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Computation of the p^6 order low-energy constants with tensor sources

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We present the results of calculations of the p^4 and p^6 order low-energy constants for the chiral Lagrangian with tensor sources for both two and three flavors of pseudoscalar mesons. This is a generalization of our previous work on similar calculations without tensor sources in terms of the quark self-energy $\Sigma(p^2)$, based on the first principle derivation of the low-energy effective Lagrangian and computation of the low-energy constants with some rough approximations. With the help of partial integration and some epsilon relations, we find that some p^6 order operators with tensor sources appearing in the literature are related to each other. That leaves 98 independent terms for n-flavor, 92 terms for three-flavor, and 65 terms for two-flavor cases. We also find that the odd-intrinsic-parity chiral Lagrangian with tensor sources cannot independently exist in any order of low-energy expansion.

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I. INTRODUCTION

In the low-energy region of the strong interaction, conventional perturbation theory is ineffective. If we focus on the pseudoscalar mesons (π, K, η), chiral perturbation theory provides us with an effective way to deal with the system. It can be applied not only to the strong interaction, but also to the weak and electromagnetic interactions. It was first introduced by Weinberg [1]. The idea is to expand the meson part Lagrangian in terms of powers of external momenta and the quark masses. Gasser and Leutwyler [2,3] then extended it to the p^4 order and built up the path integral formalism which enables us to compute the various Green's functions of the light-quark scalar, pseudoscalar, vector, and axial-vector currents in terms of the chiral Lagrangian. The formulation was later generalized to the p^6 order. The form of the normal part of the p^6 order chiral Lagrangian had been obtained [4–6] soon after the anomalous parts were given [7,8]. The latest and general review on the topic can be found in [9]. Missing in this series of work are the antisymmetric tensor currents, although this could be partly because of the fact that tensor currents do not appear in the Standard Model Lagrangian, as discussed in Ref. [10]. Research on hadron matrix elements and the study of interactions beyond the Standard Model may need these tensor currents. Further, antisymmetric tensor currents not only generate the conventional 1^{--} vector mesons, but also more exotic 1^{+-} mesons. Therefore, studies involving both of these and their interactions would bring in the antisymmetric currents. More importantly, for the structure of the general currents $\bar{\psi}\Gamma\psi$, the 4×4 Γ matrices generally have 16 degrees of freedom, and one usually chooses the 16 γ matrices 1, γ_5, γ_μ, $\gamma_\mu\gamma_5$, $\sigma_{\mu\nu}$ to represent these freedoms. This implies that Γ can be expanded in terms of the 16 γ matrices, and one is used to calling the currents according to their γ matrices' structures. Taking just scalar, pseudoscalar, vector, and axial-vector currents cannot give the most general bilinear light-quark currents because of incompleteness. Adding in the tensor currents, we can get the set of the currents completely. The results of the Green's functions among the currents would then be general. Six years ago, the form of the chiral Lagrangian involving tensor currents had been discussed first in Ref. [10]. The results were the normal parts with tensor sources starting from p^4 order, and both the p^4 and p^6 order chiral Lagrangian with tensor sources were obtained. While the odd-intrinsic-parity parts with tensor sources were claimed to start from p^8 order. Based on these results, more progress has been made [11–13].

Within the chiral perturbation theory, if the order of the momentum and the current quark masses' expansion is increased, the number of independent terms rises rapidly. For example, in the three-flavor case, the p^4 order Lagrangian has 10 terms plus 2 contact terms, but the p^6 order has 90 terms plus 3 contact terms. These independent terms generate a large number of unknown low-energy constants (LECs). A summary of numerical results for the LECs can be found in [14], which makes discussions about the high order effects of the chiral Lagrangians even more difficult and complex. Compared to dealing with higher order chiral Lagrangians, adding tensor sources is relatively less heavy and realistic work is possible. Originally, LECs were fixed via the experimented data. Now, because more LECs have appeared for high orders, and sufficient experimental data are lacking, we can no longer solely rely on experiment to determine these LECs. Calculations of LECs from various models or underlying

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QCD have subsequently developed and become popular. Indeed, not only experimented data, but theoretical calculations are also needed. With these calculations, we can check the correctness of the models or the theory. As members in the community of calculating LECs, we have calculated the p^4 order LECs in [15,16], and then the p^6 order calculations in [17,18], including the normal and the anomalous parts, for two- and three-flavor cases. In this paper, we will extend our work to the tensor sources, based on the first principle derivation of the low-energy effective Lagrangian [19] and computation of the LECs with some rough approximations[1] [15] and present all LECs up to the p^6 order [21].

In Euclidean space, the chiral Lagrangian includes the real and the imaginary parts. The real part is related to the even-intrinsic-parity sector, and the imaginary part is related to the odd-intrinsic-parity sector. As the tensor source terms always appear with $\sigma_{\mu\nu}$, we can use the following Eq. (1) to interchange even- and odd-intrinsic-parity sectors:

$$\sigma^{\mu\nu}\gamma_5 = \frac{i}{2}\epsilon^{\mu\nu\lambda\rho}\sigma_{\lambda\rho}. \tag{1}$$

It implies that when calculating the tensor source parts, one needs to include both real and imaginary parts [22].[2]

This paper is organized as follows: In Sec. II, we review our previous calculations on the real part of the chiral Lagrangian from p^2 to p^6 order and add in the tensor sources. In Sec. III, we review our previous calculations on the imaginary part of the chiral Lagrangian up to p^6 order and add the tensor sources. In Sec. IV, we collect the differences in convention between our paper and Ref. [10] and discuss the possible dependent operators. In Sec. V, we give our p^4 order results with tensor sources, and Sec. VI presents our p^6 order results. Section VII is a summary.

II. CALCULATIONS INVOLVING THE REAL PART OF THE CHIRAL LAGRANGIAN FOR ORDER p^2 UP TO p^6 WITH TENSOR SOURCES

We have calculated the real part of chiral Lagrangian from p^2 to p^6 order without tensor sources in Ref. [17]. Using the same method, we can also deal with the tensor source part. For convenience, we give a short introduction here, while adding in these external tensor sources.

The difference from Ref. [17] is the external tensor sources $\bar{t}^{\mu\nu}$. Adding them in the original external sources s, p, v^μ, a^μ, denoted as scalar, pseudoscalar, vector and axial-vector sources, respectively, we get the complete sources set as follows:

$$J = \not{v} + \not{a}\gamma_5 - s + ip\gamma_5 + \sigma_{\mu\nu}\bar{t}^{\mu\nu}. \tag{2}$$

From the original QCD, the Lagrangian can be written as the QCD Lagrangian, $\mathcal{L}^0_{\mathrm{QCD}}$, plus the external sources part, and the generating functional reads

$$\begin{aligned} Z[J] &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\Psi\mathcal{D}\bar{\Psi}\mathcal{D}A_\mu \\ &\quad\times \exp\left\{i\int d^4x[\mathcal{L}^0_{\mathrm{QCD}} + \bar{\psi}J\psi]\right\} \\ &= \int \mathcal{D}U \exp\left\{i\int d^4x\mathcal{L}_{\mathrm{ChPT}}(U,J)\right\} = \int \mathcal{D}U e^{iS_{\mathrm{eff}}}, \end{aligned} \tag{3}$$

where ψ, Ψ and A_μ are light-quark, heavy-quark, and gluon fields, respectively; U is the pseudoscalar meson field; $\mathcal{L}_{\mathrm{ChPT}}$ is the chiral Lagrangian; and S_{eff} is the effective action. Because this form of the chiral Lagrangian is explicitly U dependent at the high momentum orders and is hard to investigate [4,5] due to its complex U and J structures, we are used to making the chiral rotation to simplify the Lagrangian as [15,17,19]

$$\begin{aligned} J_\Omega &= [\Omega P_R + \Omega^\dagger P_L][J + i\not{\partial}][\Omega P_R + \Omega^\dagger P_L] \\ &= \not{v}_\Omega + \not{a}_\Omega\gamma_5 - s_\Omega + ip_\Omega\gamma_5 + \sigma_{\mu\nu}\bar{t}^{\mu\nu}_\Omega, \end{aligned} \tag{4}$$

$$U = \Omega^2, \qquad P_L = \frac{1-\gamma_5}{2}, \qquad P_R = \frac{1+\gamma_5}{2}. \tag{5}$$

To separate the tensor sources into even and odd parities, $t^{\mu\nu}_\pm$, we need the following tensor chiral projectors as Ref. [10]:

$$P_R^{\mu\nu\lambda\rho} = \frac{1}{4}(g^{\mu\lambda}g^{\nu\rho} - g^{\nu\lambda}g^{\mu\rho} + i\epsilon^{\mu\nu\lambda\rho}), \tag{6}$$

$$P_L^{\mu\nu\lambda\rho} = (P_R^{\mu\nu\lambda\rho})^\dagger = \frac{1}{4}(g^{\mu\lambda}g^{\nu\rho} - g^{\nu\lambda}g^{\mu\rho} - i\epsilon^{\mu\nu\lambda\rho}), \tag{7}$$

$$t^{\mu\nu} = \frac{1}{2}(t^{\mu\nu}_+ + t^{\mu\nu}_-), \qquad t^{\dagger\mu\nu} = \frac{1}{2}(t^{\mu\nu}_+ - t^{\mu\nu}_-), \tag{8}$$

$$\bar{t}^{\mu\nu} = P_L^{\mu\nu\lambda\rho}t_{\lambda\rho} + P_R^{\mu\nu\lambda\rho}t^\dagger_{\lambda\rho}, \tag{9}$$

$$\begin{aligned} \sigma_{\mu\nu}\bar{t}^{\mu\nu} &= \frac{1}{2}\sigma_{\mu\nu}t^{\mu\nu}_+ - \frac{i}{4}\sigma_{\mu\nu}\epsilon^{\mu\nu\lambda\rho}t_{-,\lambda\rho} \\ &= \frac{1}{2}\sigma_{\mu\nu}(t^{\mu\nu}_+ - t^{\mu\nu}_-\gamma_5) = \sigma_{\mu\nu}\bar{t}'^{\mu\nu}, \end{aligned} \tag{10}$$

[1]The detailed approximations in these computations of LECs are taking the ladder approximation, modeling the low-energy behavior of the gluon propagator, neglecting angle dependence in the running coupling constant in the kernel of the Schwinger-Dyson equation for the quark self-energy, assuming the ansatz solution for the external source dependent Schwinger-Dyson equation in terms of the quark self-energy, and taking the large N_c limit. The final effective action before taking the momentum expansion was shown to be equivalent to the result of a phenomenological, gauge invariant, nonlocal, and dynamical (GND) quark model [20].

[2]In a private communication, Y.-L. Ma in Ref. [16] had already obtained a similar but unpublished result in 2003.

$$\bar{t}'^{\mu\nu} \equiv \frac{1}{2}(t_+^{\mu\nu} - t_-^{\mu\nu}\gamma_5). \tag{11}$$

To obtain Eq. (10), we have used the γ-matrix identity Eq. (1) and introduced $\bar{t}'^{\mu\nu}$ for calculational convenience. After this operation, our definitions have the simple relations as found in [5,10] (see Appendix A). Using the same method as presented in [17,19], we can obtain the effective action $S_{\rm eff}$ introduced in Eq. (3) from the first principle of QCD,

$$S_{\rm eff} = -iN_c\mathrm{Tr}\ln[i\not{\partial} + J_\Omega - \Pi_{\Omega c}] + iN_c\mathrm{Tr}\ln[i\not{\partial} + J_\Omega] - iN_c\mathrm{Tr}\ln[i\not{\partial} + J] + N_c\mathrm{Tr}[\Phi_{\Omega c}\Pi_{\Omega c}^T] + N_c\sum_{n=2}^{\infty}\int d^4x_1\ldots d^4x'_n\frac{(-i)^n(N_cg_s^2)^{n-1}}{n!}\bar{G}^{\sigma_1\ldots\sigma_n}_{\rho_1\ldots\rho_n}(x_1,x'_1,\ldots,x_n,x'_n)\Phi^{\sigma_1\rho_1}_{\Omega c}(x_1,x'_1)\cdots\Phi^{\sigma_n\rho_n}_{\Omega c}(x_n,x'_n) + O\Big(\frac{1}{N_c}\Big). \tag{12}$$

Equation (12) is the same as Eq. (1) in [17], but J_Ω (the external source J, including currents and densities after Goldstone fields dependent chiral rotation Ω) includes the tensor sources. $\Phi_{\Omega c}$ and $\Pi_{\Omega c}$ are, respectively, the two-point rotated quark Green's function and the interaction part of the two-point rotated quark vertices in the presence of the external sources; $\Pi_{\Omega c}$ is defined by

$$\Phi^{\sigma\rho}_{\Omega c}(x,y) \equiv \frac{1}{N_c}\langle\overline{\psi}^\sigma_\Omega(x)\psi^\rho_\Omega(y)\rangle = -i[(i\not{\partial} + J_\Omega - \Pi_{\Omega c})^{-1}]^{\rho\sigma}(y,x), \qquad \psi_\Omega(x) \equiv [\Omega(x)P_L + \Omega^\dagger(x)P_R]\psi(x), \tag{13}$$

with subscript c denoting the corresponding classical field. $\bar{G}^{\sigma_1\ldots\sigma_n}_{\rho_1\ldots\rho_n}(x_1,x'_1,\ldots,x_n,x'_n)$ is the effective gluon n-point Green's function, including gluon and heavy-quark contributions, and g_s is the strong coupling constant of QCD. Note that the last two terms in the rhs of Eq. (12) can be shown to be independent of the pseudoscalar meson field U or Ω and therefore are just irrelevant constants in the effective action, while the second and third terms represent the variations of the path integral measure of the light-quark field ψ. The remaining first term relies on $\Pi_{\Omega c}$. $\Phi_{\Omega c}$ and $\Pi_{\Omega c}$ are related by the first equation of (13) and determined by

$$[\Phi_{\Omega c} + \tilde{\Xi}]^{\sigma\rho} + \sum_{n=1}^{\infty}\int d^4x_1d^4x'_1\ldots d^4x_nd^4x'_n\frac{(-i)^{n+1}(N_cg_s^2)^n}{n!}\bar{G}^{\sigma\sigma_1\ldots\sigma_n}_{\rho\rho_1\ldots\rho_n}(x,y,x_1,x'_1,\ldots,x_n,x'_n) \times\Phi^{\sigma_1\rho_1}_{\Omega c}(x_1,x'_1)\cdots\Phi^{\sigma_n\rho_n}_{\Omega c}(x_n,x'_n) = O\Big(\frac{1}{N_c}\Big), \tag{14}$$

where $\tilde{\Xi}$ is a Lagrangian multiplier which ensures the constraint $\mathrm{tr}_l[\gamma_5\Phi^T_{\Omega c}(x,x)] = 0$. Equation (14) is the Schwinger-Dyson equation (SDE) in the presence of the rotated external source. In Ref. [15], we have assumed the ansatz solution of (14) to be in the approximate form

$$\Pi^{\sigma\rho}_{\Omega c}(x,y) = [\Sigma(\bar{\nabla}_x^2)]^{\sigma\rho}\delta^4(x-y), \qquad \bar{\nabla}^\mu_x = \partial^\mu_x - iv^\mu_\Omega(x), \tag{15}$$

where Σ is the quark self-energy which satisfies the SDE (14) with a vanishing rotated external source. Under the ladder approximation, this SDE in Euclidean space-time is reduced to the standard form of

$$\Sigma(p^2) - 3C_2(R)\int\frac{d^4q}{4\pi^3}\frac{\alpha_s[(p-q)^2]}{(p-q)^2}\frac{\Sigma(q^2)}{q^2+\Sigma^2(q^2)} = 0, \tag{16}$$

where $C_2(R)$ is the second order Casimir operator of the quark representation R. In our case, quarks belong to the $SU(N_c)$ fundamental representation; therefore, $C_2(R) = (N_c^2-1)/2N_c$, and in the large N_c limit, we will neglect the second term of it. $\alpha_s(p^2)$ is the running coupling constant of QCD which depends on N_c and the number of quark flavors. With these approximations, the action (12) of the chiral Lagrangian becomes

$$S_{\rm eff} \approx -iN_c\mathrm{Tr}\ln[i\not{\partial} + J_\Omega - \Sigma(\bar{\nabla}_x^2)] + iN_c\mathrm{Tr}\ln[i\not{\partial} + J_\Omega] - iN_c\mathrm{Tr}\ln[i\not{\partial} + J] + O\Big(\frac{1}{N_c}\Big). \tag{17}$$

We have proved that the second and the third terms in (17) only provide the contribution correlating to the Wess-Zumino-Witten term. In the large N_c limit, if we do not focus on Wess-Zumino-Witten terms, we can neglect them:

$$S_{\rm eff} \approx -iN_c\mathrm{Tr}\ln[i\not{\partial} + J_\Omega - \Sigma(\bar{\nabla}_x^2)]. \tag{18}$$

Because in Minkowski space, it is not convenient to perform the computation, we perform the Wick rotation to change Eq. (17) to Euclidean space, with the metric tensor $g^{\mu\nu} = \mathrm{diag}(1,1,1,1)$:

$$
\begin{aligned}
&x^0|_M \to -ix^4|_E, \quad x^i|_M \to x^i|_M, \quad \gamma^0|_M \to \gamma^4|_E, \\
&\gamma^i|_M \to i\gamma^i|_E, \quad \gamma_5|_M \to \gamma_5|_E, \quad s_\Omega|_M \to -s_\Omega|_E, \\
&p_\Omega|_M \to -p_\Omega|_E, \quad \bar{t}_{\Omega,00}|_M \to -\bar{t}_{\Omega,44}|_E, \quad \bar{t}_{\Omega,ij}|_M \to \bar{t}_{\Omega,ij}|_E, \\
&\bar{t}_{\Omega,0i}|_M \to i\bar{t}_{\Omega,4i}|_E, \quad \bar{t}_{\Omega,i0}|_M \to i\bar{t}_{\Omega,i4}|_E. \qquad (19)
\end{aligned}
$$

Here v_Ω^μ, a_Ω^μ transform as x^μ, whereas $\bar{t}_{\Omega,\mu\nu}$ are considered (axial-)vector-(axial-)vector combined. Equation (17) in Euclidean space is

$$
\begin{aligned}
S_{\rm eff} &\approx N_c \mathrm{Tr}\ln[\not\partial + J_{\Omega,E} + \Sigma(-\bar{\nabla}_x^2)] \\
&= N_c \mathrm{Tr}\ln[\not\partial - i\not{v}_\Omega - i\not{a}_\Omega\gamma_5 - s_\Omega + ip_\Omega\gamma_5 \\
&\quad + \sigma_{\mu\nu}\bar{t}_\Omega^{\dagger\mu\nu} + \Sigma(-\bar{\nabla}_x^2)]. \qquad (20)
\end{aligned}
$$

With the help of the Schwinger proper time method [23], the real part (or, equivalently, the even-intrinsic-parity part or the normal part) of $\mathrm{Tr}\ln[\cdots]$ in Euclidean space-time can be written as

$$
\begin{aligned}
&\mathrm{ReTr}\ln[\not\partial - i\not{v}_\Omega - i\not{a}_\Omega\gamma_5 - s_\Omega + ip_\Omega\gamma_5 + \sigma_{\mu\nu}\bar{t}_\Omega^{\dagger\mu\nu} + \Sigma(-\bar{\nabla}_x^2)] \\
&\quad \equiv \mathrm{ReTr}\ln[D - \sigma_{\mu\nu}\bar{t}_\Omega^{\mu\nu} + \Sigma(-\bar{\nabla}_x^2)] \\
&\quad = \frac{1}{2}\mathrm{Tr}\ln[[D^\dagger + \sigma_{\mu\nu}\bar{t}_\Omega^{\dagger\mu\nu} + \Sigma(-\bar{\nabla}_x^2)][D + \sigma_{\mu\nu}\bar{t}_\Omega^{\mu\nu} + \Sigma(-\bar{\nabla}_x^2)]] = \frac{1}{2}\mathrm{Tr}\ln[O+N] \\
&\quad = -\frac{1}{2}\lim_{\Lambda\to\infty}\int_{\frac{1}{\Lambda^2}}^\infty \frac{d\tau}{\tau}\mathrm{Tr}e^{-\tau(O+N)}(\text{remove const term}) \\
&\quad = -\frac{1}{2}\lim_{\Lambda\to\infty}\int_{\frac{1}{\Lambda^2}}^\infty \frac{d\tau}{\tau}\int d^4x \mathrm{tr}_f\langle x|e^{-\tau(O+N)}|x\rangle \qquad (21)
\end{aligned}
$$

$$
D \equiv \not\partial - i\not{v}_\Omega - i\not{a}_\Omega\gamma_5 - s_\Omega + ip_\Omega\gamma_5. \qquad (22)
$$

Where a cutoff Λ is introduced into the theory to regularize the possible ultraviolet divergences. O is the old operator without tensor sources in [17], and N is the new operator with tensor sources:

$$
O = [D^\dagger + \Sigma(-\bar{\nabla}_x^2)][D + \Sigma(-\bar{\nabla}_x^2)], \qquad (23)
$$

$$
\begin{aligned}
N = &-\bar{\nabla}_x^\lambda\bar{t}_\Omega^{\mu\nu}\gamma^\lambda\sigma^{\mu\nu} + \bar{t}_\Omega^{\dagger\mu\nu}\bar{\nabla}_x^\lambda\sigma^{\mu\nu}\gamma^\lambda - i\not{a}_\Omega\bar{t}_\Omega^{\mu\nu}\sigma^{\mu\nu}\gamma_5 - i\bar{t}_\Omega^{\dagger\mu\nu}\sigma^{\mu\nu}\not{a}_\Omega\gamma_5 - s_\Omega\bar{t}_\Omega^{\mu\nu}\sigma^{\mu\nu} - \bar{t}_\Omega^{\dagger\mu\nu}s_\Omega\sigma^{\mu\nu} - ip_\Omega\bar{t}_\Omega^{\mu\nu}\sigma^{\mu\nu}\gamma_5 \\
&+ i\bar{t}_\Omega^{\dagger\mu\nu}p_\Omega\sigma^{\mu\nu}\gamma_5 + \Sigma(-\bar{\nabla}_x^2)\bar{t}_\Omega^{\mu\nu}\sigma^{\mu\nu} + \bar{t}_\Omega^{\dagger\mu\nu}\Sigma(-\bar{\nabla}_x^2)\sigma^{\mu\nu} + \bar{t}_\Omega^{\dagger\mu\nu}\bar{t}_\Omega^{\lambda\rho}\sigma_{\mu\nu}\sigma_{\lambda\rho}. \qquad (24)
\end{aligned}
$$

If we calculate Eq. (21) directly, it is not explicitly chiral covariant for each term in the calculation. In order to recover the chiral covariant form to get the LECs, we would need to collect the relevant terms together by hand, which consumes too much time. Fortunately, we found a method of keeping the chiral covariance at each step in the low-energy expansion computation [24], and we used it successfully to obtain the LECs of the real part without tensor sources [17]. We introduce it here briefly. Use the relations

$$
k^\mu + i\bar{\nabla}_x^\mu = e^{i\bar{\nabla}_x\cdot\frac{\partial}{\partial k}}\left(k^\mu + \tilde{F}^\mu\left(\bar{\nabla}, \frac{\partial}{\partial k}\right)\right)e^{-i\bar{\nabla}_x\cdot\frac{\partial}{\partial k}}, \qquad (25)
$$

$$
\begin{aligned}
\tilde{F}^\mu = &\frac{1}{2}[\bar{\nabla}_x^\nu, \bar{\nabla}_x^\mu]\partial_k^\nu - \frac{i}{3}[\bar{\nabla}_x^\lambda, [\bar{\nabla}_x^\nu, \bar{\nabla}_x^\mu]]\partial_k^\lambda\partial_k^\nu - \frac{1}{8}[\bar{\nabla}_x^\rho, [\bar{\nabla}_x^\lambda, [\bar{\nabla}_x^\nu, \bar{\nabla}_x^\mu]]]\partial_k^\rho\partial_k^\lambda\partial_k^\nu + \frac{i}{30}[\bar{\nabla}_x^\sigma, [\bar{\nabla}_x^\rho, [\bar{\nabla}_x^\lambda, [\bar{\nabla}_x^\nu, \bar{\nabla}_x^\mu]]]]\partial_k^\sigma\partial_k^\rho\partial_k^\lambda\partial_k^\nu \\
&+ \frac{1}{144}[\bar{\nabla}_x^\delta, [\bar{\nabla}_x^\sigma, [\bar{\nabla}_x^\rho, [\bar{\nabla}_x^\lambda, [\bar{\nabla}_x^\nu, \bar{\nabla}_x^\mu]]]]]\partial_k^\delta\partial_k^\sigma\partial_k^\rho\partial_k^\lambda\partial_k^\nu + O(p^7). \qquad (26)
\end{aligned}
$$

Substituting (25) into the integrand in (21), we change it to

$$\begin{aligned}
\mathrm{tr}_f\langle x|e^{-\tau(O(i\bar{\nabla}_x)+N(i\bar{\nabla}_x))}|x\rangle &= \mathrm{tr}_f\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4k'}{(2\pi)^4}e^{ik'\cdot x}\langle k'|e^{-\tau(O(i\bar{\nabla}_x)+N(i\bar{\nabla}_x))}|k\rangle e^{-ik\cdot x}\\
&= \mathrm{tr}_f\int\frac{d^4k}{(2\pi)^4}e^{-\tau(O(k+i\bar{\nabla}_x)+N(k+i\bar{\nabla}_x))}\\
&= \mathrm{tr}_f\int\frac{d^4k}{(2\pi)^4}e^{-\tau(e^{i\bar{\nabla}_x\cdot\frac{\partial}{\partial k}}\tilde{O}(k+i\bar{\nabla}_x)e^{-i\bar{\nabla}_x\cdot\frac{\partial}{\partial k}}+e^{i\bar{\nabla}_x\cdot\frac{\partial}{\partial k}}\tilde{N}(k+i\bar{\nabla}_x)e^{-i\bar{\nabla}_x\cdot\frac{\partial}{\partial k}})}\\
&= \mathrm{tr}_f\int\frac{d^4k}{(2\pi)^4}e^{i\bar{\nabla}_x\cdot\frac{\partial}{\partial k}}e^{-\tau(\tilde{O}(k+i\bar{\nabla}_x)+\tilde{N}(k+i\bar{\nabla}_x))}e^{-i\bar{\nabla}_x\cdot\frac{\partial}{\partial k}}\\
&= \mathrm{tr}_f\int\frac{d^4k}{(2\pi)^4}e^{-\tau(\tilde{O}(k+i\bar{\nabla}_x)+\tilde{N}(k+i\bar{\nabla}_x))},
\end{aligned}\tag{27}$$

where $\tilde{O}$ is the original exponent without tensor sources, which can be found in Eqs. (14) and (15) in Ref. [17], and $\tilde{N}$ is the new operator with tensor sources. We have

$$\begin{aligned}
\tilde{N} = {}& i(k^\lambda+\tilde{F}^\lambda)\tilde{\bar{t}}_\Omega^{\mu\nu}\gamma^\lambda\sigma^{\mu\nu} - i\tilde{\bar{t}}_\Omega^{\dagger\mu\nu}(k^\lambda+\tilde{F}^\lambda)\sigma^{\mu\nu}\gamma^\lambda - i\tilde{a}_\Omega^\lambda\tilde{\bar{t}}^{\mu\nu}\gamma^\lambda\sigma_\Omega^{\mu\nu}\gamma_5 - i\tilde{\bar{t}}_\Omega^{\dagger\mu\nu}\tilde{a}_\Omega^\lambda\sigma^{\mu\nu}\gamma^\lambda\gamma_5 - \tilde{s}_\Omega\tilde{\bar{t}}_\Omega^{\mu\nu}\sigma^{\mu\nu}\\
&- \tilde{\bar{t}}_\Omega^{\dagger\mu\nu}\tilde{s}_\Omega\sigma^{\mu\nu} - i\tilde{p}_\Omega\tilde{\bar{t}}_\Omega^{\mu\nu}\sigma^{\mu\nu}\gamma_5 + i\tilde{\bar{t}}_\Omega^{\dagger\mu\nu}\tilde{p}_\Omega\sigma^{\mu\nu}\gamma_5 + \Sigma([k^\mu+\tilde{F}^\mu]^2)\tilde{\bar{t}}_\Omega^{\mu\nu}\sigma^{\mu\nu}\\
&+ \tilde{\bar{t}}_\Omega^{\dagger\mu\nu}\Sigma([k^\mu+\tilde{F}^\mu]^2)\sigma^{\mu\nu} + \tilde{\bar{t}}_\Omega^{\dagger\mu\nu}\tilde{\bar{t}}_\Omega^{\lambda\rho}\sigma_{\mu\nu}\sigma_{\lambda\rho},
\end{aligned}\tag{28}$$

$$\begin{aligned}
\tilde{\mathcal{O}} = {}& \mathcal{O} - i[\bar{\nabla}_x^\nu,\mathcal{O}]\partial_k^\nu - \frac{1}{2}[\bar{\nabla}_x^\lambda,[\bar{\nabla}_x^\nu,\mathcal{O}]]\partial_k^\nu\partial_k^\lambda + \frac{i}{6}[\bar{\nabla}_x^\rho,[\bar{\nabla}_x^\lambda,[\bar{\nabla}_x^\nu,\mathcal{O}]]]\partial_k^\nu\partial_k^\lambda\partial_k^\rho + \frac{1}{24}[\bar{\nabla}_x^\sigma,[\bar{\nabla}_x^\rho,[\bar{\nabla}_x^\lambda,[\bar{\nabla}_x^\nu,\mathcal{O}]]]]\partial_k^\sigma\partial_k^\nu\partial_k^\lambda\partial_k^\rho\\
&- \frac{i}{120}[\bar{\nabla}_x^\delta,[\bar{\nabla}_x^\sigma,[\bar{\nabla}_x^\rho,[\bar{\nabla}_x^\lambda,[\bar{\nabla}_x^\nu,\mathcal{O}]]]]]\partial_k^\delta\partial_k^\sigma\partial_k^\nu\partial_k^\lambda\partial_k^\rho + O(p^7),
\end{aligned}\tag{29}$$

where $\tilde{\mathcal{O}} \equiv (\tilde{a}^\mu, \tilde{s}, \tilde{p}, \tilde{\bar{t}}^{\alpha\beta})^T$ and $\mathcal{O} \equiv (a_\Omega^\mu, s_\Omega, p_\Omega, \bar{t}_\Omega^{\prime\alpha\beta})^T$. With Eqs. (21) and (27), we get

$$\mathrm{Re}\,\mathrm{Tr}\ln[\not{\partial} - i\not{v}_\Omega - i\not{a}_\Omega\gamma_5 - s_\Omega + ip_\Omega\gamma_5 + \sigma_{\mu\nu}\bar{t}_\Omega^{\mu\nu} + \Sigma(-\bar{\nabla}^2)] = -\frac{1}{2}\lim_{\Lambda\to\infty}\int_{\frac{1}{\Lambda^2}}^\infty\frac{d\tau}{\tau}\int d^4x\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}e^{B+B_t}\cdot 1. \tag{30}$$

B can be found in Eq. (17) of Ref. [17], and $B_t = -\tau\tilde{N}$. Expanding Eq. (30) in powers of momentum, theoretically, we can get all orders of the chiral Lagrangian. Before giving our result, we need to discuss the difference between our paper and Ref. [10]; this will be done in Sec. IV.

III. CALCULATIONS INVOLVING THE IMAGINARY PART OF THE CHIRAL LAGRANGIAN FOR ORDER p^2 UP TO p^6 WITH TENSOR SOURCES

Because of Eq. (1) or, equivalently, (39) below in the next section, all the odd-intrinsic-parity sectors of the chiral Lagrangian can be changed to the even-intrinsic-parity sectors. If one keeps t_+ and t_- explicit, then the odd-intrinsic-parity sector is redundant and can be shown to be equivalent to even-intrinsic-parity operators. If instead only one of the two (t_+ or t_-) is used, then odd-intrinsic-parity operators are present. In Euclidean space, the odd-intrinsic-parity sectors belong to the imaginary parts. In other words, in Euclidean space, we can interchange the imaginary and real tensor source dependent parts. But in this section, we also call the parts without using Eq. (1) imaginary parts. Now we deal with the imaginary parts of Eq. (20) as a compensation of real parts discussion in Sec. II. We have calculated the p^6 order imaginary part's LECs without tensor sources in Ref. [18]. Using the same method, we can also compute the contributions involving the tensor source part. We repeat the process here, adding the tensor source.

First, to confirm the Wess-Zumino-Witten terms, we need to introduce a fifth-dimension integral. We now write Ω, in Eq. (4), as $\Omega = e^{-i\beta}$ and further introduce a parameter t dependent rotation element $\Omega(t) = e^{-i\beta t}$. With the help of relations $\Omega(1) = \Omega$ and $\Omega(0) = 1$, Eq. (20) becomes

$$S_{\mathrm{eff}}[U(t), J(t)] \approx N_c\mathrm{Tr}\ln[\not{\partial} + J_{\Omega(t)} + \Sigma(-\bar{\nabla}_t^2)]_{t=1}, \tag{31}$$

with $\bar{\nabla}_t^\mu = \partial^\mu - iv_{\Omega(t)}^\mu$. $J_{\Omega(t)}$ is J_Ω with Ω replaced by $\Omega(t)$.

Second, because we only focus on the meson terms, adding in an extra pure source makes no sense for the results. We insert an extra pure source term, setting $t = 0$ in (31), with the help of

$$\frac{\partial J_{\Omega(t)}}{\partial t} = \frac{1}{2}\left[\frac{\partial U}{\partial t}U_t^\dagger\gamma_5, \not{\partial} + J_{\Omega(t)}\right]_+, \qquad U_t = \Omega^2(t), \tag{32}$$

we can further proceed to express the chiral Lagrangian in terms of an integration over the parameter t:

$$\begin{aligned}S_{\mathrm{eff}}[U(t),J(t)] &= N_c\mathrm{Tr}\ln[\not\partial + J_{\Omega(t)} + \Sigma(-\bar{\nabla}_t^2)]_{t=0,\Sigma\ \mathrm{dependent}}^{t=1} = N_c\int_0^1 dt\frac{d}{dt}\mathrm{Tr}\ln[\not\partial + J_{\Omega(t)} + \Sigma(-\nabla_t^2)]|_{\Sigma\ \mathrm{dependent}} \\ &= N_c\int_0^1 dt\mathrm{Tr}\Big[\Big[\frac{\partial J_{\Omega(t)}}{\partial t} + \frac{\partial\Sigma(-\nabla_t^2)}{\partial t}\Big][\not\partial + J_{\Omega(t)} + \Sigma(-\nabla_t^2)]^{-1}\Big]_{\Sigma\ \mathrm{dependent}} \\ &= N_c\int_0^1 dt\mathrm{Tr}\Big[\Big(\frac{1}{2}\Big[\frac{\partial U_t}{\partial t}U_t^\dagger\gamma_5, \not\partial + J_{\Omega(t)}\Big]_+ + \frac{\partial\Sigma(-\nabla_t^2)}{\partial t}\Big)[\not\partial + J_{\Omega(t)} + \Sigma(-\nabla_t^2)]^{-1}\Big]_{\Sigma\ \mathrm{dependent}}. \end{aligned} \tag{33}$$

We only need to calculate the Σ dependent terms in Eq. (33), because the Σ independent terms are related to the contact terms [17].

Finally, as in [18], expanding (33) to p^4 order, we can get the Wess-Zumino-Witten term and terms related to the tensor source. Furthermore, to the p^6 order, including the tensor source, we can get the imaginary part we need. The particular calculation after expanding (33) will be introduced in Sec. V B.

IV. CONVENTION DIFFERENCES AND INDEPENDENT OPERATORS

To accord with our original results and for computational convenience, we make the following changes in this paper:

(i) To match Ref. [5] and our original results in Ref. [17], we define

$$\chi_{\pm,\mu} = \nabla_\mu\chi_\pm - \frac{i}{2}\{\chi_\mp, u_\mu\}, \tag{34}$$

comparing this with

$$\chi_{\pm,\mu} = \nabla_\mu\chi_\pm, \tag{35}$$

of Ref. [10]. Here χ_+, χ_- and u_μ are analogous to s_Ω, p_Ω, and a_Ω in our notation (see details in Appendix A).

(ii) To match the coefficients' dimensions for a given order, i.e., all the coefficients in a given order have the same dimensions, we change $t_\pm^{\mu\nu}$ in Table 2 in [10] to $b_0 t_\pm^{\mu\nu}$, the analog of $\tau^{\mu\nu}$ defined in [10]. Now, all the coefficients in the p^4 order are dimensionless, whereas in the p^6 order, their dimensions are GeV^{-2}.

(iii) Via partial integration and the application of the equations of motion, Y_{115} and Y_{116} are not independent. We list the new relations in Appendix B.

(iv) Reference [10] does not consider the epsilon relations

$$\epsilon^{\lambda\rho\delta\mu}\epsilon_{\lambda\rho\delta}{}^{\nu} = -6g^{\mu\nu}, \tag{36}$$

$$\epsilon^{\sigma\delta\mu\nu}\epsilon_{\sigma\delta}{}^{\lambda\rho} = -2g^{\mu\lambda}g^{\nu\rho} + 2g^{\mu\rho}g^{\nu\lambda}, \tag{37}$$

$$\begin{aligned}\epsilon^{\alpha\mu\nu\lambda}\epsilon_\alpha{}^{\rho\sigma\delta} = &-g^{\mu\rho}g^{\nu\sigma}g^{\lambda\delta} + g^{\mu\rho}g^{\nu\delta}g^{\lambda\sigma} \\ &- g^{\mu\sigma}g^{\nu\delta}g^{\lambda\rho} + g^{\mu\sigma}g^{\nu\rho}g^{\lambda\delta} \\ &- g^{\mu\delta}g^{\nu\rho}g^{\lambda\sigma} + g^{\mu\delta}g^{\nu\sigma}g^{\lambda\rho}.\end{aligned} \tag{38}$$

Combining with Eq. (5.3) in [10],

$$\epsilon_{\mu\nu\alpha\beta}t_\pm^{\alpha\beta} = 2it_{\mp,\mu\nu}, \tag{39}$$

one can reduce $t_-t_- \to t_+t_+$ and $t_-t_+ \to t_+t_-$, i.e., changing even t_- to the corresponding t_+ and exchanging the order of t_+ and t_-. For example,

$$t_-^{\mu\nu}t_{-,\mu\lambda} = \frac{1}{2}g^\nu{}_\lambda t_+^{\mu\rho}t_{+,\mu\rho} - t_{+\lambda}^{\mu}t_{+,\mu}{}^{\nu}. \tag{40}$$

Hence, even t_- terms and some $\langle\cdots t_-\cdots t_+\cdots\rangle$ terms are not independent[3]. We substitute the epsilon relations in Y_i, $i = 23$–30, 53, 56, 81, 83, 89, 91, 93, 104, 105, 109–111, 118, 119, finding that most terms lead to new relations. We list all the new relations in Appendix B. All the terms in the lhs of (B1) are considered to be dependent and reducible. We find that, in total, there are 22 additional dependent operators in the n-flavor case, 21 in the three-flavor case, and 13 in the two-flavor case, leaving 98 independent operators for n flavors, 92 for three flavors, and 65 for two flavors. In Sec. IV of Ref. [17], we found the result that without quark self-energy, all the coefficients, except contact terms, must vanish. Now in the present work, if we similarly ignore the quark self-energy, without relations (36)–(39), we cannot obtain these zero results. Instead, with relations (36)–(39), we do reproduce the vanishing result. This shows the importance of relations (36)–(39) in the computation.

[3]To avoid confusion of our notation with that used in Ref. [10], and for more convenience both in calculation and displaying results, we use both $\mathrm{tr}_f[\cdots]$ in the calculation and $\langle\cdots\rangle$ in the result to represent the tracing over flavor indices.

(v) Also, with (36)–(39), adding (41)

$$\epsilon^{\mu\nu\lambda\rho}\epsilon^{\mu'\nu'\lambda'\rho'} = -\det(g^{\alpha\alpha'}), \qquad \alpha = \mu, \nu, \lambda, \rho, \quad \alpha' = \mu', \nu', \lambda', \rho', \tag{41}$$

one can remove all epsilons for the tensor source terms in the chiral Lagrangian as follows. First, even epsilons can be changed to $g_{\mu\nu}$, and odd epsilons can be reduced to 1. Second, in one-epsilon terms, one can change t_+ to t_- or t_- to t_+ with the help of (39), leaving only two epsilons. Finally, using (36)–(38) and (41), all the epsilons can be removed. In other words, there do not exist the odd-intrinsic-parity parts with tensor sources at any order of the low-energy expansion, if one keeps t_+ and t_- as building blocks and changes them with Eq. (1).

V. THE p^4 ORDER CHIRAL LAGRANGIAN WITH TENSOR SOURCES

A. Real part

With the same method used in Ref. [17], we can expand the exponent in Eq. (30) to the order p^4. Reference [10] had given us the p^4 order Lagrangian of the form

$$\begin{aligned}\mathcal{L}_{4,t} &= \Lambda_1\langle t_+^{\mu\nu} f_{+\mu\nu}\rangle - i\Lambda_2\langle t_+^{\mu\nu} u_\mu u_\nu\rangle \\ &\quad + \Lambda_3\langle t_+^{\mu\nu} t^+_{\mu\nu}\rangle + \Lambda_4\langle t_+^{\mu\nu}\rangle^2 \\ &\equiv \sum_{n=1}^{4}\Lambda_n X_n.\end{aligned}\tag{42}$$

Considering that our computation is done under the large N_c limit, if we only expand Eq. (30) but do not consider the equations of motion,

$$\nabla_\mu u^\mu = \frac{i}{2}\left(\chi_- - \frac{\langle\chi_-\rangle}{N_f}\right), \tag{43}$$

terms in the chiral Lagrangian with two or more traces vanish. To avoid unnecessary complications, in this paper we retain in the calculation only those terms with one-trace:

$$\begin{aligned}\mathcal{L}_{4,n,t} &= \lambda_1 \mathrm{tr}_f[t_{+,\Omega,\mu\nu} t^{\mu\nu}_{+,\Omega}] + i\lambda_2 \mathrm{tr}_f[a_{\Omega,\mu} a_{\Omega,\nu} t^{\mu\nu}_{+,\Omega}] \\ &\quad + \lambda_3 \mathrm{tr}_f[V_{\Omega,\mu\nu} t^{\mu\nu}_{+,\Omega}] + O\left(\frac{1}{N}\right) \\ &\equiv \sum_{n=1}^{3}\lambda_n x_n + O\left(\frac{1}{N}\right).\end{aligned}\tag{44}$$

Expanding Eq. (30), we get the analytic results as

$$\lambda_1 = N_C \int_{\frac{1}{\Lambda^2}}^{\infty}\frac{d\tau}{\tau}\int\frac{d^4k}{(2\pi)^4} e^{-\tau(k^2+\Sigma_k^2)}(-2\tau^2\Sigma_k^2), \tag{45}$$

$$\begin{aligned}\lambda_2 &= N_C \int_{\frac{1}{\Lambda^2}}^{\infty}\frac{d\tau}{\tau}\int\frac{d^4k}{(2\pi)^4} e^{-\tau(k^2+\Sigma_k^2)} \\ &\quad \times (12\tau^2\Sigma_k - 4\tau^3 k^2\Sigma_k - 8\tau^3\Sigma_k^3),\end{aligned}\tag{46}$$

$$\lambda_3 = N_C \int_{\frac{1}{\Lambda^2}}^{\infty}\frac{d\tau}{\tau}\int\frac{d^4k}{(2\pi)^4} e^{-\tau(k^2+\Sigma_k^2)}(-2\tau^2\Sigma_k + 2\tau^2 k^2\Sigma_k'), \tag{47}$$

and the relations between λ_n and Λ_n are

$$\Lambda_1 = \frac{1}{2b_0}\lambda_3, \tag{48}$$

$$\Lambda_2 = -\frac{1}{4b_0}\lambda_2 - \frac{1}{2b_0}\lambda_3, \tag{49}$$

$$\Lambda_3 = \frac{1}{b_0^2}\lambda_1, \tag{50}$$

$$\Lambda_4 = 0. \tag{51}$$

The numerical results are listed in the second and sixth columns in Table II.

B. Imaginary part

As in Ref. [18], we can expand the exponent in Eq. (33) to the p^4 order. Using (39), we change $t_\pm$ to $t_\mp$, absorb the ϵ factors, and finally get the results

$$\mathcal{L}_{4,i,t} = \sum_{n=1}^{8} z_n \bar{o}_n. \tag{52}$$

The terms $\bar{o}_n$ are listed in Table I, and z_n are listed in Eq. (53).

TABLE I. The obtained operators of the p^4 order.

n	$\bar{o}_n$
1	$\langle\frac{\partial U_t}{\partial t}U_t^\dagger t^{\mu\nu}_{-,t}a^\mu_t a^\nu_t + \frac{\partial U_t}{\partial t}U_t^\dagger a^\mu_t a^\nu_t t^{\mu\nu}_{-,t}\rangle$
2	$\langle\frac{\partial U_t}{\partial t}U_t^\dagger a^\mu_t t^{\mu\nu}_{-,t}a^\nu_t\rangle$
3	$\langle\frac{\partial U_t}{\partial t}U_t^\dagger a^\mu_t t^{\mu\nu}_{+,t}\bar{\nabla}^\nu_t - \frac{\partial U_t}{\partial t}U_t^\dagger \bar{\nabla}^\mu_t t^{\mu\nu}_{+,t}a^\nu_t\rangle$
4	$\langle\frac{\partial U_t}{\partial t}U_t^\dagger \bar{\nabla}^\mu_t t^{\mu\nu}_{-,t}\bar{\nabla}^\nu_t\rangle$
5	$\langle\frac{\partial U_t}{\partial t}U_t^\dagger t^{\mu\nu}_{-,t}\bar{\nabla}^\mu_t\bar{\nabla}^\nu_t + \frac{\partial U_t}{\partial t}U_t^\dagger \bar{\nabla}^\mu_t\bar{\nabla}^\nu_t t^{\mu\nu}_{-,t}\rangle$
6	$\langle\frac{\partial U_t}{\partial t}U_t^\dagger t^{\mu\nu}_{+,t}a^\mu_t\bar{\nabla}^\nu_t - \frac{\partial U_t}{\partial t}U_t^\dagger \bar{\nabla}^\mu_t a^\nu_t t^{\mu\nu}_{+,t}\rangle$
7	$\langle\frac{\partial U_t}{\partial t}U_t^\dagger t^{\mu\nu}_{+,t}\bar{\nabla}^\mu_t a^\nu_t - \frac{\partial U_t}{\partial t}U_t^\dagger a^\mu_t\bar{\nabla}^\nu_t t^{\mu\nu}_{+,t}\rangle$
8	$\langle\frac{\partial U_t}{\partial t}U_t^\dagger t^{\mu\nu}_{-,t}t^{\mu\nu}_{+,t} + \frac{\partial U_t}{\partial t}U_t^\dagger t^{\mu\nu}_{+,t}t^{\mu\nu}_{-,t}\rangle$

$$z_1 = N_C \int \frac{d^4k}{(2\pi)^4}(2\Sigma_k X^2 - 4\Sigma_k^3 X^3), \quad z_2 = 0,$$
$$z_3 = N_C \int \frac{d^4k}{(2\pi)^4}(2i\Sigma_k' X - 2i\Sigma_k^2\Sigma_k' X^2 - 4i\Sigma_k^3 X^3), \quad z_4 = 0,$$
$$z_5 = N_C \int \frac{d^4k}{(2\pi)^4}(-2\Sigma_k X^2 + 2\Sigma_k' X - 2\Sigma_k^2\Sigma_k' X^2),$$
$$z_6 = N_C \int \frac{d^4k}{(2\pi)^4}(-2i\Sigma_k X^2 + 4i\Sigma_k^3 X^3),$$
$$z_7 = N_C \int \frac{d^4k}{(2\pi)^4}(-2i\Sigma_k X^2 + 2i\Sigma_k' X - 2i\Sigma_k^2\Sigma_k' X^2),$$
$$z_8 = N_C \int \frac{d^4k}{(2\pi)^4}(2i\Sigma_k^2 X^2), \tag{53}$$

where $X \equiv 1/(k^2 + \Sigma_k^2)$. Theoretically, we can integrate (52) to confirm (42), but it is too hard to do the integral, and even worse to extend to the p^6 order. Oppositely, we differentiate (42) to compare with Eq. (52). With the relation in Table VI, X_n in Eq. (42) can be represented by Ω fields denoted in Eq. (4). Introducing a parameter t as in Eq. (31), we change $X_n \longrightarrow X_n(t)$, and $X_n = X_n(1)$. Under the large N_C limit, our results only relate to the one-trace terms $X_{1,2,3}$. With the help of Eq. (54), we can differentiate one-trace term $X_{1,2,3}(t)$ to get Eq. (55).

$$\frac{\partial a_t^\mu}{\partial t} = \frac{i}{2}\left[\nabla_t^\mu \frac{\partial U_t}{\partial t} U_t^\dagger - \frac{\partial U_t}{\partial t} U_t^\dagger \nabla_t^\mu\right]$$
$$\frac{\partial v_t^\mu}{\partial t} = \frac{1}{2}\left[a_t^\mu \frac{\partial U_t}{\partial t} U_t^\dagger - \frac{\partial U_t}{\partial t} U_t^\dagger a_t^\mu\right]$$
$$\frac{\partial s_t}{\partial t} = -\frac{i}{2}\left[p_t \frac{\partial U_t}{\partial t} U_t^\dagger + \frac{\partial U_t}{\partial t} U_t^\dagger p_t\right]$$
$$\frac{\partial p_t}{\partial t} = \frac{i}{2}\left[s_t \frac{\partial U_t}{\partial t} U_t^\dagger + \frac{\partial U_t}{\partial t} U_t^\dagger s_t\right]$$
$$\begin{aligned}\frac{\partial d^\mu a_t^\nu}{\partial t} = \frac{i}{2}\Big[&(\nabla_t^\mu\nabla_t^\nu + a_t^\nu a_t^\mu)\frac{\partial U_t}{\partial t}U_t^\dagger - \nabla_t^\nu \frac{\partial U_t}{\partial t} U_t^\dagger \nabla_t^\mu \\ &- \nabla_t^\mu \frac{\partial U_t}{\partial t} U_t^\dagger \nabla_t^\nu - a_t^\nu \frac{\partial U_t}{\partial t} U_t^\dagger a_t^\mu - a_t^\mu \frac{\partial U_t}{\partial t} U_t^\dagger a_t^\nu \\ &+ \frac{\partial U_t}{\partial t} U_t^\dagger (\nabla_t^\nu \nabla_t^\mu + a_t^\mu a_t^\nu)\Big]\end{aligned}$$
$$\begin{aligned}\frac{\partial V_t^{\mu\nu}}{\partial t} = \frac{1}{2}\Big[&-\nabla_t^\mu \frac{\partial U_t}{\partial t} U_t^\dagger a_t^\nu + \frac{\partial U_t}{\partial t} U_t^\dagger \nabla_t^\mu a_t^\nu - \frac{\partial U_t}{\partial t} U_t^\dagger d_t^\mu a_t^\nu \\ &+ d_t^\mu a_t^\nu \frac{\partial U_t}{\partial t} U_t^\dagger + a_t^\nu \nabla_t^\mu \frac{\partial U_t}{\partial t} U_t^\dagger - a_t^\nu \frac{\partial U_t}{\partial t} U_t^\dagger \nabla_t^\mu \\ &+ \nabla_t^\nu \frac{\partial U_t}{\partial t} U_t^\dagger a_t^\mu - \frac{\partial U_t}{\partial t} U_t^\dagger \nabla_t^\nu a_t^\mu + \frac{\partial U_t}{\partial t} U_t^\dagger d_t^\nu a_t^\mu \\ &- d_t^\nu a_t^\mu \frac{\partial U_t}{\partial t} U_t^\dagger - a_t^\mu \nabla_t^\nu \frac{\partial U_t}{\partial t} U_t^\dagger + a_t^\mu \frac{\partial U_t}{\partial t} U_t^\dagger \nabla_t^\nu\Big]\end{aligned}$$
$$\frac{\partial t_+^{\mu\nu}}{\partial t} = -\frac{1}{2}\frac{\partial U_t}{\partial t} U_t^\dagger t_{-,t}^{\mu\nu} - \frac{1}{2} t_{-,t}^{\mu\nu} \frac{\partial U_t}{\partial t} U_t^\dagger$$
$$\frac{\partial t_-^{\mu\nu}}{\partial t} = -\frac{1}{2}\frac{\partial U_t}{\partial t} U_t^\dagger t_{+,t}^{\mu\nu} - \frac{1}{2} t_{+,t}^{\mu\nu} \frac{\partial U_t}{\partial t} U_t^\dagger, \tag{54}$$

$$(X_{1,t}, X_{2,t}, X_{3,t})^T = A_4(\bar{o}_1, \bar{o}_2, \bar{o}_3, \bar{o}_4, \bar{o}_5, \bar{o}_6, \bar{o}_7)^T,$$
$$X_{i,t} = dX_i(t)/dt, \tag{55}$$

$$A_4 = \begin{pmatrix} 2i & 0 & 0 & 0 & -2i & 2 & 2 & 0 \\ 2i & 0 & -2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{56}$$

Combined with Eqs. (42), (52), (55), and (56), we get

$$\begin{aligned}&(\Lambda_1, \Lambda_2, \Lambda_3)(X_{1,t}, X_{2,t}, X_{3,t})^T \\ &\quad = (\Lambda_1, \Lambda_2, \Lambda_3)A_4(\bar{o}_1, \bar{o}_2, \bar{o}_3, \bar{o}_4, \bar{o}_5, \bar{o}_6, \bar{o}_7, \bar{o}_8)^T \\ &\quad = (z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8) \\ &\qquad \times (\bar{o}_1, \bar{o}_2, \bar{o}_3, \bar{o}_4, \bar{o}_5, \bar{o}_6, \bar{o}_7, \bar{o}_8)^T\end{aligned} \tag{57}$$

$$\Rightarrow (\Lambda_1, \Lambda_2, \Lambda_3)A_4 = (z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8). \tag{58}$$

In Eq. (58), z_i and A_4 are obtained; they are eight linear equations with three unknown variables. Calculating the reduced row echelon form of A_4, we get

$$A_4' = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & -i & -i & 0 \\ 0 & 0 & 1 & 0 & -i & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{59}$$

We choose $\bar{o}_{1,3,8}$ as independent operators to calculate Λ_n, leaving the other five equations as strict constraints. The analytical results are

$$\begin{aligned}\Lambda_1 &= \frac{N_C}{b_0}\int \frac{d^4k}{(2\pi)^4}[\Sigma_k' X - \Sigma_k X^2 - \Sigma_k^2\Sigma_k' X^2], \\ \Lambda_2 &= \frac{N_C}{b_0}\int \frac{d^4k}{(2\pi)^4}[-\Sigma_k' X + \Sigma_k^2\Sigma_k' X^2 + 2\Sigma_k^3 X^3], \\ \Lambda_3 &= \frac{N_C}{b_0^2}\int \frac{d^4k}{(2\pi)^4}[-2\Sigma_k^2 X^2], \qquad \Lambda_4 = 0.\end{aligned} \tag{60}$$

In Eq. (60), we find that unlike those terms without tensors, which result in a fifth-dimension integral as Wess-Zumino-Witten terms, the tensor source terms can be fully worked out in the fourth dimension for the p^4 order. The numerical results are listed in the third and seventh columns in Table II.

As for the definition of b_0, we take the same spirit as that for B_0, where we have taken $\chi = 2B_0(s + ip)$ with $\chi_\pm = u^\dagger \chi u^\dagger \pm u\chi^\dagger u$, and let the final p^2 order of the chiral Lagrangian appear as $\mathcal{L}_2 = \frac{F_0^2}{4}(u_\mu u^\mu + \chi_+)$ [5]. In other words, with dimensional coefficient B_0 in front of operator $u^\dagger(s + ip)u^\dagger \pm u(s - ip)u$, the LEC of this operator becomes $\frac{F_0^2}{4}$ which is the same as that of another p^2 order operator, $u_\mu u^\mu$. Now we choose b_0 in such a way that it makes Λ_2 equal to the value of $\hat{L}_3$. Here, $\hat{L}_3$ is the LEC of another p^4 order operator $\langle(u^\mu u_\mu)^2\rangle$ and it is the p^4 order analogue of dimension coefficient F_0^2 in the p^2 order chiral Lagrangian. Although there may exist ambiguities for this

TABLE II. The obtained values of the p^4 order coefficients. The definition has some difference from [10]. The details can be found in Sec. IV. $\Lambda_{r,n}$ come from the real part, and $\Lambda_{i,n}$ come from the imaginary part; Λ_n are their sum.

	$N_f = 3$				$N_f = 2$		
n	$10^3\Lambda_{r,n}$	$10^3\Lambda_{i,n}$	$10^3\Lambda_n$	m	$10^3\Lambda_{r,m}$	$10^3\Lambda_{i,m}$	$10^3\Lambda_m$
1	$-7.62^{-0.44}_{+0.62}$	$-9.01^{-0.14}_{+0.17}$	$-16.62^{-0.58}_{+0.79}$	1	$-4.98^{-0.30}_{+0.42}$	$-5.93^{-0.10}_{+0.12}$	$-10.92^{-0.40}_{+0.53}$
2	$6.85^{+0.14}_{-0.21}$	$6.94^{+0.11}_{-0.13}$	$13.79^{+0.25}_{-0.34}$	2	$4.46^{+0.10}_{-0.14}$	$4.52^{+0.07}_{-0.09}$	$8.99^{+0.17}_{-0.23}$
3	$-1.42^{-0.09}_{+0.12}$	$-1.55^{-0.05}_{+0.06}$	$-2.97^{-0.14}_{+0.18}$	3	$-0.60^{-0.04}_{+0.05}$	$-0.66^{-0.02}_{+0.03}$	$-1.26^{-0.06}_{+0.08}$
4	$\equiv 0$	$\equiv 0$	$\equiv 0$	4	$\equiv 0$	$\equiv 0$	$\equiv 0$

kind of definition (because instead of Λ_2, one can also choose Λ_1 by fixing its values to $\hat{L}_3$ or other $\hat{L}_i$ to define b_0), we emphasize that no matter what value of b_0 we choose, the final interaction strength is the same. For example, for operator $\langle t_+^{\mu\nu} u_\mu u_\nu \rangle$, the coefficient is $\Lambda_1 b_0$, but from Eq. (60), we know Λ_1 is proportional to $1/b_0$; therefore, the interaction strength $\Lambda_1 b_0$ is independent of the value of b_0.

TABLE III. The p^6 order operators $\bar{O}_n$.

n	$\bar{O}_n$	n	$\bar{O}_n$	n	$\bar{O}_n$
1	$i\{a_{\Omega,\mu}a^\mu_\Omega, a_{\Omega,\nu}a_{\Omega,\lambda}\}t^{\nu\lambda}_{+,\Omega}$	27	$ia_{\Omega,\mu}[d_\nu a_{\Omega,\lambda}, d^\mu t^{\nu\lambda}_{+,\Omega}]$	53	$iV_{\Omega,\mu\nu}\{d_\lambda a^\mu_\Omega, t^{\nu\lambda}_{\Omega,-}\}$
2	$ia_{\Omega,\mu}a_{\Omega,\nu}(a^\mu_\Omega a_{\Omega,\lambda}t^{\nu\lambda}_{+,\Omega} + a_{\Omega,\lambda}a^\nu_\Omega t^{\mu\lambda}_{+,\Omega})$	28	$ia_{\Omega,\mu}[d_\nu a_{\Omega,\lambda}, d^\nu t^{\mu\lambda}_{\Omega,+}]$	54	$iV_{\Omega,\mu\nu}\{d_\lambda a^\lambda_\Omega, t^{\mu\nu}_{-,\Omega}\}$
3	$ia_{\Omega,\mu}a_{\Omega,\nu}a^\nu_\Omega a_{\Omega,\lambda}t^{\mu\lambda}_{+,\Omega}$	29	$ia_{\Omega,\mu}[d_\nu a_{\Omega,\lambda}, d^\lambda t^{\mu\nu}_{+,\Omega}]$	55	$s_\Omega[d_\mu a_{\Omega,\nu}, t^{\mu\nu}_{-,\Omega}]$
4	$ia_{\Omega,\mu}a_{\Omega,\nu}a_{\Omega,\lambda}a^\mu_\Omega t^{\nu\lambda}_{+,\Omega}$	30	$ia_{\Omega,\mu}\{d^\nu t_{+,\Omega,\nu\lambda}, t^{\mu\lambda}_{-,\Omega}\}$	56	$ip_\Omega[t^{\mu\nu}_{+,\Omega}, d_\mu a_{\Omega,\nu}]$
5	$a_{\Omega,\mu}a^\mu_\Omega[d_\nu a_{\Omega,\lambda}, t^{\nu\lambda}_{-,\Omega}]$	31	$ia_{\Omega,\mu}\{d_\nu t^\mu_{+,\Omega\lambda}, t^{\nu\lambda}_{-,\Omega}\}$	57	$iV_{\Omega,\mu\nu}V^\mu_{\Omega\lambda}t^{\nu\lambda}_{+,\Omega}$
6	$a_{\Omega,\mu}a_{\Omega,\nu}(d^\mu a_{\Omega,\lambda}t^{\nu\lambda}_{-,\Omega} - t^\mu_{-,\Omega\lambda}d^\nu a^\lambda_\Omega)$	32	$ia_{\Omega,\mu}\{d^\mu t_{+,\Omega,\nu\lambda}, t^{\nu\lambda}_{-,\Omega}\}$	58	$iV_{\Omega,\mu\nu}t^\mu_{+,\Omega\lambda}t^{\nu\lambda}_{+,\Omega}$
7	$a_{\Omega,\mu}a_{\Omega,\nu}(d^\nu a_{\Omega,\lambda}t^{\mu\lambda}_{-,\Omega} - t^\nu_{-,\Omega\lambda}d^\mu a^\lambda_\Omega)$	33	$ia_{\Omega,\mu}a_{\Omega,\nu}\{t^{\mu\nu}_{+,\Omega}, s_\Omega\}$	59	$iV_{\Omega,\mu\nu}t^\mu_{-,\Omega\lambda}t^{\nu\lambda}_{-,\Omega}$
8	$a_{\Omega,\mu}a_{\Omega,\nu}(d_\lambda a^\mu_\Omega t^{\nu\lambda}_{-,\Omega} - t^\mu_{-,\Omega\lambda}d^\lambda a^\nu_\Omega)$	34	$ia_{\Omega,\mu}s_\Omega a_{\Omega,\nu}t^{\mu\nu}_{+,\Omega}$	60	$iV_{\Omega,\mu\nu}\{p_\Omega, t^{\mu\nu}_{-,\Omega}\}$
9	$a_{\Omega,\mu}a_{\Omega,\nu}(d_\lambda a^\nu_\Omega t^{\mu\lambda}_{-,\Omega} - t^\nu_{-\Omega\lambda}d^\lambda a^\mu_\Omega)$	35	$id_\mu a^\mu_\Omega[d_\mu a_{\Omega,\lambda}, t^{\nu\lambda}_{+,\Omega}]$	61	$V_{\Omega,\mu\nu}\{t^{\mu\nu}_{+,\Omega}, s_\Omega\}$
10	$a_{\Omega,\mu}a_{\Omega,\nu}\{d_\lambda a^\lambda_\Omega, t^{\mu\nu}_{-,\Omega}\}$	36	$id_\mu a_{\Omega,\nu}d^\mu a_{\Omega,\lambda}t^{\nu\lambda}_{+,\Omega}$	62	$it_{\Omega,\mu\nu}t^\mu_{+,\Omega\lambda}t^{\nu\lambda}_{+,\Omega}$
11	$a_{\Omega,\mu}(d^\mu a_{\Omega,\nu}a_{\Omega,\lambda}t^{\nu\lambda}_{-,\Omega} + d_\nu a_{\Omega,\lambda}a^\nu_\Omega t^{\mu\lambda}_{-,\Omega})$	37	$id_\mu a_{\Omega,\nu}[d^\nu a_{\Omega,\lambda}, t^{\mu\lambda}_{+,\Omega}]$	63	$it_{\Omega,\mu\nu}t^\mu_{-,\Omega\lambda}t^{\nu\lambda}_{-,\Omega}$
12	$a_{\Omega,\mu}(d_\nu a^\mu_\Omega a_{\Omega,\lambda}t^{\nu\lambda}_{-,\Omega} + d_\nu a_{\Omega,\lambda}a^\lambda_\Omega t^{\mu\nu}_{-,\Omega})$	38	$id_\mu a_{\Omega,\nu}d_\lambda a^\nu_\Omega t^{\mu\lambda}_{+,\Omega}$	64	$s_\Omega t_{+,\Omega,\mu\nu}t^{\mu\nu}_{+,\Omega}$
13	$a_{\Omega,\mu}d_\nu a^\nu_\Omega a_{\Omega,\lambda}t^{\mu\lambda}_{-,\Omega}$	39	$V_{\Omega,\mu\nu}(a^\mu_\Omega a_{\Omega,\lambda}t^{\nu\lambda}_{+,\Omega} - t^\mu_{+,\Omega\lambda}a^\lambda_\Omega a^\nu_\Omega)$	65	$s_\Omega t_{-,\Omega,\mu\nu}t^{\mu\nu}_{-,\Omega}$
14	$a_{\Omega,\mu}a^\mu_\Omega t_{+,\Omega,\nu\lambda}t^{\nu\lambda}_{+,\Omega}$	40	$V_{\Omega,\mu\nu}(a_{\Omega,\lambda}a^\mu_\Omega t^{\nu\lambda}_{+,\Omega} - t^\mu_{+,\Omega\lambda}a^\nu_\Omega a^\lambda_\Omega)$	66	$d_\mu V^\mu_{\Omega\nu}d_\lambda t^{\nu\lambda}_{+,\Omega}$
15	$a_{\Omega,\mu}a_{\Omega,\nu}t^\mu_{+,\Omega\lambda}t^{\nu\lambda}_{+,\Omega}$	41	$V_{\Omega,\mu\nu}\{a_{\Omega,\lambda}a^\lambda_\Omega, t^{\mu\nu}_{+,\Omega}\}$	67	$d_\mu V_{\Omega,\nu\lambda}d^\mu t^{\nu\lambda}_{+,\Omega}$
16	$a_{\Omega,\mu}a_{\Omega,\nu}t^\nu_{+,\Omega\lambda}t^{\mu\lambda}_{+,\Omega}$	42	$V_{\Omega,\mu\nu}(a^\mu_\Omega t^\nu_{+,\Omega\lambda}a^\lambda_\Omega - a_{\Omega,\lambda}t^{\mu\lambda}_{+,\Omega}a^\nu_\Omega)$	68	$d_\mu V_{\Omega,\nu\lambda}d^\nu t^{\nu\mu\lambda}_{+,\Omega}$
17	$a_{\Omega,\mu}a^\mu_\Omega t_{-,\Omega,\nu\lambda}t^{\nu\lambda}_{-,\Omega}$	43	$V_{\Omega,\mu\nu}a_{\Omega,\lambda}t^{\mu\nu}_{+,\Omega}a^\lambda_\Omega$	69	$d_\mu t^\mu_{+,\Omega\nu}d_\lambda t^{\nu\lambda}_{+,\Omega}$
18	$a_{\Omega,\mu}a_{\Omega,\nu}t^\mu_{-,\Omega\lambda}t^{\nu\lambda}_{-,\Omega}$	44	$ia_{\Omega,\mu}[t^\mu_{+,\Omega\nu}, d^\nu p_\Omega]$	70	$d_\mu t_{+,\Omega,\nu\lambda}d^\mu t^{\nu\lambda}_{+,\Omega}$
19	$a_{\Omega,\mu}a_{\Omega,\nu}t^\nu_{-,\Omega\lambda}t^{\mu\lambda}_{-,\Omega}$	45	$a_{\Omega,\mu}[t^\mu_{-,\Omega\nu}, d^\nu s_\Omega]$	71	$d_\mu t_{+,\Omega,\nu\lambda}d^\nu t^{\mu\lambda}_{+,\Omega}$
20	$a_{\Omega,\mu}(t^\mu_{+,\Omega\nu}a_{\Omega,\lambda}t^{\nu\lambda}_{+,\Omega} - t_{\Omega,\nu\lambda}a^\nu_\Omega t^{\mu\lambda}_{+,\Omega})$	46	$id_\mu a^\mu_\Omega\{t_{-,\Omega,\nu\lambda}, t^{\nu\lambda}_{+,\Omega}\}$	72	$d_\mu t^\mu_{-,\Omega\nu}d_\lambda t^{\nu\lambda}_{-,\Omega}$
21	$a_{\Omega,\mu}t_{+,\Omega,\nu\lambda}a^\mu_\Omega t^{\nu\lambda}_{+,\Omega}$	47	$id_\mu a_{\Omega,\nu}\{t^\mu_{-,\Omega\lambda}, t^{\nu\lambda}_{+,\Omega}\}$	73	$d_\mu t_{-,\Omega,\nu\lambda}d^\mu t^{\nu\lambda}_{-,\Omega}$
22	$a_{\Omega,\mu}(t^\mu_{-,\Omega\nu}a_{\Omega,\lambda}t^{\nu\lambda}_{-,\Omega} - t_{\Omega,\nu\lambda}a^\nu_\Omega t^{\mu\lambda}_{-,\Omega})$	48	$id_\mu a_{\Omega,\nu}\{t^\nu_{-,\Omega\lambda}, t^{\mu\lambda}_{+,\Omega}\}$	74	$d_\mu t_{-,\Omega,\nu\lambda}d^\nu t^{\mu\lambda}_{-,\Omega}$
23	$a_{\Omega,\mu}t_{-,\Omega,\nu\lambda}a^\mu_\Omega t^{\nu\lambda}_{-,\Omega}$	49	$iV_{\Omega,\mu\nu}\{a^\mu_\Omega, d_\lambda t^{\nu\lambda}_{-,\Omega}\}$	75	$a_{\Omega,\mu}a_{\Omega,\nu}\{t^{\mu\nu}_{-,\Omega}, p_\Omega\}$
24	$ia_{\Omega,\mu}[d^\mu a_{\Omega,\nu}d_\lambda, t^{\nu\lambda}_{+,\Omega}]$	50	$iV_{\Omega,\mu\nu}\{a_{\Omega,\lambda}, d^\mu t^{\nu\lambda}_{-,\Omega}\}$	76	$a_{\Omega,\mu}p_\Omega a_{\Omega,\nu}t^{\mu\nu}_{-,\Omega}$
25	$ia_{\Omega,\mu}[d_\nu a^\mu_\Omega, d_\lambda t^{\nu\lambda}_{+,\Omega}]$	51	$iV_{\Omega,\mu\nu}\{a_{\Omega,\lambda}, d^\lambda t^{\mu\nu}_{-,\Omega}\}$	77	$ip_\Omega\{t_{+,\Omega\mu\nu}, t^{\mu\nu}_{-,\Omega}\}$
26	$ia_{\Omega,\mu}[d_\nu a^\nu_\Omega, d_\lambda t^{\mu\lambda}_{+,\Omega}]$	52	$iV_{\Omega,\mu\nu}\{d^\mu a_{\Omega,\lambda}, t^{\nu\lambda}_{-,\Omega}\}$		

C. Numerical results

In Table II, we list our p^4 order LECs with tensor sources for cutoff $\Lambda = 1000^{+100}_{-100}$ MeV in Eq. (21). The 10% variation of the cutoff is considered in our calculation to examine the effects of cutoff dependence and the result change can be treated as the error of our calculations. The results are taken as the values at $\Lambda = 1$ GeV. The superscript is the difference caused at $\Lambda = 1.1$ GeV and the subscript is the difference caused at $\Lambda = 0.9$ GeV, i.e.,[4]

$$\Lambda_{n,\Lambda=1\ \mathrm{GeV}}\Big|^{\Lambda_{n,\Lambda=1.1\ \mathrm{GeV}}-\Lambda_{n,\Lambda=1\ \mathrm{GeV}}}_{\Lambda_{n,\Lambda=0.9\ \mathrm{GeV}}-\Lambda_{n,\Lambda=1\ \mathrm{GeV}}}. \quad (61)$$

The LECs include three and two flavors both from the real part, $\Lambda_{r,n}$, and the imaginary part, $\Lambda_{i,n}$. We also list the sum $\Lambda_n = \Lambda_{r,n} + \Lambda_{i,n}$. We get $b_0 = 1.32^{-0.04}_{+0.06}$ GeV in the three-flavor case, and $b_0 = 2.01^{-0.06}_{+0.09}$ GeV in the two-flavor case. b_0 in the two-flavor cases seems larger, because of the Cayley-Hamilton relation giving more relations.

To compare with our original results, the parameters we use to get Table II are the same as Refs. [17,18].[5] We choose the running coupling constant from Ref. [25] to solve Eq. (16), and we get the same quark self-energy as Fig. 2 in Ref. [15], but adding the two-flavor case. Except for the quark self-energy, we need another input parameter F_0, the p^2 order coefficient in the chiral Lagrangian. We set $F_0 = 87$ MeV to get F_π of about 93 MeV [17].

To examine our numerical result, we compute the magnetic susceptibility of the quark condensate, which we will show is proportional to Λ_1. We first introduce an external electromagnetic field A^μ_{em} into the generating functional (3) by adding in an external field term $-\bar{\psi} q \not{A}_{\mathrm{em}} \psi$ into the Lagrangian on the exponential integrand. The generating functional is changed from $Z[J]$ to $Z[J - q\not{A}_{\mathrm{em}}]$. The magnetic susceptibility of the quark condensate χ is

$$\frac{e}{3}\chi\langle\bar{\psi}\psi\rangle F^{\mu\nu}(x) = \langle 0|\bar{\psi}(x)\sigma^{\mu\nu}\psi(x)|0\rangle = -i\frac{1}{Z[J - iq\not{A}_{\mathrm{em}}]}\frac{\delta Z[J - iq\not{A}_{\mathrm{em}}]}{\delta \bar{t}_{\mu\nu}(x)}\Bigg|_{J=0}, \quad (62)$$

which leads to $\chi\langle\bar{\psi}\psi\rangle = -4\Lambda_1 b_0$. In the case of $\Lambda = 1$ GeV, we obtain

$$\chi = -\frac{4\Lambda_1 b_0}{\langle\bar{\psi}\psi\rangle} = \frac{4\Lambda_1 b_0}{N_f F_0^2 B_0} = -7.3\ \mathrm{GeV}^{-2}. \quad (63)$$

Comparing this result with those in Refs. [26–28] gives the results $-(8.16 \pm 0.41)\ \mathrm{GeV}^{-2}$, $-2.7\ \mathrm{GeV}^{-2}$, and $-3.3\ \mathrm{GeV}^{-2}$, respectively. There is a factor of $1 \sim 3$ difference between our result and the references. Although we choose $F_0 = 87$ MeV instead of $F_\pi = 92.4$ MeV in the quark condensate, it will decrease the absolute value of our result a little bit, but the correction is not enough. We leave the investigation of this discrepancy to future studies.

VI. THE p^6 ORDER CHIRAL LAGRANGIAN WITH TENSOR SOURCES

A. Real part

Continuing our process in Sec. V, we can obtain the p^6 order results directly. Before listing our results, we first introduce the existing results. Reference [10] gave the p^6 order Lagrangian as follows:

$$S_{\mathrm{eff}}\big|_{p^6,\mathrm{tensor\ sources}} = \int d^4x \begin{cases} \sum_{n=1}^{117} K_n^T Y_n + 3\,\text{contact terms} & n\ \text{flavors} \\ \sum_{n=1}^{110} C_n^T O_n + 3\,\text{contact terms} & \text{three flavors}. \\ \sum_{n=1}^{75} c_n^T P_n + 3\,\text{contact terms} & \text{two flavors} \end{cases} \quad (64)$$

Here, we use the notation Y_n, O_n, P_n to denote n, three, and two flavors' independent monomials, which can be found in Table 2 in Ref. [10], and K_n^T, C_n^T, c_n^T for their coefficients. For reasons in Sec. IV and Appendix B, some of them are not independent, but we use the same numbers. If one monomial is not independent, we just neglect it.

In our calculation, expanding Eq. (21) as the p^4 order, we only get one-trace terms without the equation of motion

$$S_{\mathrm{eff}}\big|_{p^6,\mathrm{tensor\ sources}} = \int d^4x\left[\sum_{n=1}^{77} \mathcal{Z}_n^T \mathrm{tr}_f[\bar{O}_n] + O\left(\frac{1}{N_c}\right)\right]. \quad (65)$$

The terms $\bar{O}_n$ are the p^6 order operators we can obtain from our calculation, and $\mathcal{Z}_n^T$ are the corresponding coefficients. For those operators with more than one derivative, for example $\bar{O}_{66} = d_\mu V^\mu_{\Omega\nu} d_\lambda t^{\nu\lambda}_{+,\Omega}$, the derivatives are arranged in such a way that each $V_{\Omega,\mu}{}^\nu$ and $t_{+,\Omega,\nu\lambda}$ has a derivative and we do not put two derivatives in one operator. We list all operators in Table III. With the help of a computer, we can get the coefficients $\mathcal{Z}_n^T$, listed in Appendix C. Making use of Table VI, relations for our coefficients $\mathcal{Z}_n^T$ and K_n^T, listed in Appendix D, can be obtained directly. Combining

[4]Notice that Λ with subscript n, Λ_n, means the p^4 order coefficients in Eq. (60), but Λ without the subscript is the cutoff in our calculation introduced in Eq. (28).

[5]In order to match the $N_f = 2$ result, we choose a heavy-quark number 4 here, instead of 2 in Refs. [17,18]. (Heavy-quark fields are integrated out and absorbed into the effective gluon propagator.)

TABLE IV. The p^6 order operators in the large N_C limits, $\bar{O}_n^{T,W}$.

n	$\bar{O}_n^{T,W}$	Y_i	n	$\bar{O}_n^{T,W}$	Y_i
1	$i\langle t_{+\mu\nu}\{u_\lambda u^\lambda, u^\mu u^\nu\}\rangle/b_0$	Y_1/b_0	29	$\langle t_{-\mu\nu}[f_-^{\mu\nu}, u_\lambda u^\lambda]\rangle/b_0$	Y_{69}/b_0
2	$i\langle t_{+\mu\nu}u^\lambda u^\mu u^\nu u_\lambda\rangle/b_0$	Y_2/b_0	30	$\langle t_{-\mu\nu}(f_-^{\mu\lambda}u^\nu u_\lambda - u_\lambda u^\nu f_-^{\mu\lambda})\rangle/b_0$	Y_{70}/b_0
3	$i\langle t_{+\mu\nu}u^\mu u_\lambda u^\lambda u^\nu\rangle/b_0$	Y_3/b_0	31	$\langle t_{-\mu\nu}(f_-^{\mu\lambda}u_\lambda u^\nu - u^\nu u_\lambda f_-^{\mu\lambda})\rangle/b_0$	Y_{71}/b_0
4	$i\langle t_{+\mu\nu}\{u_\lambda, u^\mu u^\lambda u^\nu\}\rangle/b_0$	Y_4/b_0	32	$\langle t_{-\mu\nu}(u^\nu f_-^{\mu\lambda}u_\lambda - u_\lambda f_-^{\mu\lambda}u^\nu)\rangle/b_0$	Y_{72}/b_0
5	$\langle t_{+\mu\nu}t_+^{\mu\nu}u_\lambda u^\lambda\rangle/b_0^2$	Y_9/b_0^2	33	$\langle t_{+\mu\nu}\{f_+^{\mu\nu}, \chi_+\}\rangle/B_0b_0$	Y_{74}/B_0b_0
6	$\langle t_{+\mu\nu}u_\lambda t_+^{\mu\nu}u^\lambda\rangle/b_0^2$	Y_{10}/b_0^2	34	$\langle t_{-\mu\nu}\{f_+^{\mu\nu}, \chi_-\}\rangle/B_0b_0$	Y_{75}/B_0b_0
7	$\langle t_{+\mu\nu}t_+^{\mu\lambda}u_\lambda u^\nu\rangle/b_0^2$	Y_{11}/b_0^2	35	$\langle t_{+\mu\nu}[f_-^{\mu\nu}, \chi_-]\rangle/B_0b_0$	Y_{76}/B_0b_0
8	$\langle t_{+\mu\nu}t_+^{\mu\lambda}u^\nu u_\lambda\rangle/b_0^2$	Y_{12}/b_0^2	36	$i\langle t_{-\mu\nu}\{f_+^{\mu\nu}, h_\lambda^\lambda\}\rangle/b_0$	$-Y_{75}/b_0 + \frac{2}{N_f b_0}Y_{80}$
9	$\langle t_{+\mu\nu}(u^\nu t_+^{\mu\lambda}u_\lambda + u_\lambda t_+^{\mu\lambda}u^\nu)\rangle/b_0^2$	Y_{13}/b_0^2	37	$i\langle t_{+\mu\nu}\{t_-^{\nu\lambda}, h_\lambda^\mu\rangle/b_0^2$	Y_{81}/b_0^2
10	$\langle t_{+\mu\nu}t_+^{\mu\nu}\chi_+\rangle/B_0b_0^2$	$Y_{31}/B_0b_0^2$	38	$i\langle t_{+\mu\nu}f_-^{\mu\lambda}f_{-\lambda}^\nu\rangle/b_0$	Y_{84}/b_0
11	$\langle t_{+\mu\nu}t_-^{\mu\nu}\chi_- + t_-^{\mu\nu}t_+^{\mu\nu}\chi_-\rangle/B_0b_0^2$	$2Y_{32}/B_0b_0^2$	39	$\langle it_{+\mu\nu}f_+^{\mu\lambda}f_{+\lambda}^\nu\rangle/b_0$	Y_{85}/b_0
12	$i\langle t_{+\mu\nu}\{t_-^{\mu\nu}, h_\lambda^\lambda\}\rangle/B_0b_0$	$-2Y_{32}/B_0b_0 + \frac{2}{N_f B_0 b_0}Y_{35}$	40	$i\langle t_{-\mu\nu}\{f_-^{\nu\lambda}, f_{+\lambda}^\mu\}\rangle/b_0$	Y_{86}/b_0
13	$i\langle t_{+\mu\nu}\{\chi_+, u^\mu u^\nu\}\rangle/B_0b_0$	Y_{39}/B_0b_0	41	$i\langle t_{+\mu\nu}t_+^{\mu\lambda}t_{+\lambda}^\nu\rangle/b_0^3$	Y_{88}/b_0^3
14	$i\langle t_{+\mu\nu}u^\mu\chi_+u^\nu\rangle/B_0b_0$	Y_{40}/B_0b_0	42	$i\langle f_{+\mu\nu}t_+^{\nu\lambda}t_{+\lambda}^\mu\rangle/b_0^2$	Y_{90}/b_0^2
15	$i\langle t_{-\mu\nu}\{\chi_-, u^\mu u^\nu\}\rangle/B_0b_0$	Y_{43}/B_0b_0	43	$\langle\nabla_\mu t_+^{\mu\nu}\nabla^\lambda f_{+\lambda\nu}\rangle/b_0$	Y_{94}/b_0
16	$\langle it_{-\mu\nu}u^\mu\chi_-u^\nu\rangle/B_0b_0$	Y_{44}/B_0b_0	44	$i\langle\nabla_\lambda t_{+\mu\nu}[h^{\mu\lambda}, u^\nu]\rangle/b_0$	Y_{95}/b_0
17	$\langle t_{-\mu\nu}u^\mu h_\lambda^\lambda u^\nu\rangle/b_0$	$Y_{44}/b_0 - \frac{1}{N_f B_0}Y_{45}$	45	$i\langle\nabla^\mu t_{+\mu\nu}[h^{\nu\lambda}, u_\lambda]\rangle/b_0$	Y_{96}/b_0
18	$\langle t_{-\mu\nu}\{u^\mu u^\nu, h_\lambda^\lambda\}\rangle/b_0$	$Y_{43}/b_0 - \frac{2}{N_f B_0}Y_{45}$	46	$i\langle\nabla^\mu t_{+\mu\nu}[f_-^{\nu\lambda}, u_\lambda]\rangle/b_0$	Y_{97}/b_0
19	$\langle t_{-\mu\nu}(h^{\nu\lambda}u_\lambda u^\mu - u^\mu u_\lambda h^{\nu\lambda})\rangle/b_0$	Y_{47}/b_0	47	$i\langle\nabla^\mu t_{+\nu\lambda}[f_-^{\nu\lambda}, u_\mu]\rangle/b_0$	Z_1/b_0 [a]
20	$\langle t_{-\mu\nu}(h^{\nu\lambda}u^\mu u_\lambda - u_\lambda u^\mu h^{\nu\lambda})\rangle/b_0$	Y_{48}/b_0	48	$i\langle\nabla_\mu t_{+\nu\lambda}[f_-^{\mu\nu}, u^\lambda]\rangle/b_0$	Z_2/b_0 [b]
21	$\langle t_{-\mu\nu}(u_\lambda h^{\nu\lambda}u^\mu - u^\mu h^{\nu\lambda}u_\lambda)\rangle/b_0$	Y_{49}/b_0	49	$i\langle\nabla^\mu t_{-\mu\nu}\{f_+^{\nu\lambda}, u_\lambda\}\rangle/b_0$	Y_{98}/b_0
22	$\langle\nabla_\lambda t_{\mu\nu}\nabla^\lambda t_+^{\mu\nu}\rangle/b_0^2$	Y_{51}/b_0^2	50	$i\langle\nabla_\lambda t_{-\mu\nu}\{f_+^{\mu\nu}, u^\lambda\}\rangle/b_0$	Y_{99}/b_0
23	$\langle\nabla_\mu t_+^{\mu\nu}\nabla^\lambda t_{+\lambda\nu}\rangle/b_0^2$	Y_{52}/b_0^2	51	$i\langle\nabla_\lambda t_{-\mu\nu}\{f_+^{\mu\lambda}, u^\nu\}\rangle/b_0$	Y_{100}/b_0
24	$\langle t_{+\mu\nu}\{f_+^{\mu\nu}, u_\lambda u^\lambda\}\rangle/b_0$	Y_{57}/b_0	52	$i\langle\{\nabla_\mu t_+^{\mu\nu}, t_{-\nu\lambda}\}u^\lambda\rangle/b_0^2$	Y_{105}/b_0^2
25	$\langle t_{+\mu\nu}u_\lambda f_+^{\mu\nu}u^\lambda\rangle/b_0$	Y_{58}/b_0	53	$i\langle\nabla^\mu t_+^{\nu\lambda}\{t_{-\mu\lambda}, u_\nu\}\rangle/b_0^2$	Z_3/b_0^2 [c]
26	$\langle t_{+\mu\nu}(f_+^{\mu\lambda}u^\nu u_\lambda + u_\lambda u^\nu f_+^{\mu\lambda})\rangle/b_0$	Y_{59}/b_0	54	$\langle t_-^{\mu\nu}[\chi_{+\mu}, u_\nu]\rangle/B_0b_0$	Y_{112}/B_0b_0
27	$\langle t_{+\mu\nu}(f_+^{\mu\lambda}u_\lambda u^\nu + u^\nu u_\lambda f_+^{\mu\lambda})\rangle/b_0$	Y_{60}/b_0	55	$\langle t_+^{\mu\nu}[\chi_{-\mu}, u_\nu]\rangle/B_0b_0$	Y_{113}/B_0b_0
28	$\langle t_{+\mu\nu}(u^\nu f_+^{\mu\lambda}u_\lambda + u_\lambda f_+^{\mu\lambda}u^\nu)\rangle/b_0$	Y_{61}/b_0	56	$i\langle t_{+\mu\nu}h^{\mu\lambda}h_\lambda^\nu\rangle/b_0$	Y_{114}/b_0

[a] $Z_1 = -\frac{1}{2}Y_1 + Y_2 - Y_{59} + Y_{61} + \frac{1}{2}Y_{76} - Y_{84} - Y_{96} - Y_{97} + Y_{114}$.
[b] $Z_2 = \frac{1}{2}Y_1 + Y_3 - Y_4 + \frac{1}{2}Y_{39} + Y_{40} + Y_{60} - Y_{61} + Y_{84} - Y_{95} + Y_{113} - Y_{114}$.
[c] $Z_3 = -\frac{1}{4}Y_9 - \frac{1}{4}Y_{10} + Y_{11} + \frac{1}{2}Y_{13} + \frac{1}{4}Y_{32} - \frac{1}{4N_f}Y_{35} - \frac{1}{2}Y_{51} + 2Y_{52} - Y_{90} + Y_{105} - 4Y_{118} + 2Y_{119}$.

Appendixes B, C, and D, and using the parameters in Sec. V C, we obtain both two and three flavors' numerical results, and list them in the second and sixth columns in Table V.

B. Imaginary part

As in Sec. V B, continuing our process to the p^6 order, we can derive the p^6 order LECs too. In the large N_C limits, without using the equations of motion, and removing the contact terms, we get 56 independent terms, $\bar{O}_n^{T,W}$, and list them in Table IV. We also list their relation to Y_i of Ref. [10]. As in Eq. (44), in p^6 order, we get

$$\mathcal{L}_{6,i,t} = \sum_{n=1}^{56} \tilde{K}_n^{T,W}\tilde{O}_n^{T,W}, \tag{66}$$

where $\tilde{K}_n^{T,W}$ are the coefficients related to $\tilde{O}_n^{T,W}$. Introducing a parameter t to $\tilde{O}_n^{T,W}$, we change $\tilde{O}_n^{T,W} \to \tilde{O}_n^{T,W}(t)$. Then differentiating $\tilde{O}_n^{T,W}(t)$, we get, in like manner to (55),

$$(\tilde{O}_{1,t}^{T,W}, \tilde{O}_{2,t}^{T,W}, \ldots, \tilde{O}_{56,t}^{T,W})^T = A_6(\bar{O}_1^{T,W}, \bar{O}_2^{T,W}, \ldots, \bar{O}_{339}^{T,W})^T,$$
$$\tilde{O}_{i,t}^{T,W} = d\tilde{O}_i^{T,W}(t)/dt \tag{67}$$

and

TABLE V. The obtained values of the p^6 order LECs. C_n^T denote the three-flavor coefficients, and c_m^T denote two flavors. n, m are the number of independent monomials in [10] with some difference. The subscripts r, i denote the LECs from the real part and imaginary part of the chiral Lagrangian. As some monomials are not independent, we denote their coefficients by a preceding symbol "$\cdots$". The details can be found in Sec. IV. The value $\equiv 0$ means that the constants vanish in the large N_C limit.

n	10^3 GeV$^2C_{r,n}^T$	10^3 GeV$^2C_{i,n}^T$	10^3 GeV$^2C_n^T$	m	10^3 GeV$^2c_{r,m}^T$	10^3 GeV$^2c_{i,m}^T$	10^3 GeV$^2c_m^T$
1	$2.20^{+0.05}_{-0.06}$	$2.06^{+0.09}_{-0.14}$	$4.26^{+0.14}_{-0.20}$				
2	$3.93^{+0.18}_{-0.26}$	$3.94^{+0.18}_{-0.26}$	$7.87^{+0.35}_{-0.52}$	1	$3.53^{+0.11}_{-0.15}$	$3.38^{+0.16}_{-0.23}$	$6.91^{+0.26}_{-0.38}$
3	$-3.30^{-0.24}_{+0.37}$	$-3.55^{-0.16}_{+0.24}$	$-6.85^{-0.40}_{+0.61}$	2	$1.51^{-0.12}_{+0.20}$	$0.99^{+0.05}_{-0.07}$	$2.50^{-0.07}_{+0.14}$
4	$-1.71^{+0.01}_{-0.03}$	$-1.46^{-0.07}_{+0.10}$	$-3.17^{-0.05}_{+0.06}$				
5	$\equiv 0$	$\equiv 0$	$\equiv 0$				
6	$\equiv 0$	$\equiv 0$	$\equiv 0$				
7	$-0.29^{-0.03}_{+0.05}$	$-0.94^{-0.06}_{+0.08}$	$-1.23^{-0.09}_{+0.13}$	3	$-0.13^{-0.02}_{+0.02}$	$-0.41^{-0.03}_{+0.04}$	$-0.53^{-0.04}_{+0.06}$
8	$-0.08^{-0.02}_{+0.04}$	$0.18^{+0.01}_{-0.02}$	$0.10^{-0.01}_{+0.02}$	4	$-0.04^{-0.01}_{+0.02}$	$0.07^{+0.00}_{-0.01}$	$0.04^{-0.01}_{+0.01}$
9	$1.24^{+0.03}_{-0.02}$	$-0.70^{-0.04}_{+0.06}$	$0.55^{-0.02}_{+0.04}$	5	$0.49^{+0.01}_{-0.01}$	$-0.33^{-0.02}_{+0.03}$	$0.16^{-0.01}_{+0.02}$
10	$-4.91^{-0.29}_{+0.39}$	$-0.61^{-0.04}_{+0.05}$	$-5.52^{-0.32}_{+0.44}$	6	$-2.05^{-0.13}_{+0.17}$	$-0.22^{-0.01}_{+0.02}$	$-2.28^{-0.14}_{+0.19}$
11	$1.14^{+0.10}_{-0.16}$	$0.61^{+0.04}_{-0.05}$	$1.76^{+0.14}_{-0.21}$	7	$0.49^{+0.05}_{-0.07}$	$0.27^{+0.02}_{-0.02}$	$0.76^{+0.06}_{-0.09}$
12	$\equiv 0$	$\equiv 0$	$\equiv 0$				
13	$\equiv 0$	$\equiv 0$	$\equiv 0$				
14	$\equiv 0$	$\equiv 0$	$\equiv 0$				
15	$\equiv 0$	$\equiv 0$	$\equiv 0$				
16	$\equiv 0$	$\equiv 0$	$\equiv 0$				
17	$\equiv 0$	$\equiv 0$	$\equiv 0$	8	$\equiv 0$	$\equiv 0$	$\equiv 0$
18	$\equiv 0$	$\equiv 0$	$\equiv 0$	9	$\equiv 0$	$\equiv 0$	$\equiv 0$
19	$\cdots$	$\cdots$	$\cdots$	10	$\cdots$	$\cdots$	$\cdots$
20	$\cdots$	$\cdots$	$\cdots$	11	$\cdots$	$\cdots$	$\cdots$
21	$\cdots$	$\cdots$	$\cdots$	12	$\cdots$	$\cdots$	$\cdots$
22	$\cdots$	$\cdots$	$\cdots$				
23	$\cdots$	$\cdots$	$\cdots$				
24	$\cdots$	$\cdots$	$\cdots$				
25	$\cdots$	$\cdots$	$\cdots$	13	$\cdots$	$\cdots$	$\cdots$
26	$-1.44^{-0.06}_{+0.13}$	$-2.02^{+0.08}_{-0.09}$	$-3.47^{+0.02}_{+0.03}$	14	$-0.63^{-0.03}_{+0.06}$	$-0.89^{+0.04}_{-0.04}$	$-1.52^{+0.01}_{+0.01}$
27	$1.46^{-0.12}_{+0.09}$	$2.96^{-0.21}_{+0.24}$	$4.41^{-0.32}_{+0.34}$	15	$0.61^{-0.05}_{+0.04}$	$1.26^{-0.09}_{+0.11}$	$1.88^{-0.14}_{+0.14}$
28	$\equiv 0$	$\equiv 0$	$\equiv 0$	16	$\equiv 0$	$\equiv 0$	$\equiv 0$
29	$\equiv 0$	$\equiv 0$	$\equiv 0$	17	$\equiv 0$	$\equiv 0$	$\equiv 0$
30	$0.58^{+0.04}_{-0.05}$	$0.28^{+0.02}_{-0.02}$	$0.86^{+0.06}_{-0.08}$	18	$0.18^{+0.01}_{-0.02}$	$0.09^{+0.01}_{-0.01}$	$0.27^{+0.02}_{-0.03}$
31	$\equiv 0$	$\equiv 0$	$\equiv 0$	19	$\equiv 0$	$\equiv 0$	$\equiv 0$
32	$\equiv 0$	$\equiv 0$	$\equiv 0$				
33	$\equiv 0$	$\equiv 0$	$\equiv 0$				
34	$-0.43^{-0.44}_{+0.60}$	$-4.08^{-0.19}_{+0.29}$	$-4.51^{-0.63}_{+0.88}$	20	$-0.41^{-0.30}_{+0.40}$	$-2.76^{-0.13}_{+0.20}$	$-3.17^{-0.43}_{+0.60}$
35	$-5.26^{-0.64}_{+0.91}$	$-7.92^{-0.40}_{+0.59}$	$-13.18^{-1.04}_{+1.50}$	21	$-3.63^{-0.43}_{+0.62}$	$-5.35^{-0.28}_{+0.40}$	$-8.98^{-0.71}_{+1.02}$
36	$\equiv 0$	$\equiv 0$	$\equiv 0$				
37	$\equiv 0$	$\equiv 0$	$\equiv 0$				
38	$-5.34^{-0.30}_{+0.44}$	$-4.93^{-0.33}_{+0.47}$	$-10.27^{-0.63}_{+0.91}$	22	$-2.99^{-0.18}_{+0.26}$	$-2.73^{-0.20}_{+0.28}$	$-5.72^{-0.38}_{+0.54}$

TABLE 5. (*Continued*)

n	$10^3\ \mathrm{GeV}^2 C^T_{r,n}$	$10^3\ \mathrm{GeV}^2 C^T_{i,n}$	$10^3\ \mathrm{GeV}^2 C^T_{n}$	m	$10^3\ \mathrm{GeV}^2 c^T_{r,m}$	$10^3\ \mathrm{GeV}^2 c^T_{i,m}$	$10^3\ \mathrm{GeV}^2 c^T_{m}$
39	$1.91^{-0.10}_{+0.11}$	$4.82^{+0.29}_{-0.41}$	$6.73^{+0.19}_{-0.30}$	23	$2.21^{-0.02}_{+0.01}$	$4.24^{+0.24}_{-0.35}$	$6.45^{+0.22}_{-0.35}$
40	$2.15^{+0.10}_{-0.15}$	$2.15^{+0.10}_{-0.14}$	$4.30^{+0.19}_{-0.29}$				
41	$\equiv 0$	$\equiv 0$	$\equiv 0$				
42	$-2.94^{-0.13}_{+0.20}$	$-2.92^{-0.13}_{+0.19}$	$-5.87^{-0.26}_{+0.40}$	24	$-1.94^{-0.09}_{+0.14}$	$-1.93^{-0.09}_{+0.13}$	$-3.87^{-0.18}_{+0.27}$
43	$3.41^{+0.16}_{-0.25}$	$3.43^{+0.15}_{-0.23}$	$6.84^{+0.32}_{-0.47}$	25	$2.23^{+0.11}_{-0.17}$	$2.25^{+0.10}_{-0.15}$	$4.48^{+0.21}_{-0.32}$
44	$6.55^{+0.29}_{-0.43}$	$6.52^{+0.29}_{-0.43}$	$13.06^{+0.58}_{-0.86}$	26	$4.29^{+0.20}_{-0.29}$	$4.27^{+0.20}_{-0.29}$	$8.57^{+0.39}_{-0.59}$
45	$\equiv 0$	$\equiv 0$	$\equiv 0$	27	$\equiv 0$	$\equiv 0$	$\equiv 0$
46	$-0.00^{-0.07}_{+0.09}$	$-0.84^{-0.05}_{+0.07}$	$-0.84^{-0.12}_{+0.16}$	28	$0.04^{-0.03}_{+0.04}$	$-0.31^{-0.02}_{+0.03}$	$-0.27^{-0.05}_{+0.06}$
47	$-14.18^{-0.94}_{+1.33}$	$-4.59^{-0.28}_{+0.39}$	$-18.77^{-1.22}_{+1.72}$	29	$-6.11^{-0.42}_{+0.59}$	$-2.02^{-0.13}_{+0.17}$	$-8.13^{-0.55}_{+0.77}$
48	...	...	...	30	...	...	...
49	$\equiv 0$	$\equiv 0$	$\equiv 0$	31	$\equiv 0$	$\equiv 0$	$\equiv 0$
50	$\equiv 0$	$\equiv 0$	$\equiv 0$	32	$\equiv 0$	$\equiv 0$	$\equiv 0$
51	...	...	...	33	...	...	...
52	$-0.04^{-0.06}_{+0.09}$	$-0.19^{-0.01}_{+0.01}$	$-0.23^{-0.07}_{+0.11}$	34	$-0.05^{-0.04}_{+0.06}$	$-0.15^{-0.01}_{+0.01}$	$-0.21^{-0.05}_{+0.08}$
53	$1.36^{+0.06}_{-0.08}$	$1.35^{+0.06}_{-0.09}$	$2.71^{+0.12}_{-0.17}$	35	$0.91^{+0.04}_{-0.06}$	$0.91^{+0.04}_{-0.06}$	$1.81^{+0.08}_{-0.12}$
54	$-3.36^{-0.10}_{+0.15}$	$-3.23^{-0.14}_{+0.21}$	$-6.58^{-0.25}_{+0.36}$	36	$-2.11^{-0.07}_{+0.10}$	$-2.02^{-0.09}_{+0.14}$	$-4.12^{-0.16}_{+0.23}$
55	$-1.43^{-0.20}_{+0.32}$	$-1.80^{-0.08}_{+0.12}$	$-3.23^{-0.28}_{+0.44}$	37	$-1.07^{-0.14}_{+0.22}$	$-1.33^{-0.06}_{+0.09}$	$-2.40^{-0.20}_{+0.31}$
56	$-2.79^{-0.03}_{+0.03}$	$-2.53^{-0.11}_{+0.17}$	$-5.32^{-0.14}_{+0.20}$	38	$-1.74^{-0.02}_{+0.02}$	$-1.56^{-0.07}_{+0.11}$	$-3.30^{-0.09}_{+0.12}$
57	$\equiv 0$	$\equiv 0$	$\equiv 0$				
58	$\equiv 0$	$\equiv 0$	$\equiv 0$				
59	$\equiv 0$	$\equiv 0$	$\equiv 0$				
60	$\equiv 0$	$\equiv 0$	$\equiv 0$				
61	$\equiv 0$	$\equiv 0$	$\equiv 0$				
62	$-2.83^{-0.15}_{+0.24}$	$-2.89^{-0.13}_{+0.19}$	$-5.72^{-0.28}_{+0.43}$	39	$-1.87^{-0.11}_{+0.16}$	$-1.91^{-0.09}_{+0.13}$	$-3.78^{-0.19}_{+0.29}$
63	$-6.56^{-0.30}_{+0.45}$	$-6.59^{-0.29}_{+0.44}$	$-13.15^{-0.60}_{+0.88}$	40	$-4.26^{-0.20}_{+0.30}$	$-4.28^{-0.20}_{+0.29}$	$-8.54^{-0.40}_{+0.59}$
64	$3.74^{+0.16}_{-0.23}$	$8.47^{+0.38}_{-0.56}$	$12.22^{+0.54}_{-0.79}$	41	$2.42^{+0.11}_{-0.15}$	$5.55^{+0.26}_{-0.38}$	$7.97^{+0.36}_{-0.53}$
65	$0.10^{-0.01}_{+0.03}$	$0.05^{+0.00}_{-0.00}$	$0.15^{-0.01}_{+0.02}$	42	$0.04^{-0.01}_{+0.02}$	$0.01^{+0.00}_{-0.00}$	$0.05^{-0.01}_{+0.02}$
66	$\equiv 0$	$\equiv 0$	$\equiv 0$	43	$\equiv 0$	$\equiv 0$	$\equiv 0$
67	$2.26^{-0.12}_{+0.12}$	$0.35^{-0.02}_{+0.02}$	$2.62^{-0.13}_{+0.15}$	44	$1.45^{-0.07}_{+0.08}$	$0.22^{-0.01}_{+0.02}$	$1.67^{-0.09}_{+0.09}$
68	$-1.76^{-0.16}_{+0.23}$	$-2.16^{-0.13}_{+0.19}$	$-3.92^{-0.29}_{+0.42}$	45	$-0.36^{-0.07}_{+0.09}$	$-0.61^{-0.05}_{+0.07}$	$-0.96^{-0.12}_{+0.16}$
69	$-4.23^{-0.31}_{+0.46}$	$-4.76^{-0.25}_{+0.36}$	$-8.99^{-0.56}_{+0.82}$	46	$-2.81^{-0.21}_{+0.31}$	$-3.15^{-0.17}_{+0.25}$	$-5.97^{-0.38}_{+0.55}$
70	$\equiv 0$	$\equiv 0$	$\equiv 0$	47	$\equiv 0$	$\equiv 0$	$\equiv 0$
71	$\equiv 0$	$\equiv 0$	$\equiv 0$	48	$-0.82^{-0.04}_{+0.06}$	$-0.83^{-0.04}_{+0.06}$	$-1.65^{-0.08}_{+0.12}$
72	$\equiv 0$	$\equiv 0$	$\equiv 0$				
73	$1.66^{+0.08}_{-0.13}$	$1.67^{+0.07}_{-0.11}$	$3.34^{+0.16}_{-0.24}$				
74	...	...	...	49	...	...	...
75	$\equiv 0$	$\equiv 0$	$\equiv 0$	50	$\equiv 0$	$\equiv 0$	$\equiv 0$
76	...	...	...				
77	$-1.20^{-0.07}_{+0.11}$	$-1.24^{-0.06}_{+0.08}$	$-2.44^{-0.13}_{+0.19}$	51	$-0.76^{-0.05}_{+0.07}$	$-0.79^{-0.04}_{+0.05}$	$-1.56^{-0.08}_{+0.13}$

TABLE 5. (*Continued*)

n	$10^3\ \mathrm{GeV}^2 C^T_{r,n}$	$10^3\ \mathrm{GeV}^2 C^T_{i,n}$	$10^3\ \mathrm{GeV}^2 C^T_{n}$	m	$10^3\ \mathrm{GeV}^2 c^T_{r,m}$	$10^3\ \mathrm{GeV}^2 c^T_{i,m}$	$10^3\ \mathrm{GeV}^2 c^T_{m}$
78	$5.46^{+0.27}_{-0.41}$	$5.51^{+0.25}_{-0.37}$	$10.97^{+0.51}_{-0.78}$	52	$3.45^{+0.18}_{-0.27}$	$3.48^{+0.16}_{-0.24}$	$6.93^{+0.34}_{-0.51}$
79	$-4.71^{-0.29}_{+0.44}$	$-4.96^{-0.22}_{+0.33}$	$-9.67^{-0.52}_{+0.77}$	53	$-3.02^{-0.20}_{+0.29}$	$-3.19^{-0.15}_{+0.22}$	$-6.20^{-0.34}_{+0.51}$
80	$\equiv 0$	$\equiv 0$	$\equiv 0$				
81	$7.99^{+0.54}_{-0.77}$	$10.12^{+0.62}_{-0.85}$	$18.11^{+1.16}_{-1.61}$	54	$3.44^{+0.24}_{-0.34}$	$4.37^{+0.28}_{-0.38}$	$7.81^{+0.52}_{-0.72}$
82	⋯	⋯	⋯	55	⋯	⋯	⋯
83	$-7.07^{-0.47}_{+0.67}$	$-1.06^{-0.06}_{+0.09}$	$-8.12^{-0.54}_{+0.76}$	56	$-2.96^{-0.21}_{+0.29}$	$-0.40^{-0.03}_{+0.03}$	$-3.36^{-0.23}_{+0.32}$
84	⋯	⋯	⋯	57	⋯	⋯	⋯
85	$\equiv 0$	$\equiv 0$	$\equiv 0$	58	$\equiv 0$	$\equiv 0$	$\equiv 0$
86	⋯	⋯	⋯				
87	$-22.85^{-1.32}_{+2.00}$	$-23.69^{-1.06}_{+1.57}$	$-46.54^{-2.38}_{+3.57}$	59	$-15.16^{-0.90}_{+1.36}$	$-15.72^{-0.73}_{+1.07}$	$-30.88^{-1.63}_{+2.43}$
88	$-0.81^{+0.02}_{-0.03}$	$-0.63^{-0.03}_{+0.04}$	$-1.44^{-0.01}_{+0.01}$	60	$-0.43^{+0.02}_{-0.03}$	$-0.31^{-0.01}_{+0.02}$	$-0.73^{+0.00}_{-0.01}$
89	$5.07^{+0.27}_{-0.40}$	$5.19^{+0.23}_{-0.34}$	$10.26^{+0.50}_{-0.74}$	61	$3.37^{+0.18}_{-0.27}$	$3.45^{+0.16}_{-0.24}$	$6.82^{+0.34}_{-0.51}$
90	$1.55^{+0.11}_{-0.16}$	$1.69^{+0.08}_{-0.11}$	$3.24^{+0.19}_{-0.28}$	62	$1.14^{+0.08}_{-0.12}$	$1.23^{+0.06}_{-0.08}$	$2.38^{+0.14}_{-0.20}$
91	$4.82^{+0.22}_{-0.34}$	$4.78^{+0.21}_{-0.32}$	$9.60^{+0.43}_{-0.65}$	63	$3.22^{+0.15}_{-0.23}$	$3.19^{+0.15}_{-0.22}$	$6.41^{+0.30}_{-0.45}$
92	$-4.93^{-0.29}_{+0.43}$	$-5.12^{-0.23}_{+0.34}$	$-10.04^{-0.51}_{+0.77}$	64	$-3.19^{-0.19}_{+0.29}$	$-3.32^{-0.15}_{+0.23}$	$-6.51^{-0.35}_{+0.52}$
93	$9.51^{+0.44}_{-0.68}$	$9.49^{+0.42}_{-0.63}$	$19.01^{+0.86}_{-1.31}$	65	$6.31^{+0.30}_{-0.47}$	$6.30^{+0.29}_{-0.43}$	$12.61^{+0.60}_{-0.90}$
94	$\equiv 0$	$\equiv 0$	$\equiv 0$				
95	$\equiv 0$	$\equiv 0$	$\equiv 0$				
96	$\equiv 0$	$\equiv 0$	$\equiv 0$				
97	⋯	⋯	⋯	66	⋯	⋯	⋯
98	$6.92^{+0.46}_{-0.65}$	$3.35^{+0.20}_{-0.28}$	$10.28^{+0.67}_{-0.94}$	67	$2.93^{+0.20}_{-0.29}$	$1.41^{+0.09}_{-0.12}$	$4.34^{+0.29}_{-0.41}$
99	$\equiv 0$	$\equiv 0$	$\equiv 0$	68	$\equiv 0$	$\equiv 0$	$\equiv 0$
100	$\equiv 0$	$\equiv 0$	$\equiv 0$	69	$\equiv 0$	$\equiv 0$	$\equiv 0$
101	$\equiv 0$	$\equiv 0$	$\equiv 0$	70	$\equiv 0$	$\equiv 0$	$\equiv 0$
102	⋯	⋯	⋯				
103	⋯	⋯	⋯				
104	⋯	⋯	⋯				
105	$-3.06^{+0.16}_{-0.17}$	$0.00^{+0.00}_{+0.00}$	$-3.06^{+0.16}_{-0.17}$	71	$-1.96^{+0.10}_{-0.11}$	$0.00^{+0.00}_{+0.00}$	$-1.96^{+0.10}_{-0.11}$
106	$-2.07^{-0.78}_{+1.04}$	$-6.97^{-0.45}_{+0.65}$	$-9.04^{-1.23}_{+1.69}$	72	$-1.58^{-0.53}_{+0.70}$	$-4.74^{-0.31}_{+0.45}$	$-6.32^{-0.84}_{+1.15}$
107	$-0.15^{+0.06}_{-0.10}$	$0.06^{+0.00}_{-0.00}$	$-0.09^{+0.07}_{-0.10}$	73	$0.01^{+0.05}_{-0.07}$	$0.15^{+0.01}_{-0.01}$	$0.15^{+0.05}_{-0.08}$
108	⋯	⋯	⋯	74	⋯	⋯	⋯
109	⋯	⋯	⋯	75	⋯	⋯	⋯
110	$\equiv 0$	$\equiv 0$	$\equiv 0$				

$$(\tilde{K}^{T,W}_1, \tilde{K}^{T,W}_2, \ldots, \tilde{K}^{T,W}_{56})(\tilde{O}^{T,W}_{1,t}, \tilde{O}^{T,W}_{2,t}, \ldots, \tilde{O}^{T,W}_{56,t})^T = (\tilde{K}^{T,W}_1, \tilde{K}^{T,W}_2, \ldots, \tilde{K}^{T,W}_{56})A_6(\bar{O}^{T,W}_1, \bar{O}^{T,W}_2, \ldots, \bar{O}^{T,W}_{339})^T$$
$$= (\bar{K}^{T,W}_1, \bar{K}^{T,W}_2, \ldots, \bar{K}^{T,W}_{339})(\bar{K}^{T,W}_1, \bar{K}^{T,W}_2, \ldots, \bar{K}^{T,W}_{339})^T \quad (68)$$

$$\Rightarrow (\tilde{K}^{T,W}_1, \tilde{K}^{T,W}_2, \ldots, \tilde{K}^{T,W}_{56})A_6 = (\bar{K}^{T,W}_1, \bar{K}^{T,W}_2, \ldots, \bar{K}^{T,W}_{339}). \quad (69)$$

$\bar{O}_n^{T,W}$ in p^6 order are the same as $\bar{o}_n$ in p^4 order, and $\bar{K}_n^{T,W}$ are the same as z_n in p^4 order. There are 339 $\bar{O}_n^{T,W}$ and $\bar{K}_n^{T,W}$, which are too many to be listed here. Moreover, A_6 is the same as A_4 in p^4 order too. Using (69), we can also get $\tilde{O}_n^{T,W}$, which are listed in Appendix E. Combined with Table IV, we can get the analytical results from the imaginary part. The numerical results are listed in the third and seventh columns in Table V.

C. Numerical results

In Table V, we list our p^6 order LECs with tensor sources, including the three- and two-flavor cases, and the results for the real and the imaginary parts. Similar to the p^4 order, we calculated the values with $\Lambda = 1$ GeV, and use the superscript and subscript to denote the differences caused with $\Lambda = 1.1$ GeV and $\Lambda = 0.9$ GeV, respectively:

$$C^T_{n,\Lambda=1\text{ GeV}}\Big|^{C^T_{n,\Lambda=1.1\text{ GeV}}-C^T_{n,\Lambda=1\text{ GeV}}}_{C^T_{n,\Lambda=0.9\text{ GeV}}-C^T_{n,\Lambda=1\text{ GeV}}}$$
$$c^T_{n,\Lambda=1\text{ GeV}}\Big|^{c^T_{n,\Lambda=1.1\text{ GeV}}-c^T_{n,\Lambda=1\text{ GeV}}}_{c^T_{n,\Lambda=0.9\text{ GeV}}-c^T_{n,\Lambda=1\text{ GeV}}}. \tag{70}$$

$C^T_{n,\Lambda}$ and $c^T_{n,\Lambda}$ denote the three- and two-flavor cases, respectively. Because of the relations given in Appendix B, some terms are not independent; we denote their coefficients by the symbol "$\cdots$". In Ref. [10], the coefficients were multiplied by a suitable power of b_0 to express these with the same dimensional units. We choose b_0 at the end in Sec. V B.

The calculation process is too complicated; to avoid possible mistakes, the expansion in Eqs. (30) and (33) and most of the other calculations are done by computer. To check the correctness of our results, we examine them in various alternative ways. First, because these results contain the original results in [17,18], if we switch off the tensor sources, as a check, we must recover the original results. Second, some terms in Table III and the p^6 order operators in Table 53 have two parts; we calculate them separately. C, P, and Hermitian invariance constrain the two parts of the coefficients to be equal (or with a minus sign difference). Our analytical results for the separate parts must give the same coefficients. Third, if we switch off the quark self-energy, all the LECs, except the contact terms, must be zeros [17]. This places a strong restriction on our results. We found that this restriction can be realized only when we use the new relations given in Appendix B. Fourth, in the p^6 order, because of the strict constraint conditions in Eq. (59), we have $339 - 56 = 283$ constraint conditions. They are also a strong restriction on our results. With all the above assessments, we are confident of the reliability of our numerical results for LECs.

The authors of Ref. [10] told us that operators contributing to the odd-intrinsic-parity part with tensor fields start from the p^8 order, and we showed in Sec. IV that the odd-intrinsic-parity parts with tensor fields cannot independently exist. So we have obtained all the LECs to the p^6 order, with scalar, pseudoscalar, vector, axial-vector, and tensor sources, including the normal and anomalous parts, and two- and three-flavor cases. We found that in our method, all the contact terms' coefficients are divergent, except H_1 in the p^4 order normal part.

VII. SUMMARY

To summarize our results, we extended our previous computation for LECs in Refs. [15,17] include tensor sources, and obtain all LECs of order p^4 and p^6 for the chiral Lagrangian. We find that the operators given in Ref. [10] are not all independent because of certain relations involving epsilon. Adding these relations, we can reduce 22 operators for n-flavor, 21 for three-flavor, and 13 for two-flavor cases, leaving 98 independent operators for n-flavor, 92 for three-flavor, and 65 for two-flavor cases. Our LECs are presented with numerical values for both two- and three-flavor cases. We also find that the odd-intrinsic-parity parts' chiral Lagrangian with tensor sources cannot independently exist. Thus, up to the p^6 order, we have already given all the LECs' values, although, in obtaining these values, we have made many approximations. As a first step in estimating values, these results not only provide the sign and order of magnitude, but also the quantitative information of LECs. With improvements in the computation procedure, we expect more precise results to be obtained in the future. Another direction of research is applying the present chiral Lagrangian with tensor sources, adding the known LECs to various low-energy (π, K, η) processes. We hope more physical results can be obtained.

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APPENDIX A: RELATIONS AMONG OUR SYMBOLS AND THOSE USED IN REF. [10]

To help in understanding the mutual relation between the notations in our current formulation and those in Ref. [10], we provide a comparison in Table VI.

TABLE VI. Comparison between notations introduced in Ref. [10] (first and third columns) and those defined in the current paper (second and fourth columns).

Ref. [10]	Present paper	Ref. [10]	Present paper
∇^μ	d^μ	χ_-^μ	$4iB_0 d^\mu p_\Omega - 4iB_0 s_\Omega a_\Omega^\mu - 4iB_0 a_\Omega^\mu s_\Omega$
u	Ω	$f_+^{\mu\nu}$	$2V_\Omega^{\mu\nu} - 2i(a_\Omega^\mu a_\Omega^\nu - a_\Omega^\nu a_\Omega^\mu)$
u^μ	$2a_\Omega^\mu$	$\nabla^\lambda f_+^{\mu\nu}$	$2d^\lambda V_\Omega^{\mu\nu} - 2id^\lambda(a_\Omega^\mu a_\Omega^\nu - a_\Omega^\nu a_\Omega^\mu)$
χ	χ	$f_-^{\mu\nu}$	$-2(d^\mu a_\Omega^\nu - d^\nu a_\Omega^\mu)$
χ_+	$4B_0 s_\Omega$	$\nabla^\lambda f_-^{\mu\nu}$	$-2(d^\lambda d^\mu a_\Omega^\nu - d^\lambda d^\nu a_\Omega^\mu)$
χ_+^μ	$4B_0 d^\mu s_\Omega + 4B_0 p_\Omega a_\Omega^\mu + 4B_0 a_\Omega^\mu p_\Omega$	$h^{\mu\nu}$	$2(d^\mu a_\Omega^\nu + d^\nu a_\Omega^\mu)$
χ_-	$4iB_0 p_\Omega$	Γ^μ	$-iv_\Omega^\mu$
$t_+^{\mu\nu}$	$t_{+,\Omega}^{\mu\nu}$	$t_-^{\mu\nu}$	$t_{-,\Omega}^{\mu\nu}$

APPENDIX B: NEW RELATIONS

In this appendix, we list the new relations when using the epsilon relations in Sec. IV. The left-hand sides of (B1) are considered to be dependent and reducible.

$$
\begin{aligned}
&Y_{23} = \frac{1}{2}Y_9 - Y_{12}, \quad Y_{24} = \frac{1}{2}Y_9 - Y_{11}, \quad Y_{25} = Y_{10} - Y_{13}, \quad Y_{26} = \frac{1}{2}Y_{14} - Y_{15}, \quad Y_{27} = \frac{1}{2}Y_{16} - Y_{18},\\
&Y_{28} = \frac{1}{2}Y_{16} - Y_{17}, \quad Y_{29} = Y_{19} - Y_{20}, \quad Y_{30} = Y_{21} - Y_{22}, \quad Y_{53} = -\frac{1}{2}Y_{11} + \frac{1}{2}Y_{12} + \frac{1}{2}Y_{51} - Y_{52} + Y_{90},\\
&Y_{56} = \frac{1}{2}Y_{54} - Y_{55}, \quad Y_{81} = \frac{1}{2}Y_{32} - \frac{1}{2n_f}Y_{35}, \quad Y_{83} = \frac{1}{2}Y_{36} - \frac{1}{2n_f}Y_{38} - Y_{82}, \quad Y_{89} = Y_{88}, \quad Y_{91} = Y_{90},\\
&Y_{93} = Y_{92}, \quad Y_{104} = \frac{1}{2}Y_{32} - \frac{1}{2n_f}Y_{35}, \quad Y_{109} = \frac{1}{2}Y_{36} - \frac{1}{2n_f}Y_{38} - Y_{106}, \quad Y_{110} = -\frac{1}{2}Y_{82} + \frac{1}{2}Y_{92} + \frac{1}{2}Y_{106} + Y_{107},\\
&Y_{111} = -\frac{1}{4}Y_{36} + \frac{1}{4n_f}Y_{38} + \frac{1}{2}Y_{82} + \frac{1}{2}Y_{92} + \frac{1}{2}Y_{106} + Y_{108}, \quad Y_{115} = -\frac{1}{2}Y_1 + Y_2 - \frac{1}{2}Y_{57} + Y_{58} + Y_{84} - Y_{96} + Y_{97} + Y_{114},\\
&Y_{116} = \frac{1}{2}Y_{69} - Y_{70} + Y_{72} + \frac{1}{2}Y_{75} - \frac{1}{n_f}Y_{80} - Y_{86} - 2Y_{98} - Y_{99}, \quad Y_{119} = 0.
\end{aligned}
\tag{B1}
$$

APPENDIX C: Z_n COEFFICIENTS

$$
\begin{aligned}
Z_1^T &= \int dK\left[-10\tau^3\Sigma_k + \frac{10}{3}\tau^4k^2\Sigma_k + \frac{40}{3}\tau^4\Sigma_k^3 - \frac{2}{9}\tau^5k^4\Sigma_k - 2\tau^5k^2\Sigma_k^3 - \frac{8}{3}\tau^5\Sigma_k^5\right],\\
Z_2^T &= \int dK\left[+10\tau^3\Sigma_k - \frac{8}{3}\tau^4k^2\Sigma_k - \frac{40}{3}\tau^4\Sigma_k^3 + 2\tau^5k^2\Sigma_k^3 + \frac{8}{3}\tau^5\Sigma_k^5\right],\\
Z_3^T &= \int dK\left[-10\tau^3\Sigma_k + 4\tau^4k^2\Sigma_k + \frac{40}{3}\tau^4\Sigma_k^3 - \frac{2}{9}\tau^5k^4\Sigma_k - 2\tau^5k^2\Sigma_k^3 - \frac{8}{3}\tau^5\Sigma_k^5\right],\\
Z_4^T &= \int dK\left[-10\tau^3\Sigma_k + 2\tau^4k^2\Sigma_k + \frac{40}{3}\tau^4\Sigma_k^3 + \frac{2}{9}\tau^5k^4\Sigma_k - 2\tau^5k^2\Sigma_k^3 - \frac{8}{3}\tau^5\Sigma_k^5\right],\\
Z_5^T &= \int dK\left[+4\tau^3\Sigma_k - \frac{2}{3}\tau^4k^2\Sigma_k - \frac{4}{3}\tau^4\Sigma_k^3 - \frac{1}{9}\tau^4k^4\Sigma_k'\right],\\
Z_6^T &= \int dK\left[-\frac{22}{3}\tau^3\Sigma_k + \frac{1}{3}\tau^3k^2\Sigma_k' + 2\tau^4k^2\Sigma_k + \frac{8}{3}\tau^4\Sigma_k^3\right],
\end{aligned}
$$

$$Z_7^T = \int dK\left[+6\tau^3\Sigma_k + \tau^3k^2\Sigma_k' - \frac{4}{3}\tau^4k^2\Sigma_k - \frac{8}{3}\tau^4\Sigma_k^3 - \frac{2}{9}\tau^4k^4\Sigma_k'\right],$$

$$Z_8^T = \int dK\left[+4\tau^3\Sigma_k - \frac{4}{3}\tau^4k^2\Sigma_k - \frac{4}{3}\tau^4\Sigma_k^3 + \frac{1}{9}\tau^4k^4\Sigma_k'\right],$$

$$Z_9^T = \int dK\left[-4\tau^3\Sigma_k + \frac{4}{3}\tau^4k^2\Sigma_k + \frac{4}{3}\tau^4\Sigma_k^3 - \frac{1}{9}\tau^4k^4\Sigma_k'\right],$$

$$Z_{10}^T = \int dK\left[-6\tau^3\Sigma_k + \tau^3k^2\Sigma_k' + \frac{4}{3}\tau^4k^2\Sigma_k + \frac{8}{3}\tau^4\Sigma_k^3 - \frac{2}{9}\tau^4k^4\Sigma_k'\right],$$

$$Z_{11}^T = \int dK\left[-\frac{20}{3}\tau^3\Sigma_k + \frac{2}{3}\tau^3k^2\Sigma_k' + \frac{4}{3}\tau^4k^2\Sigma_k + \frac{8}{3}\tau^4\Sigma_k^3 - \frac{2}{9}\tau^4k^4\Sigma_k'\right],$$

$$Z_{12}^T = \int dK\left[+\frac{2}{3}\tau^4k^2\Sigma_k - \frac{2}{9}\tau^4k^4\Sigma_k'\right],$$

$$Z_{13}^T = \int dK\left[+4\tau^3\Sigma_k + 2\tau^3k^2\Sigma_k' - \frac{2}{3}\tau^4k^2\Sigma_k - \frac{8}{3}\tau^4\Sigma_k^3 - \frac{4}{9}\tau^4k^4\Sigma_k'\right],$$

$$Z_{14}^T = \int dK\left[+2\tau^2 - \frac{3}{2}\tau^3k^2 - 8\tau^3\Sigma_k^2 + \frac{2}{9}\tau^4k^4 + 2\tau^4k^2\Sigma_k^2 + \frac{8}{3}\tau^4\Sigma_k^4\right],$$

$$Z_{15}^T = \int dK\left[-4\tau^2 + \tau^3k^2 + 16\tau^3\Sigma_k^2 + \frac{2}{9}\tau^4k^4 - 4\tau^4k^2\Sigma_k^2 - \frac{16}{3}\tau^4\Sigma_k^4\right],$$

$$Z_{16}^T = \int dK\left[+4\tau^2 - 3\tau^3k^2 - 16\tau^3\Sigma_k^2 + \frac{2}{9}\tau^4k^4 + 4\tau^4k^2\Sigma_k^2 + \frac{16}{3}\tau^4\Sigma_k^4\right],$$

$$Z_{17}^T = \int dK\left[-2\tau^2 + \frac{3}{2}\tau^3k^2 + 2\tau^3\Sigma_k^2 - \frac{2}{9}\tau^4k^4 - \frac{4}{3}\tau^4k^2\Sigma_k^2\right],$$

$$Z_{18}^T = \int dK\left[+4\tau^2 - \tau^3k^2 - 4\tau^3\Sigma_k^2 - \frac{2}{9}\tau^4k^4 + \frac{4}{3}\tau^4k^2\Sigma_k^2\right],$$

$$Z_{19}^T = \int dK\left[-4\tau^2 + 3\tau^3k^2 + 4\tau^3\Sigma_k^2 - \frac{2}{9}\tau^4k^4 - \frac{4}{3}\tau^4k^2\Sigma_k^2\right],$$

$$Z_{20}^T = \int dK\left[+2\tau^2 - \tau^3k^2 - 8\tau^3\Sigma_k^2 + \frac{1}{9}\tau^4k^4 + 2\tau^4k^2\Sigma_k^2 + \frac{8}{3}\tau^4\Sigma_k^4\right],$$

$$Z_{21}^T = \int dK\left[+\tau^2 - 4\tau^3\Sigma_k^2 - \frac{1}{9}\tau^4k^4 + \tau^4k^2\Sigma_k^2 + \frac{4}{3}\tau^4\Sigma_k^4\right],$$

$$Z_{22}^T = \int dK\left[-2\tau^2 + \tau^3k^2 - \frac{1}{9}\tau^4k^4 + \frac{2}{3}\tau^4k^2\Sigma_k^2\right],$$

$$Z_{23}^T = \int dK\left[-\tau^2 + \frac{1}{9}\tau^4k^4 + \frac{2}{3}\tau^4k^2\Sigma_k^2\right],$$

$$Z_{24}^T = \int dK\left[+\frac{4}{3}\tau^3\Sigma_k - \tau^3k^2\Sigma_k' - \frac{2}{3}\tau^4k^2\Sigma_k + \frac{1}{9}\tau^4k^4\Sigma_k' + \frac{2}{3}\tau^4k^4\Sigma_k\Sigma_k'^2 + \frac{1}{9}\tau^5k^4\Sigma_k - \frac{4}{9}\tau^5k^4\Sigma_k^3\Sigma_k'^2\right],$$

$$Z_{25}^T = \int dK\left[-\frac{2}{3}\tau^3\Sigma_k + \frac{2}{3}\tau^3k^2\Sigma_k'\right],$$

$$Z_{26}^T = \int dK\left[-\frac{2}{3}\tau^3\Sigma_k + \frac{2}{3}\tau^4k^2\Sigma_k - \frac{4}{9}\tau^4k^4\Sigma_k\Sigma_k'^2 - \frac{1}{9}\tau^5k^4\Sigma_k + \frac{4}{9}\tau^5k^4\Sigma_k^3\Sigma_k'^2\right],$$

$$Z_{27}^T = \int dK\left[+\frac{1}{3}\tau^3k^2\Sigma_k' + \frac{2}{3}\tau^4k^2\Sigma_k - \frac{1}{9}\tau^4k^4\Sigma_k' - \frac{2}{3}\tau^4k^4\Sigma_k\Sigma_k'^2 - \frac{1}{9}\tau^5k^4\Sigma_k + \frac{4}{9}\tau^5k^4\Sigma_k^3\Sigma_k'^2\right],$$

$$Z_{28}^T = \int dK\left[-6\tau^3\Sigma_k + 10\tau^3k^2\Sigma_k\Sigma_k'^2 + \frac{8}{3}\tau^4k^2\Sigma_k + \frac{8}{3}\tau^4\Sigma_k^3 - \frac{1}{9}\tau^4k^4\Sigma_k' - \frac{10}{9}\tau^4k^4\Sigma_k\Sigma_k'^2 - \frac{40}{3}\tau^4k^2\Sigma_k^3\Sigma_k'^2 \right.$$
$$\left. - \frac{2}{9}\tau^5k^4\Sigma_k - \frac{2}{3}\tau^5k^2\Sigma_k^3 + \frac{8}{9}\tau^5k^4\Sigma_k^3\Sigma_k'^2 + \frac{8}{3}\tau^5k^2\Sigma_k^5\Sigma_k'^2\right],$$

$$Z^T_{29}=\int dK\bigg[+\frac{2}{3}\tau^3\Sigma_k+\frac{2}{3}\tau^4k^2\Sigma_k-\frac{4}{9}\tau^4k^4\Sigma_k\Sigma_k'^2-\frac{1}{9}\tau^5k^4\Sigma_k+\frac{4}{9}\tau^5k^4\Sigma_k^3\Sigma_k'^2\bigg],$$

$$Z^T_{30}=\int dK\bigg[+4\tau^2-\frac{4}{3}\tau^3k^2-4\tau^3\Sigma_k^2\bigg],\qquad Z^T_{31}=\int dK\bigg[+\frac{8}{3}\tau^2-\frac{2}{3}\tau^3k^2-\frac{4}{3}\tau^3\Sigma_k^2\bigg],$$

$$Z^T_{32}=\int dK\bigg[-3\tau^2+\frac{7}{6}\tau^3k^2+2\tau^3\Sigma_k^2\bigg],\qquad Z^T_{33}=\int dK\bigg[-4\tau^2+\tau^3k^2+16\tau^3\Sigma_k^2-\frac{8}{3}\tau^4k^2\Sigma_k^2-\frac{16}{3}\tau^4\Sigma_k^4\bigg],$$

$$Z^T_{34}=\int dK\bigg[-4\tau^2+2\tau^3k^2+16\tau^3\Sigma_k^2-\frac{8}{3}\tau^4k^2\Sigma_k^2-\frac{16}{3}\tau^4\Sigma_k^4\bigg],$$

$$Z^T_{35}=\int dK\bigg[-\frac{4}{3}\tau^3\Sigma_k+\tau^3k^2\Sigma_k'+\frac{2}{3}\tau^4k^2\Sigma_k-\frac{1}{9}\tau^4k^4\Sigma_k'-\frac{2}{3}\tau^4k^4\Sigma_k\Sigma_k'^2-\frac{1}{9}\tau^5k^4\Sigma_k+\frac{4}{9}\tau^5k^4\Sigma_k^3\Sigma_k'^2\bigg],$$

$$Z^T_{36}=\int dK\bigg[-6\tau^3\Sigma_k+10\tau^3k^2\Sigma_k\Sigma_k'^2+\frac{8}{3}\tau^4k^2\Sigma_k+\frac{8}{3}\tau^4\Sigma_k^3-\frac{2}{9}\tau^4k^4\Sigma_k'-\frac{4}{3}\tau^4k^4\Sigma_k\Sigma_k'^2-\frac{40}{3}\tau^4k^2\Sigma_k^3\Sigma_k'^2$$
$$-\frac{2}{9}\tau^5k^4\Sigma_k-\frac{2}{3}\tau^5k^2\Sigma_k^3+\frac{8}{9}\tau^5k^4\Sigma_k^3\Sigma_k'^2+\frac{8}{3}\tau^5k^2\Sigma_k^5\Sigma_k'^2\bigg],$$

$$Z^T_{37}=\int dK\bigg[+\frac{1}{3}\tau^3k^2\Sigma_k'+\frac{2}{3}\tau^4k^2\Sigma_k-\frac{1}{9}\tau^4k^4\Sigma_k'-\frac{2}{3}\tau^4k^4\Sigma_k\Sigma_k'^2-\frac{1}{9}\tau^5k^4\Sigma_k+\frac{4}{9}\tau^5k^4\Sigma_k^3\Sigma_k'^2\bigg],$$

$$Z^T_{38}=\int dK\bigg[+\frac{2}{3}\tau^3\Sigma_k-\frac{2}{3}\tau^3k^2\Sigma_k'\bigg],$$

$$Z^T_{39}=\int dK\bigg[-\frac{20}{3}\tau^3\Sigma_k+\frac{4}{3}\tau^3k^2\Sigma_k'+2\tau^4k^2\Sigma_k+\frac{8}{3}\tau^4\Sigma_k^3-\frac{2}{9}\tau^4k^4\Sigma_k'\bigg],$$

$$Z^T_{40}=\int dK\bigg[+6\tau^3\Sigma_k+\frac{2}{3}\tau^3k^2\Sigma_k'-2\tau^4k^2\Sigma_k-\frac{8}{3}\tau^4\Sigma_k^3-\frac{2}{9}\tau^4k^4\Sigma_k'\bigg],$$

$$Z^T_{41}=\int dK\bigg[+\frac{10}{3}\tau^3\Sigma_k-\frac{1}{3}\tau^3k^2\Sigma_k'-\tau^4k^2\Sigma_k-\frac{4}{3}\tau^4\Sigma_k^3+\frac{2}{9}\tau^4k^4\Sigma_k'\bigg],$$

$$Z^T_{42}=\int dK\bigg[+6\tau^3\Sigma_k+\frac{2}{3}\tau^3k^2\Sigma_k'-2\tau^4k^2\Sigma_k-\frac{8}{3}\tau^4\Sigma_k^3+\frac{2}{9}\tau^4k^4\Sigma_k'\bigg],$$

$$Z^T_{43}=\int dK\bigg[+\frac{10}{3}\tau^3\Sigma_k+\frac{2}{3}\tau^3k^2\Sigma_k'-\tau^4k^2\Sigma_k-\frac{4}{3}\tau^4\Sigma_k^3-\frac{2}{9}\tau^4k^4\Sigma_k'\bigg],$$

$$Z^T_{44}=\int dK\bigg[-4\tau^2+2\tau^3k^2+4\tau^3\Sigma_k^2\bigg],\qquad Z^T_{45}=\int dK\bigg[+\frac{8}{3}\tau^2-\frac{4}{3}\tau^3k^2-\frac{4}{3}\tau^3\Sigma_k^2\bigg],$$

$$Z^T_{46}=\int dK\bigg[-2\tau^2+\frac{5}{6}\tau^3k^2+2\tau^3\Sigma_k^2\bigg],\qquad Z^T_{47}=\int dK\bigg[-\frac{8}{3}\tau^2+\frac{5}{3}\tau^3k^2+\frac{4}{3}\tau^3\Sigma_k^2\bigg],$$

$$Z^T_{48}=\int dK\bigg[+6\tau^2-\frac{8}{3}\tau^3k^2-4\tau^3\Sigma_k^2\bigg],\qquad Z^T_{49}=\int dK\bigg[+\frac{2}{3}\tau^3\Sigma_k-\frac{1}{9}\tau^4k^4\Sigma_k'-\frac{2}{9}\tau^4k^4\Sigma_k\Sigma_k'^2\bigg],$$

$$Z^T_{50}=\int dK\bigg[+\frac{2}{3}\tau^2\Sigma_k'+\frac{4}{3}\tau^3\Sigma_k-\frac{4}{3}\tau^3\Sigma_k^2\Sigma_k'+2\tau^3k^2\Sigma_k\Sigma_k'^2-\frac{1}{9}\tau^4k^4\Sigma_k'-\frac{2}{9}\tau^4k^4\Sigma_k\Sigma_k'^2\bigg],$$

$$Z^T_{51}=\int dK\bigg[+\tau^3\Sigma_k-\frac{7}{6}\tau^3k^2\Sigma_k'+\frac{1}{9}\tau^4k^4\Sigma_k'+\frac{2}{9}\tau^4k^4\Sigma_k\Sigma_k'^2\bigg],$$

$$Z^T_{52}=\int dK\bigg[+\frac{2}{3}\tau^2\Sigma_k'+\frac{2}{3}\tau^3\Sigma_k+\frac{2}{3}\tau^3k^2\Sigma_k'-\frac{4}{3}\tau^3\Sigma_k^2\Sigma_k'-2\tau^3k^2\Sigma_k\Sigma_k'^2-\frac{1}{3}\tau^4k^2\Sigma_k+\frac{1}{9}\tau^4k^4\Sigma_k'$$
$$+\frac{2}{9}\tau^4k^4\Sigma_k\Sigma_k'^2+\frac{4}{3}\tau^4k^2\Sigma_k^3\Sigma_k'^2\bigg],$$

$$
\begin{aligned}
Z^T_{53} &= \int dK\Big[+\frac{4}{3}\tau^3\Sigma_k - \frac{5}{3}\tau^3k^2\Sigma'_k + \frac{1}{9}\tau^4k^4\Sigma'_k + \frac{2}{9}\tau^4k^4\Sigma_k\Sigma'^2_k\Big],\\
Z^T_{54} &= \int dK\Big[+\frac{1}{3}\tau^3k^2\Sigma'_k - \frac{1}{9}\tau^4k^4\Sigma'_k - \frac{2}{9}\tau^4k^4\Sigma_k\Sigma'^2_k\Big], \qquad Z^T_{55} = \int dK\Big[+\frac{8}{3}\tau^2 - \frac{1}{3}\tau^3k^2 - \frac{4}{3}\tau^3\Sigma^2_k\Big],\\
Z^T_{56} &= \int dK[+2\tau^2], \qquad Z^T_{57} = \int dK[-2\tau^3\Sigma_k + 2\tau^3k^2\Sigma'_k - 2\tau^3k^2\Sigma_k\Sigma'^2_k],\\
Z^T_{58} &= \int dK\Big[-\frac{16}{3}\tau^2 + 3\tau^3k^2 + 4\tau^3\Sigma^2_k\Big], \qquad Z^T_{59} = \int dK\Big[+\frac{4}{3}\tau^2 - \tau^3k^2\Big],\\
Z^T_{60} &= \int dK[-\tau^2],\ Z^T_{61} = \int dK[+3\tau^2 - \tau^3k^2 - 2\tau^3\Sigma^2_k], \qquad Z^T_{62} = \int dK\Big[+4\tau^2\Sigma_k - 2\tau^3k^2\Sigma_k - \frac{8}{3}\tau^3\Sigma^3_k\Big],\\
Z^T_{63} &= \int dK[-4\tau^2\Sigma_k + 2\tau^3k^2\Sigma_k], \qquad Z^T_{64} = \int dK[-6\tau^2\Sigma_k + 2\tau^3k^2\Sigma_k + 4\tau^3\Sigma^3_k],\\
Z^T_{65} &= \int dK[+2\tau^2\Sigma_k - 2\tau^3k^2\Sigma_k],\\
Z^T_{66} &= \int dK\Big[+\frac{2}{3}\tau^2\Sigma'_k + \frac{1}{3}\tau^3k^2\Sigma'_k + \frac{2}{9}\tau^3k^4\Sigma'^3_k - \frac{1}{9}\tau^4k^4\Sigma'_k - \frac{2}{9}\tau^4k^4\Sigma_k\Sigma'^2_k\Big],\\
Z^T_{67} &= \int dK\Big[+\frac{2}{3}\tau^3\Sigma_k - \frac{11}{18}\tau^3k^2\Sigma'_k - \frac{11}{9}\tau^3k^2\Sigma_k\Sigma'^2_k - \frac{2}{9}\tau^3k^4\Sigma'^3_k - \frac{1}{6}\tau^4k^2\Sigma_k\\
&\quad + \frac{1}{9}\tau^4k^4\Sigma'_k + \frac{2}{9}\tau^4k^4\Sigma_k\Sigma'^2_k + \frac{2}{3}\tau^4k^2\Sigma^3_k\Sigma'^2_k\Big],\\
Z^T_{68} &= \int dK\Big[-\frac{1}{9}\tau^3k^2\Sigma'_k + \frac{4}{9}\tau^3k^2\Sigma_k\Sigma'^2_k - \frac{2}{9}\tau^3k^4\Sigma'^3_k + \frac{1}{9}\tau^4k^4\Sigma'_k + \frac{2}{9}\tau^4k^4\Sigma_k\Sigma'^2_k\Big],\\
Z^T_{69} &= \int dK\Big[-\frac{1}{3}\tau^2 + \frac{1}{3}\tau^3k^2 - \frac{1}{18}\tau^4k^4 + \frac{2}{9}\tau^4k^4\Sigma^2_k\Sigma'^2_k\Big],\\
Z^T_{70} &= \int dK\Big[+\frac{5}{6}\tau^2 - \frac{1}{2}\tau^2k^2\Sigma'^2_k - \frac{1}{2}\tau^3k^2 - \frac{2}{3}\tau^3\Sigma^2_k + 2\tau^3k^2\Sigma^2_k\Sigma'^2_k + \frac{1}{18}\tau^4k^4 + \frac{1}{6}\tau^4k^2\Sigma^2_k\\
&\quad - \frac{2}{9}\tau^4k^4\Sigma^2_k\Sigma'^2_k - \frac{2}{3}\tau^4k^2\Sigma^4_k\Sigma'^2_k\Big],\\
Z^T_{71} &= \int dK\Big[-\frac{1}{3}\tau^2 - \frac{1}{3}\tau^3k^2 + \frac{1}{18}\tau^4k^4 - \frac{2}{9}\tau^4k^4\Sigma^2_k\Sigma'^2_k\Big],\\
Z^T_{72} &= \int dK\Big[+\frac{1}{3}\tau^2 - \frac{1}{3}\tau^3k^2 + \frac{1}{18}\tau^4k^4 - \frac{2}{9}\tau^4k^4\Sigma^2_k\Sigma'^2_k\Big],\\
Z^T_{73} &= \int dK\Big[-\frac{1}{2}\tau^2 - \frac{1}{2}\tau^2k^2\Sigma'^2_k + \frac{1}{3}\tau^3k^2 - \frac{1}{18}\tau^4k^4 + \frac{2}{9}\tau^4k^4\Sigma^2_k\Sigma'^2_k\Big],\\
Z^T_{74} &= \int dK\Big[+\frac{1}{3}\tau^2 + \frac{1}{3}\tau^3k^2 - \frac{1}{18}\tau^4k^4 + \frac{2}{9}\tau^4k^4\Sigma^2_k\Sigma'^2_k\Big], \qquad Z^T_{75} = \int dK[-4\tau^2 + \tau^3k^2 + 4\tau^3\Sigma^2_k],\\
Z^T_{76} &= \int dK[-4\tau^2 + 2\tau^3k^2], \qquad Z^T_{77} = \int dK[+2\tau^2\Sigma_k],
\end{aligned}
\tag{C1}
$$

$$
\int dK \equiv N_c \int \frac{d^4k}{(2\pi)^4} e^{-\tau(k^2+\Sigma^2_k)} \int_{\frac{1}{\Lambda^2}}^{\infty} \frac{d\tau}{\tau}. \tag{C2}
$$

APPENDIX D: Z^T_n AND K^T_n'S RELATIONS

This appendix list relations between our coefficients, Z^T_n, and those in Ref. [10], K^T_n. Some coefficients vanish because of the new relations in Appendix B.

$$K_1^T = +\frac{1}{16b_0}Z_1^T - \frac{1}{16b_0}Z_{27}^T - \frac{1}{16b_0}Z_{28}^T + \frac{1}{16b_0}Z_{29}^T - \frac{1}{32b_0}Z_{36}^T + \frac{1}{32b_0}Z_{38}^T + \frac{1}{8b_0}Z_{41}^T + \frac{1}{16b_0}Z_{42}^T$$

$$K_2^T = +\frac{1}{16b_0}Z_4^T + \frac{1}{8b_0}Z_{27}^T + \frac{1}{16b_0}Z_{36}^T - \frac{1}{16b_0}Z_{38}^T + \frac{1}{8b_0}Z_{40}^T + \frac{1}{8b_0}Z_{43}^T + \frac{1}{16b_0}Z_{57}^T + \frac{1}{4b_0}Z_{67}^T + \frac{1}{8b_0}Z_{68}^T$$

$$K_3^T = +\frac{1}{16b_0}Z_3^T - \frac{1}{8b_0}Z_{28}^T + \frac{1}{8b_0}Z_{29}^T - \frac{1}{8b_0}Z_{39}^T + \frac{1}{16b_0}Z_{57}^T + \frac{1}{4b_0}Z_{67}^T + \frac{1}{8b_0}Z_{68}^T$$

$$K_4^T = +\frac{1}{16b_0}Z_2^T + \frac{1}{8b_0}Z_{28}^T - \frac{1}{8b_0}Z_{29}^T + \frac{1}{16b_0}Z_{39}^T - \frac{1}{16b_0}Z_{40}^T - \frac{1}{16b_0}Z_{42}^T - \frac{1}{16b_0}Z_{57}^T - \frac{1}{4b_0}Z_{67}^T - \frac{1}{8b_0}Z_{68}^T$$

$$K_5^T = 0 \quad K_6^T = 0 \quad K_7^T = 0 \quad K_8^T = 0 \quad K_9^T = +\frac{1}{4b_0^2}Z_{14}^T + \frac{1}{4b_0^2}Z_{17}^T + \frac{1}{8b_0^2}Z_{18}^T + \frac{1}{8b_0^2}Z_{19}^T - \frac{1}{8b_0^2}Z_{31}^T$$

$$K_{10}^T = +\frac{1}{4b_0^2}Z_{21}^T - \frac{1}{4b_0^2}Z_{22}^T + \frac{1}{4b_0^2}Z_{23}^T - \frac{1}{8b_0^2}Z_{31}^T$$

$$K_{11}^T = +\frac{1}{4b_0^2}Z_{16}^T - \frac{1}{4b_0^2}Z_{18}^T + \frac{1}{2b_0^2}Z_{31}^T + \frac{1}{4b_0^2}Z_{58}^T + \frac{1}{4b_0^2}Z_{59}^T + \frac{1}{2b_0^2}Z_{71}^T + \frac{1}{2b_0^2}Z_{72}^T$$

$$K_{12}^T = +\frac{1}{4b_0^2}Z_{15}^T - \frac{1}{4b_0^2}Z_{19}^T - \frac{1}{4b_0^2}Z_{58}^T - \frac{1}{4b_0^2}Z_{59}^T - \frac{1}{2b_0^2}Z_{71}^T - \frac{1}{2b_0^2}Z_{72}^T$$

$$K_{13}^T = -\frac{1}{4b_0^2}Z_{20}^T + \frac{1}{4b_0^2}Z_{22}^T + \frac{1}{4b_0^2}Z_{31}^T \quad K_{14}^T = 0 \quad K_{15}^T = 0 \quad K_{16}^T = 0 \quad K_{17}^T = 0 \quad K_{18}^T = 0$$

$$K_{19}^T = 0 \quad K_{20}^T = 0 \quad K_{21}^T = 0 \quad K_{22}^T = 0 \quad K_{31}^T = +\frac{1}{4B_0b_0^2}Z_{64}^T + \frac{1}{4B_0b_0^2}Z_{65}^T$$

$$K_{32}^T = +\frac{1}{8b_0^2}Z_{31}^T + \frac{1}{4b_0^2}Z_{32}^T - \frac{1}{2b_0^2}Z_{46}^T - \frac{1}{8b_0^2}Z_{47}^T - \frac{1}{8b_0^2}Z_{48}^T + \frac{1}{2B_0b_0^2}Z_{77}^T \quad K_{33}^T = 0 \quad K_{34}^T = 0$$

$$K_{35}^T = -\frac{1}{8N_fb_0^2}Z_{31}^T - \frac{1}{4N_fb_0^2}Z_{32}^T + \frac{1}{2N_fb_0^2}Z_{46}^T + \frac{1}{8N_fb_0^2}Z_{47}^T + \frac{1}{8N_fb_0^2}Z_{48}^T \quad K_{36}^T = 0 \quad K_{37}^T = 0 \quad K_{38}^T = 0$$

$$K_{39}^T = +\frac{1}{16b_0}Z_{26}^T - \frac{1}{16b_0}Z_{28}^T + \frac{1}{16b_0}Z_{29}^T + \frac{1}{16B_0b_0}Z_{33}^T - \frac{1}{16B_0b_0}Z_{44}^T + \frac{1}{8B_0b_0}Z_{61}^T - \frac{1}{16b_0}Z_{66}^T + \frac{1}{8b_0}Z_{67}^T + \frac{1}{16b_0}Z_{68}^T$$

$$K_{40}^T = +\frac{1}{8b_0}Z_{26}^T - \frac{1}{8b_0}Z_{28}^T + \frac{1}{8b_0}Z_{29}^T + \frac{1}{16B_0b_0}Z_{34}^T - \frac{1}{8B_0b_0}Z_{44}^T - \frac{1}{8b_0}Z_{66}^T + \frac{1}{4b_0}Z_{67}^T + \frac{1}{8b_0}Z_{68}^T \quad K_{41}^T = 0 \quad K_{42}^T = 0$$

$$K_{43}^T = +\frac{1}{16b_0}Z_{10}^T - \frac{1}{16B_0b_0}Z_{45}^T + \frac{1}{16b_0}Z_{50}^T + \frac{1}{8b_0}Z_{51}^T - \frac{1}{8b_0}Z_{54}^T + \frac{1}{8B_0b_0}Z_{60}^T - \frac{1}{16B_0b_0}Z_{75}^T$$

$$K_{44}^T = +\frac{1}{16b_0}Z_{13}^T - \frac{1}{8b_0}Z_{45}^T - \frac{1}{8b_0}Z_{50}^T - \frac{1}{16B_0b_0}Z_{76}^T \quad K_{45}^T = -\frac{1}{8N_fb_0}Z_{10}^T - \frac{1}{16N_fb_0}Z_{13}^T - \frac{1}{4N_fb_0}Z_{51}^T + \frac{1}{4N_fb_0}Z_{54}^T$$

$$K_{46}^T = 0 \quad K_{47}^T = -\frac{1}{16b_0}Z_7^T - \frac{1}{16b_0}Z_9^T + \frac{1}{16b_0}Z_{49}^T + \frac{1}{8b_0}Z_{50}^T - \frac{1}{16b_0}Z_{52}^T - \frac{1}{16b_0}Z_{53}^T$$

$$K_{48}^T = -\frac{1}{16b_0}Z_6^T - \frac{1}{16b_0}Z_8^T - \frac{1}{16b_0}Z_{50}^T - \frac{1}{8b_0}Z_{51}^T + \frac{1}{16b_0}Z_{52}^T + \frac{1}{16b_0}Z_{53}^T$$

$$K_{49}^T = -\frac{1}{16b_0}Z_{11}^T - \frac{1}{16b_0}Z_{12}^T + \frac{1}{16b_0}Z_{49}^T - \frac{1}{16b_0}Z_{50}^T - \frac{1}{8b_0}Z_{51}^T \quad K_{50}^T = 0$$

$$K_{51}^T = -\frac{1}{4b_0^2}Z_{31}^T + \frac{1}{b_0^2}Z_{70}^T - \frac{1}{2b_0^2}Z_{72}^T + \frac{1}{b_0^2}Z_{73}^T + \frac{1}{2b_0^2}Z_{74}^T \quad K_{52}^T = +\frac{1}{b_0^2}Z_{31}^T - \frac{1}{b_0^2}Z_{69}^T + \frac{1}{b_0^2}Z_{71}^T + \frac{1}{b_0^2}Z_{72}^T - \frac{1}{b_0^2}Z_{74}^T$$

$$K_{54}^T = 0 \quad K_{55}^T = 0 \quad K_{57}^T = -\frac{1}{32b_0}Z_{36}^T + \frac{1}{32b_0}Z_{38}^T + \frac{1}{8b_0}Z_{41}^T \quad K_{58}^T = +\frac{1}{16b_0}Z_{36}^T - \frac{1}{16b_0}Z_{38}^T + \frac{1}{8b_0}Z_{43}^T$$

$$K^T_{59}=-\frac{1}{8b_0}Z^T_{27}-\frac{1}{8b_0}Z^T_{40}-\frac{1}{8b_0}Z^T_{57}-\frac{1}{2b_0}Z^T_{67}-\frac{1}{4b_0}Z^T_{68}$$
$$K^T_{60}=-\frac{1}{8b_0}Z^T_{28}+\frac{1}{8b_0}Z^T_{29}-\frac{1}{8b_0}Z^T_{39}+\frac{1}{8b_0}Z^T_{57}+\frac{1}{2b_0}Z^T_{67}+\frac{1}{4b_0}Z^T_{68}$$
$$K^T_{61}=+\frac{1}{8b_0}Z^T_{27}+\frac{1}{8b_0}Z^T_{28}-\frac{1}{8b_0}Z^T_{29}-\frac{1}{8b_0}Z^T_{42}$$
$$K^T_{62}=0\quad K^T_{63}=0\quad K^T_{64}=0\quad K^T_{65}=0\quad K^T_{66}=0\quad K^T_{67}=0\quad K^T_{68}=0$$
$$K^T_{69}=+\frac{1}{16b_0}Z^T_{5}-\frac{1}{16b_0}Z^T_{49}+\frac{1}{16b_0}Z^T_{52}+\frac{1}{16b_0}Z^T_{53}$$
$$K^T_{70}=+\frac{1}{16b_0}Z^T_{6}-\frac{1}{16b_0}Z^T_{8}+\frac{1}{16b_0}Z^T_{50}+\frac{1}{8b_0}Z^T_{51}-\frac{3}{16b_0}Z^T_{52}-\frac{1}{16b_0}Z^T_{53}$$
$$K^T_{71}=+\frac{1}{16b_0}Z^T_{7}-\frac{1}{16b_0}Z^T_{9}+\frac{1}{16b_0}Z^T_{49}+\frac{1}{16b_0}Z^T_{52}-\frac{1}{16b_0}Z^T_{53}$$
$$K^T_{72}=-\frac{1}{16b_0}Z^T_{11}+\frac{1}{16b_0}Z^T_{12}-\frac{1}{16b_0}Z^T_{49}-\frac{1}{16b_0}Z^T_{50}-\frac{1}{8b_0}Z^T_{51}+\frac{1}{8b_0}Z^T_{52}+\frac{1}{8b_0}Z^T_{53}\quad K^T_{73}=0$$
$$K^T_{74}=-\frac{1}{16B_0b_0}Z^T_{55}+\frac{1}{8B_0b_0}Z^T_{61}\quad K^T_{75}=+\frac{1}{16b_0}Z^T_{52}+\frac{1}{16b_0}Z^T_{53}-\frac{1}{8b_0}Z^T_{54}+\frac{1}{16B_0b_0}Z^T_{55}+\frac{1}{8B_0b_0}Z^T_{60}$$
$$K^T_{76}=+\frac{1}{16b_0}Z^T_{26}+\frac{1}{16b_0}Z^T_{27}-\frac{1}{16b_0}Z^T_{35}+\frac{1}{16B_0b_0}Z^T_{55}-\frac{1}{16B_0b_0}Z^T_{56}-\frac{1}{16b_0}Z^T_{66}+\frac{1}{8b_0}Z^T_{67}+\frac{1}{16b_0}Z^T_{68}$$
$$K^T_{77}=0\quad K^T_{78}=0\quad K^T_{79}=0\quad K^T_{80}=-\frac{1}{8N_fb_0}Z^T_{52}-\frac{1}{8N_fb_0}Z^T_{53}+\frac{1}{4N_fb_0}Z^T_{54}\quad K^T_{82}=0$$
$$K^T_{84}=-\frac{1}{8b_0}Z^T_{27}-\frac{1}{8b_0}Z^T_{28}+\frac{1}{8b_0}Z^T_{29}+\frac{1}{8b_0}Z^T_{36}-\frac{1}{8b_0}Z^T_{37}\quad K^T_{85}=+\frac{1}{4b_0}Z^T_{57}+\frac{1}{b_0}Z^T_{67}+\frac{1}{2b_0}Z^T_{68}$$
$$K^T_{86}=-\frac{1}{4b_0}Z^T_{52}\quad K^T_{87}=0\quad K^T_{88}=+\frac{1}{b_0^2}Z^T_{62}+\frac{1}{b_0^2}Z^T_{63}\quad K^T_{90}=-\frac{1}{2b_0^2}Z^T_{31}-\frac{1}{2b_0^2}Z^T_{58}-\frac{1}{2b_0^2}Z^T_{59}-\frac{1}{b_0^2}Z^T_{71}-\frac{1}{b_0^2}Z^T_{72}$$
$$K^T_{92}=0\quad K^T_{94}=-\frac{1}{2b_0}Z^T_{66}+\frac{1}{b_0}Z^T_{67}+\frac{1}{2b_0}Z^T_{68}\quad K^T_{95}=+\frac{1}{4b_0}Z^T_{28}$$
$$K^T_{96}=+\frac{1}{8b_0}Z^T_{24}+\frac{1}{8b_0}Z^T_{25}-\frac{1}{8b_0}Z^T_{27}-\frac{1}{16b_0}Z^T_{36}+\frac{1}{16b_0}Z^T_{38}+\frac{1}{8b_0}Z^T_{66}-\frac{1}{4b_0}Z^T_{67}-\frac{1}{8b_0}Z^T_{68}$$
$$K^T_{97}=+\frac{1}{8b_0}Z^T_{24}-\frac{1}{8b_0}Z^T_{25}-\frac{1}{8b_0}Z^T_{27}+\frac{1}{16b_0}Z^T_{36}-\frac{1}{16b_0}Z^T_{38}+\frac{1}{8b_0}Z^T_{66}-\frac{1}{4b_0}Z^T_{67}-\frac{1}{8b_0}Z^T_{68}$$
$$K^T_{98}=+\frac{1}{4b_0}Z^T_{49}-\frac{1}{4b_0}Z^T_{52}-\frac{1}{4b_0}Z^T_{53}\quad K^T_{99}=+\frac{1}{4b_0}Z^T_{51}-\frac{1}{8b_0}Z^T_{52}-\frac{1}{8b_0}Z^T_{53}$$
$$K^T_{100}=-\frac{1}{4b_0}Z^T_{50}\quad K^T_{101}=0\quad K^T_{102}=0\quad K^T_{103}=0\quad K^T_{105}=-\frac{1}{2b_0^2}Z^T_{30}+\frac{1}{2b_0^2}Z^T_{31}\quad K^T_{106}=0$$
$$K^T_{107}=0\quad K^T_{108}=0\quad K^T_{112}=-\frac{1}{8B_0b_0}Z^T_{45}$$
$$K^T_{113}=+\frac{1}{8b_0}Z^T_{26}-\frac{1}{8b_0}Z^T_{28}+\frac{1}{8b_0}Z^T_{29}-\frac{1}{8B_0b_0}Z^T_{44}-\frac{1}{8b_0}Z^T_{66}+\frac{1}{4b_0}Z^T_{67}+\frac{1}{8b_0}Z^T_{68}$$
$$K^T_{114}=+\frac{1}{8b_0}Z^T_{27}+\frac{1}{8b_0}Z^T_{28}-\frac{1}{8b_0}Z^T_{29}+\frac{1}{8b_0}Z^T_{36}+\frac{1}{8b_0}Z^T_{37}\quad K^T_{117}=0\quad K^T_{118}=-\frac{2}{b_0^2}Z^T_{31}$$
$$K^T_{119}=+\frac{1}{b_0^2}Z^T_{31}-\frac{1}{b_0^2}Z^T_{47}+\frac{1}{b_0^2}Z^T_{48}\quad K^T_{120}=+\frac{1}{4B_0b_0}Z^T_{55}\tag{D1}$$

APPENDIX E: $\tilde{K}^W_{t,n}$ COEFFICIENTS

$$\begin{aligned}\tilde{K}^{T,W}_1 = N_C\int\frac{d^4k}{(2\pi)^4}\bigg[&-\frac{1}{9}\Sigma'_kX^2+\frac{1}{8}\Sigma_k\Sigma'^2_kX^2+\frac{1}{36}\Sigma_kX^3+\frac{1}{9}\Sigma^2_k\Sigma'_kX^3\\&-\frac{19}{24}\Sigma^3_k\Sigma'^2_kX^3-\frac{5}{6}\Sigma^3_kX^4+\frac{4}{3}\Sigma^5_k\Sigma'^2_kX^4+\frac{3}{2}\Sigma^5_kX^5-\frac{2}{3}\Sigma^7_k\Sigma'^2_kX^5\bigg],\\
\tilde{K}^{T,W}_2 = N_C\int\frac{d^4k}{(2\pi)^4}\bigg[&-\frac{1}{18}\Sigma'^3_kX-\frac{43}{72}\Sigma'_kX^2-\frac{1}{6}\Sigma_k\Sigma'^2_kX^2+\frac{1}{9}\Sigma^2_k\Sigma'^3_kX^2+\frac{55}{72}\Sigma_kX^3\\&+\frac{37}{72}\Sigma^2_k\Sigma'_kX^3+3\Sigma^3_k\Sigma'^2_kX^3-\frac{1}{18}\Sigma^4_k\Sigma'^3_kX^3+\frac{5}{24}\Sigma^3_kX^4-\frac{11}{2}\Sigma^5_k\Sigma'^2_kX^4+\frac{8}{3}\Sigma^7_k\Sigma'^2_kX^5\bigg],\\
\tilde{K}^{T,W}_3 = N_C\int\frac{d^4k}{(2\pi)^4}\bigg[&-\frac{1}{18}\Sigma'^3_kX-\frac{11}{72}\Sigma'_kX^2+\frac{1}{12}\Sigma_k\Sigma'^2_kX^2+\frac{1}{9}\Sigma^2_k\Sigma'^3_kX^2+\frac{11}{72}\Sigma_kX^3+\frac{5}{72}\Sigma^2_k\Sigma'_kX^3+\frac{17}{12}\Sigma^3_k\Sigma'^2_kX^3\\&-\frac{1}{18}\Sigma^4_k\Sigma'^3_kX^3-\frac{11}{24}\Sigma^3_kX^4-\frac{17}{6}\Sigma^5_k\Sigma'^2_kX^4+\Sigma^5_kX^5+\frac{4}{3}\Sigma^7_k\Sigma'^2_kX^5\bigg],\\
\tilde{K}^{T,W}_4 = N_C\int\frac{d^4k}{(2\pi)^4}\bigg[&+\frac{1}{18}\Sigma'^3_kX+\frac{5}{12}\Sigma'_kX^2-\frac{1}{12}\Sigma_k\Sigma'^2_kX^2-\frac{1}{9}\Sigma^2_k\Sigma'^3_kX^2-\frac{5}{12}\Sigma_kX^3-\frac{1}{3}\Sigma^2_k\Sigma'_kX^3-\frac{17}{12}\Sigma^3_k\Sigma'^2_kX^3\\&+\frac{1}{18}\Sigma^4_k\Sigma'^3_kX^3+\frac{1}{12}\Sigma^3_kX^4+\frac{17}{6}\Sigma^5_k\Sigma'^2_kX^4-\frac{2}{3}\Sigma^5_kX^5-\frac{4}{3}\Sigma^7_k\Sigma'^2_kX^5\bigg],\end{aligned}$$

$$\tilde{K}^{T,W}_5 = N_C\int\frac{d^4k}{(2\pi)^4}\bigg[-\frac{2}{3}\Sigma^2_k\Sigma'^2_kX^2-\frac{11}{6}\Sigma^2_kX^3+\frac{4}{3}\Sigma^4_k\Sigma'^2_kX^3+\frac{17}{6}\Sigma^4_kX^4-\frac{2}{3}\Sigma^6_k\Sigma'^2_kX^4\bigg],$$

$$\tilde{K}^{T,W}_6 = N_C\int\frac{d^4k}{(2\pi)^4}\bigg[-\frac{2}{3}\Sigma^2_k\Sigma'^2_kX^2-\frac{1}{3}\Sigma^2_kX^3+\frac{4}{3}\Sigma^4_k\Sigma'^2_kX^3+\frac{5}{6}\Sigma^4_kX^4-\frac{2}{3}\Sigma^6_k\Sigma'^2_kX^4\bigg],$$

$$\tilde{K}^{T,W}_7 = N_C\int\frac{d^4k}{(2\pi)^4}\bigg[+\frac{4}{3}\Sigma^2_k\Sigma'^2_kX^2-\frac{4}{3}\Sigma^2_kX^3-\frac{8}{3}\Sigma^4_k\Sigma'^2_kX^3+\frac{13}{3}\Sigma^4_kX^4+\frac{4}{3}\Sigma^6_k\Sigma'^2_kX^4\bigg],$$

$$\tilde{K}^{T,W}_8 = N_C\int\frac{d^4k}{(2\pi)^4}\bigg[+\frac{4}{3}\Sigma^2_k\Sigma'^2_kX^2+\frac{2}{3}\Sigma^2_kX^3-\frac{8}{3}\Sigma^4_k\Sigma'^2_kX^3-\frac{11}{3}\Sigma^4_kX^4+\frac{4}{3}\Sigma^6_k\Sigma'^2_kX^4\bigg],$$

$$\tilde{K}^{T,W}_9 = N_C\int\frac{d^4k}{(2\pi)^4}\bigg[+\frac{4}{3}\Sigma^2_k\Sigma'^2_kX^2+\frac{5}{3}\Sigma^2_kX^3-\frac{8}{3}\Sigma^4_k\Sigma'^2_kX^3-\frac{8}{3}\Sigma^4_kX^4+\frac{4}{3}\Sigma^6_k\Sigma'^2_kX^4\bigg],$$

$$\tilde{K}^{T,W}_{10} = N_C\int\frac{d^4k}{(2\pi)^4}[-\Sigma_kX^2+2\Sigma^3_kX^3],\quad \tilde{K}^{T,W}_{11} = N_C\int\frac{d^4k}{(2\pi)^4}\bigg[+\frac{1}{2}\Sigma_kX^2\bigg],\quad \tilde{K}^{T,W}_{12} = N_C\int\frac{d^4k}{(2\pi)^4}\bigg[+\frac{3}{8}\Sigma^2_kX^3\bigg],$$

$$\tilde{K}^{T,W}_{13} = N_C\int\frac{d^4k}{(2\pi)^4}\bigg[-\frac{3}{8}\Sigma^2_kX^3+\Sigma^4_kX^4\bigg],\quad \tilde{K}^{T,W}_{14} = N_C\int\frac{d^4k}{(2\pi)^4}\bigg[-\frac{1}{4}\Sigma^2_kX^3+\Sigma^4_kX^4\bigg],$$

$$\tilde{K}^{T,W}_{15} = N_C\int\frac{d^4k}{(2\pi)^4}\bigg[+\frac{3}{8}\Sigma^2_kX^3\bigg],\quad \tilde{K}^{T,W}_{16} = N_C\int\frac{d^4k}{(2\pi)^4}\bigg[-\frac{1}{4}\Sigma^2_kX^3\bigg],$$

$$\tilde{K}^{T,W}_{17} = N_C\int\frac{d^4k}{(2\pi)^4}\bigg[-\frac{7}{72}\Sigma'_kX^2+\frac{1}{3}\Sigma_k\Sigma'^2_kX^2+\frac{7}{72}\Sigma_kX^3-\frac{11}{72}\Sigma^2_k\Sigma'_kX^3-\frac{1}{6}\Sigma^3_k\Sigma'^2_kX^3+\frac{17}{24}\Sigma^3_kX^4-\frac{1}{6}\Sigma^5_k\Sigma'^2_kX^4\bigg],$$

$$\tilde{K}^{T,W}_{18} = N_C\int\frac{d^4k}{(2\pi)^4}\bigg[+\frac{25}{144}\Sigma'_kX^2-\frac{1}{2}\Sigma_k\Sigma'^2_kX^2-\frac{25}{144}\Sigma_kX^3-\frac{7}{144}\Sigma^2_k\Sigma'_kX^3+\frac{3}{4}\Sigma^3_k\Sigma'^2_kX^3-\frac{23}{48}\Sigma^3_kX^4-\frac{1}{4}\Sigma^5_k\Sigma'^2_kX^4\bigg],$$

$$\tilde{K}^{T,W}_{19} = N_C\int\frac{d^4k}{(2\pi)^4}\bigg[+\frac{1}{36}\Sigma'_kX^2-\frac{1}{3}\Sigma_k\Sigma'^2_kX^2-\frac{1}{36}\Sigma_kX^3+\frac{7}{72}\Sigma^2_k\Sigma'_kX^3+\frac{5}{12}\Sigma^3_k\Sigma'^2_kX^3-\frac{5}{12}\Sigma^3_kX^4-\frac{1}{12}\Sigma^5_k\Sigma'^2_kX^4\bigg],$$

$$\tilde{K}^{T,W}_{20} = N_C\int\frac{d^4k}{(2\pi)^4}\bigg[-\frac{1}{8}\Sigma'_kX^2+\frac{5}{12}\Sigma_k\Sigma'^2_kX^2+\frac{1}{8}\Sigma_kX^3+\frac{1}{8}\Sigma^2_k\Sigma'_kX^3-\frac{5}{6}\Sigma^3_k\Sigma'^2_kX^3+\frac{1}{8}\Sigma^3_kX^4+\frac{5}{12}\Sigma^5_k\Sigma'^2_kX^4\bigg],$$

$$\tilde{K}_{21}^{T,W}=N_C\int\frac{d^4k}{(2\pi)^4}\bigg[-\frac{17}{72}\Sigma'_kX^2+\frac{5}{12}\Sigma_k\Sigma'^2_kX^2+\frac{17}{72}\Sigma_kX^3+\frac{1}{9}\Sigma^2_k\Sigma'_kX^3-\frac{7}{12}\Sigma^3_k\Sigma'^2_kX^3+\frac{7}{24}\Sigma^3_kX^4+\frac{1}{6}\Sigma^5_k\Sigma'^2_kX^4\bigg],$$

$$\tilde{K}_{22}^{T,W}=N_C\int\frac{d^4k}{(2\pi)^4}[-\Sigma'^2_kX+5\Sigma^2_k\Sigma'^2_kX^2-8\Sigma^4_k\Sigma'^2_kX^3-\Sigma^4_kX^4+4\Sigma^6_k\Sigma'^2_kX^4],\qquad \tilde{K}_{23}^{T,W}=0,$$

$$\tilde{K}_{24}^{T,W}=N_C\int\frac{d^4k}{(2\pi)^4}\bigg[-\frac{7}{144}\Sigma'_kX^2+\frac{3}{8}\Sigma_k\Sigma'^2_kX^2-\frac{29}{144}\Sigma_kX^3+\frac{7}{144}\Sigma^2_k\Sigma'_kX^3-\frac{47}{24}\Sigma^3_k\Sigma'^2_kX^3$$
$$+\frac{3}{16}\Sigma^3_kX^4+\frac{35}{12}\Sigma^5_k\Sigma'^2_kX^4+\frac{1}{3}\Sigma^5_kX^5-\frac{4}{3}\Sigma^7_k\Sigma'^2_kX^5\bigg],$$

$$\tilde{K}_{25}^{T,W}=N_C\int\frac{d^4k}{(2\pi)^4}\bigg[-\frac{1}{24}\Sigma'_kX^2-\frac{3}{4}\Sigma_k\Sigma'^2_kX^2+\frac{1}{8}\Sigma_kX^3+\frac{1}{24}\Sigma^2_k\Sigma'_kX^3+\frac{47}{12}\Sigma^3_k\Sigma'^2_kX^3+\frac{11}{24}\Sigma^3_kX^4$$
$$-\frac{35}{6}\Sigma^5_k\Sigma'^2_kX^4-\frac{2}{3}\Sigma^5_kX^5+\frac{8}{3}\Sigma^7_k\Sigma'^2_kX^5\bigg],$$

$$\tilde{K}_{26}^{T,W}=N_C\int\frac{d^4k}{(2\pi)^4}\bigg[-\frac{1}{9}\Sigma'^3_kX+\frac{49}{72}\Sigma'_kX^2-\frac{5}{6}\Sigma_k\Sigma'^2_kX^2+\frac{2}{9}\Sigma^2_k\Sigma'^3_kX^2-\frac{43}{72}\Sigma_kX^3-\frac{43}{72}\Sigma^2_k\Sigma'_kX^3$$
$$+\frac{5}{3}\Sigma^3_k\Sigma'^2_kX^3-\frac{1}{9}\Sigma^4_k\Sigma'^3_kX^3-\frac{5}{24}\Sigma^3_kX^4-\frac{5}{6}\Sigma^5_k\Sigma'^2_kX^4\bigg],$$

$$\tilde{K}_{27}^{T,W}=N_C\int\frac{d^4k}{(2\pi)^4}\bigg[+\frac{1}{9}\Sigma'^3_kX-\frac{5}{18}\Sigma'_kX^2+\frac{1}{3}\Sigma_k\Sigma'^2_kX^2-\frac{2}{9}\Sigma^2_k\Sigma'^3_kX^2+\frac{7}{36}\Sigma_kX^3+\frac{7}{36}\Sigma^2_k\Sigma'_kX^3+\frac{2}{3}\Sigma^3_k\Sigma'^2_kX^3$$
$$+\frac{1}{9}\Sigma^4_k\Sigma'^3_kX^3+\frac{7}{12}\Sigma^3_kX^4-\frac{7}{3}\Sigma^5_k\Sigma'^2_kX^4-\frac{1}{3}\Sigma^5_kX^5+\frac{4}{3}\Sigma^7_k\Sigma'^2_kX^5\bigg],$$

$$\tilde{K}_{28}^{T,W}=N_C\int\frac{d^4k}{(2\pi)^4}\bigg[+\frac{1}{8}\Sigma'_kX^2+\frac{1}{2}\Sigma_k\Sigma'^2_kX^2+\frac{5}{24}\Sigma_kX^3-\frac{1}{8}\Sigma^2_k\Sigma'_kX^3-\frac{7}{3}\Sigma^3_k\Sigma'^2_kX^3-\frac{19}{24}\Sigma^3_kX^4$$
$$+\frac{19}{6}\Sigma^5_k\Sigma'^2_kX^4+\frac{1}{3}\Sigma^5_kX^5-\frac{4}{3}\Sigma^7_k\Sigma'^2_kX^5\bigg],$$

$$\tilde{K}_{29}^{T,W}=N_C\int\frac{d^4k}{(2\pi)^4}\bigg[+\frac{1}{72}\Sigma'_kX^2-\frac{11}{36}\Sigma_kX^3+\frac{1}{9}\Sigma^2_k\Sigma'_kX^3-\frac{1}{4}\Sigma^3_k\Sigma'^2_kX^3+\frac{1}{6}\Sigma^3_kX^4+\frac{1}{4}\Sigma^5_k\Sigma'^2_kX^4\bigg],$$

$$\tilde{K}_{30}^{T,W}=N_C\int\frac{d^4k}{(2\pi)^4}\bigg[+\frac{37}{72}\Sigma'_kX^2-\frac{3}{4}\Sigma_k\Sigma'^2_kX^2-\frac{13}{72}\Sigma_kX^3-\frac{55}{72}\Sigma^2_k\Sigma'_kX^3+2\Sigma^3_k\Sigma'^2_kX^3+\frac{1}{24}\Sigma^3_kX^4-\frac{5}{4}\Sigma^5_k\Sigma'^2_kX^4\bigg],$$

$$\tilde{K}_{31}^{T,W}=N_C\int\frac{d^4k}{(2\pi)^4}\bigg[-\frac{3}{8}\Sigma'_kX^2+\frac{2}{3}\Sigma_k\Sigma'^2_kX^2+\frac{5}{24}\Sigma_kX^3+\frac{1}{4}\Sigma^2_k\Sigma'_kX^3-\frac{13}{12}\Sigma^3_k\Sigma'^2_kX^3+\frac{3}{8}\Sigma^3_kX^4+\frac{5}{12}\Sigma^5_k\Sigma'^2_kX^4\bigg],$$

$$\tilde{K}_{32}^{T,W}=N_C\int\frac{d^4k}{(2\pi)^4}\bigg[-\frac{5}{36}\Sigma'_kX^2+\frac{5}{12}\Sigma_k\Sigma'^2_kX^2-\frac{13}{36}\Sigma_kX^3+\frac{19}{72}\Sigma^2_k\Sigma'_kX^3-\frac{13}{12}\Sigma^3_k\Sigma'^2_kX^3+\frac{7}{12}\Sigma^3_kX^4+\frac{2}{3}\Sigma^5_k\Sigma'^2_kX^4\bigg],$$

$$\tilde{K}_{33}^{T,W}=N_C\int\frac{d^4k}{(2\pi)^4}\bigg[+\frac{1}{8}\Sigma^2_kX^3\bigg],\quad \tilde{K}_{34}^{T,W}=N_C\int\frac{d^4k}{(2\pi)^4}\bigg[+\frac{1}{8}\Sigma^2_kX^3\bigg],\quad \tilde{K}_{35}^{T,W}=N_C\int\frac{d^4k}{(2\pi)^4}\bigg[+\frac{1}{8}\Sigma^2_kX^3\bigg],$$

$$\tilde{K}_{36}^{T,W}=N_C\int\frac{d^4k}{(2\pi)^4}\bigg[-\frac{1}{36}\Sigma'_kX^2+\frac{1}{12}\Sigma_k\Sigma'^2_kX^2+\frac{11}{72}\Sigma_kX^3-\frac{7}{72}\Sigma^2_k\Sigma'_kX^3+\frac{1}{12}\Sigma^3_k\Sigma'^2_kX^3+\frac{1}{24}\Sigma^3_kX^4-\frac{1}{6}\Sigma^5_k\Sigma'^2_kX^4\bigg],$$

$$\tilde{K}_{37}^{T,W}=N_C\int\frac{d^4k}{(2\pi)^4}\bigg[+\frac{4}{3}\Sigma^2_k\Sigma'^2_kX^2+\frac{5}{6}\Sigma^2_kX^3-\frac{8}{3}\Sigma^4_k\Sigma'^2_kX^3-\frac{1}{3}\Sigma^4_kX^4+\frac{4}{3}\Sigma^6_k\Sigma'^2_kX^4\bigg],$$

$$\tilde{K}_{38}^{T,W}=N_C\int\frac{d^4k}{(2\pi)^4}\bigg[+\frac{1}{3}\Sigma'_kX^2-\frac{5}{2}\Sigma_k\Sigma'^2_kX^2+\frac{1}{6}\Sigma_kX^3-\frac{1}{3}\Sigma^2_k\Sigma'_kX^3+\frac{25}{2}\Sigma^3_k\Sigma'^2_kX^3+\Sigma^3_kX^4$$
$$-18\Sigma^5_k\Sigma'^2_kX^4-2\Sigma^5_kX^5+8\Sigma^7_k\Sigma'^2_kX^5\bigg],$$

$$\tilde{K}_{39}^{T,W} = N_C \int \frac{d^4k}{(2\pi)^4} \Big[+\frac{2}{3}\Sigma_k'^3 X - \frac{1}{2}\Sigma_k' X^2 + \Sigma_k \Sigma_k'^2 X^2 - \frac{4}{3}\Sigma_k^2 \Sigma_k'^3 X^2 + \frac{1}{2}\Sigma_k X^3 + \frac{1}{2}\Sigma_k^2 \Sigma_k' X^3 - 3\Sigma_k^3 \Sigma_k'^2 X^3 + \frac{2}{3}\Sigma_k^4 \Sigma_k'^3 X^3 - \frac{1}{2}\Sigma_k^3 X^4 + 2\Sigma_k^5 \Sigma_k'^2 X^4 \Big],$$

$$\tilde{K}_{40}^{T,W} = N_C \int \frac{d^4k}{(2\pi)^4} \Big[+\frac{23}{36}\Sigma_k' X^2 - \frac{2}{3}\Sigma_k \Sigma_k'^2 X^2 - \frac{5}{36}\Sigma_k X^3 - \frac{41}{36}\Sigma_k^2 \Sigma_k' X^3 + \frac{7}{3}\Sigma_k^3 \Sigma_k'^2 X^3 + \frac{5}{12}\Sigma_k^3 X^4 - \frac{5}{3}\Sigma_k^5 \Sigma_k'^2 X^4 \Big],$$

$$\tilde{K}_{41}^{T,W} = N_C \int \frac{d^4k}{(2\pi)^4} \Big[+\frac{20}{3}\Sigma_k^3 X^3 \Big], \quad \tilde{K}_{42}^{T,W} = N_C \int \frac{d^4k}{(2\pi)^4} [-2\Sigma_k^2 X^3],$$

$$\tilde{K}_{43}^{T,W} = N_C \int \frac{d^4k}{(2\pi)^4} \Big[+\frac{8}{9}\Sigma_k'^3 X + \frac{8}{9}\Sigma_k' X^2 - \frac{2}{3}\Sigma_k \Sigma_k'^2 X^2 - \frac{16}{9}\Sigma_k^2 \Sigma_k'^3 X^2 - \frac{5}{9}\Sigma_k X^3 - \frac{5}{9}\Sigma_k^2 \Sigma_k' X^3 - \frac{2}{3}\Sigma_k^3 \Sigma_k'^2 X^3 + \frac{8}{9}\Sigma_k^4 \Sigma_k'^3 X^3 - \frac{1}{3}\Sigma_k^3 X^4 + \frac{4}{3}\Sigma_k^5 \Sigma_k'^2 X^4 \Big],$$

$$\tilde{K}_{44}^{T,W} = N_C \int \frac{d^4k}{(2\pi)^4} \Big[+\frac{2}{9}\Sigma_k'^3 X + \frac{7}{24}\Sigma_k' X^2 - \frac{7}{6}\Sigma_k \Sigma_k'^2 X^2 - \frac{4}{9}\Sigma_k^2 \Sigma_k'^3 X^2 - \frac{7}{24}\Sigma_k X^3 - \frac{5}{24}\Sigma_k^2 \Sigma_k' X^3 + 4\Sigma_k^3 \Sigma_k'^2 X^3 + \frac{2}{9}\Sigma_k^4 \Sigma_k'^3 X^3 + \frac{5}{24}\Sigma_k^3 X^4 - \frac{11}{2}\Sigma_k^5 \Sigma_k'^2 X^4 - \frac{2}{3}\Sigma_k^5 X^5 + \frac{8}{3}\Sigma_k^7 \Sigma_k'^2 X^5 \Big],$$

$$\tilde{K}_{45}^{T,W} = N_C \int \frac{d^4k}{(2\pi)^4} \Big[+\frac{5}{24}\Sigma_k' X^2 + \frac{1}{12}\Sigma_k \Sigma_k'^2 X^2 - \frac{3}{8}\Sigma_k X^3 - \frac{5}{24}\Sigma_k^2 \Sigma_k' X^3 - \frac{5}{4}\Sigma_k^3 \Sigma_k'^2 X^3 - \frac{1}{24}\Sigma_k^3 X^4 + \frac{5}{2}\Sigma_k^5 \Sigma_k'^2 X^4 + \frac{1}{3}\Sigma_k^5 X^5 - \frac{4}{3}\Sigma_k^7 \Sigma_k'^2 X^5 \Big],$$

$$\tilde{K}_{46}^{T,W} = N_C \int \frac{d^4k}{(2\pi)^4} \Big[+\frac{37}{72}\Sigma_k' X^2 - \frac{17}{12}\Sigma_k \Sigma_k'^2 X^2 - \frac{25}{72}\Sigma_k X^3 - \frac{37}{72}\Sigma_k^2 \Sigma_k' X^3 + \frac{79}{12}\Sigma_k^3 \Sigma_k'^2 X^3 + \frac{13}{24}\Sigma_k^3 X^4 - \frac{55}{6}\Sigma_k^5 \Sigma_k'^2 X^4 - \Sigma_k^5 X^5 + 4\Sigma_k^7 \Sigma_k'^2 X^5 \Big],$$

$$\tilde{K}_{47}^{T,W} = N_C \int \frac{d^4k}{(2\pi)^4} \Big[+\frac{2}{9}\Sigma_k'^3 X + \frac{7}{18}\Sigma_k' X^2 + \frac{1}{6}\Sigma_k \Sigma_k'^2 X^2 - \frac{4}{9}\Sigma_k^2 \Sigma_k'^3 X^2 - \frac{17}{36}\Sigma_k X^3 - \frac{11}{36}\Sigma_k^2 \Sigma_k' X^3 - \frac{13}{6}\Sigma_k^3 \Sigma_k'^2 X^3 + \frac{2}{9}\Sigma_k^4 \Sigma_k'^3 X^3 - \frac{1}{4}\Sigma_k^3 X^4 + \frac{10}{3}\Sigma_k^5 \Sigma_k'^2 X^4 + \frac{1}{3}\Sigma_k^5 X^5 - \frac{4}{3}\Sigma_k^7 \Sigma_k'^2 X^5 \Big],$$

$$\tilde{K}_{48}^{T,W} = N_C \int \frac{d^4k}{(2\pi)^4} \Big[+\frac{2}{9}\Sigma_k'^3 X + \frac{11}{72}\Sigma_k' X^2 + \frac{13}{6}\Sigma_k \Sigma_k'^2 X^2 - \frac{4}{9}\Sigma_k^2 \Sigma_k'^3 X^2 - \frac{47}{72}\Sigma_k X^3 - \frac{5}{72}\Sigma_k^2 \Sigma_k' X^3 - \frac{37}{3}\Sigma_k^3 \Sigma_k'^2 X^3 + \frac{2}{9}\Sigma_k^4 \Sigma_k'^3 X^3 - \frac{25}{24}\Sigma_k^3 X^4 + \frac{109}{6}\Sigma_k^5 \Sigma_k'^2 X^4 + 2\Sigma_k^5 X^5 - 8\Sigma_k^7 \Sigma_k'^2 X^5 \Big],$$

$$\tilde{K}_{49}^{T,W} = N_C \int \frac{d^4k}{(2\pi)^4} \Big[+\frac{1}{12}\Sigma_k' X^2 + \frac{1}{4}\Sigma_k X^3 - \frac{7}{12}\Sigma_k^2 \Sigma_k' X^3 + \Sigma_k^3 \Sigma_k'^2 X^3 + \frac{1}{4}\Sigma_k^3 X^4 - \Sigma_k^5 \Sigma_k'^2 X^4 \Big],$$

$$\tilde{K}_{50}^{T,W} = N_C \int \frac{d^4k}{(2\pi)^4} \Big[+\frac{5}{12}\Sigma_k' X^2 - \frac{1}{2}\Sigma_k \Sigma_k'^2 X^2 - \frac{1}{4}\Sigma_k X^3 - \frac{2}{3}\Sigma_k^2 \Sigma_k' X^3 + \frac{3}{2}\Sigma_k^3 \Sigma_k'^2 X^3 + \frac{1}{4}\Sigma_k^3 X^4 - \Sigma_k^5 \Sigma_k'^2 X^4 \Big],$$

$$\tilde{K}_{51}^{T,W} = N_C \int \frac{d^4k}{(2\pi)^4} \Big[+\frac{1}{36}\Sigma_k' X^2 + \frac{2}{3}\Sigma_k \Sigma_k'^2 X^2 + \frac{23}{36}\Sigma_k X^3 - \frac{19}{36}\Sigma_k^2 \Sigma_k' X^3 - \frac{1}{3}\Sigma_k^3 \Sigma_k'^2 X^3 + \frac{1}{12}\Sigma_k^3 X^4 - \frac{1}{3}\Sigma_k^5 \Sigma_k'^2 X^4 \Big],$$

$$\tilde{K}_{52}^{T,W} = N_C \int \frac{d^4k}{(2\pi)^4} \Big[+\frac{8}{3}\Sigma_k^2 \Sigma_k'^2 X^2 + \frac{8}{3}\Sigma_k^2 X^3 - \frac{16}{3}\Sigma_k^4 \Sigma_k'^2 X^3 - \frac{2}{3}\Sigma_k^4 X^4 + \frac{8}{3}\Sigma_k^6 \Sigma_k'^2 X^4 \Big],$$

$$\tilde{K}_{53}^{T,W} = N_C \int \frac{d^4k}{(2\pi)^4} \Big[-\frac{8}{3}\Sigma_k^2 \Sigma_k'^2 X^2 - \frac{2}{3}\Sigma_k^2 X^3 + \frac{16}{3}\Sigma_k^4 \Sigma_k'^2 X^3 + \frac{2}{3}\Sigma_k^4 X^4 - \frac{8}{3}\Sigma_k^6 \Sigma_k'^2 X^4 \Big],$$

$$\tilde{K}_{54}^{T,W}=0,\quad \tilde{K}_{55}^{T,W}=N_C\int\frac{d^4k}{(2\pi)^4}\bigg[+\frac{1}{2}\Sigma_k^2X^3\bigg],$$

$$\tilde{K}_{56}^{T,W}=N_C\int\frac{d^4k}{(2\pi)^4}\bigg[-\frac{1}{18}\Sigma_k'X^2-\frac{1}{2}\Sigma_k\Sigma_k'^2X^2+\frac{2}{9}\Sigma_kX^3+\frac{1}{18}\Sigma_k^2\Sigma_k'X^3+\frac{19}{6}\Sigma_k^3\Sigma_k'^2X^3$$
$$+\frac{1}{6}\Sigma_k^3X^4-\frac{16}{3}\Sigma_k^5\Sigma_k'^2X^4-\frac{2}{3}\Sigma_k^5X^5+\frac{8}{3}\Sigma_k^7\Sigma_k'^2X^5\bigg], \tag{E1}$$

$$X\equiv\frac{1}{k^2+\Sigma_k^2}. \tag{E2}$$

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