{T}}{2}. \tag{102}$$

Putting Eq. (102) into Eq. (100), setting $\eta=-\tau$, and using $k\tau\ll 1$, $\lambda\mathcal{T}\gg k$, we get

$$R=\frac{\lambda\mathcal{T}}{2}+\frac{k}{2}. \tag{103}$$

Therefore, by Eq. (70), using the results (95) and (103), we obtain

$$\overline{\langle\hat{P}\rangle^2}=\frac{\lambda\mathcal{T}}{2}-\frac{\sqrt{k^4+4\lambda^2}}{\sqrt{k^2-2i\lambda}+\sqrt{k^2+2i\lambda}}$$
$$=\frac{\lambda\mathcal{T}}{2}+\frac{k}{2}-(k/2)$$
$$\times\frac{(1+4(\lambda/k^2)^2)}{F(\lambda/k^2)+2(\lambda/k^2)^2F^{-1}(\lambda/k^2)-2(\lambda/k^2)(k\eta)^{-1}}. \tag{104}$$

Once more, if we turn off CSL, we find $\overline{\langle\hat{P}\rangle^2}=0$.

We see that agreement with the observed scale-invariant spectrum, $\overline{\langle\hat{P}\rangle^2}\sim k$, can be achieved if we assume that the first term dominates, and if we set

$$\lambda=\tilde{\lambda}k. \tag{105}$$

We note that this replaces the collapse rate parameter λ of dimension time^{-2} with the parameter $\tilde{\lambda}$ of dimension time^{-1}. In that case we obtain

$$\overline{\langle\hat{P}\rangle^2}=\frac{\tilde{\lambda}k\mathcal{T}}{2}+\frac{k}{2}\left[1-\frac{(1+4(\tilde{\lambda}/k)^2)}{F(\tilde{\lambda}/k)+2(\tilde{\lambda}/k)^2F^{-1}(\tilde{\lambda}/k)-2(\tilde{\lambda}/k)(k\eta)^{-1}}\right]. \tag{106}$$

The validity of our approximations shall be discussed in Sec. X D.

X. PHYSICAL QUANTITIES

As we have shown, our theory can describe the Harrison-Zel'dovich scale-invariant spectrum. Moreover, it can be used to find expressions for various physical quantities, such as α_{lm}, $\Delta T(\mathbf{n})$, $\Psi(\mathbf{x})$ (and their probabilities of taking on various values), which is not possible with the usual approach. That is, we can calculate expressions for quantities corresponding to an *individual universe*, not just for the ensemble of universes. This we shall now demonstrate, using collapse generator $\hat{A}=\hat{P}$ discussed in Sec. VIII.

First, we need the expression for $\langle\psi,\eta|\hat{P}|\psi,\eta\rangle/\langle\psi,\eta|\psi,\eta\rangle\equiv\langle\hat{P}\rangle$. To get this, we return to the

Schrödinger equation (71) for $\langle p|\psi,\eta\rangle = \exp[-Ap^2 + Bp + C]$. With use of Eq. (77) for the already-obtained variable A, we find the equation for B:

$$\begin{aligned}\frac{d}{d\eta}B &= -\frac{1}{\eta}B - 2ik^2AB + w\\ &\approx -\frac{1}{\eta}B - 2ik^2\left[\frac{i}{2k^2\eta} + \frac{\alpha}{2k^2}\right]B + w\\ &= -i\alpha B + w \quad \text{or}\\ B(t) &= \int_{-\mathcal{T}}^{\eta} d\eta' w(\eta')e^{-i\alpha(\eta-\eta')}. \end{aligned}\tag{107}$$

Thus, according to Eqs. (66), (67), and (107),

$$\begin{aligned}\langle\hat{P}\rangle(\eta) &= \frac{B+B^*}{2(A+A^*)}\\ &= \frac{k^2}{R}\int_{-\mathcal{T}}^{\eta} d\eta' w(\eta')e^{-S(\eta-\eta')}\cos R(\eta-\eta'),\end{aligned}\tag{108}$$

where we have written

$$\begin{aligned}&\alpha = k\sqrt{1-2i\lambda} \equiv R - iS, \qquad \text{so } R^2 - S^2 = k^2,\\ &RS = \lambda k^2, \quad \text{so } R = kF(\lambda), \qquad S = \frac{\lambda k}{F(\lambda)}.\end{aligned}\tag{109}$$

Now we can express physical quantities in terms of $\langle\hat{P}\rangle$. As was done in obtaining (63), and using (108) we write

$$\begin{aligned}\langle\hat{\pi}(\mathbf{k},\eta)\rangle &= \langle\hat{\pi}_S(\mathbf{k},\eta)\rangle + i\langle\hat{\pi}_A(\mathbf{k},\eta)\rangle\\ &= \frac{\langle\hat{P}_S\rangle + i\langle\hat{P}_A\rangle}{2\sqrt{d\mathbf{k}}}\\ &= \frac{k^2}{2R}\int_{-\mathcal{T}}^{\eta} d\eta'[w_S(\mathbf{k},\eta')\\ &\quad + iw_A(\mathbf{k},\eta')]e^{-S(\eta-\eta')}\cos R(\eta-\eta'),\end{aligned}\tag{110}$$

where we have introduced white noise functions $w(\mathbf{k},\eta)$. These are only defined in the upper half $\mathbf{k}$-plane. However, the Fourier transform of $\hat{\pi}(\mathbf{k})$ is $\hat{\pi}(\mathbf{x})$ which is real. This implies $w_S(-\mathbf{k}) = w_S(\mathbf{k})$ and $w_A(-\mathbf{k}) = -w_A(\mathbf{k})$, so Eq. (110) holds for all $\mathbf{k}$.

To go along with the expression (108) for $\langle\hat{P}\rangle$, we must have the probability of $w(\eta)$. To get the probability, we could go back to Schrödinger's equation (71) and calculate C. However, since the probability is $\langle\psi,\eta|\psi,\eta\rangle$, it is easier to use Eq. (71) to get

$$\begin{aligned}&\frac{d}{d\eta}\langle\psi,\eta|\psi,\eta\rangle\\ &\quad = -\frac{w^2(\eta)}{2\lambda}\langle\psi,\eta|\psi,\eta\rangle + 2w(\eta)\\ &\qquad \times\langle\psi,\eta|\hat{P}|\psi,\eta\rangle - 2\lambda\langle\psi,\eta|\hat{P}^2|\psi,\eta\rangle\\ &\quad = \left[-\frac{w^2(\eta)}{2\lambda} + 2w(\eta)\langle\hat{P}\rangle - 2\lambda\langle\hat{P}^2\rangle\right]\langle\psi,\eta|\psi,\eta\rangle\\ &\quad = \left\{-\frac{1}{2\lambda}[w(\eta) - 2\lambda\langle\hat{P}\rangle]^2 - \frac{\lambda k}{F(\lambda)}\right\}\langle\psi,\eta|\psi,\eta\rangle,\end{aligned}\tag{111}$$

where the last step follows since Eq. (70) holds without the ensemble average. Thus, we obtain the probability density,

$$P(w) = \langle\psi,-\tau|\psi,-\tau\rangle = e^{-\frac{1}{2\lambda}\int_{-\mathcal{T}}^{-\tau}d\eta[w(\eta)-2\lambda\langle\hat{P}\rangle(\eta)]^2},\tag{112}$$

where $\langle\hat{P}\rangle(\eta)$ is given by Eq. (108) [the factor $\exp -\lambda k(\tau+\mathcal{T})/F(\lambda)$ has been absorbed in the normalization appropriate to Dw].

To go to the continuum case, replace $\int d\eta$ by $\int d\eta d\mathbf{k}$ and $w(\eta)$ by $w(\mathbf{k},\eta)$ in Eq. (112).

In order to do calculations with the probability (112), it is convenient to define a new random variable:

$$\begin{aligned}v(\eta) &\equiv w(\eta) - 2\lambda\langle\hat{P}\rangle(\eta)\\ &= w(\eta) - 2S\int_{-\mathcal{T}}^{\eta} d\eta' w(\eta')e^{-S(\eta-\eta')}\cos R(\eta-\eta').\end{aligned}\tag{113}$$

The Jacobian determinant of the transformation from variables $w(\eta)$ to $v(\eta)$ is 1 (essentially, the matrix transformation has all diagonal elements = 1, and zeros to the right of the diagonal). The probability density of $v(\eta)$ is very simple,

$$\begin{aligned}&P(v) = e^{-\frac{1}{2\lambda}\int_{-\mathcal{T}}^{-\tau}d\eta v(\eta)^2}, \qquad \overline{v(\eta)} = 0,\\ &\overline{v(\eta)v(\eta')} = \lambda\delta(\eta-\eta').\end{aligned}\tag{114}$$

However, since the physical quantities are expressed in terms of w, we need to invert Eq. (113) to obtain the expression for w in terms of v. This is done in the Appendix, with the result

$$\begin{aligned}w(\eta) = v(\eta) &+ \frac{2S}{k}\int_{-\mathcal{T}}^{\eta} d\eta'[S\sin k(\eta-\eta')\\ &+ k\cos k(\eta-\eta')]v(\eta').\end{aligned}\tag{115}$$

We shall provide a few examples of the use of this formalism. We shall show, using it, that we obtain the result (86) for $\overline{\langle\hat{P}\rangle^2}$. We shall calculate the probability distribution of the temperature fluctuations (which shall prove to be a Gaussian) as well as the correlation function of the temperature fluctuations. Finally, we exhibit the expression for α_{lm}, and the ensemble average of $|\alpha_{lm}|^2$.

A. Calculation of $\overline{\langle\hat{P}\rangle^2}$—Again

As a consistency check, we employ a different way of calculating $\overline{\langle\hat{P}\rangle^2}$ than was done in Sec. VIII. Using Eqs. (108) and (115), and taking $\eta = 0$, we write

$$\langle\hat{P}\rangle = \frac{k^2}{R}\int_{-\mathcal{T}}^{0} d\eta' e^{S\eta'}\cos R(\eta') \times\Big[v(\eta') + \frac{2S}{k}\int_{-\mathcal{T}}^{\eta'} d\eta_1[S\sin k(\eta'-\eta_1) + k\cos k(\eta'-\eta_1)]v(\eta_1)\Big]. \tag{116}$$

We immediately note that $\overline{\langle\hat{P}\rangle} = 0$, since $\overline{v(\eta)} = 0$.

The order of integration of the double integral can next be exchanged, and the integral over η' performed. There is a term which cancels the single integral in Eq. (116), and the result is

$$\langle\hat{P}\rangle = \frac{k}{R}\int_{-\mathcal{T}}^{0} d\eta_1 v(\eta_1)[k\cos k\eta_1 - S\sin k\eta_1]. \tag{117}$$

Therefore, with use of Eq. (114), and neglecting terms small compared to $k\mathcal{T}$, we obtain

$$\overline{\langle\hat{P}\rangle^2} = \frac{\lambda k^2}{R^2}\int_{-\mathcal{T}}^{0} d\eta_1[k\cos k\eta_1 - S\sin k\eta_1]^2 \approx \frac{\lambda k^2\mathcal{T}}{2R^2}[S^2+k^2] = \frac{\lambda k^2\mathcal{T}}{2}. \tag{118}$$

This is the same result as in Eq. (86).

B. Temperature fluctuation

The temperature fluctuation at the end of inflation is given by Eq. (47) with $\eta = -\tau$. (Since $k\tau \ll 1$, we shall replace τ by 0.) With use of Eq. (110), we therefore have

$$\frac{\Delta T}{T} = c\int\frac{d\mathbf{k}}{k^2}e^{ikR_D\hat{\mathbf{k}}\cdot\hat{n}}\langle\hat{\pi}(\mathbf{k},0)\rangle = c\int\frac{d\mathbf{k}}{k^2}e^{ikR_D\hat{\mathbf{k}}\cdot\hat{n}}\frac{k^2}{2R}\int_{-\mathcal{T}}^{0} d\eta'[w_S(\mathbf{k},\eta') + iw_A(\mathbf{k},\eta')]e^{S\eta'}\cos R\eta'$$
$$= c\int_+\frac{d\mathbf{k}}{k^2}\frac{k^2}{R}\int_{-\mathcal{T}}^{0} d\eta' e^{S\eta'}\cos R(\eta')[\cos(kR_D\hat{\mathbf{k}}\cdot\hat{n})w_S(\mathbf{k},\eta') - \sin(kR_D\hat{\mathbf{k}}\cdot\hat{n})w_A(\mathbf{k},\eta')]. \tag{119}$$

To conveniently calculate probabilities, we need to replace the w's by v's, using Eq. (115). Just as in the previous section, we may then exchange the order of the double integral, obtaining

$$\frac{\Delta T(\hat{n})}{T} = c\int_+\frac{d\mathbf{k}}{k^2}\frac{k}{R}\int_{-\mathcal{T}}^{0} d\eta_1[\cos(kR_D\hat{\mathbf{k}}\cdot\hat{n})v_S(\mathbf{k},\eta_1) - \sin(kR_D\hat{\mathbf{k}}\cdot\hat{n})v_A(\mathbf{k},\eta_1)][k\cos(k\eta_1) - S\sin(k\eta_1)]. \tag{120}$$

1. Probability distribution of the temperature fluctuations

The probability that $\Delta T(\hat{n})/T = C$ is given by

$$P(C) = \int Dv e^{-\int_{\mathcal{T}}^{0} d\eta\int_+ d\mathbf{k}\frac{1}{2\lambda(k)}[v_S^2(\mathbf{k})+v_A^2(\mathbf{k})]}\delta(\Delta T(\hat{n})/T - C). \tag{121}$$

Writing Eq. (120) as

$$\frac{\Delta T(\hat{n})}{T} = \int_{-\mathcal{T}}^{0} d\eta_1\int_+ d\mathbf{k}[f_S(\mathbf{k},\eta_1)v_S(\mathbf{k},\eta_1) + f_A(\mathbf{k},\eta_1)v_A(\mathbf{k},\eta_1)], \tag{122}$$

Eq. (121) becomes

$$P(C) = \int Dv e^{-\int_{\mathcal{T}}^{0} d\eta\int_+ d\mathbf{k}\frac{1}{2\lambda(k)}[v_S^2(\mathbf{k})+v_A^2(\mathbf{k})]}\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{i\omega(\int_{-\mathcal{T}}^{0} d\eta_1\int_+ d\mathbf{k}[f_S(\mathbf{k},\eta_1)v_S(\mathbf{k},\eta_1)+f_A(\mathbf{k},\eta_1)v_A(\mathbf{k},\eta_1)]-C)}$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega C}e^{-\omega^2\int_{-\mathcal{T}}^{0} d\eta_1\int_+ d\mathbf{k}\frac{\lambda(k)}{2}[f_S^2(\mathbf{k},\eta_1)+f_A^2(\mathbf{k},\eta_1)]} = e^{-\frac{1}{2}\frac{C^2}{\int_{-\mathcal{T}}^{0} d\eta_1\int_+ d\mathbf{k}\lambda(k)[f_S^2(\mathbf{k},\eta_1)+f_A^2(\mathbf{k},\eta_1)]}}. \tag{123}$$

Thus, we see that the probability is Gaussian, with variance

$$\overline{\Big[\frac{\Delta T(\hat{n})}{T}\Big]^2} = c^2\int_+ d\mathbf{k}\lambda(k)\frac{1}{(Rk)^2}\int_{-\mathcal{T}}^{0} d\eta_1[k\cos(k\eta_1) - S\sin(k\eta_1)]^2 \approx c^2\int_+ d\mathbf{k}\lambda(k)\frac{\mathcal{T}}{2k^2} = \pi c^2\tilde{\lambda}\mathcal{T}\int_0^\infty\frac{dk}{k}. \tag{124}$$

We have set $\lambda(k) = \tilde{\lambda}/k$ in the observed range of k, 10^{-3} Mpc$^{-1} < k < 10^2$ Mpc^{-1}, in order to obtain agreement with the Harrison-Zel'dovich spectrum. However, $\overline{[\Delta T(\hat{n})/T]^2}$ is the ensemble average (over possible universes) of a measured quantity so it seems reasonable to expect that the integral in Eq. (124) should converge, and it does not.

One might think the issue of the divergence of the above integral could be resolved simply by taking into account the fact that the temperature fluctuations seen in the CMB, are the result of modifications by "late time"[11] processes of the primordial seeds of cosmic structure we have calculated. The fact however, is that such modifications[12] occur after the reheating stage, and are therefore of no help at the time just prior to reheating, which is the era to which our calculations refer.[13]

If there were no modification in this estimate for the inflationary regime itself, the resulting ensuing fluctuations would be predicted to be of arbitrarily large magnitude and a perturbative treatment would, of course, be invalid.

The issue is intimately tied with the universal scale invariance of the H-Z spectrum and is thus a problem that, although often unrecognized, afflicts the standard treatment of inflationary perturbations. Let us consider for instance the treatment presented in Ref. [19].

Consider the expression for the Newtonian potential at a point $\mathbf{x}$. To do this, one uses Eq. (8.5) of that book and writes

$$\Phi(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{k}\, \Phi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{125}$$

The ensemble average of the square is then

$$\overline{\Phi^2(\mathbf{x})} = \frac{1}{(2\pi)^3} \int d\mathbf{k}\, d\mathbf{k}' \overline{\Phi_{\mathbf{k}} \Phi^*_{\mathbf{k}'}} e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}. \tag{126}$$

Next one uses the definition (8.11) of Ref. [19] which indicates that we might write the ensemble average as

$$\overline{\Phi_{\mathbf{k}} \Phi^*_{\mathbf{k}'}} = \delta(\mathbf{k} - \mathbf{k}') 2\pi^2 \delta^2_\Phi(k)/k^3, \tag{127}$$

where in their notation $\delta^2_\Phi(k)$ is the power spectrum (for the gravitational potential Φ) and is taken to be a function only of $k \equiv \|\mathbf{k}\|$. Thus, substituting in (126) we find

$$\overline{\Phi^2(\mathbf{x})} = \frac{1}{4\pi} \int d\mathbf{k}\, \delta^2_\Phi(k)/k^3. \tag{128}$$

Now we take Eq. (8.103) of Ref. [19] which indicates that the spectrum is scale independent. This requires $\delta^2_\Phi(k)$ to be constant, independent of k and, in fact, given by $4(\varepsilon + p)/C_s$, where ε and p are the energy density and pressure of the background inflaton field, respectively, and where C_s is the effective "speed of sound" [see Eq. (8.50) of Ref. [19]] which for a canonical scalar field is just the speed of light. We thus obtain

$$\begin{aligned} \overline{\Phi(\mathbf{x})^2} &= \frac{2(\varepsilon + p)}{C_s} \int_0^\infty k^2 dk/k^3 \\ &= \frac{2(\varepsilon + p)}{C_s} \int_0^\infty dk/k \end{aligned} \tag{129}$$

which diverges just as our estimate of the temperature fluctuation in (124).

So, it seems that the problem we face here is not intrinsic to the CSL theory or, more generally, to the hypothesis that some kind of collapse of the wave function plays an important role in inflationary cosmology. What the analysis in the present paper does is show the problem more explicitly. After all what the collapse does, roughly speaking, is provide a mechanism to convert quantum mechanical uncertainties already there in the initial state into a range of actual values at a later time.

Of course, if the problem lies beyond the collapse approach and seems intrinsic to the inflationary proposal for generation of primordial cosmological inhomogeneities and anisotropies, that does not mean it can be ignored. One approach might be to consider a similar problem that appears in any treatment of quantum fields: the uncertainty in the value of a field in its vacuum state at any point in Minkowski space-time is infinite. This is often taken to indicate that one cannot consider such a quantity as the field at a point and must, rather, focus attention on things like smeared fields over space-time regions. After all, quantum fields are distribution-valued operators rather than ordinary operators on Hilbert space. In the inflationary context however it is rather unclear what would be the appropriate smearing one should consider.

For our problem, we might rather consider the possibility that (124) be made finite as the result of a more complicated behavior of $\lambda(k) \neq \tilde{\lambda}/k$ as $k \to 0$ and $k \to \infty$. For example, we might set

$$\lambda(k) = \frac{\tilde{\lambda}}{k_1 + k + (k^2/k_2)} \quad \text{so that} \quad \int_0^\infty dk \lambda(k) \approx \tilde{\lambda} \ln \frac{k_2}{k_1} \tag{130}$$

for $k_2 \gg k_1$ and the interval (k_2, k_1) very much wider than the range of k appropriate to the CMB. That would in effect make a prediction, that the H-Z scale invariant spectrum not hold outside a particular range of k.

On the other hand, this issue might be resolved by things unrelated to the CSL collapse rate and therefore using the above argument to modify it might be premature. One

[11]That refers to the regime well after inflation has ended.

[12]These are due to well understood physics connected with the behavior of the plasmas of ordinary matter (quark-gluon plasma before hadronization forms the nucleons, nucleon-electron-photon plasma before nucleosynthesis, and hydrogen-helium plasma after nucleosynthesis) codified in the so-called transfer functions which describe how each mechanism alters the temperature structure. For example, transfer functions are known to exhibit a damping effect known as "Silk Damping" at short distances (large k) due to viscosity/photon diffusion.

[13]Strictly speaking at the end of inflation the conditions are still far from thermal equilibrium, there is therefore no temperature and in fact no radiation. It is only after the reheating that the decay of the inflation is supposed to lead the system towards a state of "thermal equilibrium." However, the quantities we are computing have their counterpart as the inflaton-energy-density fluctuations and Newtonian potential fluctuations.

possibility that would certainly help in this regard is the known fact that, during the slow-roll inflation, the expansion rate is not exactly de Sitter like. This leads, in the usual analysis, to a modification of the H-Z spectrum corresponding to a factor $k^{(n_s-1)}$ where the "spectral index" n_s is known both theoretically and empirically to be smaller than 1 [24]. This would remove the divergence coming from the $k \to \infty$ behavior. Regarding the divergence arising from the $k \to 0$ behavior it seems that the most natural resolution would come from noting that the region that undergoes inflation would in all likelihood have started as a very small patch, which despite the nearly exponential expansion resulting from inflation will remain finite. Thus, the range of physically relevant values of k would be bounded from below by some $k_{\rm finite} > 0$.

2. Correlation of the temperature fluctuation

The correlation function of the temperature fluctuation can be found from Eq. (120):

$$\overline{\frac{\Delta T(\hat{n})}{T}\frac{\Delta T(\hat{n}')}{T}} = c^2 \int_+ d\mathbf{k}\lambda(k)\frac{1}{(Rk)^2}\cos[kR_D\hat{\mathbf{k}}\cdot(\hat{n}-\hat{n}')]\int_{-\mathcal{T}}^0 d\eta_1[k\cos(k\eta_1) - S\sin(k\eta_1)]^2$$
$$= 4(\pi c)^2\mathcal{T}\int_0^\infty dk\lambda(k)\sum_{lm} j_l^2(kR_D)Y_{lm}(\hat{n})Y_{lm}^*(\hat{n}') = \pi c^2\mathcal{T}\int_0^\infty dk\lambda(k)\sum_l (2l+1)j_l^2(kR_D)P_l(\hat{n}\cdot\hat{n}'). \quad (131)$$

[Note, $\sum_l (2l+1)j_l^2(x) = 1$, so Eq. (131) agrees with the divergent Eq. (124) when $\hat{n} = \hat{n}'$.]

C. α_{lm}

From the definition (44) of α_{lm} and Eq. (120) we can exhibit the expression for α_{lm}:

$$\alpha_{lm} = c\int_+ d\mathbf{k}\frac{1}{kR}\int_{-\mathcal{T}}^0 d\eta_1\sum_l (2l+1)j_l(kR_D)P_l(\hat{k}\cdot\hat{n}')$$
$$\times[\mathcal{E}(l)v_S(\mathbf{k},\eta_1) + i\mathcal{O}(l)v_A(\mathbf{k},\eta_1)]$$
$$\times[k\cos(k\eta_1) - S\sin(k\eta_1)], \quad (132)$$

where $\mathcal{E}(l)$ is 1 if l is even and 0 if l is odd, and $\mathcal{O}(l)$ is 0 if l is even and 1 if l is odd.

However, it is easiest to use the second equation of (131) to find

$$\overline{|\alpha_{lm}|^2} = (2\pi c)^2\mathcal{T}\int_0^\infty dk\lambda(k)j_l^2(kR_D)$$
$$= (2\pi c)^2\tilde{\lambda}\mathcal{T}\int_0^\infty \frac{dk}{k}j_l^2(kR_D)$$
$$= \frac{(2\pi c)^2\tilde{\lambda}\mathcal{T}}{2l(l+1)}, \quad (133)$$

which is to be compared with the scale invariant observations expressed in Eqs. (51) and (52). In Eq. (133) we have set $\lambda(k) = \tilde{\lambda}/k$, although we earlier specified that it might behave differently as $k \to 0$ and $k \to \infty$. This different behavior has been ignored in calculating the integral in (133), which converges perfectly well without it. Thus, such possibly different behavior is now constrained to have a negligible effect on the integral in Eq. (133).

This concludes our demonstration that the dynamics produces agreement with the observed Harrison-Zel'dovich spectrum.

D. Estimates

It is necessary to make some order of magnitude estimates to justify the approximations we have made. We start from the fact that the temperature fluctuations in the CMB are $\frac{\Delta T}{T} = 1/3\Psi \sim 10^{-5}$. We can now use that together with our other results to check the self-consistency of assumptions made in earlier sections. We note that we took the dominant term in Eq. (88) to be

$$\overline{\langle\hat{P}\rangle^2} \approx \frac{\tilde{\lambda}k\mathcal{T}}{2}. \quad (134)$$

This led to the result (124),

$$\left[\frac{\Delta T}{T}\right]^2 = \frac{\pi}{4}\left[\frac{4\pi G\phi_0'}{3a}\right]^2\tilde{\lambda}\mathcal{T}\mathcal{I}, \quad (135)$$

where $\mathcal{I} \equiv 1/\tilde{\lambda}\int dk\lambda(k)$ is at least as large as $\int_{10^{-3}}^{10^2} dk/k \approx 11.5$.

Using the definition $\mathcal{H} \equiv a'(\eta)/a(\eta)$ in the second equation of (5) and taking the derivative with respect to η we find

$$6\mathcal{H}\mathcal{H}' = 4\pi G(2\phi_0''\phi_0' + 4aa'V[\phi_0]$$
$$+ 2a^2\partial_{\phi_0}V[\phi_0]\phi_0'). \quad (136)$$

Substituting here the expression for ϕ_0'' from the first equation of (5) gives

$$6\mathcal{H}\mathcal{H}' = 4\pi G(-4\mathcal{H}(\phi_0')^2 + 4aa'V[\phi_0])$$
$$= 16\pi G\mathcal{H}(-(\phi_0')^2 + a^2V[\phi_0]). \quad (137)$$

Therefore

$$\mathcal{H}' = \frac{8\pi G}{3}(-(\phi_0')^2 + a^2V[\phi_0]). \quad (138)$$

Now, using the definition of the slow-roll parameter $\epsilon \equiv 1 - \mathcal{H}'/\mathcal{H}^2$, the expression above for $\mathcal{H}'$ and the second equation of (5) again to give $\mathcal{H}^2$, we have

$$\epsilon = \frac{3(\phi_0')^2}{(\phi_0')^2 + a^2 V[\phi_0])} \approx \frac{3(\phi_0')^2}{(a^2 V[\phi_0])}, \tag{139}$$

where in the last step we have used the standard inflationary requirement that the potential term is much larger than the kinetic term. Therefore we have $(\phi_0')^2 \approx \epsilon a^2 V/3$.

Using this result, we can rewrite the quantity c^2 appearing in Eq. (135) as

$$\left[\frac{4\pi G \phi_0'}{3a}\right]^2 = \frac{(4\pi)^2}{27} G^2 \epsilon V \approx \epsilon \frac{V}{M_{\rm Pl}^4} \approx \epsilon \left(\frac{M_{\rm GUT}}{M_{\rm Pl}}\right)^4. \tag{140}$$

Thus, $(\frac{\Delta T}{T})^2 \approx \epsilon [V/(M_{\rm Pl})^4] \tilde\lambda \mathcal{T} I$.

However as discussed in Ref. [25], the effect of reheating (unless some very unusual coincidences occur) is to multiply this result by $1/\epsilon^2$. Thus, we have

$$\left(\frac{\Delta T}{T}\right)^2 \approx \frac{1}{\epsilon} \frac{V}{(M_{Pl})^4} \tilde\lambda \mathcal{T} I. \tag{141}$$

The observations yield $(\frac{\Delta T}{T})^2 \sim 10^{-10}$. The small departure from flatness of the spectrum is taken to indicate $\epsilon \approx 10^{-2}$. With $V^{1/4}$ given by the GUT scale as $\approx 10^{15}$ GeV $\approx 10^{-4} M_{\rm Pl}$, and taking $I \approx 10$, we conclude that

$$\tilde\lambda \mathcal{T} \quad \text{is of order } 10^3 \gg 1. \tag{142}$$

In Sec. VIII, following Eq. (77), we used $\lambda k \mathcal{T} = \tilde\lambda \mathcal{T} \gg 1$, so this justifies the approximation made there.

Since we had estimated $\mathcal{T}$ to be of order 10^8 Mpc $= 10^{22}$ sec, we have

$$\tilde\lambda \approx 10^{-5}\ \text{Mpc}^{-1} \approx 10^{-19}\ \text{sec}^{-1}. \tag{143}$$

Curiously, this is not far removed from the value 10^{-16} sec^{-1} suggested by GRW [13] in their theory of instantaneous collapses on position, and adopted in the CSL [14] theory of continuous dynamical collapse on mass density.

Thus, for the modes of interest we have $\tilde\lambda/k$ in the range of 10^{-2} to 10^{-7}. We use this estimate to consider the magnitude of the subleading term in Eq. (88),

$$\overline{\langle \hat P\rangle^2} = (k/2)\left(\tilde\lambda \mathcal{T} + 1 - \frac{\sqrt{2}}{\sqrt{1+\sqrt{1+4(\tilde\lambda/k)^2}}}\right), \tag{144}$$

where we indeed see that it is much smaller than the leading one.

Therefore, we have a self-consistent assignment of values for the parameters which satisfies the approximations made and matches the theory with the observations.

XI. COLLAPSE ON FIELD OPERATORS

We have presented two examples that lead to the H-Z spectrum. In one case, the collapse-generating operator is $\hat P \sim \hat\pi(\mathbf{k})$ (Sec. VIII), whose Fourier transform is the field operator $\hat\pi(\mathbf{x})$. In the other case it is $\hat X \sim \hat{\mathbf{y}}(\mathbf{k})$ (Sec. IX), whose Fourier transform is the field operator $\hat{\mathbf{y}}(\mathbf{x})$. In this section we shall take a look at the state vector evolution written in terms of these field operators. We shall give the argument in detail in the first case: the second case proceeds exactly similarly.

The state vector evolution given by Eq. (55), applied to the full set of commuting collapse-generating operators we have discussed, is

$$\begin{aligned}|\psi,\eta\rangle &= \mathcal{T} e^{-i\int_{-\mathcal{T}}^{\eta} d\eta' \hat H - \int_{-\mathcal{T}}^{\eta} d\eta' \int_+ d\mathbf{k} \frac{1}{4\lambda(k)}[w_S(\mathbf{k},\eta') - 2\lambda(k)\sqrt{2}\hat\pi_S(\mathbf{k})]^2 - \int_{-\mathcal{T}}^{\eta} d\eta' \int_+ d\mathbf{k} \frac{1}{4\lambda(k)}[w_A(\mathbf{k},\eta') - 2\lambda(k)\sqrt{2}\hat\pi_A(\mathbf{k})]^2}|\psi,-\mathcal{T}\rangle \\ &= \mathcal{T} e^{-i\int_{-\mathcal{T}}^{\eta} d\eta' \hat H - \frac{1}{4\tilde\lambda}\int_{-\mathcal{T}}^{\eta} d\eta' \int_+ d\mathbf{k}[\sqrt{k} w_S(\mathbf{k},\eta') - 2\tilde\lambda\sqrt{2/k}\,\hat\pi_S(\mathbf{k})]^2 - \frac{1}{4\tilde\lambda}\int_{-\mathcal{T}}^{\eta} d\eta' \int_+ d\mathbf{k}[\sqrt{k} w_A(\mathbf{k},\eta') - 2\tilde\lambda\sqrt{2/k}\,\hat\pi_A(\mathbf{k})]^2}|\psi,-\mathcal{T}\rangle, \end{aligned} \tag{145}$$

where we have written $\lambda = \tilde\lambda/k$ and, following Eq. (20), we have written $\hat P_{S,A}(\mathbf{k}) = \sqrt{2}\hat\pi_{S,A}(\mathbf{k})$. We now define $w_S(\mathbf{k},t') \equiv \sqrt{2/k}\, w_R(\mathbf{k},t')$, $w_A(\mathbf{k},t') \equiv \sqrt{2/k}\, w_I(\mathbf{k},t')$, and $w(\mathbf{k},t') \equiv w_R(\mathbf{k},t') + i w_I(\mathbf{k},t')$. We also recall that $\hat\pi(\mathbf{k}) = \pi_S(\mathbf{k}) + i\pi_A(\mathbf{k})$. Therefore, Eq. (145) may be written as

$$\begin{aligned}|\psi,\eta\rangle &= \mathcal{T} e^{-i\int_{-\mathcal{T}}^{\eta} d\eta' \hat H - 2\frac{1}{4\tilde\lambda}\int_{-\mathcal{T}}^{\eta} d\eta' \int_+ d\mathbf{k}[w(\mathbf{k},\eta') - 2\tilde\lambda k^{-1/2}\hat\pi(\mathbf{k})][w^*(\mathbf{k},\eta') - 2\tilde\lambda k^{-1/2}\hat\pi^\dagger(\mathbf{k})]}|\psi,-\mathcal{T}\rangle \\ &= \mathcal{T} e^{-i\int_{-\mathcal{T}}^{\eta} d\eta' \hat H - \frac{1}{4\tilde\lambda}\int_{-\mathcal{T}}^{\eta} d\eta' \int d\mathbf{k}[w(\mathbf{k},\eta') - 2\tilde\lambda k^{-1/2}\hat\pi(\mathbf{k})][w^*(\mathbf{k},\eta') - 2\tilde\lambda k^{-1/2}\hat\pi^\dagger(\mathbf{k})]}|\psi,-\mathcal{T}\rangle, \end{aligned} \tag{146}$$

where we have replaced $2\times$ the integral by an integral which includes the lower half $\mathbf{k}$-plane by defining $w_R(-\mathbf{k},t') \equiv w_R(\mathbf{k},t')$, $w_I(-\mathbf{k},t') \equiv -w_I(\mathbf{k},t')$.

We may now convert to a real noise function and a Hermitian field operator defined as

$$w(\mathbf{x},\eta') \equiv \frac{1}{(2\pi)^{3/2}}\int d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} w(\mathbf{k},\eta'),\ \tilde\pi(\mathbf{x}) \equiv \frac{1}{(2\pi)^{3/2}}\int d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} \frac{1}{k^{1/2}}\hat\pi(\mathbf{k}) = (-\nabla^2)^{-1/4}\hat\pi(\mathbf{x}). \tag{147}$$

Putting the inverse Fourier transforms of Eq. (147) into (146), we get the result we have been seeking:

$$|\psi,\eta\rangle = \mathcal{T} e^{-i\int_{-\mathcal{T}}^{\eta} d\eta' \hat H - \frac{1}{4\tilde\lambda}\int_{-\mathcal{T}}^{\eta} d\eta' \int d\mathbf{x}'[w(\mathbf{x}',\eta') - 2\tilde\lambda\,\tilde\pi(\mathbf{x}')]^2}|\psi,-\mathcal{T}\rangle. \tag{148}$$

This is just the standard CSL state vector evolution, where the collapse-generating operators (toward whose joint eigenstates collapse tends) are $\tilde{\pi}(\mathbf{x})$ for all $\mathbf{x}$.

Similarly, in the second case, defining

$$\tilde{y}(\mathbf{x}) \equiv \frac{1}{(2\pi)^{3/2}} \int d\mathbf{k}\, e^{i\mathbf{k}\cdot\mathbf{x}} k^{1/2} y(\mathbf{k}) = (-\nabla^2)^{1/4}\hat{y}(\mathbf{x}), \tag{149}$$

the result is

$$|\psi, t\rangle = \mathcal{T} e^{-i\int_{-\mathcal{T}}^{\eta} d\eta' \hat{H} - \frac{1}{4\tilde{\lambda}}\int_{-\mathcal{T}}^{\eta} d\eta' \int d\mathbf{x}[w(\mathbf{x},\eta') - 2\tilde{\lambda}\tilde{y}(\mathbf{x})]^2} |\psi, -\mathcal{T}\rangle. \tag{150}$$

This is just the standard CSL state vector evolution, where the collapse-generating operators (toward whose joint eigenstates collapse tends) are $\tilde{y}(\mathbf{x})$ for all $\mathbf{x}$.

Are there fundamental reasons determining the appearance of the operators $(-\nabla^2)^{-1/4}\hat{\pi}(\mathbf{x})$ [or $(-\nabla^2)^{1/4}\hat{y}(\mathbf{x})$] rather than the more natural $\hat{\pi}(\mathbf{x})$ [or $\hat{y}(\mathbf{x})$]? Perhaps a truly satisfactory answer will have to wait for a general theory expressing, in all situations, from particle physics to cosmology, the exact form of the CSL-type of modification to the evolution of quantum states. However, we might look at the particular situation at hand and offer a two part response.

First, note that the dimensions of $w(\mathbf{x}', \eta')$ and $\tilde{\lambda}\,\tilde{\pi}(\mathbf{x}')$ have to be the same to achieve the CSL form exhibited in (148). In order that the exponent in (148) be dimensionless, the dimension of $w(\mathbf{x}', \eta')$ is $(\text{time})^{-5/2}$, so $\tilde{\pi}(\mathbf{x}')$ must have the dimension $(\text{time})^{-3/2}$. Now, in order that the Hamiltonian in (10) have the dimension $(\text{time})^{-1}$, $\hat{\pi}(\mathbf{x}')$ has the dimension $(\text{time})^{-2}$. Therefore, if the collapse-generating operator is to be of the form (operator) $\times$ $\hat{\pi}(\mathbf{x}')$, that operator must have the dimension $(\text{time})^{1/2}$, which of course is the dimension of $(-\nabla^2)^{-1/4}$.

But, second, then it follows that the collapse-generating operator is *effectively* $\hat{\pi}(\mathbf{x})$. As we have seen, if we set $\hat{H} = 0$ in Eq. (148), then at infinite time the collapse takes the system into some eigenstate of the complete commuting set of operators $\tilde{\pi}(\mathbf{x})$. But, that state will also be an eigenstate of the complete commuting set of operators $\hat{\pi}(\mathbf{x})$, since if $\tilde{\pi}(\mathbf{x})|\phi\rangle = u(\mathbf{x})|\phi\rangle$, then $\hat{\pi}(\mathbf{x})|\phi\rangle = (-\nabla^2)^{1/4}u(\mathbf{x})|\phi\rangle$.

Therefore, we may regard the collapse-generating operator $(-\nabla^2)^{-1/4}\hat{\pi}(\mathbf{x})$ as the theory's way of giving us, in the most natural way, collapse toward eigenstates of the inflaton fluctuation momentum field $\hat{\pi}(\mathbf{x})$ consistent with the CSL state vector evolution form. It is therefore quite remarkable that such collapse leads to a prediction that agrees with the scale invariant spectrum, in agreement with the current cosmological observations.

A similar case may be made for Eq. (150) and its *effective* collapse-generating operator, the inflaton fluctuation field $\hat{y}(\mathbf{x})$.

XII. DISCUSSION

In this manuscript, we treat the emergence of the seeds of cosmic structure from the dynamics of the fluctuation of the inflaton field. This is the approach employed in the usual quantum treatment of this problem but, as previously discussed, that approach does not really account for the emergence of primordial inhomogeneities and anisotropies of the universe. Our approach differs from the standard one in that the evolution of the state vector from the initial Bunch-Davies vacuum to the end of the inflation era invokes a version of the CSL dynamical collapse theory adapted to this setting. We have taken as the basic theoretical setting the approximation known as semiclassical gravity. Here, gravitation is treated at the classical level while the matter fields are treated at the quantum level, and their energy momentum is taken to appear in Einstein's equations as the corresponding expectation value.

In this treatment, the expectation value of an operator differs from that given in the standard quantum mechanical treatment, as a result of the CSL theory's modification of Schrödinger's equation. To apply the CSL theory, it is necessary to select the collapse-generating operator, toward whose eigenstates the collapse occurs. We have considered two cases. One is where the collapse-generating operator represents the Fourier mode of the inflaton field perturbation. The other is where the collapse-generating operator represents the Fourier mode of the momentum conjugate to the inflaton field perturbation.

As we have indicated, the quantity of direct observational interest, $|\alpha_{lm}|^2$, which characterizes the distribution of the temperature fluctuations across the celestial sphere in terms of an expansion in spherical harmonics, refers to the time of the decoupling η_D. Instead, we evaluated that quantity at the end of inflation, at $\eta = -\tau \approx 0$. The changes over this interval, from $-\tau$ to η_D, are codified in transfer functions $T_k(\eta_r, \eta_D)$: when taken into account, effectively give direct observational access to the spectrum at the end of inflation. This turns out to be the Harrison-Zel'dovich scale-invariant spectrum. That is what the theory must give.

We have seen that agreement between observations and theory will result if $\overline{\langle\hat{P}\rangle^2}(k)$ turns out to be proportional to k. This is achieved in the two cases of collapse-generating operators we considered, by choosing the collapse rate parameter λ of the CSL theory to have a simple dependence upon the mode's momentum magnitude k.

In the case where we take as "collapse-generating operator" the operator $\hat{P}$, it is necessary that $\lambda = \tilde{\lambda}/k$. λ in this case is dimensionless, so $\tilde{\lambda}$ has the dimension of rate.

In the case where we take as "collapse-generating operator" the operator $\hat{X}$, it is necessary that $\lambda = \tilde{\lambda}k$. λ in this case has dimensions $(\text{time})^{-2}$ so, again, $\tilde{\lambda}$ has the dimension of rate.

If we take seriously the idea that the choice of $\lambda(k)$ at large and small values of k must be chosen to make the integral in

(124) finite, we would expect deviations in the H-Z spectral form, for large and small k that, although unobservable today, could be potentially detected in future experiments.

Finally, one could ask which one of the two options we have discussed (or perhaps another one) might be the correct one? Why, in order to be consistent with observations (the H-Z spectrum), does the CSL parameter λ have (in each case) a particular, simple, dependence on k, leading to the collapse-generating operators $(-\nabla^2)^{-1/4}\hat{\pi}(\mathbf{x})$ and $(-\nabla^2)^{1/4}\hat{y}(\mathbf{x})$? We have no deeper reason, other than "it works out that way."

One may argue, utilizing the remarks above, that the H-Z spectrum arises from choosing $\lambda(k)$ in the simplest way consistent with the constraints that the effective collapse parameter $\tilde{\lambda}$ ought to have the dimension of rate, that the evolution of each mode contains just one dimensional parameter, k, and our results for $\overline{\langle\hat{P}\rangle^2}(k)$. This is suggestive, but it is not a fundamental or conclusive argument. For example, in the more general context of the inflationary problem we have considered, there are other quantities with the same dimension as k, such as the Planck mass, the GUT scale, the (inflationary potential)$^{1/4}$, the mass of the inflaton field, or a combination of these. If they are included along with k, there are other options to give $\tilde{\lambda}$ the dimension of rate which do not yield the H-Z spectrum. Therefore, one still wishes for a deeper reason for the choices we have uncovered. A response to this important question may have to wait for the so-far phenomenologically motivated inclusion of dynamical collapse in quantum physics to be replaced by a general and fundamental theory, perhaps for a natural relationship between dynamical collapse and gravitation [2,26] to be uncovered.

We conclude that the CSL theory, with a suitable choice of the operator controlling the collapse, is capable of addressing the shortcomings in the standard inflationary account of the emergence of the seeds of cosmic structure. That is, it can choose a universe possessing inhomogeneities and anisotropies. Our results are in agreement with observation in the regimes so far investigated empirically. We have argued that they are also consistent with and, indeed, require deviations at much smaller angular scales, which could be uncovered when the required technology becomes available.

While this paper was being written, we learned of the results of a similar study that came to a different conclusion [27]. We believe that the reason their uncertainty, or spread of the final wave function for the relevant modes $\mathbf{k}$, becomes large and ours small is connected with the fact that we are considering a different observable. They looked at (or took as "focus" operator) the field amplitude for the v field (the Mukahnov-Sasaki variable, which is a combination of the perturbed scalar field and the Newtonian potential). We took as focus operator the momentum conjugate to the field variable, as discussed at the start of Sec. VI.

This difference arises due to the different ways that gravitation is considered within the two approaches. The work of Ref. [27] follows the now-traditional approach of treating the gravitational perturbations just as any other field which can be subjected to direct quantization. The present work follows an approach initiated in Ref. [15], which is based on the idea that here one is dealing with a situation where gravitation must be considered as emergent and not suitable for the standard quantization procedure. This view led us to adopt a semiclassical treatment of the gravitational perturbations (for a more in-depth discussion of this point see Sec. VIII of Ref. [7]).

It is well known that the exponential expansion of the scale factor produces an extreme squeezing in the quantum states. Expressed in terms of suitable canonical variables, that leads to an enormous growth in the uncertainty of the field amplitude, and an enormous decrease in the uncertainty of the conjugate momentum. The paper [27] focuses on an amplitude operator which, in the absence of collapse, exhibits a large increase in the uncertainty due to the cosmic expansion. On the contrary, we have been led to focus on the operator $\hat{P}$ that, in the absence of collapse, exhibits no increase in the uncertainty (it remains constant). It seems therefore that the results can be understood as follows: The localizing effect of CSL is not enough to overcome the large increase in the uncertainty of the operator considered in Ref. [27], but on the other hand it is enough to produce the desired localization in the operator whose uncertainty would have remained constant in the absence of the localization effects of CSL. This is consistent with our finding that the fractional dispersion in $\hat{P}$ decreases like $\mathcal{T}^{-1}$. (We re-emphasize that the focus operator, the object for which we compute expectation values, should not be confused with the "collapse generating operator" $\hat{A}$, which is the object driving the CSL dynamics.)

In fact, as an exercise (although there is no physical motivation in our considerations for doing so), we have carried out an analysis similar to the one performed in Secs. VIII and IX for $\hat{P}$, but taking the focus operator to be $\hat{X}$. In that case we find that the uncertainty in the value of $\hat{X}$ for the state that results from the CSL evolution diverges when the conformal time for the end of inflation $\tau \rightarrow 0$.

ACKNOWLEDGMENTS

We thank Jerome Martin and Vincent Vennin for very useful exchanges. The work of D. S. is supported in part by the CONACYT Grant No. 101712. and by UNAM-PAPIIT Grant No. IN107412.

APPENDIX: FINDING $w(\eta)$ IN TERMS OF $v(\eta)$

In this Appendix, we invert Eq. (113),

$$v(\eta) = w(\eta) - 2S\int_{-\mathcal{T}}^{\eta} d\eta' w(\eta') e^{-S(\eta-\eta')}\cos R(\eta-\eta'), \tag{A1}$$

solving for $w(\eta)$ in terms of $v(\eta)$.

We extend the integral's lower limit to $-\infty$, keeping in mind that $w(\eta) = v(\eta) = 0$ for $\eta < -\mathcal{T}$, and extend the upper limit to ∞ by putting a factor $\Theta(\eta - \eta')$ in the integrand [$\Theta(x)$ is the step function]. Upon expressing $w(\eta')$ in the integrand in terms of its Fourier transform $\tilde{w}(\omega)$, and changing the integration variable to $x \equiv \eta' - \eta$, we obtain

$$v(\eta) = w(\eta) - 2S\int_{-\infty}^{\infty} dx\Theta(-x)e^{Sx}\cos Rx \times \int_{-\infty}^{\infty} d\omega e^{i\omega(x+\eta)}\tilde{w}(\omega) = w(\eta) - \int_{-\infty}^{\infty} d\omega e^{i\omega\eta}\frac{2S(S+i\omega)}{(S+i\omega)^2+R^2}\tilde{w}(\omega). \quad \text{(A2)}$$

Taking the Fourier transform yields

$$\tilde{v}(\omega) = \tilde{w}(\omega)\frac{k^2-\omega^2}{(S+i\omega)^2+R^2} \quad \text{or} \quad \tilde{w}(\omega) = \tilde{v}(\omega) - \frac{2S(S+i\omega)}{\omega^2-k^2}\tilde{v}(\omega), \quad \text{(A3)}$$

where we have used (109). When taking the Fourier transform of (A3), the integration path is taken below the poles on the real axis, so that $w(\eta)$ depends upon $v(\eta')$ for $\eta' \le \eta$. This results in

$$w(\eta) = v(\eta) - 2S\int_{-\infty}^{\infty} d\eta' v(\eta')\left[S + \frac{d}{d\eta}\right]\frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{i\omega(\eta-\eta')}\frac{1}{(\omega-k-i\epsilon)(\omega+k-i\epsilon)} = v(\eta) + \frac{2S}{k}\int_{-\infty}^{\infty} d\eta' v(\eta')\left[S + \frac{d}{d\eta}\right]\Theta(\eta-\eta')\sin k(\eta-\eta'), \quad \text{(A4)}$$

which is the result reported in Eq. (115).

[1] For reviews about the various approaches to the measurement problem in quantum mechanics, see for instance the classic references, M. Jammer, *Philosophy of Quantum Mechanics. The Interpretations of Quantum Mechanics in Historical Perspective* (John Wiley and Sons, New York, 1974); B. D'Espagnat, *Conceptual Foundations Of Quantum Mechanics* (W.A. Benjamin, Menlo Park, CA, 1970).
[2] R. Penrose, *The Emperor's New Mind* (Oxford University, New York, 1989).
[3] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic Publishers, Dordrecht, 1993); R. Omnes, *The Interpretation of Quantum Mechanics* (Princeton University, Princeton, NJ, 1994); and the more specific critiques S. L. Adler, Stud. Hist. Phil. Mod. Phys. **34**, 135 (2003); A. Elby, Found. Phys. Lett. **6**, 5 (1993); the review in A. Bassi and G. C. Ghirardi, Phys. Rep. **379**, 257 (2003).
[4] J. Bell, in *Sixty-Two Years of Uncertainty*, edited by A. Miller (Plenum, New York, 1990), pp. 17–31.
[5] J. B. Hartle, arXiv:gr-qc/9701022.
[6] D. Bohm and J. Bub, Rev. Mod. Phys. **38**, 453 (1966); F. Karolyhazy, Nuovo Cimento A **42**, 390 (1966); P. Pearle, Phys. Rev. D **13**, 857 (1976).
[7] D. Sudarsky, Int. J. Mod. Phys. D **20**, 509 (2011).
[8] N. F. Mott, Proc. R. Soc. A **126**, 79 (1929).
[9] A. Albrecht, P. Ferreira, M. Joyce, and T. Prokopec, Phys. Rev. D **50**, 4807 (1994).
[10] D. Polarski and A. A. Starobinsky, Classical Quantum Gravity **13**, 377 (1996).
[11] R. Laflamme and A. Matacz, Int. J. Mod. Phys. D **02**, 171 (1993); F. C. Lombardo and D. Lopez Nacir, Phys. Rev. D **72**, 063506 (2005).
[12] C. Kiefer and D. Polarski, Adv. Sci. Lett. **2**, 164 (2009).
[13] G. C. Ghirardi, A. Rimini, and T. Weber, Phys. Rev. D **34**, 470 (1986); **36**, 3287 (1987); Found. Phys. **18**, 1 (1988).
[14] P. Pearle, Phys. Rev. A **39**, 2277 (1989); G. C. Ghirardi, P. Pearle, and A. Rimini, Phys. Rev. A **42**, 78 (1990).
[15] A. Perez, H. Sahlmann, and D. Sudarsky, Classical Quantum Gravity **23**, 2317 (2006); A. De Unanue and D. Sudarsky, Phys. Rev. D **78**, 043510 (2008); S. J. Landau, C. G. Scoccola, and D. Sudarsky, Phys. Rev. D **85**, 123001 (2012).
[16] B. S. De Witt, Phys. Rev. **160**, 1113 (1967); J. A. Wheeler, in *Battelle Reencontres 1987*, edited by C. De Witt and J. A. Wheeler (Benjamin, New York, 1968); C. J. Isham, arXiv:gr-qc/9210011.
[17] A. Diez-Tejedor and D. Sudarsky, J. Cosmol. Astropart. Phys. 07 (2012) 045.
[18] S. Weinberg, *Cosmology* (Oxford University Press, New York, 2008).
[19] V. Mukhanov, *Physical Foundations of Cosmology* (Cambridge University Press, Cambridge, England, 2005).
[20] C. Moller, in *Les Theories Relativistes de la Gravitation, Colloques Internationaux CNRS 91*, edited by A. Lichnerowicz and M.-A. Tonnelat (CNRS, Paris, 1962).
[21] L. Rosenfeld, Nucl. Phys. **40**, 353 (1963).

[22] T. W. B. Kibble, in *Quantum Gravity 2: A Second Oxford Symposium*, edited by C. J. Isham, R. Penrose, and D. W. Sciama (Clarendon, Oxford, 1981), p. 63; D. N. Page and C. D. Geilker, Phys. Rev. Lett. **47**, 979 (1981).
[23] P. Pearle, Phys. Rev. A **78**, 022107 (2008).
[24] See for instance Ref. [19], p. 347.
[25] G. León and D. Sudarsky, Classical Quantum Gravity **27**, 225017 (2010).
[26] P. Pearle and E. Squires, Found. Phys. **26**, 291 (1996).
[27] J. Martin, V. Vennin, and P. Peter, Phys. Rev. D **86**, 103524 (2012).