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Black holes in six-dimensional conformal gravity

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We study conformally invariant theories of gravity in six dimensions. In four dimensions, there is a unique such theory that is polynomial in the curvature and its derivatives, namely, Weyl-squared, and furthermore all solutions of Einstein gravity are also solutions of the conformal theory. By contrast, in six dimensions there are three independent conformally invariant polynomial terms one could consider. There is a unique linear combination (up to overall scale) for which Einstein metrics are also solutions, and this specific theory forms the focus of our attention in this paper. We reduce the equations of motion for the most general spherically symmetric black hole to a single fifth-order differential equation. We obtain the general solution in the form of an infinite series, characterized by five independent parameters, and we show how a finite three-parameter truncation reduces to the already known Schwarzschild-AdS metric and its conformal scaling. We derive general results for the thermodynamics and the first law for the full five-parameter solutions. We also investigate solutions in extended theories coupled to conformally invariant matter, and in addition we derive some general results for conserved charges in cubic-curvature theories in arbitrary dimensions.

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I. INTRODUCTION

Higher-derivative gravity theories are of interest for a variety of reasons. They arise naturally in string theory and M-theory, in the form of higher-order corrections to the leading Einstein-Hilbert term in the low-energy effective action. In this context, the corrections take the form of an infinite series of terms that involve derivatives of arbitrarily high order. There are also situations where it is of interest to consider theories where there are just a finite number of higher-derivative terms. Examples include topologically massive gravity in three dimensions [1,2], where a gravitational Chern-Simons term gives a three-derivative contribution, proportional to the Cotton tensor; new massive gravity in three dimensions [3], where there is a four-derivative contribution arising from a curvature-squared term in the action; and numerous higher-dimensional examples involving curvature-squared or higher modifications to Einstein gravity. Recent examples that have been considered in four dimensions include Einstein gravity with a cosmological constant, with an additional Weyl-squared term whose coefficient may be tuned to give ‘‘critical gravity’’ for which the additional normally massive spin-2 excitations around an AdS background become massless [4]; and pure Weyl-squared conformal gravity, which has been argued to be equivalent to Einstein gravity with a cosmological constant [5]. In dimensions $D \leq 6$, supersymmetric extensions of certain higher-derivative theories are also known. These can arise because the supersymmetry is realized off shell, with the added higher-derivative bosonic terms being extended to complete and independent superinvariants. Thus, unlike the situation in the string or M-theory effective actions, where supersymmetry is on shell and works order by order, requiring an infinity of higher-order terms, in the off-shell supergravities only a finite number of terms are required.

In four dimensions there is a unique conformally invariant pure gravity theory that is polynomial in the curvature, for which the action is given by the square of the Weyl tensor. It has the important feature that any Einstein metric is also a solution of the conformal theory. Furthermore, since any conformal scaling of a solution is also a solution, this means that any conformally Einstein metric is automatically a solution of four-dimensional conformal gravity. This is a useful property when one is looking for solutions to the theory, since previously known ones from Einstein gravity will be solutions too. Of course, since the equations of motion of the conformal gravity are of higher order than those in Einstein gravity, there will exist further solutions over and above those of Einstein gravity. In a recent paper [6], various classes of solutions in four-dimensional conformal gravity were investigated in detail, including spherically symmetric asymptotically AdS black holes, and black holes obeying asymptotically Lifshitz boundary conditions. The general spherically symmetric asymptotically AdS black hole solution was already known [7]. It has one additional parameter, over and above the mass and the cosmological constant of the Schwarzschild-AdS black hole. This parameter can be understood as coming from the freedom to make a (spherically symmetric) conformal rescaling of Schwarzschild-AdS black hole.

It does, nevertheless, provide an interesting extension of the usual Schwarzschild-AdS black holes, in which the additional parameter can be interpreted as a characterization of massive spin-2 "hair." In [6], the thermodynamics and the first law for these extended solutions was studied. In four dimensions, charged rotating black holes [8] and the generalized Plebanski solutions [9] were also obtained. The neutral solutions are all conformal to Einstein metrics [8].

In the present paper, we carry out some analogous investigations in six-dimensional conformal gravity. The situation is more complicated in six dimensions because there is no longer a unique choice of conformal theory. In fact, there is a three-parameter family of conformal gravities in six dimensions that have actions polynomial in the curvature and its derivatives (see [10,11]), described by the action $I = \beta_1 I_1 + \beta_2 I_2 + \beta_3 I_3$, where[1]

$$I_1 = C_{\mu\rho\sigma\nu}C^{\mu\alpha\beta\nu}C_\alpha{}^{\rho\sigma}{}_\beta, \qquad I_2 = C_{\mu\nu\rho\sigma}C^{\rho\sigma\alpha\beta}C_{\alpha\beta}{}^{\mu\nu},$$
$$I_3 = C_{\mu\rho\sigma\lambda}\Big(\delta^\mu_\nu\Box + 4R^\mu{}_\nu - \frac{6}{5}R\delta^\mu_\nu\Big)C^{\nu\rho\sigma\lambda} + \nabla_\mu J^\mu,$$
$$J^\mu = 4R_\mu{}^{\lambda\rho\sigma}\nabla^\nu R_{\nu\lambda\rho\sigma} + 3R^{\nu\lambda\rho\sigma}\nabla_\mu R_{\nu\lambda\rho\sigma} - 5R^{\nu\lambda}\nabla_\mu R_{\nu\lambda} + \frac{1}{2}R\nabla_\mu R - R_\mu{}^\nu\nabla_\nu R + 2R^{\nu\lambda}\nabla_\nu R_{\lambda\mu}, \qquad (1.1)$$

and the coefficients β_1, β_2, and β_3 are arbitrary. In general, Einstein metrics will not be solutions of the theory, except in the special case where $\beta_1 = 4\beta_2 = -12\beta_3$. In particular, with this choice of parameters the theory allows Schwarzschild-AdS black holes as solutions, and this has the advantage that at least some explicit spherically symmetric solutions are available for investigation.

Accordingly, we shall consider the Lagrangian

$$e^{-1}\mathcal{L}_{\rm conf} = \beta\Big(4I_1 + I_2 - \frac{1}{3}I_3\Big) = \beta\Big(RR^{\mu\nu}R_{\mu\nu} - \frac{3}{25}R^3 - 2R^{\mu\nu}R^{\rho\sigma}R_{\mu\rho\nu\sigma} - R^{\mu\nu}\Box R_{\mu\nu} + \frac{3}{10}R\Box R\Big) + \text{total derivative}. \qquad (1.2)$$

The equations of motion of this system are given by

$$E_{\mu\nu} \equiv E^{(1)}_{\mu\nu} - \frac{3}{25}E^{(2)}_{\mu\nu} - 2E^{(3)}_{\mu\nu} - E^{(4)}_{\mu\nu} + \frac{3}{10}E^{(5)}_{\mu\nu} = 0, \qquad (1.3)$$

where the individual contributions $E^{(n)}_{\mu\nu}$ coming from the variation of each term in (1.2) are given in Appendix B.

In Sec. II, we study the equations of motion for spherically symmetric black hole solutions. These can be reduced to a fifth-order ordinary differential equation for a single undetermined metric function. As mentioned above, the Schwarzschild-AdS metric of six-dimensional Einstein gravity is a solution, and furthermore, any conformal scaling is also a solution. This provides us with an explicit three-parameter family of spherically symmetric black hole solutions, but, unlike the situation in four-dimensional conformal gravity, this does not exhaust the space of solutions, which should be characterized by a total of five parameters. We have not been able to construct the most general such solution explicitly, but we have constructed it as an infinite series expansion for the metric function, with explicit expansion coefficients.

In Sec. III, we use the Noether procedure to construct a conserved charge which, when integrated over a compact spatial surface at infinity, provides an expression for the mass of the black hole. Only the first few terms in our series expansion for the metric function contribute in this asymptotic formula, and so we are able to obtain an explicit expression for the mass of the general five-parameter solution. The same conserved charge, when integrated over the horizon, yields the expression for the product TS of the temperature and the entropy. Furthermore, the temperature itself can be calculated via a computation of the surface gravity. By this means, we are able to obtain explicit expressions for the temperature and entropy of the exact three-parameter family of black holes whose expression can be given in closed form.

In Sec. IV, using the general methods developed by Wald [13,14], we use the conserved charge mentioned above to derive the first law of thermodynamics for the general five-parameter spherically symmetric black holes. We also derive a Smarr-type formula for these solutions.

In Sec. V, we discuss extensions of the conformal gravity theory in which conformally invariant "matter" is added also. In particular, this can include a 2-form potential, and also an electromagnetic field whose field strength couples quadratically to the Weyl tensor. In Sec. VI we discuss various further explicit solutions of conformal gravity and these conformal matter extensions.

In Sec. VII, we give a general discussion of the calculation of conserved charges in curvature-cubed theories of gravity in arbitrary dimensions, using the general conformal methods developed by Ashtekar, Magnon, and Das (AMD) [15,16]. In Sec. VIII we discuss *tricritical gravity* in six dimensions, which was first constructed in [11].[2] This is obtained by appending an Einstein-Hilbert term, a cosmological term, and a Weyl-squared term to the

[1]We use the same conventions as in [12].

[2]The unitarity problem and consistent truncation of ghostlike logarithmic modes in multicritical gravity theories were studied in [17–19]. It was shown that at the level of the free theory, in special cases they could admit a unitary subspace. However, as pointed out later by [20], the analysis carried out for the free theory is invalid at the nonlinear level, and the would-be unitary subspace suffers from a linearization instability and is absent in the full nonlinear theory. Including the ghostlike logarithmic modes seems to be indispensable for the consistency of the theory. As a consequence, these multicritical gravity theories were conjectured to be the gravity duals of multirank logarithmic conformal field theories (CFTs).

conformal theory that we have been studying in this paper. The coefficients of the additional terms are tuned so that the additional spin-2 modes around an AdS background, which are generically massive, become massless. We include a discussion of consistent boundary conditions that can be imposed in this theory. Following the conclusions in Sec. IX, we then include two appendices. In Appendix A, we review the derivation of a useful necessary condition [21,22] that must be satisfied by any metric that is conformal to an Einstein metric. This provides a valuable tool when investigating whether a given solution in the conformal gravity might be "new," as opposed to merely being a conformal scaling of a previously known Einstein metric. Finally, in Appendix B, we give expressions for the contributions to the field equations that result from the various six-derivative terms that arise in the six-dimensional theories that we are considering.

II. STATIC BLACK HOLE SOLUTIONS

We shall consider the *Ansatz* for static solutions of the form

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_{4,k}^2, \qquad (2.1)$$

where for $k = 1, -1$, or 0 the metric $d\Omega_{4,k}^2$ describes a unit 4-sphere, hyperbolic 4-space, or the 4-torus, respectively, and f is a function of r. (The metric functions in g_{tt} and g_{rr} can be taken to be inversely related, as we have done here, by using the conformal symmetry.) Since we shall typically be concentrating on the $k = 1$ case we shall commonly refer to the metric as being "spherically symmetric," even when k is unspecified. Substituting the *Ansatz* into the equations of motion (1.3), we find that all the equations are satisfied provided that the equation $E_{rr} = 0$ is satisfied. This gives rise to a fifth-order differential equation for the function $f(r)$. (Analogous solutions for an action using just I_1 and I_2 were obtained in [23].)

It is in fact possible to exploit the conformal symmetry of the problem to obtain a simpler parametrization. Passing to the conformally related metric $d\hat{s}^2$, and introducing a new radial coordinate ρ and metric function $h(\rho)$ defined by

$$d\hat{s}^2 = r^{-2} ds^2, \qquad \rho = 1/r, \qquad h(\rho) = r^{-2} f(r), \qquad (2.2)$$

we obtain

$$d\hat{s}^2 = -h(\rho) dt^2 + \frac{d\rho^2}{h(\rho)} + d\Omega_{4,k}^2. \qquad (2.3)$$

Now in the new metric, the equations of motion imply simply

$$\begin{aligned} &-216k^3 + 42kh''^2 + 6h''^3 - 84kh'h^{(3)} - 18h'h''h^{(3)} \\ &\quad + 5h(h^{(3)})^2 + 20h'^2h^{(4)} - 10hh''h^{(4)} + 10hh'h^{(5)} = 0, \end{aligned} \qquad (2.4)$$

where a prime means a derivative with respect to ρ, and $h^{(n)}$ denotes the nth derivative of h.

If Eq. (2.4) is differentiated once more, it yields a rather simple sixth-order equation[3]

$$10hh^{(6)} + 30h'h^{(5)} + 12h''h^{(4)} - 13(h^{(3)})^2 - 84kh^{(4)} = 0. \qquad (2.5)$$

Using on Eqs. (2.4) and (2.5), we can obtain the general spherically symmetric solution as a series expansion of the form

$$h(\rho) = \sum_{n\geq 0} \frac{b_n}{n!} \rho^n, \qquad (2.6)$$

where $\{b_0, b_1, b_2, b_3, b_4\}$ are free parameters, while b_5 and b_n, $(n \geq 6)$ are determined by

$$\begin{aligned} &6b_2^3 - 18b_1b_2b_3 + 5b_0b_3^2 + 42kb_2^2 - 84kb_1b_4 - 216k^3 + 20b_1^2b_4 - 10b_0b_2b_4 + 10b_0b_1b_5 = 0, \\ &10b_0b_{2n} + \sum_{m=1}^{n-1} \alpha(2n, m) b_m b_{2n-m} + \beta(2n, n) b_n^2 - 84kb_{2n-2} = 0, \qquad n \geq 3, \\ &10b_0b_{2n+1} + \sum_{m=1}^{n} \alpha(2n+1, m) b_m b_{2n+1-m} - 84kb_{2n-1} = 0, \qquad n \geq 3, \end{aligned} \qquad (2.7)$$

with the coefficients $\alpha(n, m)$ and $\beta(2n, n)$ given by

$$\begin{aligned} \alpha(n, m) &= 2C_{n-6}^{m-6} + 30C_{n-6}^{m-5} + 12C_{n-6}^{m-4} - 26C_{n-6}^{m-3} \\ &\quad + 12C_{n-6}^{m-2} + 30C_{n-6}^{m-1} + 10C_{n-6}^{m}, \\ \beta(2n, n) &= 10C_{2n-6}^{n} + 30C_{2n-6}^{n-1} + 12C_{2n-6}^{n-2} - 13C_{2n-6}^{n-3}. \end{aligned} \qquad (2.8)$$

Here $C_n^k = n!/(k!(n-k)!)$ is the binomial coefficient, and it is understood that the factorial of a negative integer is infinity. The first equation in (2.7) relating b_0 to b_5 is obtained by substituting Eq. (2.6) into Eq. (2.4) and setting ρ to zero. Similarly, the second (and third) equation in Eq. (2.7) are obtained by inserting Eq. (2.6) into the $(2n-6)$th (or $(2n-5)$th) derivative of Eq. (2.5) and

[3]In the case of four-dimensional conformal gravity, the analogous equation that results from differentiating the third-order equation for h is simply $h^{(4)} = 0$, showing that the general spherically symmetric static solution of four-dimensional conformal gravity is given by a third-order polynomial.

choosing $\rho = 0$. Contained within the five-parameter general solution Eq. (2.6) are black hole solutions.

In terms of the previous spherically symmetric *Ansatz* Eq. (2.1), the five-parameter solution takes the form

$$a(r) = f(r) = r^2\left(a_0 + \frac{a_1}{r} + \frac{a_2}{r^2} + \frac{a_3}{r^3} + \frac{a_4}{r^4} + \frac{a_5}{r^5} + \sum_{n\geq 6}\frac{a_n}{r^n}\right). \tag{2.9}$$

Here we use parameters $a_n = b_n/n!$ for later convenience.

We find that there exists a three-parameter subset of solutions that corresponds to a finite truncation of the five-parameter general solutions. In terms of the usual parametrization Eq. (2.1), it is given by

$$f = r^2\left(a_0 + \frac{a_1}{r} + \frac{a_2}{r^2} + \frac{a_3}{r^3} + \frac{a_4}{r^4} + \frac{a_5}{r^5}\right), \tag{2.10}$$

where

$$a_1 = \frac{a_4(a_4^3 + 50ka_5^2)}{125a_5^3}, \qquad a_2 = k + \frac{2a_4^3}{25a_5^2}, \qquad a_3 = \frac{2a_4^2}{5a_5}, \qquad a_n = 0 \quad \text{for } n \geq 6. \tag{2.11}$$

In fact this three-parameter subset of the general solutions admits of a very simple interpretation. As we already noted, any solution of the Einstein equations is also a solution of the specific conformally invariant theory we are considering here. Furthermore, any conformal scaling of an Einstein metric will also be a solution. The solutions given by (2.10) and (2.11) are in fact precisely the family of conformal scalings of the Schwarzschild-AdS metric that can be cast within the form of the *Ansatz* (2.1). To see this, we start from the Schwarzschild-AdS metric in the standard form

$$ds^2_{\rm SAdS} = -\left(k + y^2/L^2 - \frac{m}{y^3}\right)dt^2 + \left(k + y^2/L^2 - \frac{m}{y^3}\right)^{-1}dy^2 + y^2 d\Omega^2_{4,k}, \tag{2.12}$$

which satisfies $R_{\mu\nu} = -5L^2 g_{\mu\nu}$. The metrics (2.1) with f given by (2.10) and (2.11) are conformally related, with $ds^2_{\rm SAdS} = \Omega^2 ds^2$, where

$$\Omega^2 = \frac{1}{(cr+1)^2}, \quad y = \frac{r}{1+cr}, \quad a_0 = c^2k + \frac{1}{L^2} - c^5 m, \quad a_1 = 2ck - 5c^4 m, \quad a_2 = k - 10c^3 m, \quad a_3 = -10c^2 m, \quad a_4 = -5cm, \quad a_5 = -m. \tag{2.13}$$

The "thermalized vacuum" corresponds to solutions with $\mu = 0$ (see [6] for the analogous discussion in four-dimensional conformal gravity). The thermodynamic quantities for the Schwarzschild-AdS black hole in six-dimensional conformal gravity are given by

$$E = -96\beta\frac{m}{L^4}, \qquad T = \frac{5m - 2ky_+^3}{4\pi y_+^4}, \qquad S = -96\pi\beta\left(\frac{y_+^4}{L^4} - k\right). \tag{2.14}$$

These quantities satisfy the first law of thermodynamics

$$dE = TdS. \tag{2.15}$$

A. Spherically symmetric solutions that are not conformally Einstein metrics

In [6], it was shown that the general spherically symmetric solution of four-dimensional conformal gravity is conformal to the Schwarzschild-AdS (dS) metric. By contrast, we find that the general five-parameter solutions given in Eq. (2.6) are not conformal to any Einstein metric. To see this, let us suppose that $e^{2\phi}d\hat{s}^2$ was in fact an Einstein metric. By using the necessary condition for a metric to be conformally Einstein in six dimensions [22] (see Appendix A), we find ϕ must be a function of ρ and that

$$\phi' = \frac{h'''}{3(2k - h'')}. \tag{2.16}$$

Combining this equality with the requirement that $e^{2\phi}d\hat{s}^2$ be an Einstein metric implies that h should satisfy

$$3h''h^{(4)} - 2(h^{(3)})^2 - 6kh^{(4)} = 0, \tag{2.17}$$

which then implies that h is a certain fifth-order polynomial in ρ. Substituting back into the equations of motion for conformal gravity then leads us back to the closed-form three-parameter solution given by (2.10) and (2.11). Thus, we have proved that the general spherically symmetric solution of six-dimensional conformal gravity is not conformally Einstein.

III. ENERGY OF ADS BLACK HOLES IN $D = 6$ CONFORMAL GRAVITY

To calculate the energy of the black hole solutions in Eq. (2.6), we start from the conformally invariant Lagrangian in Eq. (1.2) and derive the Noether charge associated with the Killing vector ξ^μ. We consider the variation of the Lagrangian 6-form induced by ξ^μ,

$$\mathcal{L}_\xi L = E^{\alpha\beta}\mathcal{L}_\xi g_{\alpha\beta} + d\Theta(g_{\alpha\beta}, \mathcal{L}_\xi g_{\alpha\beta}), \tag{3.1}$$

where $E^{\alpha\beta}$ represents the equations of motion. When $E^{\alpha\beta} = 0$ is satisfied, then using the identity

$$\mathcal{L}_\xi = di_\xi + i_\xi d, \tag{3.2}$$

for the Lie derivative of a differential form, we find a conserved current defined by

$$J = \Theta - i_\xi L, \qquad dJ = 0 \Rightarrow J = dQ[\xi]. \tag{3.3}$$

Explicitly, in six-dimensional conformal gravity, the conserved charge is a 4-form

$$Q[\xi] = \frac{1}{2!4!} \int \epsilon_{\alpha\beta\mu\nu\lambda\rho} Q^{\alpha\beta} dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\rho, \tag{3.4}$$

which consists of two parts, $Q_1^{\alpha\beta} + Q_2^{\alpha\beta}$. $Q_1^{\alpha\beta}$ and $Q_2^{\alpha\beta}$ depend on $\nabla_\mu \xi_\nu$ and ξ_μ, respectively:

$$\begin{aligned}
Q_1^{\alpha\beta} &= X^{\alpha\beta\mu\nu} \nabla_\mu \xi_\nu, \\
X^{\alpha\beta\mu\nu} &= -\beta\Big(24 C^{[\alpha}{}_\lambda{}^{|\nu|}{}_\rho C^{\beta]\lambda\mu\rho} - 6 C^{[\alpha}{}_{\lambda\rho\sigma} g^{\beta][\nu} C^{\mu]\lambda\rho\sigma} + \frac{3}{5} C^{\lambda\rho}{}_{\sigma\delta} C^{\sigma\delta}{}_{\lambda\rho} g^{\alpha[\mu} g^{\nu]\beta}\Big) \\
&\quad - \beta\Big(6 C^{\alpha\beta}{}_{\lambda\rho} C^{\lambda\rho\mu\nu} - 2 C^{[\alpha}{}_{\lambda\rho\sigma} g^{\beta][\nu} C^{\mu]\lambda\rho\sigma} + \frac{1}{5} C^{\lambda\rho}{}_{\sigma\delta} C^{\sigma\delta}{}_{\lambda\rho} g^{\alpha[\mu} g^{\nu]\beta}\Big) + \frac{2}{3}\beta(2\Box C^{\alpha\beta\mu\nu} + 4 C^{[\alpha}{}_{\lambda\rho\sigma} g^{\beta][\nu} C^{\mu]\lambda\rho\sigma} \\
&\quad + R^{\lambda[\alpha} C_\lambda{}^{\beta]\mu\nu} + R^{\lambda[\mu} C_\lambda{}^{\nu]\alpha\beta} - R^{\lambda\rho} C_\lambda{}^\alpha{}_\rho{}^{[\mu} g^{\nu]\beta} + R^{\lambda\rho} C_\lambda{}^\beta{}_\rho{}^{[\mu} g^{\nu]\alpha}) - \frac{2}{5}\beta(4 R C^{\alpha\beta\mu\nu} + C^{\lambda\rho}{}_{\sigma\delta} C^{\sigma\delta}{}_{\lambda\rho} g^{\alpha[\mu} g^{\nu]\beta}); \\
Q_2^{\alpha\beta} &= \xi_\nu \nabla_\mu X^{\alpha\beta\mu\nu} + \frac{2}{3}\beta \xi^{[\alpha} J^{\beta]} - \frac{2}{3}\beta(2\xi^\lambda C^{[\beta}{}_{\nu\rho\sigma} \nabla^{\alpha]} R_\lambda{}^{\nu\rho\sigma} - 2\xi^\lambda C^{[\alpha}{}_{\nu\rho\sigma} \nabla_\lambda R^{\beta]\nu\rho\sigma} - 2\xi_\lambda C^{\lambda\nu\rho\sigma} \nabla^{[\alpha} R^{\beta]}{}_{\nu\rho\sigma} \\
&\quad - 2\xi^\lambda R_{\lambda\nu\rho\sigma} \nabla^{[\alpha} C^{\beta]\nu\rho\sigma} + 2\xi^\lambda \nabla_\lambda C^{[\alpha}{}_{\nu\rho\sigma} R^{\beta]\nu\rho\sigma} + 2\xi^\lambda \nabla^{[\alpha} C_{\lambda\nu\rho\sigma} R^{\beta]\nu\rho\sigma} - C^{\lambda\rho\sigma\delta} \xi^{[\alpha} \nabla^{\beta]} C_{\lambda\rho\sigma\delta}).
\end{aligned} \tag{3.5}$$

When evaluated at infinity, $Q[\xi]$ gives the mass of black hole solutions in Eq. (2.6)

$$E = \beta V(\Omega_k)\Big(96 a_0^2 a_5 - \frac{12}{25}(17k^2 a_1 - 14k a_1 a_2 - 3a_1 a_2^2 + 54 k a_0 a_3 + 5a_1^2 a_3 + 46 a_0 a_2 a_3 - 60 a_0 a_1 a_4)\Big), \tag{3.6}$$

after using the on-shell relations among the a_is. It should be emphasized that this expression for the mass is valid for the full five-parameter family of solutions.

When evaluated on the horizon, $Q[\xi]$ is equal to TS. Since $\nabla_\mu \xi_\nu = \kappa \epsilon_{\mu\nu}$, where κ is the surface gravity on the horizon, $\epsilon_{\mu\nu}$ is the binormal vector horizon normalized to satisfy $\epsilon_{\mu\nu}\epsilon^{\mu\nu} = -2$ and the Killing vector ξ vanishes on the horizon, it follows that the entropy formula can be simplified to give

$$S = \pi \int_{\mathcal{H}} X^{\alpha\beta\mu\nu} \epsilon_{\alpha\beta} \epsilon_{\mu\nu} d\Sigma, \tag{3.7}$$

where $X^{\alpha\beta\mu\nu}$ is defined in the first line of Eq. (3.5). Explicit calculation for the three-parameter closed-form solutions (2.10) and (2.11) leads to

$$S = -\frac{\beta 96\pi V(\Omega_k)(5a_5 + a_4 r_+)^3 (125 a_5^3 + 15 a_4^2 a_5 r_+^2 + a_4^3 r_+^3 + 75 a_4 a_5^2 r_+ + 250 a_5^2 r_+^3)}{15625 a_5^4 r_+^6}, \tag{3.8}$$

where r_+ is the largest positive root of $f(r) = 0$, i.e., it is the radius of outer horizon. The temperature is given by

$$T = -\frac{(5a_5 + a_4 r_+)(125 a_5^3 + 75 a_4 a_5^2 r_+ + 15 a_4^2 a_5 r_+^2 + a_4^3 r_+^3 + 50 a_5^2 r_+^3)}{500\pi a_5^3 r_+^4}. \tag{3.9}$$

By using the parameter relations in Eq. (2.13), it can be shown that the entropy and temperature of the three-parameter black holes in six-dimensional conformal gravity are equal to those of the conformally related Schwarzschild-AdS black hole. In other words, the entropy and temperature are conformal invariants, as is also the case in four dimensions [6].[4] This is related to the fact that conformal factor Ω^2 in (2.13) is regular on the horizon and the near-horizon geometry is preserved. By contrast, the asymptotic region of the Schwarzschild black hole is altered by the conformal transformation and hence the Schwarzschild black hole energy in (2.14) becomes (3.6). It follows that the first law of thermodynamics (2.15) of the Schwarzschild black hole no longer applies. The three-parameter black hole is a globally distinct spacetime even though it is locally conformal to the Schwarzschild black hole. We shall derive the first law of thermodynamics in the next section.

If we define the Helmholtz energy to be $F = -T I_E$, where I_E is the Euclidean action, then we find that

$$F = E - TS. \tag{3.10}$$

A simple way to see this is to calculate the Euclidean action of the conformally related Schwarzschild-AdS metric with

[4] It should be emphasized that the expression for the mass of the black holes, given by (3.6), is valid for the general five-parameter solutions, since it is evaluated on a surface at infinity where only the leading orders in the radial falloff contribute. By contrast, the entropy (3.8) and temperature (3.9) are evaluated on the horizon, and so without having closed-form expressions for the general solutions, these can only be evaluated explicitly for the three-parameter closed-form truncation.

$y \in [y_+, 1/c]$ [see Eq. (2.13)]. In general, to obtain a finite action, certain counterterms are needed. However, because of the conformality of the action of conformal gravity, it turns out that without the addition of counterterms, the on-shell action for the asymptotically AdS solutions discussed in this paper is finite. This phenomenon has been observed previously in [5,6,11].

IV. BLACK HOLE THERMODYNAMICS

In the previous section, we derived the conserved quantities in six-dimensional conformal gravity by the Noether method. The expressions for the entropy and temperature of the general spherical solution are given by

$$T = -\frac{h'(\rho_+)}{4\pi},$$
$$S = \frac{4\pi\beta V(\Omega_k)}{25}(204 - 84h''(\rho_+) - 9h''(\rho_+)^2 + 10h'(\rho_+)h^{(3)}(\rho_+)). \tag{4.1}$$

From Eq. (3.3), one can see that

$$dQ = -i_\xi L. \tag{4.2}$$

Evaluating this equation in the region bounded by horizon and infinity just gives

$$F = E - TS. \tag{4.3}$$

To study the first law of thermodynamics, we follow the construction of [13,14]. We do this by considering the difference between $J[\xi, g_{\alpha\beta} + \delta g_{\alpha\beta}]$ and $J[\xi, g_{\alpha\beta}]$, where $g_{\alpha\beta} + \delta g_{\alpha\beta}$ also solves the equation of motion, in other words, where $\delta g_{\alpha\beta}$ satisfies the linearized equations of motion. We have

$$\delta J = \mathcal{L}_\xi \Theta - i_\xi \delta L. \tag{4.4}$$

Utilizing the identity in Eq. (3.2) and the on-shell condition $d\Theta = \delta L$, we find

$$d(\delta Q - i_\xi \Theta(g_{\alpha\beta}, \delta g_{\alpha\beta})) = 0, \tag{4.5}$$

where the definition of δ has been given in the previous section. Evaluating this equation in the region bounded by the horizon and infinity leads to the first law of thermodynamics

$$dE = TdS - \int_\infty \Theta(g_{\alpha\beta}, \delta g_{\alpha\beta}). \tag{4.6}$$

In the context of conformal gravity, the cosmological constant a_0 is a parameter of the solution, rather than of the theory, and hence we may treat a_0 as a further thermodynamic variable. Treating the cosmological constant as a thermodynamic variable has been considered previously. See, for example, [6,24–26]. Specific to the six-dimensional conformal gravity, by calculating the second term in the above equation, we obtain for the general five-parameter solutions

$$dE = TdS + \Psi_0 da_0 + \Psi_1 da_1 + \Psi_2 da_2 + \Psi_3 da_3, \tag{4.7}$$

where

$$\Psi_0 = \frac{24\beta V(\Omega_k)}{25}(50a_0a_5 + 20a_1a_4 - 19a_2a_3 - 6ka_3),$$
$$\Psi_1 = \frac{12\beta V(\Omega_k)}{25}(3a_2^2 + 14ka_2 - 17k^2 - 5a_1a_3 - 20a_0a_4),$$
$$\Psi_2 = -\frac{48\beta V(\Omega_k)}{5}a_0a_3,$$
$$\Psi_3 = -\frac{72\beta V(\Omega_k)}{5}a_0(a_2 - k). \tag{4.8}$$

These quantities satisfy the Smarr-like formula

$$E = 2\Psi_0 a_0 + \Psi_1 a_1 - \Psi_3 a_3, \tag{4.9}$$

which coincides with the result from dimensional analysis. Since the solution is asymptotically AdS, a_0 plays the role of a cosmological constant, with Ψ_0 being its conjugate variable. As was discussed in [6], in Einstein gravity, where the entropy is simply one quarter of the horizon area, without explicit dependence on the cosmological constant, Ψ_0 has the interpretation of being the volume of the black hole. In conformal gravity, on the other hand, the entropy has a manifest dependence on the cosmological constant, and hence Ψ_0 is not simply proportional to the volume. a_1, a_2, and a_3 are the extra integration constants of the fifth-order equations of six-dimensional conformal gravity, as compared with the second-order equations in Einstein gravity. The extra canonical-conjugate pairs (Ψ_1, a_1), (Ψ_2, a_2), and (Ψ_3, a_3) can be interpreted as "massive spin-2 hair," because the spectrum of six-dimensional conformal gravity contains in addition to the massless graviton, two massive gravitons (strictly speaking, one of them is partially massless).

It is straightforward to verify that the explicit three-parameter black holes given by (2.10) and (2.11) indeed satisfy the first law (4.7) and Smarr relation (4.9). It should be emphasized, however, that the more general five-parameter solutions, which we are only able to present as infinite series expansions, will also satisfy the first law and Smarr relation.

V. COUPLING TO CONFORMAL MATTER

In four dimensions, Maxwell theory is conformally invariant, and so is the enlarged system when it is coupled to conformal gravity. Charged black hole solutions in this theory can be used for studying some strongly coupled fermionic systems, such as non-Fermi liquids. In particular, the charged massless Dirac equation can be solved exactly for a generic frequency ω and wave number k. Using this, an explicit expression for the Green function $G(\omega, k)$ was obtained for general ω and k in Refs. [27,28]. By contrast, such a Green function in the Reissner-Nordstrøm black hole geometry can only be obtained explicitly for small ω, and in the extremal or near-extremal limit [29,30].

The analogous conformal "matter" in six dimensions is described by a 2-form potential B whose field strength is $H = dB$. It is also possible to write down a conformally invariant coupling of a vector potential A coupled quadratically to the Weyl tensor through its field strength $F = dA$. Slightly more generally, we may consider a conformally invariant matter Lagrangian of the form

$$\mathcal{L}_{\rm mat} = \sqrt{-g}\Big(\gamma C^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} - \frac{1}{12}H^2\Big) + \sigma B\wedge F\wedge F, \tag{5.1}$$

where the 3-form field strength is now given by

$$H = dB + \sigma A\wedge F, \tag{5.2}$$

with σ being a constant. The Bianchi identity and the equation of motion for H are given by

$$dH = \sigma F\wedge F = d*H. \tag{5.3}$$

It follows that it is consistent to impose the self-duality condition $H = *H$. Note that the Maxwell field A can be replaced, more generally, by a Yang-Mills field without breaking the conformal symmetry. The rich structure of conformal matter suggests that there should be a variety of applications of six-dimensional conformal gravity in the AdS/CFT correspondence.

VI. FURTHER SOLUTIONS

In this section, we present various further solutions of conformal gravity and of the conformal theories with additional fields that we discussed in the previous section. Specifically, Sec. VI A contains solutions of the pure conformal gravity theory, Sec. VI B contains solutions of the conformal theory including a Maxwell field, and Sec. VI C contains solutions in the conformal theory with instead a 2-form potential.

A. Neutral solutions

1. Lifshitz black holes:

There are Lifshitz vacuum solutions in the theory described by (1.2), given by

$$ds^2 = -r^{2z}\Big(1+\frac{1}{r^2}\Big)dt^2 + \frac{\sigma dr^2}{r^2}\Big(1+\frac{1}{r^2}\Big)^{-1} + r^2 d\Omega_4^2, \tag{6.1}$$

with $z = 0$ or $z = \frac{8}{3}$:

$$z = 0\colon\ \sigma = 4, \qquad z = \frac{8}{3}\colon\ \sigma = \frac{4}{9}. \tag{6.2}$$

We can find explicit black hole solutions that are asymptotic to these Lifshitz geometries, and that are conformally related to the Schwarzschild-AdS solution. For $z = \frac{8}{3}$, we find the black hole solution

$$ds^2 = -r^{16/3} f dt^2 + \frac{4dr^2}{9r^2 f} + r^2 d\Omega_{4,k}^2, \tag{6.3}$$

where

$$f = r^{-10/3}\Big[-\frac{1}{5}\Lambda c^2 + k(r^{2/3}+a)^2 - mc^{-3}(r^{2/3}+a)^5\Big]. \tag{6.4}$$

It is conformally related to the Schwarzschild-AdS metric

$$d\hat{s}^2 = -c^2\Big(k - \frac{m}{\rho^3} - \frac{1}{5}\Lambda\rho^2\Big)dt^2 + \frac{d\rho^2}{(1-\frac{m}{\rho^3}-\frac{1}{5}\Lambda\rho^2)} + \rho^2 d\Omega_4^2, \tag{6.5}$$

by $ds^2 = \Omega^{-2}d\hat{s}^2$ with $\rho = \Omega r$ and

$$\Omega = \frac{c}{r^{5/3}+ar}. \tag{6.6}$$

For the case $z = 0$, we find the black hole solution

$$ds^2 = -f dt^2 + \frac{4dr^2}{r^2 f} + r^2 d\Omega_{4,k}^2, \tag{6.7}$$

where

$$f = r^2\Big[-\frac{1}{5}\Lambda c^2 + k(r^{-2}+a)^2 - mc^{-3}(r^{-2}+a)^5\Big]. \tag{6.8}$$

It is conformally related to the Schwarzschild-AdS metric (6.5) by $ds^2 = \Omega^{-2}d\hat{s}^2$ with $\rho = \Omega r$ and

$$\Omega = \frac{c}{r^{-1}+ar}. \tag{6.9}$$

2. String solution:

$$ds^2 = f(r)(-dt^2+dx^2) + f(r)^{-1}(dr^2 + r^2 d\Omega_3^2),$$
$$f = \Big(1-\frac{m}{r^2}\Big)^{-\frac{1}{2}+\sqrt{\frac{3}{2}}}\Big(1+\frac{m}{r^2}\Big)^{-\frac{1}{2}-\sqrt{\frac{3}{2}}}. \tag{6.10}$$

This solution has a power-law singularity at $r = \sqrt{m}$.

B. Charged black hole

The neutral spherically symmetric black hole solutions (2.6) can be generalized by turning on the vector field A that enters the Lagrangian (5.1). We may consider the black hole *Ansatz*

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_{4,k}^2, \qquad A = \phi dt, \tag{6.11}$$

where f and ϕ are functions of r only. Letting $\rho = 1/r$ and $h(\rho) = r^{-2}f(r)$, we find that the function ϕ satisfies

$$\phi'(\rho) = \frac{q}{h''(\rho)}, \tag{6.12}$$

where a derivative is with respect to ρ. The function h then satisfies

$$\beta(10hh^{(6)} + 30h'h^{(5)} + 12h''h^{(4)} - 13(h^{(3)})^2 - 84kh^{(4)}) = 15\gamma(\phi''^2 + \phi'\phi'''). \tag{6.13}$$

The general solution that is asymptotic to AdS takes the form

$$f = r^2\left(c_0 + \frac{c_1}{r} + \frac{c_2}{r^2} + \frac{c_3}{r^3} + \frac{c_4}{r^4} + \frac{c_5}{r^5} + \sum_{i=6}\frac{c_i}{r^i}\right), \tag{6.14}$$

and the coefficients c_i with $i \geq 6$ can be expressed in terms of the c_i with $i = 0, 1, \ldots, 5$, in a manner analogous to the way the b_i coefficients in (2.6) were solved in the neutral case.

We have found two special solutions:

1. Solution 1

We find a truncated solution

$$ds^2 = -(c_0 r^2 + c_1 r + c_2)dt^2 + \frac{dr^2}{c_0 r^2 + c_1 r + c_2} + r^2 d\Omega_{4,k}^2,$$
$$A = \frac{q}{2c_2 r}dt. \tag{6.15}$$

This solution is analogous to the four-dimensional "BPS" black hole obtained in [31]. In particular, when the c_i are chosen such that $(-g_{tt})$ is a perfect square, the metric has an AdS_2 factor in the near-horizon geometry, as in the four-dimensional example.

2. Solution 2

If $\beta = 0$, we find that the equations can be solved exactly, giving

$$a = a_0\left(1 + \frac{Q}{r}\right)^{3/2}, \qquad f = r^2\left(d_0 + \frac{d_1}{r} + \tilde{d}_0 a\right). \tag{6.16}$$

C. Black dyonic string solutions

We consider the Lagrangian density of six-dimensional conformal gravity coupled to a 2-form potential:

$$e^{-1}\mathcal{L} = \beta\left(4I_1 + I_2 - \frac{1}{3}I_3\right) - \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda}. \tag{6.17}$$

The associated equations of motion are

$$0 = \beta\left(E_{\mu\nu}^{(1)} - \frac{3}{25}E_{\mu\nu}^{(2)} - 2E_{\mu\nu}^{3} - E_{\mu\nu}^{(4)} + \frac{3}{10}E_{\mu\nu}^{(5)}\right) - \frac{1}{4}\left(H_{\mu}{}^{\lambda\rho}H_{\nu\lambda\rho} - \frac{1}{6}g_{\mu\nu}H_{\lambda\rho\sigma}H^{\lambda\rho\sigma}\right). \tag{6.18}$$

1. Type I

$$ds^2 = -f(r)dt^2 + r^2 dx^2 + f(r)^{-1}dr^2 + r^2 d\Omega_{3,k}^2,$$
$$H_{(3)} = Qr^{-2}dt \wedge dx \wedge dr + P\Omega_{(3,k)}. \tag{6.19}$$

The solution is given by

$$f = r^2\left(a_0 + \frac{a_1}{r} + \frac{a_2}{r^2} + \frac{a_3}{r^3} + \frac{a_4}{r^4} + \frac{a_5}{r^5}\right), \tag{6.20}$$

where a_0 and a_1 can take arbitrary values, $a_3 = a_4 = a_5 = 0$ and

$$\frac{24\beta}{25}(2a_2 - k)(a_2 + 2k)^2 - (P^2 + Q^2) = 0,$$
$$\frac{24\beta}{25}(a_2 + 2k)(4a_2^2 - a_2 k + 2k^2) + (P^2 + Q^2) = 0. \tag{6.21}$$

The above two equations lead to

$$a_2 + 2k = 0, \qquad (P^2 + Q^2) = 0, \qquad 3a_2 + k = 0,$$
$$\frac{40\beta}{9}k^3 + (P^2 + Q^2) = 0, \qquad a_2 = 0,$$
$$\frac{96\beta}{25}k^3 + (P^2 + Q^2) = 0. \tag{6.22}$$

Among these solutions, we find that the first one, with vanishing flux, is actually conformally Einstein metric. Explicitly,

$$d\hat{s}^2 = \Omega^2 ds^2, \quad \Omega^2 = r^{-2}\mathrm{sech}^2\frac{\sqrt{k}(x-c)}{\sqrt{2}} \quad \hat{R}_{\mu\nu} = \frac{5k}{2}\hat{g}_{\mu\nu}. \tag{6.23}$$

The conformal metric describes a static soliton located at $x = c$, when $k > 0$.

2. Type II

$$ds^2 = \frac{1}{H(r)}(-f(r)dt^2 + dx^2) + H(r)(f(r)^{-1}dr^2 + r^2 d\Omega_{3,k}^2),$$
$$H_{(3)} = QH(r)^{-2}r^{-3}dt \wedge dx \wedge dr + P\Omega_{(3,k)}. \tag{6.24}$$

One class of solution is found to be

$$H(r) = 1, \qquad f(r) = a_0 r^2 + a_2,$$
$$\frac{96\beta}{25}(a_2 - k)^2(a_2 + k) + (P^2 + Q^2) = 0. \tag{6.25}$$

3. Other Ansätze

We may consider the following *Ansatz* [32]:

$$ds^2 = f(r)(-dt^2 + dx^2) + f(r)^{-1}(dr^2 + r^2 d\Omega_3^2),$$
$$H_{(3)} = Qf^2 r^{-3}dt \wedge dx \wedge dr + P\Omega_{(3)}, \tag{6.26}$$

where $\Omega_{(3)}$ is the volume form of the unit S^3. Adopting the above *Ansatz*, the equations of motion for the 2-form potential are satisfied automatically. Before presenting the equation following from the metric variations, it is useful to make the field redefinitions and coordinate transformation

$$r = e^{\rho}, \qquad f(r) = e^{h(\rho)+\rho}, \qquad \dot{h}(\rho) = W(\rho), \tag{6.27}$$

where a dot denotes a derivative with respect to ρ. In terms of $W(\rho)$ we find

$$\begin{aligned} 0 = {} & 8 - 8W^2 - 8W^4 + 8W^6 - 24W^2\dot{W} - 136W^4\dot{W} \\ & + 6\dot{W}^2 + 14W^2\dot{W}^2 - 16\dot{W}^3 - 12W\ddot{W} - 28W^3\ddot{W} \\ & + 48W\dot{W}\ddot{W} + 5\ddot{W}^2 + 40W^2\dddot{W} - 10\dot{W}\dddot{W} \\ & + 10W\ddddot{W} + \frac{25}{12\beta}(P^2 + Q^2). \end{aligned} \tag{6.28}$$

A class of solutions of this equation is given by

$$f(r) = r^a, \quad \frac{96}{25}\beta(a-2)^2a^2(a^2-2a+2) + (P^2+Q^2) = 0. \tag{6.29}$$

For this solution to be real β must be negative, coinciding with the condition under which the energy and entropy of the AdS black holes are positive. Especially, when $a = 2$, the solution is $AdS_3 \times S_3$ with vanishing string charges. By a conformal scaling, the solutions can be mapped to

$$\begin{aligned} d\hat{s}^2 &= r^{-2a}ds^2 \\ &= (-dt^2 + dx^2) + d\rho^2 + (a-1)^2\rho^2 d\Omega_3^2. \end{aligned} \tag{6.30}$$

This has a conical singularity at the origin of the transverse space of the string.

To obtain $AdS_3 \times S_3$ solutions with nontrivial flux, we reparametrize the metric and $H_{(3)}$ as

$$\begin{aligned} ds^2 &= r^2(-dt^2 + dx^2) + r^{-2}dr^2 + a^2 d\Omega_3^2, \\ H_{(3)} &= Qf^2r^{-3}dt\wedge dx\wedge dr + Pa^3\Omega_{(3)}. \end{aligned} \tag{6.31}$$

The equations of motion are solved provided that

$$\frac{96}{25}\beta(a^2-1)^2(a^2+1) + a^6(P^2+Q^2) = 0. \tag{6.32}$$

The theory also admits $AdS_3 \times \tilde{S}_3$ as a solution, where $\tilde{S}_3$ is a squashed 3-sphere. The $AdS_3 \times \tilde{S}_3$ metric is given by

$$ds^2 = r^2(-dt^2 + dx^2) + r^{-2}dr^2 + a^2\sigma_3^2 + (\sigma_1^2 + \sigma_2^2). \tag{6.33}$$

If we choose the vielbeins to be

$$\begin{aligned} e^0 &= rdt, & e^1 &= rdx, & e^2 &= r^{-1}dr, \\ e^3 &= a\sigma_3, & e^4 &= \sigma_2, & e^5 &= \sigma_1, \end{aligned} \tag{6.34}$$

and $H_{(3)}$ to be given by

$$H_{(3)} = Qe^0\wedge e^1\wedge e^2 + Pe^3\wedge e^4\wedge e^5, \tag{6.35}$$

then we obtain a solution when

$$\frac{96}{25}\beta(a^2-1)(41-63a^2) + (P^2+Q^2) = 0. \tag{6.36}$$

VII. AMD CHARGE FOR GENERAL CUBIC CURVATURE THEORIES

In this section we apply the conformal methods developed by Ashtekar, Magnon, and Das (AMD) [15,16] for calculating conserved charges in asymptotically AdS backgrounds to the case of cubic-curvature theories in arbitrary dimensions. The AMD conserved quantities are extracted from the leading falloff of the electric part of the Weyl tensor. The falloff rate of the curvature is weighted by a smooth function Ω, with the conformal boundary $\mathcal{I}$ being defined at $\Omega = 0$. For further details about the conditions on the choice of Ω, the reader is referred to Refs. [15,16]; here we only mention some necessary points. For a d-dimensional asymptotically AdS spacetime ($d \geq 4$), on the boundary $\mathcal{I}$ we require

$$\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{7.1}$$

$$\text{At } \Omega = 0, \qquad \hat{n}_\mu = \partial_\mu\Omega \neq 0, \tag{7.2}$$

$$\hat{n}_\mu\hat{n}^\mu = \frac{1}{\ell^2}, \qquad \hat{\nabla}_\mu\hat{n}_\nu = 0. \tag{7.3}$$

Since $\mathcal{I}$ is defined to be at $\Omega = 0$, it follows that n_μ is a normal vector on the boundary $\mathcal{I}$. Near the boundary,

$$R_{\mu\nu\lambda\rho} \to -\frac{1}{\ell^2}(g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}), \tag{7.4}$$

$$T_{\mu\nu} \to \Omega^{d-2}\tau_{\mu\nu}, \tag{7.5}$$

$$C_{\mu\nu\lambda\rho} \to \Omega^{d-5}K_{\mu\nu\lambda\rho}, \tag{7.6}$$

where $g_{\mu\nu}$ is the physical metric, the hatted quantities are referred to the conformal metric $\hat{g}_{\mu\nu}$, and $T_{\mu\nu}$ is the energy momentum tensor. As first noticed in [33], the condition (7.4) is required in higher-curvature theories in order to ensure that the metric that satisfies the equations of motion is indeed asymptotically AdS. [In Einstein gravity, by contrast, as discussed in [15,16], Eq. (7.4) is *implied* by the Einstein equations together with Eqs. (7.5) and (7.6).]

For any theory of gravity with the equations

$$E_{\mu\nu} = 8\pi G_{(d)}T_{\mu\nu}, \tag{7.7}$$

one can show that

$$\Omega^{-(d-3)}(\nabla_{[\lambda}P_{\mu]\nu})\hat{n}^\lambda\hat{n}^\nu\xi^\mu = \frac{d-2}{2\ell^2}8\pi G_{(d)}\tau_{\mu\nu}\hat{n}^\nu\xi^\mu + O(\Omega), \tag{7.8}$$

where

$$P_{\mu\nu} \equiv E_{\mu\nu} - \frac{1}{d-1}g_{\mu\nu}E_{\lambda\rho}g^{\lambda\rho}. \tag{7.9}$$

In general, the leading falloff of $(\nabla_{[\lambda}P_{\mu]\nu})\hat{n}^\lambda\hat{n}^\nu\xi^\mu$ is of the order Ω^{d-3} and can be expressed as

$-\frac{(d-2)}{2(d-3)}\Xi\hat{\nabla}^{\rho}(K_{\lambda\mu\nu\rho}\hat{n}^{\lambda}\hat{n}^{\nu}\xi^{\mu})\Omega^{d-3}$. The conserved quantity associated with the Killing vector ξ^{μ} can be defined, when $\tau_{\mu\nu}$ vanishes on the boundary, as

$$Q_{\xi}[C] = -\frac{\ell\Xi}{8\pi G_{(d)}(d-3)}\int_C d\hat{S}^m_{(d-2)}\hat{\mathcal{E}}_{mn}\xi^n, \quad \hat{\mathcal{E}}_{mn} \equiv \ell^2 K_{\lambda m\rho n}\hat{n}^{\lambda}\hat{n}^{\rho}, \tag{7.10}$$

where the indices m and n label the coordinates on the $(d-1)$-dimensional boundary $\mathcal{I}$, since the electric part of the Weyl tensor, $\hat{\mathcal{E}}_{mn}$, has no components in the normal direction. C is a $(d-2)$ dimensional spherical cross section on $\mathcal{I}$.

Consider the Lagrangian for the general class of cubic-curvature theories of the form

$$\begin{aligned}16\pi G_{(d)}e^{-1}\mathcal{L} = {}& -2\Lambda + R + \alpha_1\mathcal{L}_{\rm GB} + \alpha_2 R^2 + \alpha_3 R_{\mu\nu}R^{\mu\nu} + \beta_1 RR_{\mu\nu}R^{\mu\nu} + \beta_2 R^3 + \beta_3 R_{\mu\lambda\nu\rho}R^{\mu\nu}R^{\lambda\rho}\\ &+ \beta_4 R_{\mu\nu}\Box R^{\mu\nu} + \beta_5 R\Box R + \beta_6 R_{\mu}{}^{\nu}R_{\nu}{}^{\lambda}R_{\lambda}{}^{\mu} + \beta_7 R_{\mu\nu}R^{\mu\lambda\rho\sigma}R^{\nu}{}_{\lambda\rho\sigma} + \beta_8 RR^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho}\\ &+ \beta_9 R^{\mu\nu}{}_{\lambda\rho}R^{\lambda\rho}{}_{\sigma\delta}R^{\sigma\delta}{}_{\mu\nu} + \beta_{10}R^{\mu}{}_{\lambda}{}^{\nu}{}_{\rho}R^{\lambda}{}_{\sigma}{}^{\rho}{}_{\delta}R^{\sigma}{}_{\mu}{}^{\delta}{}_{\nu}.\end{aligned} \tag{7.11}$$

The AMD formula for quadratic-curvature theories has been obtained in [34]. By repeating the procedure (the corrections to the equations of motion from cubic curvature terms are presented in Appendix B), we can obtain the contributions from the cubic-curvature terms to the AMD charges. In the general cubic-curvature theories Eq. (7.11), the AMD charges take the same form as in Eq. (7.10), with the coefficient of proportionality Ξ given by

$$\begin{aligned}\Xi = {}& 1 + R_0\Big[2\alpha_1\frac{(d-3)(d-4)}{d(d-1)} + 2\alpha_2 + \frac{2\alpha_3}{d}\Big] + R_0^2\Big[\frac{3\beta_1}{d} + 3\beta_2 + \frac{3\beta_3}{d^2} + \frac{3\beta_6}{d^2} + \beta_7\frac{2(9-2d)}{d^2(d-1)} + \beta_8\frac{2(9-2d)}{d(d-1)}\\ &+ \beta_9\frac{12(7-2d)}{d^2(d-1)^2} + \beta_{10}\frac{3(3d-8)}{d^2(d-1)^2}\Big],\\ R_0 = {}& -\frac{d(d-1)}{\ell^2}.\end{aligned} \tag{7.12}$$

We notice that the terms $R_{\mu\nu}\Box R^{\mu\nu}$ and $R\Box R$ do not contribute to the charge, for solutions whose asymptotic behavior obeys Eqs. (7.4) and (7.6).

As a check of the above formula, we can calculate the charge for the case of the six-dimensional Euler density

$$E_6 = \frac{1}{8}\epsilon_{\mu_1\nu_1\mu_2\nu_2\mu_3\nu_3}\epsilon^{\rho_1\sigma_1\rho_2\sigma_2\rho_3\sigma_3}R^{\mu_1\nu_1}{}_{\rho_1\sigma_1}R^{\mu_2\nu_2}{}_{\rho_2\sigma_2}R^{\mu_3\nu_3}{}_{\rho_3\sigma_3}. \tag{7.13}$$

In terms of the quantities in Eq. (7.11), the Euler density E_6 corresponds to the combination of cubic-curvature terms with coefficients

$$\begin{aligned}&\beta_1 = -12, \quad \beta_2 = 1, \quad \beta_3 = 24, \quad \beta_4 = 0,\\ &\beta_5 = 0, \quad \beta_6 = 16, \quad \beta_7 = -24, \quad \beta_8 = 3,\\ &\beta_9 = 4, \quad \beta_{10} = -8.\end{aligned} \tag{7.14}$$

Inserting these coefficients into Eq. (7.12), we find that the coefficient Ξ for E_6 is given by

$$\Xi_{E_6} = R_0^2\frac{24(d-3)(d-4)(d-5)(d-6)}{d^2(d-1)^2}, \tag{7.15}$$

which indeed vanishes for $d = 6$ as expected.

The AMD formula for general cubic-curvature theories can also be used for finding the criticality condition and computing the conserved quantities in quasitopological gravity [35,36]. The d-dimensional quasitopological gravity is defined by the action

$$\begin{aligned}I = {}& \frac{1}{16\pi G_d}\int d^dx\Big(\frac{(d-1)(d-2)}{L^2} + R + \frac{\lambda L^2}{(d-3)(d-4)}\mathcal{X}_4\\ &- \frac{8(2d-3)}{(d-6)(d-3)(3d^2-15d+16)}\mu L^4\mathcal{Z}_D\Big),\end{aligned} \tag{7.16}$$

where $\mathcal{X}_4$ is the Gauss-Bonnet combination and $\mathcal{Z}_D$ is the quasitopological combination consisting of cubic-curvature terms with

$$\begin{aligned}&\beta_1 = \frac{-3(3d-4)}{2(2d-3)(d-4)}, \quad \beta_2 = \frac{3d}{8(2d-3)(d-4)},\\ &\beta_3 = \frac{3d}{(2d-3)(d-4))}, \quad \beta_4 = 0, \quad \beta_5 = 0,\\ &\beta_6 = \frac{6(d-2)}{(2d-3)(d-4)}, \quad \beta_7 = -\frac{3(d-2)}{(2d-3)(d-4)},\\ &\beta_8 = \frac{3(3d-8)}{8(2d-3)(d-4)}, \quad \beta_9 = 0, \quad \beta_{10} = 1.\end{aligned} \tag{7.17}$$

This theory has as a solution the asymptotically AdS metric

$$ds^2 = -\Big(k + \frac{r^2}{L^2} f\Big)dt^2 + \frac{dr^2}{k + \frac{r^2}{L^2} f} + r^2 d\Omega_k^2, \qquad (7.18)$$

where $f(r)$ satisfies the cubic equation

$$\Big(1 - \frac{\omega^{d-1}}{r^{d-1}}\Big) - f + \lambda f^2 + \mu f^3 = 0. \qquad (7.19)$$

To compute the mass of these black holes using the AMD formula Eq. (7.12), we choose $\Omega = 1/r$. We then find that

$$C_{t\Omega t\Omega} \to -\frac{1}{2}(d-2)(d-3)\Omega^{d-5}\frac{\omega^{d-1}}{L^2(1-2\lambda f_\infty - 3\mu f_\infty^2)},$$
$$\Xi = 1 - 2\lambda f_\infty - 3\mu f_\infty^2, \qquad (7.20)$$

where f_∞ denotes the asymptotic value of $f(r)$ as r tends to infinity. Therefore, Eq. (7.10) gives the mass of the black holes in quasitopological gravity as

$$M = \frac{(d-2)\omega^{d-1} V(\Omega_k)}{16\pi G_{(d)} L^2}. \qquad (7.21)$$

The temperature and entropy (using Wald's formula) are [36]

$$T = \frac{(d-1)}{4\pi}\Big(\frac{\omega^{d-1} r_+^{6-d}}{L^2(r_+^4 + 2\lambda k L^2 r_+^2 - 3\mu k^2 L^4)} - \frac{2k}{(d-1)r_+}\Big),$$
$$S = \frac{V(\Omega_k)}{4G_{(d)}}\Big(r_+^{d-2} + \frac{2(d-2)}{d-4}\lambda k L^2 r_+^{d-4} - \frac{3(d-2)}{d-6}\mu k^2 L^4 r_+^{d-6}\Big). \qquad (7.22)$$

It is straightforward to check that the first law of thermodynamics holds in this case.

VIII. TRICRITICAL GRAVITY IN SIX DIMENSIONS

"Critical gravity" is the name given to higher-derivative theories of gravity that admit AdS backgrounds, and which generically describe massive as well as massless spin-2 modes, in the special case where the parameters of the theory are tuned such that the massive spin-2 modes become massless. One example is the chiral point in three-dimensional topologically massive gravity, discussed in [2], and another is the four-dimensional critical theory discussed in [4], where a Weyl-squared term with an appropriately tuned coefficient is added to cosmological Einstein gravity. In that case, with a fourth-order Lagrangian, there is one massive spin-2 excitation in addition to the usual massless spin-2. In theories of the kind we are considering in this paper, with sixth-order Lagrangians, there are in general two massive spin-2 excitations in addition to the massless spin-2, and so the possibility of tuning the parameters so that all three are massless arises. This is known as tricritical gravity.

A. The theory

In six dimensions, one tricritical gravity model has been constructed in which the scalar modes do not propagate [11]. The Lagrangian for this theory is given by

$$16\pi G_{(6)}\sigma^{-1}e^{-1}\mathcal{L}_6 = -2\Lambda + R + \frac{1}{2}\alpha C^{\mu\nu\lambda\rho}C_{\mu\nu\lambda\rho} - \mathcal{L}_{\rm conf}, \qquad (8.1)$$

where $\mathcal{L}_{\rm conf}$ is defined in Eq. (1.2) and σ is the overall sign. In the AdS_6 background, the spin-2 modes satisfy

$$-(\Box+2)\Big(1 + \frac{3}{2}\alpha(\Box+6) + \beta(\Box+6)(\Box+8)\Big)h_{\mu\nu} = 0. \qquad (8.2)$$

The tricriticality condition is achieved when

$$\alpha = -\frac{5}{12}, \qquad \beta = \frac{1}{16}, \qquad (8.3)$$

where the AdS "radius" has been set to 1.

Another tricritical model has the Lagrangian

$$16\pi G_{(6)}\sigma^{-1}e^{-1}\mathcal{L}_6 = -2\Lambda + R + \frac{1}{4}\alpha\Big(R^{\mu\nu}R_{\mu\nu} - \frac{3}{10}R^2\Big) - \mathcal{L}_{\rm conf}. \qquad (8.4)$$

This also admits all Einstein metrics as solutions. The spin-2 modes in this case satisfy

$$-(\Box+2)\Big(1 + \frac{1}{4}\alpha(\Box+10) + \beta(\Box+6)(\Box+8)\Big)h_{\mu\nu} = 0, \qquad (8.5)$$

and the tricriticality condition is achieved when

$$\alpha = -\frac{5}{7}, \qquad \beta = \frac{1}{56}. \qquad (8.6)$$

In both models, at the tricritical point the spin-2 modes satisfy

$$-(\Box + 2)^3 h_{\mu\nu} = 0. \qquad (8.7)$$

Massless, massive, and log modes of the spin-2 field $h_{\mu\nu}$ in AdS_6 were obtained in [37].

B. Consistent boundary conditions in tricritical gravity

Our starting point is the AdS_6 metric coordinatized as

$$ds^2 = \ell^2(-\cosh^2\rho dt^2 + d\rho^2 + \sinh^2\rho(d\theta_1^2 + \sin^2\theta_1(d\theta_2^2 + \sin^2\theta_2 d\phi_1^2 + \cos^2\theta_2 d\phi_2^2))). \qquad (8.8)$$

Near the AdS_6 boundary, the solutions to Eq. (8.7) [37] have the falloff behavior

$$h_{55} = \mathcal{O}(\rho^2 e^{-7\rho}), \qquad h_{5i} = \mathcal{O}(\rho^2 e^{-5\rho}),$$
$$h_{ij} = \mathcal{O}(\rho^2 e^{-3\rho}), \qquad (8.9)$$

where $x^i = \{t, \theta_1, \theta_2, \phi_1, \phi_2\}$ for $0 \le i \le 4$ and $x^5 = \rho$. This implies that near the AdS_6 boundary, the Weyl tensor has the asymptotic behavior

$$C_{\mu\nu\lambda\rho} \to \Omega(\log^2\Omega J_{\mu\nu\lambda\rho} + \log\Omega L_{\mu\nu\lambda\rho} + K_{\mu\nu\lambda\rho}), \tag{8.10}$$

where Ω approaches $e^{-2\rho}$ near the boundary. In this case, we find that for the two tricritical models, $(\nabla_{[\lambda} P_{\mu]\nu})\hat{n}^\lambda \hat{n}^\nu \xi^\mu$ remains of order Ω, and therefore one can obtain finite AMD charges for these two models.

Explicitly, for the first tricritical model, the AMD charge at the tricritical point is given by

$$Q_\xi[C]_1 = -\frac{25\ell\sigma}{192\pi G_{(6)}} \int_C d\hat{S}_4^m \hat{\mathcal{E}}_{mn} \xi^n, \qquad \hat{\mathcal{E}}_{mn} \equiv \ell^2 J_{\lambda m\rho n} \hat{n}^\lambda \hat{n}^\rho. \tag{8.11}$$

Similarly, for the second tricritical model, the AMD charge at the tricritical point is given by

$$Q_\xi[C]_2 = -\frac{25\ell\sigma}{672\pi G_{(6)}} \int_C d\hat{S}_4^m \hat{\mathcal{E}}_{mn} \xi^n, \qquad \hat{\mathcal{E}}_{mn} \equiv \ell^2 J_{\lambda m\rho n} \hat{n}^\lambda \hat{n}^\rho. \tag{8.12}$$

Asymptotic Killing vectors should be compatible with the boundary conditions (8.9), implying that they should obey

$$\mathcal{L}_\xi g_{55} = \mathcal{O}(\rho^2 e^{-7\rho}), \qquad \mathcal{L}_\xi g_{5i} = \mathcal{O}(\rho^2 e^{-5\rho}), \qquad \mathcal{L}_\xi g_{ij} = \mathcal{O}(\rho^2 e^{-3\rho}). \tag{8.13}$$

Vector fields ξ satisfying these equations (modulo "trivial" diffeomorphisms) generate the asymptotic symmetry group. We denote the Killing vector fields by U^μ_{ab} ($a, b = 1, \ldots, 7$). Since in the coordinate system used in Eq. (8.8), these obey $U^\mu_{ab} = \mathcal{O}(1)$, we find that the asymptotic Killing vector fields can only differ from the Killing vectors at the order

$$\xi^\mu = \frac{1}{2}\xi^{ab}_\infty U_{ab} + \mathcal{O}(\rho^2 e^{-7\rho}), \tag{8.14}$$

where ξ^{ab}_∞ is constant. The boundary conditions Eq. (8.9) can be verified to be consistent, yielding well-defined charges that are finite, integrable, and conserved. It can be shown that the associated asymptotic symmetry group is still $SO(2, 5)$.

IX. CONCLUSIONS

In this paper, we have studied some aspects of conformally invariant gravities in six dimensions. Unlike in four dimensions where there is a unique theory that is polynomial in the curvature or its derivatives (described by a Weyl-squared Lagrangian), in six dimensions there are three such independent conformally invariant terms that could be considered. However, if we impose the additional requirement that, like in four dimensions, Einstein metrics should also be solutions of the theory, then this implies that a unique linear combination of the three terms is singled out. It is this specific theory that has formed the focus of most of our attention in this paper, since it has the advantage that at least some solutions, namely, Einstein metrics and their conformal scalings, can be obtained explicitly.

Using the freedom to perform coordinate transformations and conformal scalings, the general *Ansatz* for spherically symmetric black holes can be expressed in terms of a single function of the radius. This function obeys a fifth-order differential equation which, unfortunately, we have not been able to solve in closed form in general. We were, however, able to construct the general solution as an infinite series expansion, characterized by the expected number of five independent parameters. Within this class of solutions is a three-parameter subset for which the series expansion terminates. This closed-form class of solutions corresponds precisely to the standard Schwarzschild-AdS metrics, and their spherically symmetric conformal scalings. We studied the thermodynamics of the black holes, obtaining a first law for the five-parameter family of solutions, and verifying that this was indeed satisfied by the explicit closed-form subset of solutions.

We considered also some more general conformal theories in six dimensions, in which conformally invariant "matter" is coupled to conformal gravity. Specifically, we looked at a bilinear coupling of a Maxwell field strength to the Weyl tensor, and also kinetic and Chern-Simons terms involving a 2-form potential. We obtained a variety of further solutions for these theories, and also for the pure conformal gravity.

In our work, we concentrated on the particular choice of six-dimensional conformal gravity for which conformally Einstein metrics are also solutions. It would be of interest also to study the broader class of conformal gravities in six dimensions for which this is no longer the case. It may not be easy, within the broader class of theories, to obtain explicit closed-form solutions, but nevertheless it could be of interest to investigate black hole solutions, and their thermodynamics.

A further interesting question is whether any of the six-dimensional conformal gravities could be supersymmetrized. As far as we are aware, there are no known obstacles to doing this, other than the complexity of the problem. If it could be achieved, then it would presumably be an off-shell theory, since experience suggests that this is the only way in which one is likely to be able to construct a higher-derivative supergravity that does not require an infinity of higher-order terms (such as in string theory).

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APPENDIX A: NECESSARY CONDITION FOR CONFORMALLY EINSTEIN

In this appendix, we present a detailed derivation of a necessary condition for a d-dimensional metric to be conformally Einstein metric. This condition was derived first in four dimensions in [21], and subsequently, in arbitrary dimensions, in [22].

A d-dimensional spacetime with metric g_{ab} is conformally Einstein if there exists a conformal transformation to a new metric $\hat{g}_{ab} = \Omega^2 g_{ab}$ such that

$$\hat{R}_{ab} - \frac{1}{d}\hat{g}_{ab}\hat{R} = 0, \tag{A1}$$

or, equivalently,

$$\hat{P}_{ab} - \frac{1}{d}\hat{g}_{ab}\hat{P} = 0, \tag{A2}$$

where

$$P_{ab} \equiv -\frac{1}{(d-2)}R_{ab} + \frac{1}{2(d-1)(d-2)}Rg_{ab}. \tag{A3}$$

Defining $\Upsilon_a \equiv \nabla_a \ln\Omega$, then from the conformal transformation of the Ricci scalar and Ricci tensor we have

$$\Omega^{-2}(R + 2(d-1)\nabla^c\Upsilon_c + (d-1)(d-2)\Upsilon_c\Upsilon^c) = \text{constant}, \tag{A4}$$

$$R_{ab} - \frac{1}{d}g_{ab}R + (d-2)\nabla_a\Upsilon_b - \frac{d-2}{d}g_{ab}\nabla_c\Upsilon^c - (d-2)\Upsilon_a\Upsilon_b + \frac{d-2}{d}g_{ab}\Upsilon^c\Upsilon_c = 0, \tag{A5}$$

and (A2) becomes

$$P_{ab} - \frac{1}{d}g_{ab}P - \nabla_a\Upsilon_b + \frac{1}{d}g_{ab}\nabla_c\Upsilon^c + \Upsilon_a\Upsilon_b - \frac{1}{d}g_{ab}\Upsilon^c\Upsilon_c = 0. \tag{A6}$$

Taking a derivative of (A4) gives

$$\begin{aligned}0 &= \nabla_a R - 2R\Upsilon_a - 4(d-1)\Upsilon_a\nabla^c\Upsilon_c - 2(d-1)(d-2)\Upsilon_a\Upsilon^c\Upsilon_c + 2(d-1)\nabla_a\nabla_c\Upsilon^c + 2(d-1)(d-2\Upsilon^c\nabla_a\Upsilon_c\\ &= \nabla_a P - 2P\Upsilon_a + 2\Upsilon_a\nabla^c\Upsilon_c + (d-2)\Upsilon_a\Upsilon^c\Upsilon_c - \nabla_a\nabla_c\Upsilon^c - (d-2)\Upsilon^c\nabla_a\Upsilon_c.\end{aligned} \tag{A7}$$

Using this, we obtain

$$\nabla_{[a}P_{b]c} + \frac{1}{2}C_{abcd}\Upsilon^d = 0. \tag{A8}$$

Using

$$\nabla^d C_{abcd} = 2(d-3)\nabla_{[a}P_{b]c}, \tag{A9}$$

we finally obtain

$$\nabla^d C_{abcd} + (d-3)\Upsilon^d C_{abcd} = 0. \tag{A10}$$

This necessary condition must be satisfied by any conformally Einstein metric.

APPENDIX B: EQUATION OF GENERAL CUBIC CURVATURE

In this appendix, we present the detailed results for the variations of each of the terms in the Lagrangian (7.11). In particular, this includes the results needed for obtaining the equations of motion (1.3) for the conformally invariant theory that forms the focus of most of our attention in this paper.

$$\begin{aligned}(1)\colon RR^{\mu\nu}R_{\mu\nu} \Rightarrow E^{(1)}_{\mu\nu} &= (\Box(R_{\lambda\sigma}R^{\lambda\sigma}) + \nabla_\lambda\nabla_\sigma(RR^{\lambda\sigma}) - \frac{1}{2}RR_{\lambda\sigma}R^{\lambda\sigma})g_{\mu\nu} + R_{\lambda\sigma}R^{\lambda\sigma}R_{\mu\nu} + 2RR_{\lambda\mu}R^{\lambda}{}_{\nu} + (\Box RR_{\mu\nu})\\ &\quad - \nabla_\mu\nabla_\nu(R_{\lambda\sigma}R^{\lambda\sigma}) - \nabla_\lambda\nabla_\mu(RR^{\lambda}{}_{\nu}) - \nabla_\lambda\nabla_\nu(RR^{\lambda}{}_{\mu}),\end{aligned}$$

$$(2):\ R^3 \Rightarrow E^{(2)}_{\mu\nu} = \Big(3\Box R^2 - \frac{1}{2}R^3\Big)g_{\mu\nu} + 3R^2 R_{\mu\nu} - 3\nabla_\mu\nabla_\nu R^2,$$

$$(3):\ R^{\mu\nu}R^{\lambda\rho}R_{\mu\lambda\nu\rho} \Rightarrow E^{(3)}_{\mu\nu} = -\frac{1}{2}R^{\sigma\delta}R^{\lambda\rho}R_{\sigma\lambda\delta\rho}g_{\mu\nu} + \frac{3}{2}R^{\rho\sigma}R_{\rho\mu\sigma\lambda}R^\lambda{}_\nu + \frac{3}{2}R^{\rho\sigma}R_{\rho\nu\sigma\lambda}R^\lambda{}_\mu + \Box(R^{\rho\sigma}R_{\rho\mu\sigma\nu})$$
$$+ \nabla^\sigma\nabla^\delta(R^{\lambda\rho}R_{\lambda\sigma\rho\delta})g_{\mu\nu} - \nabla^\lambda\nabla_\mu(R^{\rho\sigma}R_{\rho\lambda\sigma\nu}) - \nabla^\lambda\nabla_\nu(R^{\rho\sigma}R_{\rho\lambda\sigma\mu}) - \nabla_{(\sigma}\nabla_{\lambda)}(R_\mu{}^\sigma R_\nu{}^\lambda)$$
$$+ \nabla_\sigma\nabla_\lambda(R_{\mu\nu}R^{\sigma\lambda}),$$

$$(4):\ R^{\mu\nu}\Box R_{\mu\nu} = -g^{\mu\nu}\nabla_\mu R^{\lambda\rho}\nabla_\nu R_{\lambda\rho} \Rightarrow E^{(4)}_{\mu\nu}$$
$$= \frac{1}{2}g_{\mu\nu}(g^{\sigma\delta}\nabla_\sigma R^{\lambda\rho}\nabla_\delta R_{\lambda\rho}) - (2\nabla^\sigma R_{\mu\lambda}\nabla_\sigma R_\nu{}^\lambda + \nabla_\mu R_{\sigma\lambda}\nabla_\nu R^{\sigma\lambda}) + 2\nabla_\lambda(R_{\sigma(\mu}\nabla_{\nu)}R^{\lambda\sigma})$$
$$+ 2\nabla_\lambda(\nabla^\lambda R^\sigma{}_{(\nu}R_{\mu)\sigma}) - 2\nabla_\sigma(\nabla_{(\mu}R_{\nu)\lambda}R^{\sigma\lambda}) + \Box^2 R_{\mu\nu} + \nabla_\sigma\nabla_\lambda R^{\sigma\lambda}g_{\mu\nu} - \nabla_\lambda\nabla_\nu(R^\lambda{}_\mu) - \nabla_\lambda\nabla_\mu(R^\lambda{}_\nu),$$

$$(5):\ R\Box R = -g^{\mu\nu}\nabla_\mu R\nabla_\nu R \Rightarrow E^{(5)}_{\mu\nu} = \frac{1}{2}g_{\mu\nu}(g^{\sigma\lambda}\nabla_\sigma R\nabla_\lambda R) - \nabla_\mu R\nabla_\nu R + 2(\Box R)R_{\mu\nu}$$
$$+ 2(\Box^2 R)g_{\mu\nu} - 2\nabla_\mu\nabla_\nu\Box R,$$

$$(6):\ R_\mu{}^\nu R_\nu{}^\lambda R_\lambda{}^\mu \Rightarrow E^{(6)}_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}R_\lambda{}^\rho R_\rho{}^\sigma R_\sigma{}^\lambda + 3R_{\lambda\mu}R_{\rho\nu}R^{\lambda\rho} + \frac{3}{2}\Box(R_\mu{}^\lambda R_{\lambda\nu}) + \frac{3}{2}\nabla_\rho\nabla_\sigma(R_\lambda{}^\rho R^{\lambda\sigma})g_{\mu\nu}$$
$$- \frac{3}{2}\nabla_\lambda\nabla_\nu(R^\lambda{}_\rho R^{\rho\mu}) - \frac{3}{2}\nabla_\lambda\nabla_\mu(R^\lambda{}_\rho R^{\rho\nu}),$$

$$(7):\ R_{\mu\nu}R^{\mu\lambda\rho\sigma}R^\nu{}_{\lambda\rho\sigma} \Rightarrow E^{(7)}_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}R_{\delta\tau}R^{\delta\lambda\rho\sigma}R^\tau{}_{\lambda\rho\sigma} + \frac{1}{2}g_{\mu\nu}\nabla_\delta\nabla_\tau(R^{\delta\lambda\rho\sigma}R^\tau{}_{\lambda\rho\sigma})$$
$$+ \frac{1}{2}\Box(R_\mu{}^{\lambda\rho\sigma}R_{\nu\lambda\rho\sigma}) - \frac{1}{2}\nabla_\delta\nabla_\mu(R^{\delta\lambda\rho\sigma}R_{\nu\lambda\rho\sigma}) - \frac{1}{2}\nabla_\delta\nabla_\nu(R^{\delta\lambda\rho\sigma}R_{\mu\lambda\rho\sigma})$$
$$+ R_{\delta\sigma}R^{\delta\lambda\rho}{}_\mu R^\sigma{}_{\nu\lambda\rho} + 2R_{\delta\sigma}R^{\delta\lambda\rho}{}_\mu R^\sigma{}_{\lambda\rho\nu} - 2\nabla_\rho\nabla_\sigma(R_{\lambda(\mu}R_{\nu)}{}^{\lambda\rho\sigma}) - 2\nabla_\rho\nabla_\sigma(R_{\lambda(\mu}R^{\lambda\rho\sigma}{}_{\nu)})$$
$$+ 2\nabla_\rho\nabla_\sigma(R^\rho{}_\lambda R^\lambda{}_{(\mu}{}^\sigma{}_{\nu)}),$$

$$(8):\ RR^{\mu\nu\lambda\rho}R_{\mu\nu\lambda\rho} \Rightarrow E^{(8)}_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}RR^{\lambda\rho\sigma\delta}R_{\lambda\rho\sigma\delta} + R_{\mu\nu}R^{\lambda\rho\sigma\delta}R_{\lambda\rho\sigma\delta} + 2RR_{\mu\lambda\rho\sigma}R_\nu{}^{\lambda\rho\sigma}$$
$$+ g_{\mu\nu}\Box(R^{\lambda\rho\sigma\delta}R_{\lambda\rho\sigma\delta}) - \nabla_\mu\nabla_\nu(R^{\lambda\rho\sigma\delta}R_{\lambda\rho\sigma\delta}) - 4\nabla_\lambda\nabla_\rho(RR^\lambda_{(\mu\nu)}{}^\rho),$$

$$(9):\ R^{\mu\nu}{}_{\lambda\rho}R^{\lambda\rho}{}_{\sigma\delta}R^{\sigma\delta}{}_{\mu\nu} \Rightarrow E^{(9)}_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}R^{\tau\eta}{}_{\lambda\rho}R^{\lambda\rho}{}_{\sigma\delta}R^{\sigma\delta}{}_{\tau\eta} + \frac{3}{2}R^\tau{}_{\mu\lambda\rho}R^{\lambda\rho}{}_{\sigma\delta}R^{\sigma\delta}{}_{\tau\nu}$$
$$+ \frac{3}{2}R^\tau{}_{\nu\lambda\rho}R^{\lambda\rho}{}_{\sigma\delta}R^{\sigma\delta}{}_{\tau\mu} + \frac{3}{2}\nabla^\sigma\nabla^\delta(R_{\sigma\mu}{}^{\lambda\rho}R_{\lambda\rho\delta\nu}) + \frac{3}{2}\nabla^\sigma\nabla^\delta(R_{\sigma\nu}{}^{\lambda\rho}R_{\lambda\rho\delta\mu}),$$

$$(10):\ R^\mu{}_\lambda{}^\nu{}_\rho R^\lambda{}_\sigma{}^\rho{}_\delta R^\sigma{}_\mu{}^\delta{}_\nu \Rightarrow E^{(10)}_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}R^\tau{}_\lambda{}^\eta{}_\rho R^\lambda{}_\sigma{}^\rho{}_\delta R^\sigma{}_\tau{}^\delta{}_\eta + \frac{3}{2}R^\sigma{}_\delta{}^\rho{}_\tau R^\delta{}_\lambda{}^\tau{}_\mu R^\lambda{}_{\sigma\nu\rho} + \frac{3}{2}R^\sigma{}_\delta{}^\rho{}_\tau R^\delta{}_\lambda{}^\tau{}_\nu R^\lambda{}_{\sigma\mu\rho}$$
$$- \frac{3}{2}\nabla_\delta\nabla_\sigma(R_\mu{}^{\lambda\sigma\rho}R^\delta{}_{\lambda\nu\rho}) - \frac{3}{2}\nabla_\delta\nabla_\sigma(R_\nu^{\lambda\sigma\rho}R^\delta{}_{\lambda\mu\rho}) + 3\nabla_\delta\nabla_\sigma(R_{(\mu}{}^\lambda{}_{\nu)}{}^\rho R^\delta{}_\lambda{}^\sigma{}_\rho). \qquad \text{(B1)}$$

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