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# Evolution of magnetic fields through cosmological perturbation theory

Héctor J. Hortua,* Leonardo Castañeda,† and J. M. Tejeiro‡

*Grupo de Gravitación y Cosmología, Observatorio Astronómico Nacional, Universidad Nacional de Colombia, cra 45 #26-85, Edificio Uriel Gutierréz, Bogotá, D.C., Colombia*

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The origin of galactic and extragalactic magnetic fields is an unsolved problem in modern cosmology. A possible scenario comes from the idea that these fields emerged from a small field, *a seed*, which was produced in the early universe (phase transitions, inflation, etc.) and it evolves in time. Cosmological perturbation theory offers a natural way to study the evolution of primordial magnetic fields. The dynamics for this field in the cosmological context is described by a cosmic dynamolike equation, through the dynamo term. In this paper we get the perturbed Maxwell's equations and compute the energy-momentum tensor to second order in perturbation theory in terms of gauge-invariant quantities. Two possible scenarios are discussed. First we consider a Friedmann-Lemaître-Robertson-Walker background without magnetic field, and we study the perturbation theory introducing the magnetic field as a perturbation. In the second scenario, we consider a magnetized Friedmann-Lemaître-Robertson-Walker and build up the perturbation theory from this background. We compare the cosmological dynamolike equation in both scenarios.

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## I. INTRODUCTION

Magnetic fields have been observed on several scales in the Universe. Galaxies and clusters of galaxies contain magnetic fields with strengths of $\sim 10^{-6}$ G [1]; fields within clusters are also likely to exist, with strengths of comparable magnitude [2]. There is also evidence of magnetic fields on scales of superclusters [3]. On the other hand, the possibility of a cosmological magnetic field has been addressed comparing the cosmic microwave background (CMB) quadrupole with one induced by a constant magnetic field (in coherence scales of $\sim$1 Mpc), constraining the field magnitude to $B < 6.8 \times 10^{-9}(\Omega_m h^2)^{1/2}$ Gauss [4]. However, the origin of such large-scale magnetic fields is still unknown. These fields are increased and maintained by a dynamo mechanism, but it needs a *seed* before the mechanism takes place [5]. Astrophysical mechanisms such as the Biermann battery have been used to explain how the magnetic field is maintained in objects such as galaxies, stars, and supernova remnants [6], but they are not likely correlated beyond galactic sizes [7]. It makes it difficult to use astrophysical mechanisms to explain the origin of magnetic fields on cosmological scales. In order to overcome this problem, the primordial origin should be found in other scenarios from which the astrophysical mechanism starts. For example, magnetic fields could be generated during primordial phase transitions (such as QCD, the electroweak phase transition, or grand unified theories), parity-violating processes that generate magnetic helicity, or during inflation [8]. Magnetic fields also are generated during the radiation era in regions with nonvanishing vorticity. This seed was proposed by Harrison [8]. Magnetic fields generation from density fluctuations in the prerecombination era has been investigated in [8]. The advantage of these primordial processes is that they offer a wide range of coherence lengths (many of which are strongly constrained by nucleosynthesis [9]), while the astrophysical mechanisms produce fields at the same order of the astrophysical size of the object. Recently a lower limit of the large-scale correlated magnetic field was found. It constrains models for the origin of cosmic magnetic fields, giving possible evidence for their primordial origin [10].

One way to describe the evolution of magnetic fields is through cosmological perturbation theory. This theory [11] is a powerful tool for understanding the present properties of the large-scale structure of the Universe and their origin. It has been mainly used to predict effects on the temperature distribution in the CMB [12]. Futhermore, linear perturbation theory combined with inflation suggests that primordial fluctuations of the Universe are adiabatic and Gaussian [13]. However, due to the high-precision measurements reached in cosmology, higher order cosmological perturbation theory is required to test the current cosmological framework [14,15]. There are mainly two approaches to studying higher order perturbative effects; one uses nonlinear theory and different manifestations of the separate universe approximation, using the $\Delta N$ formalism [16,17], and the other is the Bardeen approach where metric and matter fields are expanded in a power series [18]. Within the Bardeen approach, a set of variables are determined in such a way that has no gauge dependence. These are known in the literature as gauge-invariant variables, which have been widely used in different cosmological scenarios [19]. One

*hjhortuao@unal.edu.co
†lcastanedac@unal.edu.co
‡jmtejeiros@unal.edu.co

 

important result of cosmological perturbation theory is the coupling between gravity and electromagnetic fields, which have shown a magnetogeometrical interaction that could change the evolution of the fields on large scales. An effect is the amplification of cosmic fields. Indeed, large-scale magnetic fields in perturbed spatially open Friedmann-Lemaître-Robertson-Walker (FLRW) models decay as $a^{-1}$, a rate considerably slower than the standard $a^{-2}$ [20]. The hyperbolic geometry of these open FLRW models leads to the superadiabatic amplification on large scales [21]. The main goal in this paper is to study the late evolution of magnetic fields that were generated in early stages of the Universe. We use the cosmological perturbation theory following the gauge-invariant formalism to find the perturbed Maxwell equations up to second order, and also we obtain a dynamolike equation written in terms of gauge, invariant variables to first and second order. Furthermore, we discuss the importance that both curvature and the gravitational potential plays in the evolution of these fields. The paper is organized as follows. In the next section we briefly give an introduction of cosmological perturbation theory and we address the gauge problem in this theory. Section III presents the matter equations in the homogeneous and isotropic universe, which was used to generate the first- and second-order dynamical equations. In Sec. IV, we define the first-order gauge-invariant variables for the perturbations not only in the matter (energy density, pressure, magnetic, and electric field) but also in the geometrical quantities (gravitational potential, curvature, shear, etc.). The first-order perturbation of Maxwell's equations is reviewed in Sec. V and together with Ohm's law allows us to find the cosmological dynamo equation to describe the evolution of the magnetic field. The derivation of second-order Maxwell's equations is given in Sec. VII, and following the same methodology for the first-order case, we find the cosmological dynamo equation at second order written in terms of gauge-invariant variables. In Sec. IX, we use an alternative approximation to the model considering a magnetic field in the FLRW background. It is found that amplification effects of magnetic field appear at first order in the equations, besides the absence of fractional orders. Also a discussion between both approaches is done. Finally, Sec. X is devoted to a discussion of the main results and the connection with future works.

## II. THE GAUGE PROBLEM IN PERTURBATION THEORY

Perturbation theory helps us to find approximate solutions of the Einstein field equations through small deviations from an exact solution [22]. In this theory one works with two different space-times; one is the real space-time $(\mathcal{M}, g_{\alpha\beta})$ that describes the perturbed universe and the other is the background space-time $(\mathcal{M}_0, g^{(0)}_{\alpha\beta})$ that is an idealization and is taken as reference to generate the real space-time. Then, the perturbation of any quantity $\Gamma$ (e.g., energy density $\mu(x,t)$, 4-velocity $u^\alpha(x,t)$, magnetic field $B^i(x,t)$, or metric tensor $g_{\alpha\beta}$) is the difference between the value that the quantity $\Gamma$ takes in the real space-time and the value in the background at a given point.[1] In order to determine the perturbation in $\Gamma$, we must have a way to compare $\Gamma$ (tensor on the real space-time) with $\Gamma^{(0)}$ [$\Gamma^{(0)}$ being the value on $\mathcal{M}_0$]. This requires the assumption to identify points of $\mathcal{M}$ with those of $\mathcal{M}_0$. This is accomplished by assigning a mapping between these space-times called gauge choice given by a function $\mathcal{X}$: $\mathcal{M}_0(p) \to \mathcal{M}(\bar{p})$ for any point $p \in \mathcal{M}_0$ and $\bar{p} \in \mathcal{M}$, which generate a pullback

$$\begin{aligned}\mathcal{X}^*: \mathcal{M} &\to \mathcal{M}_0\\ T^*(\bar{p}) &\mapsto T^*(p),\end{aligned} \tag{1}$$

and thus points on the real and background space-time can be compared through $\mathcal{X}$. Then, the perturbation for $\Gamma$ is defined as

$$\delta\Gamma(p) = \Gamma(\bar{p}) - \Gamma^{(0)}(p). \tag{2}$$

We see that the perturbation $\delta\Gamma$ is completely dependent of the gauge choice because the mapping determines the representation on $\mathcal{M}_0$ of $\Gamma(\bar{p})$. However, one can also choose another correspondence $\mathcal{Y}$ between these space-times so that $\mathcal{Y}$: $\mathcal{M}_0(q) \to \mathcal{M}(\bar{p})$, $(p \neq q)$.[2] In the literature a change of this identification map is called gauge transformation. The freedom to choose between different correspondences is due to the general covariance in general relativity, which states that there is no preferred coordinate system in Nature [23]. Hence, this freedom will generate an arbitrariness in the value of $\delta\Gamma$ at any space-time point $p$, which is called gauge problem in the general relativistic perturbation theory and has been treated by [24]. This problem generates an unphysical degree of freedom to the solutions in the theory and therefore one should fix the gauge or build up nondependent quantities of the gauge.

### A. Gauge transformations and gauge-invariant variables

To define the perturbation to a given order, it is necessary to introduce the concept of Taylor expansion on a manifold and thus the metric and matter fields are expanded in a power series. Following [25], we consider a family of four-dimensional submanifolds $\mathcal{M}_\lambda$ with $\lambda \in \mathbb{R}$, embedded in a five-dimensional manifold $\mathcal{N} = \mathcal{M} \times \mathbb{R}$. Each submanifold in the family represents a perturbed space-time and the background space-time is represented by the manifold $\mathcal{M}_0$ ($\lambda = 0$). On these manifolds we consider that the Einstein field and Maxwell's equations are satisfied

[1]This difference should be taken in the same physical point.
[2]This is the active approach where transformations of the perturbed quantities are evaluated at the same coordinate point.

$$E[g_\lambda, T_\lambda] = 0 \quad \text{and} \quad M[F_\lambda, J_\lambda] = 0; \tag{3}$$

each tensor field $\Gamma_\lambda$ on a given manifold $\mathcal{M}_\lambda$ is extended to all manifold $\mathcal{N}$ through $\Gamma(p, \lambda) \equiv \Gamma_\lambda(p)$ to any $p \in \mathcal{M}_\lambda$; likewise the above equations are extended to $\mathcal{N}$.[3] We used a diffeomorphism such that the difference in the right side of Eq. (2) can be done. We introduce a one-parameter group of diffeomorphisms $\mathcal{X}_\lambda$ that identifies points in the background with points in the real space-time labeled with the value $\lambda$. Each $\mathcal{X}_\lambda$ is a member of a flow $\mathcal{X}$ on $\mathcal{N}$ and it specifies a vector field $X$ with the property $X^4 = 1$ everywhere (transverse to the $\mathcal{M}_\lambda$),[4] then points which lie on the same integral curve of $X$ have to be regarded as the same point [24]. Therefore, according to the above, one gets a definition for the tensor perturbation

$$\Delta\Gamma_\lambda \equiv \mathcal{X}^*_\lambda \Gamma|_{\mathcal{M}_0} - \Gamma_0. \tag{4}$$

At higher orders the Taylor expansion is given by [25]

$$\Delta^{\mathcal{X}}\Gamma_\lambda = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\delta^{(k)}_{\mathcal{X}}\Gamma - \Gamma_0 = \sum_{k=1}^{\infty}\frac{\lambda^k}{k!}\delta^{(k)}_{\mathcal{X}}\Gamma, \tag{5}$$

where

$$\delta^{(k)}_{\mathcal{X}}\Gamma = \left[\frac{d^k \mathcal{X}^*_\lambda \Gamma}{d\lambda^k}\right]_{\lambda=0,\mathcal{M}_0}. \tag{6}$$

Now, rewriting Eq. (4), we get

$$\mathcal{X}^*_\lambda\Gamma|_{\mathcal{M}_0} = \Gamma_0 + \lambda\delta^{(1)}_{\mathcal{X}}\Gamma + \frac{\lambda^2}{2}\delta^{(2)}_{\mathcal{X}}\Gamma + \mathcal{O}(\lambda^3). \tag{7}$$

Notice in Eqs. (6) and (7) the representation of $\Gamma$ on $\mathcal{M}_0$ is splitting in the background value $\Gamma_0$ plus $\mathcal{O}(k)$ perturbations in the gauge $\mathcal{X}_\lambda$. Therefore, the $k$th order $\mathcal{O}(k)$ in $\Gamma$ depends on gauge $\mathcal{X}$. With this description the perturbations are fields that lie in the background. The first term in Eq. (4) admits an expansion around $\lambda = 0$ given by [25]

$$\mathcal{X}^*_\lambda\Gamma|_{\mathcal{M}_0} = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\mathcal{L}^k_X\Gamma|_{\mathcal{M}_0} = \exp(\lambda\mathcal{L}_X)\Gamma|_{\mathcal{M}_0}, \tag{8}$$

where $\mathcal{L}_X\Gamma$ is the Lie derivative of $\Gamma$ with respect to a vector field $X$ that generates the flow $\mathcal{X}$. If we define $\mathcal{X}^*_\lambda\Gamma|_{\mathcal{M}_0} \equiv \Gamma^{\mathcal{X}}_\lambda$ and proceeding in the same way for another gauge choice $\mathcal{Y}$, using Eqs. (4)–(8), the tensor fields $\Gamma^{\mathcal{X},\mathcal{Y}}_\lambda$ can be written as

$$\Gamma^{\mathcal{X}}_\lambda = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\delta^{(k)}_{\mathcal{X}}\Gamma = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\mathcal{L}^k_X\Gamma|_{\mathcal{M}_0}, \tag{9}$$

$$\Gamma^{\mathcal{Y}}_\lambda = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\delta^{(k)}_{\mathcal{Y}}\Gamma = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\mathcal{L}^k_Y\Gamma|_{\mathcal{M}_0}, \tag{10}$$

if $\Gamma^{\mathcal{X}}_\lambda = \Gamma^{\mathcal{Y}}_\lambda$ for any arbitrary gauge $\mathcal{X}$ and $\mathcal{Y}$. From here it is clear that $\Gamma$ is totally gauge invariant. It is also clear that $\Gamma$ is gauge invariant to order $n \geq 1$ if and only if $\delta^{(k)}_{\mathcal{Y}}\Gamma = \delta^{(k)}_{\mathcal{X}}\Gamma$ is satisfied, or in another way

$$\mathcal{L}_X\delta^{(k)}\Gamma = 0, \tag{11}$$

for any vector field $X$ and $\forall\ k < n$. To first order ($k = 1$) any scalar that is constant in the background or any tensor that vanished in the background are gauge invariant. This result is known as the Stewart-Walker lemma [26], i.e., Eq. (11) generalizes this lemma. However, when $\Gamma$ is not gauge invariant and there are two gauge choices $\mathcal{X}_\lambda$, $\mathcal{Y}_\lambda$, the representation of $\Gamma|_{\mathcal{M}_0}$ is different depending of the used gauge. To transform the representation from a gauge choice $\mathcal{X}^*_\lambda\Gamma|_{\mathcal{M}_0}$ to another $\mathcal{Y}^*_\lambda\Gamma|_{\mathcal{M}_0}$ as with the map $\Phi_\lambda$: $\mathcal{M}_0 \to \mathcal{M}_0$ given by

$$\Phi_\lambda \equiv \mathcal{X}_{-\lambda}\circ\mathcal{Y}_\lambda \Rightarrow \Gamma^{\mathcal{Y}}_\lambda = \Phi^*_\lambda\Gamma^{\mathcal{X}}_\lambda, \tag{12}$$

as a consequence, the diffeomorphism $\Phi_\lambda$ induces a pull-back $\Phi^*_\lambda$ that changes the representation $\Gamma^{\mathcal{X}}_\lambda$ of $\Gamma$ in a gauge $\mathcal{X}_\lambda$ to the representation $\Gamma^{\mathcal{Y}}_\lambda$ of $\Gamma$ in a gauge $\mathcal{Y}_\lambda$. Now, following [27] and using the Baker-Campbell-Haussdorf formula [28], one can generalize Eq. (8) to write $\Phi^*_\lambda\Gamma^{\mathcal{X}}_\lambda$ in the following way:

$$\Phi^*_\lambda\Gamma^{\mathcal{X}}_\lambda = \exp\left(\sum_{k=1}^{\infty}\frac{\lambda^k}{k!}\mathcal{L}_{\xi_k}\right)\Gamma^{\mathcal{X}}_\lambda, \tag{13}$$

where $\xi_k$ is any vector field on $\mathcal{M}_\lambda$. Substituting Eq. (13) in Eq. (12), we have explicitly that

$$\Gamma^{\mathcal{Y}}_\lambda = \Gamma^{\mathcal{X}}_\lambda + \lambda\mathcal{L}_{\xi_1}\Gamma^{\mathcal{X}}_\lambda + \frac{\lambda^2}{2}(\mathcal{L}^2_{\xi_1} + \mathcal{L}_{\xi_2})\Gamma^{\mathcal{X}}_\lambda + \mathcal{O}(\lambda^3). \tag{14}$$

Replacing Eqs. (9) and (10) into Eq. (14), the relations to first- and second-order perturbations of $\Gamma$ in two different gauge choices are given by

$$\delta^{(1)}_{\mathcal{Y}}\Gamma - \delta^{(1)}_{\mathcal{X}}\Gamma = \mathcal{L}_{\xi_1}\Gamma_0, \tag{15}$$

$$\delta^{(2)}_{\mathcal{Y}}\Gamma - \delta^{(2)}_{\mathcal{X}}\Gamma = 2\mathcal{L}_{\xi_1}\delta^{(1)}_{\mathcal{X}}\Gamma_0 + (\mathcal{L}^2_{\xi_1} + \mathcal{L}_{\xi_2})\Gamma_0, \tag{16}$$

where the generators of the gauge transformation $\Phi$ are

$$\xi_1 = Y - X \quad \text{and} \quad \xi_2 = [X, Y]. \tag{17}$$

This vector field can be split in their time and space part

$$\xi^{(r)}_\mu \to (\alpha^{(r)}, \partial_i\beta^{(r)} + d^{(r)}_i). \tag{18}$$

Here $\alpha^{(r)}$ and $\beta^{(r)}$ are arbitrary scalar functions, and $\partial^i d^{(r)}_i = 0$. The function $\alpha_{(r)}$ determines the choice of

[3] In Eq. (3), $g_\lambda$ and $T_\lambda$ are the metric and the matter fields on $\mathcal{M}_\lambda$; similarly $F_\lambda$ and $J_\lambda$ are the electromagnetic field and the four-current on $\mathcal{M}_\lambda$.

[4] Here we introduce a coordinate system $x^\alpha$ through a chart on $\mathcal{M}_\lambda$ with $\alpha = 0, 1, 2, 3$ thus giving a vector field on $\mathcal{N}$, which has the property that $X^4 = \lambda$ in this chart, while the other components remain arbitrary.

constant time hypersurfaces, while $\partial_i\beta^{(r)}$ and $d_i^{(r)}$ fix the spatial coordinates within these hypersurfaces. The choice of coordinates is arbitrary and the definitions of perturbations are thus gauge dependent. The gauge transformation given by Eqs. (15) and (16) are quite general. To first order $\Gamma$ is gauge invariant if $\mathcal{L}_{\xi_1}\Gamma_0 = 0$, while to second order one must have other conditions $\mathcal{L}_{\xi_1}\delta_X^{(1)}\Gamma_0 = \mathcal{L}^2_{\xi_1}\Gamma_0 = 0$ and $\mathcal{L}_{\xi_2}\Gamma_0 = 0$, and so on at high orders. We will apply the formalism described above to the Robertson-Walker metric, where $k$ does mention the expansion order.

## III. FLRW BACKGROUND

At zero order (background), the Universe is well described by a spatially flat Friedman-Lemaître-Robertson-Walker metric

$$ds^2 = a^2(\tau)(-d\tau^2 + \delta_{ij}dx^i dx^j), \tag{19}$$

with $a(\tau)$ the scale factor and $\tau$ the conformal time. Hereafter the Greek indices run from 0 to 3, and the Latin ones run from 1 to 3 and a prime denotes the derivative with respect to $\tau$. The Einstein tensor components in this background are given by

$$G^0_0 = -\frac{3H^2}{a^2}, \tag{20a}$$

$$G^i_j = -\frac{1}{a^2}\left(2\frac{a''}{a} - H^2\right)\delta^i_j, \tag{20b}$$

with $H = \frac{a'}{a}$ the Hubble parameter. We consider the background filled with a single barotropic fluid where the energy-momentum tensor is

$$T^{\mu}_{(\mathrm{fl})\nu} = (\mu_{(0)} + P_{(0)})u^{\mu}_{(0)}u_{\nu}^{(0)} + P_{(0)}\delta^{\mu}_{\nu}, \tag{21}$$

with $\mu_{(0)}$ the energy density and $P_{(0)}$ the pressure. The comoving observers are defined by the four-velocity $u^\nu = (a^{-1}, 0, 0, 0)$ with $u^\nu u_\nu = -1$ and the conservation law for the fluid is

$$\mu'_{(0)} + 3H(\mu_{(0)} + P_{(0)}) = 0. \tag{22}$$

To deal with the magnetic field, the space-time under study is the fluid permeated by a weak magnetic field,[5] which is a stochastic field and can be treated as a perturbation on the background [29,30]. Since the magnetic field has no background contribution, the electromagnetic energy-momentum tensor is automatically gauge invariant at first order [see Eq. (15)]. The spatial part of Ohm's law that is the proyected current is written by

$$(g_{\mu i} + u_\mu u_i)j^\mu = \sigma g_{\lambda i}g_{\alpha\mu}F^{\lambda\alpha}u^\mu, \tag{23}$$

where $j^\mu = (\varrho, J^i)$ is the 4-current and $F^{\lambda\alpha}$ is the electromagnetic tensor given by

[5] With the property $B^2_{(0)} \ll \mu_{(0)}$.

$$F^{\lambda\alpha} = \frac{1}{a^2(\tau)}\begin{pmatrix} 0 & E^i & E^j & E^k \\ -E^i & 0 & B^k & -B^j \\ -E^j & -B^k & 0 & B^i \\ -E^k & B^j & -B^i & 0 \end{pmatrix}. \tag{24}$$

At zero order in Eq. (23) the usual Ohms law is found that gives us the relation between the 3-current and the electric field

$$J_i = \sigma E_i, \tag{25}$$

where $\sigma$ is the conductivity. Under magnetohydrodynamics (MHD) approximation, on large scales the plasma is globally neutral and charge density is neglected ($\varrho = 0$) [2]. If the conductivity is infinite ($\sigma \to \infty$) in the early universe [31], then Eq. (23) states that the electric field must vanish ($E_i = 0$) in order to keep the current density finite [32]. However, the current also should be zero ($J_i = 0$) because a nonzero current involves a movement of charge particles that breaks down the isotropy in the background.

## IV. GAUGE-INVARIANT VARIABLES AT FIRST ORDER

We write down the perturbations on a spatially flat FLRW background. The perturbative expansion at $k$th order of the matter quantities is given by

$$\mu = \mu_{(0)} + \sum_{k=1}^{\infty}\frac{1}{k!}\mu_{(k)}, \tag{26}$$

$$B^2 = \sum_{k=1}^{\infty}\frac{1}{k!}B^2_{(k)}, \tag{27}$$

$$E^2 = \sum_{k=1}^{\infty}\frac{1}{k!}E^2_{(k)}, \tag{28}$$

$$P = P_{(0)} + \sum_{k=1}^{\infty}\frac{1}{k!}P_{(k)}, \tag{29}$$

$$B^i = \frac{1}{a^2(\tau)}\left(\sum_{k=1}^{\infty}\frac{1}{k!}B^i_{(k)}\right), \tag{30}$$

$$E^i = \frac{1}{a^2(\tau)}\left(\sum_{k=1}^{\infty}\frac{1}{k!}E^i_{(k)}\right), \tag{31}$$

$$u^\mu = \frac{1}{a(\tau)}\left(\delta^\mu_0 + \sum_{k=1}^{\infty}\frac{1}{k!}v^\mu_{(k)}\right), \tag{32}$$

$$j^{\mu} = \frac{1}{a(\tau)}\left(\sum_{k=1}^{\infty}\frac{1}{k!}j^{\mu}_{(k)}\right), \tag{33}$$

where the fields used in the above formulas are the average ones (i.e., $B^2 = \langle B^2\rangle$).[6] We also consider the perturbations about a FLRW background, so that the metric tensor is given by

$$g_{00} = -a^2(\tau)\left(1 + 2\sum_{k=1}^{\infty}\frac{1}{k!}\psi^{(k)}\right), \tag{34}$$

$$g_{0i} = a^2(\tau)\sum_{k=1}^{\infty}\frac{1}{k!}\omega_i^{(k)}, \tag{35}$$

$$g_{ij} = a^2(\tau)\left[\left(1 - 2\sum_{k=1}^{\infty}\frac{1}{k!}\phi^{(k)}\right)\delta_{ij} + \sum_{k=1}^{\infty}\frac{\chi_{ij}^{(k)}}{k!}\right]. \tag{36}$$

The perturbations are split into a scalar, transverse vector part, and transverse trace-free tensor

$$\omega_i^{(k)} = \partial_i\omega^{(k)\Vert} + \omega_i^{(k)\perp}, \tag{37}$$

with $\partial^i\omega_i^{(k)\perp} = 0$. Similarly we can split $\chi_{ij}^{(k)}$ as

$$\chi_{ij}^{(k)} = D_{ij}\chi^{(k)\Vert} + \partial_i\chi_j^{(k)\perp} + \partial_j\chi_i^{(k)\perp} + \chi_{ij}^{(k)\top}, \tag{38}$$

for any tensor quantity.[7] Following [34], one can find the scalar gauge-invariant variables at first order given by

$$\Psi^{(1)} \equiv \psi^{(1)} + \frac{1}{a}(\mathcal{S}_{(1)}^{\Vert}a)', \tag{39}$$

$$\Phi^{(1)} \equiv \phi^{(1)} + \frac{1}{6}\nabla^2\chi^{(1)} - H\mathcal{S}_{(1)}^{\Vert}, \tag{40}$$

$$\Delta^{(1)} \equiv \mu_{(1)} + (\mu_{(0)})'\mathcal{S}_{(1)}^{\Vert}, \tag{41}$$

$$\Delta_P^{(1)} \equiv P_{(1)} + (P_{(0)})'\mathcal{S}_{(1)}^{\Vert}, \tag{42}$$

with $\mathcal{S}_{(1)}^{\Vert} \equiv (\omega^{\Vert(1)} - \frac{(\chi^{\Vert(1)})'}{2})$ the scalar contribution of the shear. The vector modes are

$$v_{(1)}^i \equiv v_{(1)}^i + (\chi_{\perp(1)}^i)', \tag{43}$$

$$\vartheta_i^{(1)} \equiv \omega_i^{(1)} - (\chi_i^{\perp(1)})', \tag{44}$$

$$\mathcal{V}_{(1)}^i \equiv \omega_{(1)}^i + v_{(1)}^i. \tag{45}$$

Other gauge-invariant variables are the 3-current, the charge density, and the electric and magnetic fields, because they vanish in the background. The tensor quantities are also gauge invariant because they are null in the background [see Eq. (15)].

### A. The Ohm law and the energy-momentum tensor

Using Eq. (23) the Ohm law at first order is

$$J_i^{(1)} = \sigma E_i^{(1)}. \tag{46}$$

As the conductivity of the medium finite (real MHD), the electric field and the 3-current are nonzero. Now, the electromagnetic energy-momentum tensor is

$$T^0_{(\mathrm{em})0} = -\frac{1}{8\pi}(B^2_{(1)} + E^2_{(1)}), \tag{47}$$

$$T^i_{(\mathrm{em})0} = 0, \qquad T^0_{(\mathrm{em})i} = 0, \tag{48}$$

$$T^i_{(\mathrm{em})l} = \frac{1}{4\pi}\left[\frac{1}{6}(B^2_{(1)} + E^2_{(1)})\delta^i{}_l + \Pi^{i(1)}_{l(\mathrm{em})}\right], \tag{49}$$

where $\Pi^{i(1)}_{l(\mathrm{em})} = \frac{1}{3}(B^2 + E^2)\delta^i{}_l - B_lB^i - E_lE^i$ is the anisotropic stress tensor that is gauge invariant by definition of Eq. (15). This term is important to constrain the total magnetic energy because it is a source of gravitational waves [9]. We can see that the electromagnetic energy density appears like a quadratic term in the energy-momentum tensor, which means that the electromagnetic field should be regarded as one-half-order perturbation.[8] Using Eq. (21) and considering the fluctuations of the matter fields, Eqs. (26) and (29), the energy-momentum tensor for the fluid is given by

$$T^0_{(\mathrm{fl})0} = -\Delta^{(1)} + (\mu_{(0)})'\mathcal{S}_{(1)}^{\Vert}, \tag{50}$$

$$T^i_{(\mathrm{fl})0} = (\mu_0 + P_0)(\mathcal{V}^i_{(1)} - \vartheta^i_{(1)} - (\chi^i_{\perp(1)})'), \tag{51}$$

$$T^0_{(\mathrm{fl})i} = -(\mu_0 + P_0)\mathcal{V}_i^{(1)}, \tag{52}$$

$$T^i_{(\mathrm{fl})j} = (\Delta_P^{(1)} - (P_{(0)})'\mathcal{S}_{(1)}^{\Vert})\delta^i{}_j + \Pi^{i(1)}_{j(\mathrm{fl})}, \tag{53}$$

where $\Pi^{i(1)}_{j(f)}$ is the anisotropic stress tensor [35]. The above equations are written in terms of gauge-invariant variables plus terms as $\mathcal{S}_{(1)}^{\Vert}$ that depend of the gauge choice.

### B. The conservation equations

The total energy-momentum conservation equation $\mathcal{T}^{\alpha}_{\beta;\alpha} = 0$ can be split in each component that is not necessarily conserved independently

$$\mathcal{T}^{\alpha}_{\beta;\alpha} = T^{\alpha(f)}_{\beta;\alpha} + T^{\alpha(\mathrm{E.M})}_{\beta;\alpha} = 0, \tag{54}$$

where

[6] This happens because the average evolves exactly like $B^2$ [33].

[7] With $\partial^i\chi_{ij}^{(k)\top} = 0$, $\chi_i^{(k)i} = 0$, and $D_{ij} \equiv \partial_i\partial_j - \frac{1}{3}\delta_{ij}\partial_k\partial^k$.

[8] Therefore the magnetic field should be split as $B^i = \frac{1}{a(\tau)^2}(B^i_{(\frac{1}{2})} + B^i_{(1)} + B^i_{(\frac{3}{2})} + \ldots)$; see [35].

$$T^{\alpha(\mathrm{E.M})}_{\beta;\alpha} = F_{\beta\alpha} j^{\alpha}. \tag{55}$$

Using Eqs. (50) and (53), the continuity equation $\mathcal{T}^{\alpha}_{0;\alpha} = 0$ is given by

$$(\Delta^{(1)})' + 3H(\Delta^{(1)}_P + \Delta^{(1)}) - 3(\Phi^{(1)})'(P_{(0)} + \mu_{(0)}) + (P_{(0)} + \mu_{(0)})\nabla^2 v^{(1)} - 3H(P_{(0)} + \mu_{(0)})'\mathcal{S}^{\parallel}_{(1)} - ((\mu_{(0)})'\mathcal{S}^{\parallel}_{(1)})' + (P_{(0)} + \mu_{(0)})\left(-\frac{1}{2}\nabla^2\chi^{(1)} + 3H\mathcal{S}^{\parallel}_{(1)}\right)' - (P_{(0)} + \mu_{(0)})\nabla^2\left(\frac{1}{2}\chi^{\parallel(1)}\right)' = 0. \tag{56}$$

The Navier-Stokes equation $\mathcal{T}^{\alpha}_{i;\alpha} = 0$ is

$$(\mathcal{V}^{(1)}_i)' + \frac{(\mu_{(0)} + P_{(0)})'}{(\mu_{(0)} + P_{(0)})}\mathcal{V}^{(1)}_i + 4H\mathcal{V}^{(1)}_i + \partial_i\Psi^{(1)} + \frac{\partial_i(\Delta^{(1)}_P - (P_{(0)})'\mathcal{S}^{\parallel}_{(1)}) + \partial_l\Pi^{(1)l}_{(\mathrm{fl})i}}{(\mu_{(0)} + P_{(0)})} - \partial_i\frac{1}{a}(\mathcal{S}^{\parallel}_{(1)}a)' = 0. \tag{57}$$

The last equations are written is terms of gauge-invariant variables in accordance with [35,36]. It is shown there that contribution of electromagnetic terms to the conservation equations does not exist. In [8,36] the energy-momentum tensor of each component is not conserved independently and the divergence has a source term that takes into account the energy and momentum transfer between the components of the photon, electron, proton, and the electromagnetic field $T^{\alpha(f)}_{\beta;\alpha} = K_{\beta}$.

## V. MAXWELL EQUATIONS AND THE COSMOLOGICAL DYNAMO EQUATION

Maxwell's equations are written as

$$\nabla_{\alpha}F^{\alpha\beta} = j^{\beta}, \qquad \nabla_{[\gamma}F_{\alpha\beta]} = 0. \tag{58}$$

Using Eq. (58) and the perturbation equations for the metric and electromagnetic fields, the nonhomogeneous Maxwell equations are

$$\partial_i E^i_{(1)} = a\varrho_{(1)}, \tag{59}$$

$$\epsilon^{ilk}\partial_l B^{(1)}_k = (E^i_{(1)})' + 2HE^i_{(1)} + aJ^i_{(1)}, \tag{60}$$

and the homogeneous Maxwell equations

$$B'_{k(1)} + 2HB^{(1)}_k + \epsilon^{ij}{}_k\partial_i E^{(1)}_j = 0, \tag{61}$$

$$\partial^i B^{(1)}_i = 0, \tag{62}$$

written also by [37]. Now using the last equations together with Ohm's law, Eq. (46), we get an equation that describes the evolution of magnetic field at first order; this relation is the dynamo equation:

$$(B^{(1)}_k)' + 2HB^{(1)}_k + \eta[\nabla\times(\nabla\times B^{(1)} - (E^{(1)})' - 2HE^{(1)})]_k = 0, \tag{63}$$

with $\eta = \frac{1}{4\pi\sigma}$ the diffusion coefficient. Equation (63) is similar to the dynamo equation in MHD but it is in the cosmological context [2]. This equation has one term that depends on $\eta$ that takes into account the dissipation phenomena of the magnetic field (the electric field in this term in general is dropped if we neglect the displacement current). Notice that $\eta$ is a expansion parameter (due to $\sigma$ being large). From Eq. (63) we see that for finite $\eta$, the diffusion term should not be neglected. Care should be taken with the assumption that $\eta = 0$, because it could break at small scales [31]. In the frozen condition of magnetic field lines, where amplification of the field is not taken into account, the last equation has the solution $\mathbf{B} = \frac{\mathbf{B}_0}{a^2(\tau)}$ where $\mathbf{B}_0$ is the actual magnetic field, the actual value of the scale factor $a_0(\tau) = 1$, and $\mathbf{B}$ is the magnetic field when the scale factor was $a(\tau)$.

## VI. GENERALIZATION AT SECOND ORDER

Following [25] the variable $\delta^{(2)}\boldsymbol{T}$ defined by

$$\delta^{(2)}_{\mathcal{X}}\boldsymbol{T} \equiv \delta^{(2)}_{\mathcal{X}}\Gamma - 2L_X(\delta^{(1)}_{\mathcal{X}}\Gamma) + L^2_X\Gamma_0 \tag{64}$$

is introduced. Inspecting the gauge transformation Eq. (16) one can see that $\delta^{(2)}\boldsymbol{T}$ is transformed as

$$\delta^{(2)}_{\mathcal{Y}}\boldsymbol{T} - \delta^{(2)}_{\mathcal{X}}\boldsymbol{T} = L_{\sigma}\Gamma_0, \tag{65}$$

with $\sigma = \xi_2 + [\xi_1, X]$ and $X$ is the gauge dependence part in linear order perturbation. The gauge transformation rule Eq. (65) is identical to the gauge transformation at linear order Eq. (15). This property is general and is the key to extend this theory to second order

$$L[\delta^2\boldsymbol{T}] = S[\delta\boldsymbol{T}, \delta\boldsymbol{T}]. \tag{66}$$

Notice that first- and second-order equations are similar; however, the last have as sources the coupling between linear perturbations variables. Using Eqs. (16) and (65) we arrive at the gauge-invariant quantities at second order. This coupling appearing as the quadratic terms of the linear perturbation is due to the nonlinear effects of the Einstein field equations; besides one can classify them again in scalar, vector, and tensor modes where these modes couple with each other. Now, to clarify the physical behaviors of perturbations at this order we should obtain the gauge-invariant quantities and express these equations of movements in terms of these quantities.

The scalar gauge invariants are given by

$$\Psi^{(2)} \equiv \psi^{(2)} + \frac{1}{a}(\mathcal{S}^{\parallel}_{(2)}a)' + \mathcal{T}^1(\mathcal{O}^{(2)}), \tag{67}$$

$$\Phi^{(2)} \equiv \phi^{(2)} + \frac{1}{6}\nabla^2\chi^{(2)} - H\mathcal{S}^{\parallel}_{(2)} + \mathcal{T}^2(\mathcal{O}^{(2)}), \quad (68)$$

$$\Delta^{(2)}_{\mu} \equiv \mu_{(2)} + (\mu_{(0)})'\mathcal{S}^{\parallel}_{(2)} + \mathcal{T}^3(\mathcal{O}^{(2)}), \quad (69)$$

$$\Delta^{(2)}_{\varrho} \equiv \varrho^{(2)} + \mathcal{T}^4(\mathcal{O}^{(2)}), \quad (70)$$

$$\Delta^{(2)}_{B} \equiv B^2_{(2)} + \mathcal{T}^4(\mathcal{O}^{(2)}), \quad (71)$$

$$\Delta^{(2)}_{E} \equiv E^2_{(2)} + \mathcal{T}^6(\mathcal{O}^{(2)}), \quad (72)$$

$$v^{(2)} \equiv v^{(2)} + \left(\frac{1}{2}\chi^{\parallel(2)}\right)' + \mathcal{T}^7(\mathcal{O}^{(2)}), \quad (73)$$

with $\mathcal{S}^{\parallel}_{(2)} \equiv (\omega^{\parallel(2)} - \frac{(\chi^{\parallel(2)})'}{2}) + \mathcal{T}^8(\mathcal{O}^{(2)})$. The expression for $\mathcal{T}^8(\mathcal{O}^{(2)})$ is given in Appendix A. In this case $\mathcal{S}^{\parallel}_{(2)}$ can be interpreted like shear at second order. Again it is shown that it is similar to being found at first order but it has a source term that is quadratic in the first-order functions of the transformations. The vector modes found are as follows:

$$v^i_{(2)} \equiv v^i_{(2)} + (\chi^i_{\perp(2)})' + \mathcal{T}^9(\mathcal{O}^{(2)}), \quad (74)$$

$$\vartheta^{(2)}_i \equiv \omega^{(2)}_i - (\chi^{\perp(2)}_i)' + \mathcal{T}^{10}(\mathcal{O}^{(2)}), \quad (75)$$

$$\mathcal{V}^i_{(2)} \equiv \omega^i_{(2)} + v^i_{(2)} + \mathcal{T}^{11}(\mathcal{O}^{(2)}), \quad (76)$$

$$\Pi^{(2)T}_{ij} \equiv \Pi^{(2)\mathrm{fl}}_{ij} + \Pi^{(2)\mathrm{em}}_{ij} + \mathcal{T}^{13}(\mathcal{O}^{(2)}). \quad (77)$$

The electromagnetic fields modes (from $F^{\lambda\alpha}$) are then given by

$$\mathcal{E}^{(2)}_i = E^{(2)}_i + 2\left[\frac{1}{a^2}(a^2E^{(1)}_i\alpha^{(1)})' + (\xi'_{(1)} \times B^{(1)})_i + \xi^l_{(1)}\partial_l E^{(1)}_i + E^{(1)}_l\partial_i\xi^l_{(1)}\right], \quad (78)$$

$$\mathcal{B}^{(2)}_i = B^{(2)}_i + 2\left[\frac{\alpha^{(1)}}{a^2}(a^2B^{(1)}_i)' + \xi^l_{(1)}\partial_l B^{(1)}_i + B^{(1)}_i\partial_l\xi^l_{(1)} + (E^{(1)} \times \nabla\alpha^{(1)})_i - B^{(1)}_l\partial^l\xi^{(1)}_i\right], \quad (79)$$

$$\varrho^{(2)}_{(\mathrm{Inv})} = \varrho^{(2)} + 2[(\varrho'_{(1)} - H\varrho^{(1)})\alpha^{(1)} + \xi^i_{(1)}\partial_i\varrho^{(1)} - \alpha'_{(1)}\varrho^{(1)} - J^i_{(1)}\partial_i\alpha^{(1)}], \quad (80)$$

$$\mathcal{J}^i_{(2)} = J^i_{(2)} + 2[((J^i_{(1)})' - H\mathcal{J}^i_{(1)})\alpha^{(1)} + \xi^l_{(1)}\partial_l J^i_{(1)} - \varrho^{(1)}(\xi^i_{(1)})' - J^l_{(1)}\partial_l\xi^i], \quad (81)$$

which are gauge-invariant quantities for electromagnetic fields. All these variables are similar to the quantities obtained at first order, but in the second-order case appear as sources as $T^k(\mathcal{O}^{(2)})$ that depend of the gauge choice and the coupling with terms of first order. The explicit calculation of $\mathcal{T}^k(\mathcal{O}^{(2)})$ is shown in [25,34].

### A. The Ohm law and the energy-momentum tensor

Using Eqs. (23), (30), and (31), we get the Ohm law at second order

$$\mathcal{J}^{(2)}_i = 4J^{(1)}_i\Phi^{(1)} + S^1_i(\mathcal{O}^{(2)}) + \varrho^{(1)}v^{(1)}_i + 2\sigma\Big((\mathcal{V}_{(1)} \times B^{(1)})_i + \frac{1}{2}\mathcal{E}^{(2)}_i - 2E^{(1)}_i\Big(\Phi^{(1)} - \frac{1}{2}\Psi^{(1)}\Big) + S^2_i(\mathcal{O}^{(2)})\Big). \quad (82)$$

In this case we see that the 3-current has a type of Lorentz term and shows coupling between first-order terms that affect the evolution of the current. Hereafter the functions $S^n_i(\mathcal{O}^{(2)})$ with $n \in \mathbb{Z}$ and $i$ being the component gives us the gauge dependence. The last equation shows also a coupling between the electric field and terms like $(\Phi^{(1)} - \frac{1}{2}\Psi^{(1)})$ that is associated with tidal forces (this quantity is similar to the scalar part of the electric part of the Weyl tensor) and the first right-hand term between the current and perturbation in the curvature. There exist models where the coupling of the charge particles and the field is important for explaining some phenomena like collapse or generation of magnetic field during a recombination period. In this case the Ohm law shown in Eq. (82) should be generalized and terms like Biermann battery and Hall effect should appear. Doing the expansion at second order in the fluid energy-momentum tensor, one finds the following expressions:

$$T^0_{(2)0} = -\frac{\Delta^{(2)}_{\mu}}{2} - (\mu_{(0)} + P_{(0)})(v^{(1)}_l v^l_{(1)} + \vartheta^{(1)}_l v^l_{(1)}) + S^3(\mathcal{O}^{(2)}), \quad (83)$$

$$T^i_{(2)0} = -(\mu_{(0)} + P_{(0)})\Big(\frac{\mathcal{V}^i_{(2)} - \vartheta^i_{(2)}}{2} + \Psi^{(1)}v^i_{(1)}\Big) - (\Delta^{(1)}_{\mu} + \Delta^{(1)}_{P})v^i_{(1)} + S^i_4(\mathcal{O}^{(2)}), \quad (84)$$

$$T^0_{(2)i} = -(\mu_{(0)} + P_{(0)})\Big(\frac{\mathcal{V}^{(2)}_i}{2} - 2\vartheta^{(2)}_i\Psi^{(1)} - 2v^{(1)}_i\Phi^{(1)} + v^j_{(1)}\chi^{(1)}_{ij} - v^{(1)}_i\Psi^{(1)}\Big) - (\Delta^{(1)}_{\mu} + \Delta^{(1)}_{P})\mathcal{V}^{(1)}_i + S^5_i(\mathcal{O}^{(2)}), \quad (85)$$

$$T^i_{(2)j} = \frac{1}{2}\Delta^{(2)}_P \delta^i{}_j + \frac{1}{2}\Pi^{i(2)}_j + S^{6i}_j(\mathcal{O}^{(2)}) + (\mu_{(0)} + P_{(0)})(v^{(1)}_j v^i_{(1)} + \vartheta^{(1)}_j v^i_{(1)}), \quad (86)$$

similar to [36]. Now consider Eq. (61) the electromagnetic momentum tensor at second order is

$$T^0_{(\mathrm{em})0} = -\frac{1}{8\pi}(\Delta^{(2)}_E + \Delta^{(2)}_B + S^8(\mathcal{O}^{(2)})), \quad (87)$$

$$T^i_{(\mathrm{em})0} = \frac{1}{4\pi}[-\epsilon^{ikm}E^{(1)}_k B^m_{(1)} + S^i_9(\mathcal{O}^{(2)})], \quad (88)$$

$$T^0_{(\mathrm{em})i} = \frac{1}{4\pi}[\epsilon_i^{km}E^{(1)}_k B^m_{(1)} + S_{10i}(\mathcal{O}^{(2)})], \quad (89)$$

$$T^i_{(\mathrm{em})l} = \frac{1}{4\pi}\left[\frac{1}{6}(\Delta^{(2)}_E + \Delta^{(2)}_B + S^i_{4l}(\mathcal{O}^{(1)}))\delta^i{}_l + \Pi^{i(2)}_{l(\mathrm{em})} + S^i_{11l}(\mathcal{O}^{(2)})\right]. \quad (90)$$

Using Eq. (54) the continuity equation is given by

$$(\Delta^{(2)}_\mu)' + 3H(\Delta^{(2)}_P + \Delta^{(2)}_\mu) - 3(\Phi^{(2)})'(P_{(0)} + \mu_{(0)}) + (P_{(0)} + \mu_{(0)})\nabla^2 v^{(2)} = -a^4(2E^{(1)}_i J^i_{(1)}) - S_{12}(\mathcal{O}^{(2)}), \quad (91)$$

and the Navier-stokes equation

$$\frac{1}{2}\frac{[\mu_{(0)}(1+w)\mathcal{V}^{(2)}_i]'}{\mu_{(0)}(1+w)} + 2H\mathcal{V}^{(2)}_i + \frac{1}{2}\frac{\partial_i P^{(2)} + 2\partial_j \Pi^{j(2)}_i}{\mu_{(0)}(1+w)} + \frac{1}{2}\partial_i \Psi^{(2)} + S^{13}_i(\mathcal{O}^{(2)}) = \frac{a^4(E^{(1)}_i \varrho_{(1)} + \epsilon_{ijk}J^j_{(1)}B^{(1)}_k)}{\mu_{(0)}(1+w)}, \quad (92)$$

where $w = \frac{P_{(0)}}{\mu_{(0)}}$ and $S^{13}_i$ is shown in Appendix B. Therefore, electromagnetic fields affect the evolution of matter energy density $\Delta^{(2)}_\mu$ and the peculiar velocity $\mathcal{V}^{(2)}_i$. Also, these fields influence the large structure formation and can leave imprints on the temperature anisotropy pattern of the CMB [36,37].

## VII. THE MAXWELL EQUATIONS AND THE COSMOLOGICAL DYNAMO AT SECOND ORDER

Using Eq. (46), the nonhomogeneous Maxwell equations are

$$\partial_i \mathcal{E}^i_{(2)} = -4E^i_{(1)}\partial_i(\Psi^{(1)} - 3\Phi^{(1)}) + a\Delta^{(2)}_\varrho - S_{14}(\mathcal{O}^{(2)}), \quad (93)$$

$$(\nabla\times\mathcal{B}^{(2)})^i = 2E^i_{(1)}(2(\Psi^{(1)})' - 6(\Phi^{(1)})') + (\mathcal{E}^i_{(2)})' + 2H\mathcal{E}^i_{(2)} + 2(2\Psi^{(1)} - 6\Phi^{(1)}) \times (\nabla\times B_{(1)})^i + a\mathcal{J}^i_{(2)} + S^i_{15}(\mathcal{O}^{(2)}). \quad (94)$$

While the homogeneous Maxwell equations are

$$\frac{1}{a^2}(a^2\mathcal{B}^{(2)}_k)' + (\nabla\times\mathcal{E}_{j(2)})_k = -S^{17}_k(\mathcal{O}^{(2)}), \quad (95)$$

$$\partial_i \mathcal{B}^{i(2)} = 0. \quad (96)$$

Again the $S^n_k$ terms carry out the gauge dependence. Using the Maxwell equations together with the Ohm law at second order and following the same methodology for the first-order case, we get the cosmological dynamo equation that describes the evolution of the magnetic field at second order:

$$(\mathcal{B}^{(2)}_k)' + 2H(\mathcal{B}^{(2)}_k) + \eta\Big[\nabla\times\Big(\frac{1}{a}((\nabla\times\mathcal{B}^{(2)}) - 2E_{(1)}(2(\Psi^{(1)})' - 6(\Phi^{(1)})') - (\mathcal{E}_{(2)})' - 2H\mathcal{E}_{(2)} - 2(\nabla\times B_{(1)}(2\Psi^{(1)} - 6\Phi^{(1)})) - S_{15}(\mathcal{O}^{(1)})) - \varrho^{(1)}v^{(1)} + S^1(\mathcal{O}^{(2)})\Big)\Big]_k + (\nabla\times[-2(\mathcal{V}_{(1)}\times B^{(1)}) - 2E^{(1)}\Psi^{(1)} - 2S^2(\mathcal{O}^{(1)})])_k = -S^{17}_k(\mathcal{O}^{(2)}), \quad (97)$$

where the value of $\varrho^{(1)}$ can be found to resolve the differential equation given in Appendix B. Thus the perturbations in the space-time play an important role in the evolution of primordial magnetic fields. Equations (63) and (97) are dependent on geometrical quantities (perturbation in the gravitational potential, curvature, velocity, etc.). These quantities evolve according to the Einstein field equations (the Einstein field equation to second order are given in [24]). In this way, Eq. (97) tells us how the magnetic field evolves according to the scale of the perturbation. In a subhorizon scale, the contrast density and the geometrical quantities grow. Hence, the dynamo term should amplify the magnetic field. As a final comment we point out that in order to solve the dynamolike equation for the magnetic field it is necessary to solve the Einstein field equations to the second order together with the conservation equations.

## VIII. SPECIFYING TO POISSON GAUGE

It is possible to fix the 4 degrees of freedom by imposing gauge conditions. If we impose the gauge restrictions

$$\partial^i \omega_i^{(r)} = \partial^i \chi_{ij}^{(r)} = 0, \tag{98}$$

all equation can be written in terms in quantities independent of the coordinates [38]. This gauge is called Poisson gauge and it is the gravitational analogue of the Coulomb gauge in electromagnetism (see Appendix A). The perturbed metric in the Poisson gauge reads

$$\begin{aligned} g_{00} &= -a^2(\tau)(1 + 2\psi^{(1)} + \psi^{(2)}), \\ g_{ij} &= a^2(\tau)\left[(1 - 2\phi^{(1)} - \phi^{(2)})\delta_{ij} + \frac{2\chi_{ij}^{(1)\top} + \chi_{ij}^{(2)\top}}{2}\right], \\ g_{0i} &= a^2(\tau)\left(\omega_i^{(1)\perp} + \frac{\omega_i^{(2)\perp}}{2}\right), \end{aligned} \tag{99}$$

where $\omega^{\parallel}$, $\chi^{\parallel}$, $\chi_i^{\perp}$ are null. In this case the dynamo equation in the Poisson gauge is given by

$$\begin{aligned} &B'_{k(2)} + 2HB_k^{(2)} + \eta\Big[\nabla\times(\nabla\times B_{(2)}) - (\nabla\times E'_{(2)}) - 2H(\nabla\times E^{(2)}) - 4(\Psi'_{(1)} - 3\Phi'_{(1)})(\nabla\times E_{(1)}) - 4\nabla(\Psi'_{(1)} - 3\Phi'_{(1)})\times E^{(1)} \\ &\quad + 4(\nabla\times(\nabla(\Psi^{(1)} - 3\Phi^{(1)})\times B^{(1)})) - 4((\nabla\times(\nabla\times B^{(1)}) - \nabla\times E'_{(1)} - 2H(\nabla\times E^{(1)}))\Phi^{(1)} \\ &\quad + \nabla\Phi^{(1)}\times(\nabla\times B^{(1)} - E'_{(1)} - 2HE_{(1)})) - (\nabla\varrho^{(1)})\times v^{(1)} + 2\nabla\times\left(\left(\nabla\times B^{(1)} - \frac{1}{a^2}(a^2E_{(1)})'\right)\cdot\chi_{(1)}^{\top}\right) - \varrho^{(1)}(\nabla\times v^{(1)})\Big]_k \\ &\quad - 2(\nabla\times((v^{(1)} + \omega_{(1)}^{\perp})\times B^{(1)}))_k + 4\left(\left(\nabla\left(\Phi^{(1)} - \frac{\Psi^{(1)}}{2}\right)\times E^{(1)}\right) + \left(\Phi^{(1)} - \frac{\Psi^{(1)}}{2}\right)(\nabla\times E^{(1)})\right)_k - 2\nabla\times(E^{(1)}\cdot\chi_{(1)}^{\top})_k = 0, \end{aligned} \tag{100}$$

where $E^{(1)}\cdot\chi_{(1)}^{\top} = E_i^{(1)}\chi_{\top(1)}^{ij}$. The last equation is a specific case of Eq. (97) where we fix the gauge (coordinate fixing). It is important to notice the relevance of the geometrical perturbation quantities in the evolution of the magnetic fields; again we see the influence of the tidal and Lorentz forces in the amplification of the fields. In some sense, the above equation differs from Eq. (97) due to the fact that we fix the adequate choice of the perturbation functions (we choose a gauge for writing the equation of motion without the presence of unphysical modes) while before we just wrote the equations in terms of gauge-invariant quantities that were built up with the formalism explained in the fist sections, plus terms which have taken into account the dependence of the gauge and where we need to fix them.

## IX. WEAKLY MAGNETIZED FLRW BACKGROUND

In this section we work a magnetized FLRW, i.e., we allow the presence of a weak magnetic field into our FLRW background with the property $B_{(0)}^2 \ll \mu_{(0)}$ which must to be sufficiently random to satisfy $\langle B_i\rangle = 0$ and $\langle B_{(0)}^2\rangle = \langle B_i^{(0)} B_{(0)}^i\rangle \neq 0$ to ensure that symmetries and the evolution of the background remain unaffected. Again we work under MHD approximation, and thus in large scales the plasma is globally neutral, charge density is neglected, and the electric field with the current should be zero. Thus the only zero-order magnetic variable is $B_{(0)}^2$ [30]. The evolution of the density magnetic field can be found contracting the induction equation with $B_i$ arriving at

$$(B_{(0)}^2)' = -4HB_{(0)}^2, \tag{101}$$

showing $B^2 \sim a^{-4}$ in the background. Bianchi models are often used to describe the presence of a magnetic field in the Universe due to anisotropic properties of this metric. However, as we are dealing with weak magnetic fields, it is worth assuming the presence of a magnetic field in a FLRW metric as background. Indeed, the authors in [39] found that, although there is a profound distinction between the Bianchi I equations and the FLRW approximation, at the weak field limit, these differences are reduced dramatically, and therefore the linearized Bianchi equations are the same as the FLRW ones. Under these conditions, we find that to zero order the electromagnetic energy-momentum tensor in the background is given by

$$T^0_{(\mathrm{em})0} = -\frac{1}{8\pi}B_{(0)}^2, \tag{102}$$

$$T^0_{(\mathrm{em})i} = T^i_{(\mathrm{em})0} = 0, \tag{103}$$

$$T^i_{(\mathrm{em})l} = \frac{1}{24\pi}B_{(0)}^2\delta^i{}_l. \tag{104}$$

The magnetic anisotropic stress is treated as a first-order perturbation due to stochastic properties of the field; therefore, it does not contribute to the above equations. We can see in Eqs. (21) and (102)–(104) that fluid and electromagnetic energy-momentum tensors are diagonal tensors, that is, are consistent with the condition of an isotropic and homogeneous background [30]. If we consider the average magnetic density of the background different to zero, the perturbative expansion at $k$th order of the magnetic density is given by

$$B^2 = B^2_{(0)} + \sum_{k=1}^{\infty} \frac{1}{k!} B^2_{(k)}, \tag{105}$$

where at first order we get a gauge-invariant term that describes the magnetic energy density

$$\Delta^{(1)}_{\rm mag} \equiv B^2_{(1)} + (B^2_{(0)})' \mathcal{S}^{\parallel}_{(1)}; \tag{106}$$

one can find that average density of the background field decays as $B^2_{(0)} \sim \frac{1}{a^4(\tau)}$ [40]. At first order we work with finite conductivity (real MHD). In this case the electric field and the current becomes nonzero; therefore, using Eq. (23) and assuming the Ohmic current is not neglected, we find the Ohm's law

$$J^{(1)}_i = \sigma[E^{(1)}_i + (\mathcal{V}^{(1)} \times B^{(0)})_i]. \tag{107}$$

In the last equation the Lorentz force appears at first order when a magnetic field is consider as a part of the background. Again doing the same procedure described before, but taking a weak magnetic field as a contribution from the background, we shall show the implication of this supposition afterword. The electromagnetic energy-momentum tensor at first order is given by

$$T^0_{({\rm em})0} = -\frac{1}{16\pi} F^2_{(1)}, \tag{108}$$

$$T^i_{({\rm em})0} = \frac{1}{4\pi}[B^2_{(0)} \vartheta^i_{(1)} - \epsilon^{ikm} E^{(1)}_k B^m_{(0)} + B^2_{(0)} (\chi^i_{\perp(1)})'], \tag{109}$$

$$T^0_{({\rm em})i} = \frac{1}{4\pi}[\epsilon^{km}_i E^{(1)}_k B^m_{(0)}], \tag{110}$$

$$T^i_{({\rm em})l} = \frac{1}{4\pi}\left[\frac{1}{12} F^2_{(1)} \delta^i{}_l + \Pi^{i(1)}_{l({\rm em})}\right], \tag{111}$$

where

$$F^2_{(1)} = 2\Delta^{(1)}_{({\rm mag})} - 8\Phi^{(1)} B^2_{(0)} - 2(B^2_{(0)})' \mathcal{S}^{\parallel}_{(1)} + \frac{4}{3}\nabla^2 \chi^{(1)} B^2_{(0)} - 8H\mathcal{S}^{\parallel}_{(1)} B^2_{(0)}, \tag{112}$$

and $\Pi^{i(1)}_{l({\rm em})} = \frac{1}{3}(\Delta^{(1)}_{\rm mag} + E^2)\delta^i{}_l - B_l B^i - E_l E^i$ is the anisotropic stress that appears as a perturbation of the background. This term is important to constraining the total magnetic energy because it is a source of gravitational waves [9]. The above equations are written in terms of gauge-invariant variables plus terms as $\mathcal{S}_{(1)}$ that are gauge dependent. Now, using the above Eqs. (59), (61), (60), and (62) with the Ohm's law Eq. (107), we arrive at the dynamo equation that gives us the evolution of the magnetic field to first order:

$$(B^{(1)}_k)' + 2HB^{(1)}_k + \eta[\nabla \times (\nabla \times B^{(1)} - (E^{(1)})' - 2HE^{(1)})]_k + (\nabla \times (B_{(0)} \times \mathcal{V}_{(1)}))_k = 0. \tag{113}$$

When we suppose a weak magnetic field on the background, in the dynamo equation a new term called dynamo term appears that could amplify the magnetic field. This term depends of the evolution in $\mathcal{V}_{(1)}$, see Eq. (57), and also from Eq. (57), it seems likely that when matter and velocity perturbation grow, the dynamo term amplifies the magnetic field; this is different from the first approach where the dynamo term just appears at second order. For convenience it is better to use the Lagrangian coordinates that are comoving with the local Hubble flow. Therefore, we use the convective derivative that is evaluated according to the operator formula (i.e., $\frac{d}{dt} = \frac{\partial}{\partial t} + \mathcal{V}^i_{(1)}\partial_i$). In this picture the magnetic field lines are frozen into the fluid. Using the well-known identity formula

$$\nabla \times (a \times b) = a(\nabla \cdot b) - b(\nabla \cdot a) + (b \cdot \nabla)a - (a \cdot \nabla)b, \tag{114}$$

we obtain the following result:

$$\frac{dB_i}{dt} + 2HB_i = B_j\left(\frac{\partial \mathcal{V}^{(1)}_i}{\partial x_j} - \frac{1}{3}\delta_{ij}\frac{\partial \mathcal{V}^{(1)}_k}{\partial x_k}\right) + \frac{2}{3}B_i \frac{\partial \mathcal{V}^{(1)}_j}{\partial x_j}, \tag{115}$$

where the diffusion term will not be considered for the moment. The first term on the right-hand side is associated with the shear and the last term describes the expansion of the region where $\mathcal{V}^{(1)}$ is not zero. In the case of a homogeneous collapse, $B \sim \mathcal{V}^{-\frac{2}{3}}$ gives rise to amplification of the magnetic field in places where gravitational collapse takes place. Now we write Eq. (113) in the Poisson gauge getting the following:

$$\frac{dB^k_{(1)}}{dt} + 2HB^k_{(1)} + \eta\Big[-\nabla^2 B^k_{(1)} - \left(\nabla \times \left(\frac{1}{a^2}\frac{d(a^2 E_{(1)})}{dt} - \mathcal{V}^i_{(1)}\partial_i E_{(1)}\right)\right)^k - B^k_{(0)}\nabla^2(\Psi^{(1)} - 3\Phi^{(1)}) + (B^{(0)} \cdot \nabla)\partial^k(\Psi^{(1)} - 3\Phi^{(1)}) - (\nabla(\Psi^{(1)} - 3\Phi^{(1)}) \cdot \nabla)B^k_{(0)}\Big] = B^{(0)}_l \sigma^{lk}_{(1)} - \frac{2}{3}B^k_{(0)}\partial_l \mathcal{V}^l_{(1)}, \tag{116}$$

where $\sigma^{lk}_{(1)}$ is the shear found in Eq. (115). The last term on the left-hand side in Eq. (116) should vanish due to the background isotropy. The evolution of the magnetic field following the last equation is highly dependent of term $\Psi^{(1)} - 3\Phi^{(1)}$. If the perturbations are turned off, one can check that the last equation recovers to the dynamo equation found in the literature. It should be noted that terms such as $\langle B^k_{(0)}\rangle$ are zero due to statistical field properties; therefore, contracting Eq. (116) with magnetic field $B^{(1)}_k$, we arrive at an equation at second order in which we can physically study the evolution of the density magnetic field:

$$\frac{d\Delta^{(2)}_{(\mathrm{mag})}}{dt} + 4H\Delta^{(2)}_{(\mathrm{mag})} + 2\eta\Big[-B^{(1)}\cdot\nabla^2 B^{(1)} - B^{(1)}\cdot\Big(\nabla\times\Big(\frac{1}{a^2}\frac{d(a^2E_{(1)})}{dt} - \mathcal{V}^i_{(1)}\partial_i E_{(1)}\Big)\Big) - \frac{1}{2}\Delta^{(1)}_{(\mathrm{mag})}\nabla^2(\Psi^{(1)} - 3\Phi^{(1)}) + B^k_{(1)}(B^{(0)}\cdot\nabla)\partial_k(\Psi^{(1)} - 3\Phi^{(1)})\Big] = -2\Pi^{(1)}_{ij(\mathrm{em})}\sigma^{ij}_{(1)} - \frac{2}{3}\Delta^{(1)}_{(\mathrm{mag})}\partial_l\mathcal{V}^l_{(1)}, \tag{117}$$

where using Eqs. (105) and (16) the energy density magnetic field at second order transforms as

$$\Delta^{(2)}_{(\mathrm{mag})} = B^2_{(2)} + B^{2\prime}_{(0)}\alpha_{(2)} + \alpha_{(1)}(B^{2\prime\prime}_{(0)}\alpha_{(1)} + B^{2\prime}_{(0)}\alpha'_{(1)} + 2B^{2\prime}_{(1)}) + \xi^i_{(1)}(B^{2\prime}_{(0)}\partial_i\alpha^{(1)} + 2\partial_i B^2_{(1)}). \tag{118}$$

The parameters $\alpha$ and $\xi$ are set using the Poisson gauge calculated in Appendix A. Equation (117) shows how the field acts as an anisotropic radiative fluid, which is important in times where the Universe is permeated by anisotropic components. In addition, the second term on the right-hand side describes the perturbation at first order in the volume expansion. Equations (113) and (117) show the important role of a magnetized FLRW model. The set of Eqs. (113)–(116) directly offers a first estimation of how a perturbed four-velocity coupling to a magnetic field gives a common dependence of $B \sim \mathcal{V}^{-\frac{2}{3}}$ under an ideal assumption of infinity conductivity. However, for a real MHD a complete solution should be calculated together with the case of Eq. (117). The right-hand side in Eq. (117) provides new phenomenology about the role of the shear and the anisotropic magnetic stress tensor together with a kinematical effect driven for the last term, reinforcing the claim in [8]. In the paper from Matarrese *et al.* [8] an estimation of the magnetic field to second order dropping the matter anisotropic stress tensor is given by Eq. (16), and from this equation they are to able to compute a solution for the magnetic field, although in our case we suppose the presence of stress and vector modes at first order possibly generated in early stages from the Universe.

## X. DISCUSSION

A problem in modern cosmology is to explain the origin of cosmic magnetic fields. The origin of these fields is still in debate but they must affect the formation of large-scale structure and the anisotropies in the CMB [41–43]. We can see this effect in Eq. (91) where the evolution of $\Delta^{(2)}_{\mu}$ depends on the magnetic field. In this paper we show that the perturbed metric plays an important role in the global evolution of magnetic fields. From our analysis, we wrote a dynamolike equation for cosmic magnetic fields to second order in perturbation theory in a gauge-invariant form. We get the dynamo equation from two approaches. First, using the FLRW as a background space-time and the magnetic fields as a perturbation, the results are Eqs. (63) and (97) to second order. In the second approach (see Sec. IX) a weak magnetic field was introduced in the background space-time and due to its statistical properties that allow us to write down the evolution of magnetic field Eqs. (113) and (117) and fluid variables in accordance with [30]. We observe that essentially, the functional form is the same in the two approaches, the coupling between geometrical perturbations and fields variables appear as sources in the magnetic field evolution giving a new possibility to explain the amplification of primordial cosmic magnetic fields. One important distinction between both approximations is the fractional order in the fields that appears when we consider the magnetic variables as perturbations on the background at difference when the fields are from the beginning of the background (Sec. IX). Although the first alternative is often used in studies of GWs production in the early universe [35], the physical explanation of these fractional orders is sometimes confused, while if we consider a universe permeated with a magnetic density from the background, the perturbative analysis is more straightforward. Further studies as anisotropic (Bianchi I) and inhomogeneous (Lemaitre-Tolman-Bondi) models should be addressed to see the implications from the metric behavior in the evolution of the magnetic field and relax the assumption in the weakness of the field. An important effect of this model is in an additional mechanism of generation of magnetic field from cosmological density perturbation during the radiation era. Following [8], we discussed some details about the mechanism that was proposed by Harrison (see [8] and some reviews therein). At high temperature $T > 230$ eV there is a strong interaction between electrons and photons getting two fluids: electron-photon and protons plasma (coupling between protons and photons is weak in this stage), while for temperatures below 230 eV, the Coulomb scattering between protons and electrons is more effective than electron-photon and proton-photon interaction. In any case electric fields are induced due to velocity differences between the components (more specifically differences in the strength of the interactions and mass of the components), which give rise to magnetic fields through Maxwell equations. To study this phenomena we can use the energy-momentum conservation equations, adding terms that take into account the momentum transfer in the components of the primordial plasma (electron, photons, protons, and electromagnetic field). The momentum equation for photons ($w = 1/3$) is found using Eq. (92) and adding the momentum transfer between electron-photons given by $K_i^{(\gamma)} = \frac{4}{3} n \mu_{(0)} \tau_\gamma (v_i^{(e)} - v_i^{(\gamma)})$ with $\tau_\gamma = an\sigma$, $\sigma$ being the conductivity, $n$ the density number, and $a$ the scale factor. For nonrelativistic protons and electrons ($w = 0$) we should care about the momentum transfer due to Coulomb scattering given by $K_i^{(ep)} = n(v_i^{(e)} - v_i^{(p)})/\tau_e$ where $\tau_e = \frac{m_e \sigma}{ne^2}$. Thus, we arrive at writing the momentum equation for proton, electron, and photon that are shown in Eqs. (B4)–(B6) in Appendix B. If we choose the 3-current as $J_i = qn \mathcal{V}_i$ for each component, we recover the Eqs. (4) and (5) from Matarrese *et al.* [8] considering the suppositions done by the authors. Combining the momentum equations together with the Maxwell equation found in Sec. VII, one can find a system of equations that relate the motion of the fluid to the magnetic field, considering the interaction between the species. The properties of this mechanism have been deeply studied by authors in [8], getting a generation of the field only at second order under the tight coupling approximation and other important results of the presence of magnetic fields in radiation era. Further work should be done in order to solve the cosmic dynamo equation and it is the next step that must be addressed.

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## APPENDIX A

For removing the degrees of freedom we fix the gauge conditions as

$$\partial^i \omega_i^{(r)} = \partial^i \chi_{ij}^{(r)} = 0. \tag{A1}$$

This lead to some functions being dropped

$$\omega^{\|(r)} = \chi_i^{(r)\perp} = \chi^{(r)\|} = 0, \tag{A2}$$

with the functions defined in Eqs. (37) and (38). The perturbed metric in the Poisson gauge is given by (99) thus, using the last constraints together with Eqs. (5.18)–(5.21) in [25] and following the procedure made in [11], the vector that determines the gauge transformation at first order $\xi_i^{(1)} = (\alpha^{(1)}, \partial_i \beta^{(1)} + d_i^{(1)})$ is given by

$$\alpha^{(1)} \to \omega_{(1)}^{\|} + \beta'_{(1)}, \qquad \beta^{(1)} \to -\frac{\chi^{\|(1)}}{2}, \qquad d_i^{(1)} \to -\chi_i^{\perp(1)}. \tag{A3}$$

Now to second order, when we use Eq. (5.37) in [25] with Eq. (38) we obtain the following transformations:

$$\tilde{\chi}^{(2)\|} = \chi^{(2)\|} + 2\beta^{(2)} + \frac{3}{2} \nabla^{-2} \nabla^{-2} \mathrm{X}^{(2)\|}, \tag{A4}$$

with

$$\begin{aligned}
\mathrm{X}^{(2)\|} = {} & 2(\partial^i \partial^j D_{ij} \chi'_{(1)\|} + 2H \partial^i \partial^j D_{ij} \chi^{(1)\|}) \alpha^{(1)} + \frac{2}{a^2} (a^2 \chi_{ij}^{(1)})' \partial^i \partial^j \alpha^{(1)} \\
& + 2\xi_{(1)}^k \partial^i \partial^j \partial_k D_{ij} \chi^{(1)\|} + 2\partial_k \chi_{ij}^{(1)} \partial^i \partial^j \xi_{(1)}^k + 2(-4\partial^i \partial^j \phi_{(1)} + \partial^i \partial^j \alpha^{(1)} \partial_0 + \partial^i \partial^j \xi_{(1)}^k \partial_k \\
& + 4H \partial^i \partial^j \alpha^{(1)})(\partial_{(j} d_{i)}^{(1)} + D_{ij} \beta^{(1)}) + 2(-4\phi^{(1)} + \alpha^{(1)} \partial_0 + \xi_{(1)}^k \partial_k + 4H\alpha^{(1)})(\partial^i \partial^j D_{ij} \beta_{(1)}) \\
& + 2\Big[ 2\omega_i^{(1)} \partial^i \nabla^2 \alpha_{(1)} + 2\partial^j \nabla^2 \omega_{(1)}^{\|} \partial_j \alpha^{(1)} - \partial_j \alpha^{(1)} \partial^j \nabla^2 \alpha_{(1)} + \partial^j \nabla^2 \beta'_{(1)} \partial_j \alpha^{(1)} + \xi_i^{(1)\prime} \partial^i \nabla^2 \alpha^{(1)} \\
& - \nabla^2 \Big[ (2\omega_{(1)}^k - \partial^k \alpha_{(1)} + \xi'_{(1)}) \partial_k \frac{\alpha_{(1)}}{3} \Big] \Big] + 2\Big[ 2\partial^i \partial^j (D_{ij} \chi_{(1)}^{\|} + \partial_i \chi_{k(1)}^{\perp}) \partial_j \xi_{(1)}^k + 2\chi_{ik}^{(1)} \partial^i \nabla^2 \xi_{(1)}^k \\
& + 2\partial_j \xi_{(1)}^k \partial^j \nabla^2 \xi_k^{(1)} + \partial_k \xi_i^{(1)} \partial^i \nabla^2 \xi_{(1)}^k + \partial_j \xi_{(1)}^k \partial^j \nabla^2 \beta_{(1)} - \frac{1}{3} \nabla^2 [(2\chi_{kl}^{(1)} + 2\partial_{(l} \xi_{k)}^{(1)}) \partial^l \xi_{(1)}^k] \Big].
\end{aligned} \tag{A5}$$

Now if we fix the Poisson gauge, $\tilde{\chi}^{(2)\|} = 0$ we can fix the scalar part of the space gauge

$$\beta^{(2)} = -\frac{\chi^{(2)\|}}{2} - \frac{3}{4} \nabla^{-2} \nabla^{-2} \mathrm{X}^{(2)\|}. \tag{A6}$$

For the vector space part we should know the transformation rule for the vector part

$$\tilde{\chi}_i^{(2)\perp} = -\partial_i(\nabla^{-2}\nabla^{-2}\mathrm{X}^{(2)\parallel}) + \chi_i^{(2)\perp} + d_i^{(2)} + \nabla^{-2}\mathrm{X}_i^{(2)\perp}, \tag{A7}$$

with

$$\begin{aligned}\mathrm{X}_i^{(2)\perp} = {} & 2(\partial^j D_{ij}\chi'^{\parallel}_{(1)} + \nabla^2\chi'^{\perp}_{i(1)} + 2H(\partial^j D_{ij}\chi^{\parallel}_{(1)} + \nabla^2\chi^{\perp}_{i(1)}))\alpha^{(1)} + \frac{2}{a^2}(a^2\chi_{ij}^{(1)})'\partial^j\alpha_{(1)} + 2\xi^k_{(1)}\partial_k(\partial^j D_{ij}\chi^{\parallel}_{(1)} + \nabla^2\chi_i^{\perp(1)}) \\ & + 2\partial_k\chi_{ij}^{(1)}\partial^j\xi^k_{(1)} + 2(-4\partial^j\phi_{(1)} + \partial^j\alpha_{(1)}\partial_0 + \partial^j\xi^k_{(1)}\partial_k + 4H\partial^j\alpha^{(1)})\cdot(\partial_{(j}d_{i)}^{(1)} + D_{ij}\beta^{(1)}) \\ & + (\alpha^{(1)}\partial_0 - 4\phi^{(1)} + \xi^k_{(1)}\partial_k + 4H\alpha^{(1)})\cdot(\nabla^2 d_i^{(1)} + 2\partial^j D_{ij}\beta^{(1)}) \\ & + 2\Big(\partial^j\omega_i^{(1)}\partial_j\alpha^{(1)} + \omega_i^{(1)}\nabla^2\alpha^{(1)} + \nabla^2\omega^{\parallel}_{(1)}\partial_i\alpha^{(1)} + \omega_j^{(1)}\partial^j\partial_i\alpha^{(1)} - \partial^j\partial_i\alpha^{(1)}\partial_j\alpha^{(1)} \\ & - \nabla^2\alpha^{(1)}\partial_i\alpha^{(1)} + \partial^j\xi'_{i(1)}\partial_j\frac{\alpha^{(1)}}{2} + \xi'_{i(1)}\nabla^2\frac{\alpha^{(1)}}{2} + \frac{1}{2}\xi'_{j(1)}\partial_i\partial^j\alpha^{(1)} - \frac{1}{3}\partial_i[(2\omega^k_{(1)} - \partial^k\alpha_{(1)} + \xi'_{(1)})\partial_k\alpha_{(1)}]\Big) \\ & + 2\Big(\chi_{ik}^{(1)}\nabla^2\xi^k_{(1)} + \partial_i\xi^k_{(1)}(\partial^j D_{jk}\chi^{\parallel}_{(1)} + \nabla^2\chi_k^{(1)\perp}) + \chi_{jk}^{(1)}\partial^j\partial_i\xi^k_{(1)} + \partial_j\xi^k_{(1)}\partial^j\partial_i\xi_k^{(1)} + \nabla^2\xi^k_{(1)}\partial_i\xi_k^{(1)} \\ & + \frac{1}{2}\partial^j\partial_k\xi_i^{(1)}\partial_j\xi^k_{(1)} + \frac{1}{2}\partial_k\nabla^2\beta^{(1)}\partial_i\xi^k_{(1)} + \frac{1}{2}\partial_k\xi_j^{(1)}\partial^j\partial_i\xi^k_{(1)} + \nabla^2\beta'_{(1)}\partial_i\frac{\alpha^{(1)}}{2} + \frac{1}{2}\partial_k\xi_i^{(1)}\nabla^2\xi^k_{(1)} \\ & + \partial^j\chi_{ik}^{(1)}\partial_j\xi^k_{(1)} - \frac{1}{3}\partial_i[(2\chi_{kl}^{(1)} + 2\partial_{(l}\xi_{k)}^{(1)})\partial^l\xi^k_{(1)}]\Big).\end{aligned} \tag{A8}$$

Now we use the condition $\tilde{\chi}_i^{(2)\perp} = 0$, for instance,

$$d_i^{(2)} = \partial_i(\nabla^{-2}\nabla^{-2}\mathrm{X}^{(2)\parallel}) - \chi_i^{(2)\perp} - \nabla^{-2}\mathrm{X}_i^{(2)\perp}. \tag{A9}$$

To find the temporal part of the gauge transformation, we use Eq. (5.35) in [25] and Eq. (37). With some algebra, the scalar part transforms like

$$\tilde{\omega}^{(2)\parallel} = \omega^{(2)\parallel} - \alpha^{(2)} + \beta'_{(2)} + \nabla^{-2}\mathrm{W}^{(2)\parallel}, \tag{A10}$$

with

$$\begin{aligned}\mathrm{W}^{(2)\parallel} = {} & -4(\partial^i\psi^{(1)}\partial_i\alpha^{(1)} + \psi^{(1)}\nabla^2\alpha^{(1)}) + \partial^i\alpha^{(1)}[2\omega_i^{(1)\prime} + 4H\omega_i^{(1)} - \partial_i\alpha'_{(1)} + \xi''_{(1)i} - 4H(\partial_i\alpha^{(1)} - \xi'_{i(1)})] \\ & + \alpha^{(1)}[2\nabla^2\omega'_{(1)\parallel} + 4H\nabla^2\omega^{(1)}_{\parallel} - \nabla^2\alpha'_{(1)} + \nabla^2\beta'' - 4H(\nabla^2\alpha^{(1)} - \nabla^2\beta'_{(1)})] \\ & + \partial^i\xi^j_{(1)}(2\partial_j\omega_i^{(1)} - \partial_i\partial_j\alpha^{(1)} + \partial_j\xi'_{i(1)}) + \xi^j_{(1)}(2\partial_j\nabla^2\omega^{\parallel}_{(1)} - \partial_j\nabla^2\alpha^{(1)} + \partial_j\nabla^2\beta'_{(1)}) \\ & + \alpha'_{(1)}(2\nabla^2\omega^{\parallel}_{(1)} - 3\nabla^2\alpha^{(1)} + \nabla^2\beta'_{(1)}) + \partial^i\alpha'_{(1)}(2\omega_i^{(1)} - 3\partial_i\alpha^{(1)} + \xi'_{(1)i}) + \nabla^2\xi^j_{(1)}(2\omega_j^{(1)} - \partial_j\alpha^{(1)}) \\ & + \partial_i\xi^j_{(1)}(2\partial^i\omega_j^{(1)} - \partial^i\partial_j\alpha^{(1)}) + \partial^i\xi'^j_{(1)}[\partial_j\xi_i^{(1)} + 2\chi_{ij}^{(1)} + 2\partial_i\xi_j^{(1)} - 4\phi^{(1)}\delta_{ij}] \\ & + \xi'^j_{(1)}[-4\partial_j\phi^{(1)} + 2(\partial^i D_{ij}\chi^{\parallel}_{(1)} + \nabla^2\chi_j^{(1)\perp} + 2\nabla^2\xi_j^{(1)} + \partial^i\partial_j\xi_i^{(1)})],\end{aligned} \tag{A11}$$

in this way we fix the temporal part of the gauge using $\tilde{\omega}^{(2)\parallel} = 0$ in the last equation finding the following:

$$\alpha^{(2)} = \omega^{\parallel}_{(2)} + \partial_0\beta_{(2)} + \nabla^{-2}\mathrm{W}^{(2)\parallel}. \tag{A12}$$

Therefore, we found explicitly the set of functions that fix the gauge dependence given by Eqs. (A3), (A6), (A9), and (A12). Thus, using the above equations we can calculate the gauge dependence in the scalar perturbations at second order that were shown in Eq. (67)

$$\mathcal{S}^{\parallel}_{(2)} = \omega^{\parallel}_{(2)} - \frac{\chi^{\parallel(2)\prime}}{2} - \frac{3}{4}\nabla^{-2}\nabla^{-2}\mathrm{X}_{\parallel}^{(2)\prime} + \nabla^{-2}\mathrm{W}^{(2)\parallel}, \tag{A13}$$

which can be interpreted like shear to second order; again we see the last equation is a generalization for the first-order scalar shear plus quadratic terms in the perturbed functions.

## APPENDIX B

To find the charge evolution, we use the fact that $j^{\alpha}_{;\alpha} = 0$; therefore, the temporal part of this equation drive us to the charge conservation

$$\varrho^{(1)\prime} + 3H\varrho^{(1)} + \partial_i J^i_{(1)} = 0, \tag{B1}$$

at first order in the approximation and

$$\varrho^{(2)\prime} + 3H\varrho^{(2)} + \partial_i J^i_{(2)} + (\Psi^{(1)\prime} - 3\Phi^{(1)\prime})\varrho^{(1)} + \partial_i(\Psi^{(1)\prime} - 3\Phi^{(1)\prime})J^i_{(1)} = 0, \tag{B2}$$

to second order. These equations are important for resolving the dynamo equation. In the Sec. VI was found the momentum equation at second order, where $S_i^{13}$ is given by

$$\begin{aligned} S_i^{13} &= \frac{[\Delta^{(1)}(1+c_s^2)\mathcal{V}_i^{(1)}]'}{\mu_{(0)}(1+w)} + 4H\frac{\Delta^{(1)}(1+c_s^2)\mathcal{V}_i^{(1)}}{\mu_{(0)}(1+w)} + 2\frac{[\mu_{(0)}(1+w)\chi_{ij}^{(1)}v^j_{(1)}]'}{\mu_{(0)}(1+w)} \\ &+ 8H\chi_{ij}^{(1)}v^j_{(1)} - 8H\Phi^{(1)}v_i^{(1)} - \frac{[\mu_{(0)}(1+w)(\omega_i^{(1)}+\mathcal{V}_i^{(1)})]'\Psi^{(1)}}{\mu_{(0)}(1+w)} + \frac{\Delta^{(1)}(1+c_s^2)\partial_i\Psi^{(1)}}{\mu_{(0)}(1+w)} \\ &- \omega_i^{(1)}\Psi'_{(1)} - 4H\Psi^{(1)}(\mathcal{V}_i^{(2)} + \omega_i^{(1)}) + v^j(\partial_j v_i^{(1)}) - 3\Phi'_{(1)}(\mathcal{V}_i^{(1)} + v_i^{(1)}) - v^j_{(1)}\partial_{[i}\omega_{j]}^{(1)} - 2\Psi^{(1)}\partial_i\Psi^{(1)} \\ &- 2\Phi^{(1)}[\mu_{(0)}(1+w)v_i^{(1)}]' + \frac{(\partial_j\Psi^{(1)} + H\omega_j^{(1)})\Pi_i^{j(1)}}{\mu_{(0)}(1+w)} + \mathcal{V}_i^{(1)}\partial_\alpha\partial^\alpha v_{(1)} - \frac{6\partial_j\Phi^{(1)}\Pi_i^{j(1)} - \frac{1}{2}\partial_j\chi_i^{k(1)}\Pi_k^{j(1)}}{\mu_{(0)}(1+w)}, \end{aligned} \tag{B3}$$

where $w = \frac{P_{(0)}}{\mu_{(0)}}$ is the state equation ($w = 0$ for dust and $w = 1/3$ for radiation era) and $c_s^2 = \frac{P_{(1)}}{\mu_{(1)}}$ the adiabatic sound speed. Using the expression for the momentum exchange among particles and the momentum conservation, we obtain the following equations for protons, electrons, and photons during radiation era

$$\mu_{(0)}^{(p)}[\mathcal{V}_i^{(2)(p)\prime} + H\mathcal{V}_i^{(2)(p)} + \partial_i\Psi^{(2)} + 2S_i^{13(p)}] + \partial_i\Delta_P^{(2)(p)} + \frac{4}{3}\partial_i\nabla^2\Pi^{(2)}_{(p)} = a^4(E_i^{(1)}\varrho^{(p)}_{(1)} + \epsilon_{ij}^k J^{j(p)}_{(1)}B_k^{(1)}) + K_i^{(ep)}, \tag{B4}$$

$$\mu_{(0)}^{(e)}[\mathcal{V}_i^{(2)(e)\prime} + H\mathcal{V}_i^{(2)(e)} + \partial_i\Psi^{(2)} + 2S_i^{13(e)}] + \partial_i\Delta_P^{(2)(e)} + \frac{4}{3}\partial_i\nabla^2\Pi^{(2)}_{(e)} = a^4(E_i^{(1)}\varrho^{(e)}_{(1)} + \epsilon_{ij}^k J^{j(e)}_{(1)}B_k^{(1)}) - K_i^{(ep)} + K_i^{(\gamma)}, \tag{B5}$$

$$\frac{4}{3}\mu_{(0)}^{(\gamma)}[\mathcal{V}_i^{(2)(\gamma)\prime} + \partial_i\Psi^{(2)} + 2S_i^{13(\gamma)}] + \partial_i\Delta_P^{(2)(\gamma)} + \frac{4}{3}\partial_i\nabla^2\Pi^{(2)}_{(\gamma)} = -K_i^{(\gamma)}. \tag{B6}$$

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[1] R. Beck, AIP Conf. Proc. **1085**, 83 (2008); J. A. Eilek and F. N. Owen, Astrophys. J. **567**, 202 (2002).

[2] M. Giovannini, Int. J. Mod. Phys. D **13**, 391 (2004); L. M. Widrow, Rev. Mod. Phys. **74**, 775 (2002).

[3] K. Kojima and K. Ichiki, arXiv:0902.1367; Y. Xu, P. P. Kronberg, S. Habib, and Q. W. Dufton, Astrophys. J. **637**, 19 (2006); J. R. Shaw and A. Lewis, Phys. Rev. D **86**, 043510 (2012); M. L. Bernet, F. Miniati, S. J. Lilly, P. P. Kronberg, and M. Dessauges-Zavadsky, Nature (London) **454**, 302 (2008).

[4] R. Durrer, New Astron. Rev. **51**, 275 (2007); J. D. Barrow, P. G. Ferreira, and J. Silk, Phys. Rev. Lett. **78**, 3610 (1997); D. G. Yamazaki, K. Ichiki, T. Kajino, and G. J. Mathews, Astrophys. J. **646**, 719 (2006).

[5] R. M. Kulsrud and E. G. Zweibel, Rep. Prog. Phys. **71**, 046901 (2008).

[6] L. Biermann and A. Schlüter, Phys. Rev. **82**, 863 (1951); H. Hanayama, K. Takahashi, K. Kotake, M. Oguri, K. Ichiki, and H. Ohno, Astrophys. J. **633**, 941 (2005); K. Ichiki, K. Takahashi, N. Sugiyama, H. Hanayama, and H. Ohno, arXiv:astro-ph/0701329; R. M. Kulsrud, R. Cen, J. P. Ostriker, and D. Ryu, Astrophys. J. **480**, 481 (1997).

[7] K. Dolag, M. Bartelmann, and H. Lesch, Astron. Astrophys. **348**, 351 (1999).

[8] D. Grasso and A. Riotto, Phys. Lett. B **418**, 258 (1998); K. Bamba and M. Sasaki, J. Cosmol. Astropart. Phys. 02 (2007) 030; K. E. Kunze, Phys. Rev. D **77**, 023530 (2008); R. Emami, H. Firouzjahi, and M. S. Movahed, Phys. Rev. D **81**, 083526 (2010); D. Grasso and H. R. Rubinstein, Phys. Rep. **348**, 163 (2001); L. Campanelli and M. Giannotti, Phys. Rev. D **72**, 123001 (2005); A. Berera, T. W. Kephart, and S. D. Wick, Phys. Rev. D **59**, 043510 (1999); R. S. de Souza and R. Opher, Phys. Rev. D **77**, 043529 (2008); A. Díaz-Gil, J. García-Bellido, M. García Pérez, and A. González-Arroyo, Phys. Rev. Lett. **100**, 241301 (2008); R. Durrer, L. Hollenstein, and R. K. Jain, J. Cosmol. Astropart. Phys. 03 (2011) 037; E. R. Harrison, Mon. Not. R. Astron. Soc. **147**, 279 (1970); S. Matarrese, S. Mollerach, A. Notari, and A. Riotto, Phys. Rev. D **71**, 043502 (2005); L. Hollenstein, C. Caprini, R. Crittenden, and R. Maartens, Phys. Rev. D **77**, 063517 (2008); K. Takahashi, K. Ichiki, and N. Sugiyama, Phys. Rev. D **77**, 124028 (2008).

[9] C. Caprini and R. Durrer, Phys. Rev. D **65**, 023517 (2001); T. Kahniashvili, A. G. Tevzadze, and B. Ratra, Astrophys. J. **726**, 78 (2011); T. Kahniashvili, A. G. Tevzadze, S. K. Sethi, K. Pandey, and B. Ratra, Phys. Rev. D **82**, 083005 (2010).

[10] A. Neronov and I. Vovk, Science **328**, 73 (2010); K. Takahashi, S. Inoue, K. Ichiki, and T. Nakamura, Mon. Not. R. Astron. Soc. **410**, 2741 (2011); K. Dolag, M. Kachelriess, S. Ostapchenko, and R. Toms, Astrophys. J. Lett. **727**, L4 (2011).

[11] R. Durrer, in *The Physics of the Early Universe*, edited by K. Tamvakis, Lecture Notes in Physics Vol. 653 (Springer-Verlag, Berlin, 2005), p. 31; K. A. Malik and D. Wands, Phys. Rep. **475**, 1 (2009); C. G. Tsagas, in *Cosmological Crossroads*, edited by S. Cotsakis and E. Papantonopoulos, Lecture Notes in Physics Vol. 592 (Springer-Verlag, Berlin, 2002), p. 223.

[12] R. Durrer, Fund. Cosmic Phys. **15**, 209 (1994); C. G. Tsagas, A. Challinor, and R. Maartens, Phys. Rep. **465**, 61 (2008).

[13] D. N. Spergel, L. Verde, H. V. Peiris, E. Komatsu, M. R. Nolta, C. L. Bennett, M. Halpern, G. Hinshaw, N. Jarosik, A. Kogut, M. Limon, S. S. Meyer, L. Page, G. S. Tucker, J. L. Weiland, E. Wollack, and E. L. Wright, Astrophys. J. Suppl. Ser. **148**, 175 (2003).

[14] D. Nitta, E. Komatsu, N. Bartolo, S. Matarrese, and A. Riotto, J. Cosmol. Astropart. Phys. 05 (2009) 014.

[15] K. A. Malik and D. Wands, Classical Quantum Gravity **21**, L65 (2004).

[16] D. Wands, K. A. Malik, D. H. Lyth, and A. R. Liddle, Phys. Rev. D **62**, 043527 (2000).

[17] Y.-i. Takamizu, S. Mukohyama, M. Sasaki, and Y. Tanaka, J. Cosmol. Astropart. Phys. 06 (2010) 019.

[18] J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980).

[19] H. Kodama and M. Sasaki, Prog. Theor. Phys. Suppl. **78**, 1 (1984).

[20] M. Giovannini and Z. Rezaei, Phys. Rev. D **83**, 083519 (2011); C. G. Tsagas, Classical Quantum Gravity **22**, 393 (2005); M. Bruni, R. Maartens, and C. G. Tsagas, Mon. Not. R. Astron. Soc. **338**, 785 (2003); C. G. Tsagas and R. Maartens, Phys. Rev. D **61**, 083519 (2000).

[21] J. D. Barrow and C. G. Tsagas, Mon. Not. R. Astron. Soc. **414**, 512 (2011).

[22] M. Bruni and S. Sonego, Classical Quantum Gravity **16**, L29 (1999).

[23] R. K. Sachs, *Gravitational Radiation, in Relativity, Groups and Topology*, edited by C. M. DeWitt and B. DeWitt (Gordon and Breach, New York, 1964); K. Nakamura, Phys. Rev. D **80**, 124021 (2009).

[24] G. F. R. Ellis and M. Bruni, Phys. Rev. D **40**, 1804 (1989); K. Nakamura, Prog. Theor. Phys. **110**, 723 (2003).

[25] J. M. Stewart, Classical Quantum Gravity **7**, 1169 (1990); M. Bruni, S. Matarrese, S. Mollerach, and S. Sonego, Classical Quantum Gravity **14**, 2585 (1997); K. Nakamura, Adv. Astron. **2010**, 576273 (2010).

[26] J. M. Stewart and M. Walker, Proc. R. Soc. A **341**, 49 (1974).

[27] K. A. Malik and D. R. Matravers, Classical Quantum Gravity **25**, 193001 (2008).

[28] C. F. Sopuerta, M. Bruni, and L. Gualtieri, Phys. Rev. D **70**, 064002 (2004).

[29] C. A. Clarkson, A. A. Coley, R. Maartens, and C. G. Tsagas, Classical Quantum Gravity **20**, 1519 (2003).

[30] J. D. Barrow, R. Maartens, and C. G. Tsagas, Phys. Rep. **449**, 131 (2007).

[31] K. Takahashi, K. Ichiki, H. Ohno, and H. Hanayama, Phys. Rev. Lett. **95**, 121301 (2005); M. Giovannini, Phys. Rev. D **79**, 103007 (2009).

[32] J. Jackson, *Classical Electrodynamics* (Wiley, New York, 1999); L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Pergamon, New York, 1959).

[33] T. Buchert, Gen. Relativ. Gravit. **33**, 1381 (2001).

[34] C. Pitrou, Classical Quantum Gravity **24**, 6127 (2007).

[35] R. Durrer, P. G. Ferreira, and T. Kahniashvili, Phys. Rev. D **61**, 043001 (2000); C. Caprini, R. Durrer, and T. Kahniashvili, Phys. Rev. D **69**, 063006 (2004); R. Durrer, *The Cosmic Microwave Background* (Cambridge University Press, Cambridge, England, 2008).

[36] C. Bonvin and C. Caprini, J. Cosmol. Astropart. Phys. 05, (2010) 022; K. Nakamura, Phys. Rev. D **80**, 124021 (2009); E. Fenu, C. Pitrou, and R. Maartens, Mon. Not. R. Astron. Soc. **414**, 2354 (2011).

[37] I. A. Brown, arXiv:0812.1781; A. Mack, T. Kahniashvili, and A. Kosowsky, Phys. Rev. D **65**, 123004 (2002); M. Giovannini, Classical Quantum Gravity **27**, 105011 (2010); M. Giovannini and K. E. Kunze, Phys. Rev. D **77**, 063003 (2008).

[38] E. Bertschinger, NASA STI/Recon Technical Report N **96**, 22249 (1995).

[39] C. G. Tsagas and R. Maartens, Classical Quantum Gravity **17**, 2215 (2000).

[40] K. Subramanian, Astron. Nachr. **331**, 110 (2010).

[41] J. Adams, U. H. Danielsson, D. Grasso, and H. Rubinstein, Phys. Lett. B **388**, 253 (1996).

[42] A. Kosowsky and A. Loeb, Astrophys. J. **469**, 1 (1996).

[43] R. Durrer, T. Kahniashvili, and A. Yates, Phys. Rev. D **58**, 123004 (1998).