

Ideas of four-fermion operators in electromagnetic form factor calculationsChueng-Ryong Ji,¹ Bernard L. G. Bakker,² Ho-Meoyng Choi,³ and Alfredo T. Suzuki^{1,4,*}¹*Department of Physics, North Carolina State University, Raleigh, North Carolina 27695-8202, USA*²*Department of Physics and Astrophysics, Vrije Universiteit, De Boelelaan 1081, NL-1081 HV Amsterdam, Netherlands*³*Department of Physics, Teachers College, Kyungpook National University, Daegu 702-701, Republic of Korea*⁴*Instituto de Física Teórica, UNESP Universidade Estadual Paulista, Rua Dr. Bento Teobaldo Ferraz, 271, Bloco II, 01140-070 São Paulo, SP, Brazil*

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Four-fermion operators have been used in the past to link the quark-exchange processes in the interaction of hadrons with the effective meson-exchange amplitudes. In this paper, we apply the similar idea of a Fierz rearrangement to the self-energy and electromagnetic processes and focus on the electromagnetic form factors of the nucleon and the electron. We explain the motivation of using four-fermion operators and discuss the advantage of this method in computing electromagnetic processes.

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I. INTRODUCTION

Although the calculation of the nucleon form factors based on a quark-diquark model certainly differs from the calculation of the electron form factors using quantum electrodynamics (QED), one may still discern commonalities between the two apparently different calculations. For example, both calculations on the one-loop level share essentially the same type of triangle diagram as shown in Fig. 1 for the computation of amplitudes. While the contents of the lines drawn in the two triangle diagrams are certainly different, both calculations share the same type of one-loop integration for the amplitudes given by three vertices connected by three propagators. In particular, the structure of the two fermion lines intermediated by a boson exchange is common in the two triangle diagrams and may be generically identified as the four-fermion operator that we discuss in this work. Due to this commonality, it may be conceivable to compute the two apparently different triangle amplitudes in a unified way. Such a unified way of computation is possible since the four-fermion operator can be Fierz rearranged.

A similar idea of Fierz-rearranged four-fermion operators was developed in a rather different context of applications in the early 1980s. The basic idea of these developments was to provide a basis for the one-boson-exchange interactions of baryons at low energy in the gluon exchange, which mediates quark-exchange scattering in conjunction with quark interchange in a nonperturbative bag-model framework [1–5]. In elastic nucleon-nucleon (NN) scattering, the four-fermion operator appears from the gluon-exchange mediating quark-exchange scattering and becomes bilocal when it is dressed with long-range quark-gluon correlations by means of bag-model wave

functions [1]. As this four-fermion operator is Fierz rearranged, the quark-interchange amplitude takes on the usual local form for each nucleon that is expected from the wealth of empirical knowledge at low energy [1]. The same idea was applied to πN and $\pi\pi$ scattering as well as the scattering involving hyperons [2]. A partial-wave helicity-state analysis of elastic NN scattering was carried out in momentum space [3], and a mesonic NN potential from an effective quark interchange mechanism for non-overlapping nucleons was obtained from the constituent quark model [5]. Also, meson exchanges were introduced into the harmonic oscillator quark model along with a lower component of the quark spinor [4].

In this paper, we apply the Fierz-rearranged four-fermion operator in the form factors shown in Fig. 1 and present a global formula to cover the triangle diagrams frequently used for the form factor calculations. The basic idea is presented in the next section, Sec. II, and a simple illustration of this idea is given in Sec. III via the self-energy calculation. In Sec. IV, we apply it to the form factor calculations that involve triangle diagrams and present a corresponding global formula. The conclusion and outlook follow in Sec. V. Appendices A and B detail the four-fermion invariants in comparison with the well-known Fierz identities [6,7] and the manifestly covariant calculation of form factors, respectively.

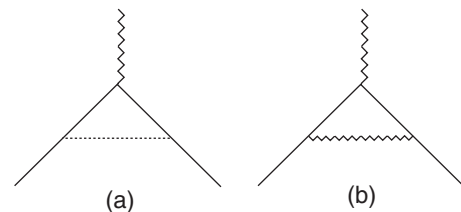


FIG. 1. Triangle diagrams for (a) nucleon form factors in quark-diquark model and (b) electron form factors in QED.

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II. BASIC IDEA

The basic idea of the four-fermion operator in electromagnetic processes is depicted in Fig. 2, where a single-photon process for a target nucleon is drawn as an illustration. The left and right portions of Fig. 2 correspond to the amplitude intended for computation and the equivalent amplitude after the four-fermion operator is Fierz rearranged, respectively. In the left portion, a photon is attached to the hadronic part, which has the two intermediate quarks denoted by the spinor indices k and ℓ that inherit the fermion number from the external nucleons, \bar{u}_i and u_j , with the corresponding spinor indices i and j , respectively. Here, the two vertices, O_{ik} and $O_{\ell j}$, connecting the nucleon and the corresponding quark are linked to the scattering amplitude $T_{k\ell}$ where the photon interacts with the constituents from the target nucleon; the rest of the constituents beside the quark is denoted by a wiggly line below the corresponding quark, and the loop integration over the internal momentum is understood. From this configuration of the integrand in the amplitude, we may identify the four-fermion operator as the multiplication of two vertices $O_{ik}O_{\ell j}$ and rearrange it as

$$\begin{aligned}
 (O_\beta)_{ik}(O^\beta)_{\ell j} &= C_S^\beta \delta_{ij} \delta_{\ell k} + C_V^\beta (\gamma_\mu)_{ij} (\gamma^\mu)_{\ell k} \\
 &\quad + C_T^\beta (\sigma_{\mu\nu})_{ij} (\sigma^{\mu\nu})_{\ell k} \\
 &\quad + C_A^\beta (\gamma_\mu \gamma_5)_{ij} (\gamma^\mu \gamma_5)_{\ell k} + C_P^\beta (\gamma_5)_{ij} (\gamma_5)_{\ell k} \\
 &= \sum_\alpha C_\alpha^\beta (\Omega_\alpha)_{ij} (\Omega^\alpha)_{\ell k}, \quad (1)
 \end{aligned}$$

where the index β specifies the nature of the operator O in the vertex, whether it is scalar (S), pseudoscalar (P), vector (V), axial-vector (A), or tensor (T). Similarly, the index α specifies the nature of the rearranged operator Ω such that $\Omega_S = \mathbf{1}$, $\Omega_P = \gamma_5$, $\Omega_V = \gamma_\mu$, $\Omega_A = \gamma_\mu \gamma_5$, and $\Omega_T = \sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$, where the Lorentz indices appear obviously for V , A , and T as denoted by μ and/or ν . Although there are only six independent tensor operators in the full Dirac algebra, we prefer to sum over the full number of 12σ tensors in Eq. (1) using Einstein's summation convention. Appendix A details the comparison among different conventions [6,7] regarding, in particular, the compensating factor $\frac{1}{2}$ for this

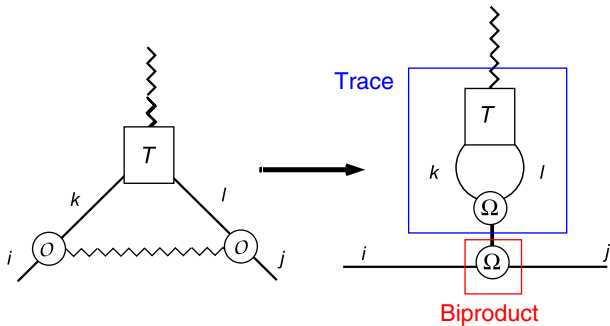


FIG. 2 (color online). Basic idea of Fierz-rearranged four-fermion operator in electromagnetic processes.

TABLE I. Fierz transformation coefficients of Eq. (1) [6].

	S	V	T	A	P
S	1/4	1/4	1/8	-1/4	1/4
V	1	-1/2	0	-1/2	-1
T	3	0	-1/2	0	3
A	-1	-1/2	0	-1/2	1
P	1/4	-1/4	1/8	1/4	1/4

double counting in tensor operators as well as the location of γ_5 in the axial-vector operator whether it be $\gamma_\mu \gamma_5$ or $\gamma_5 \gamma_\mu$.

The Fierz coefficients C_α^β ($\alpha, \beta = S, P, V, A, T$) depend on the nature of the vertices $(O_\beta)_{ik}$ and $(O^\beta)_{\ell j}$. The operator O is defined the same way as Ω is defined, i.e., $O_S = \mathbf{1}$, $O_V = \gamma_\mu$, $O_T = \sigma_{\mu\nu}$, $O_A = \gamma_\mu \gamma_5$, $O_P = \gamma_5$. With this definition of operators O and Ω , we shall use the Fierz coefficients C_α^β in Table I for different couplings O_β (and Ω_α) (see also Eq. (A5) in our Appendix A).¹

For example, one can take the following coefficients from Table I: $(C_S, C_V, C_T, C_A, C_P) = (3, 0, -1/2, 0, 3)$ and $(-1, -1/2, 0, -1/2, 1)$ for tensor vertices $(\sigma_{\mu\nu})_{ik} \times (\sigma^{\mu\nu})_{\ell j}$ and the axial-vector vertices $(\gamma_\mu \gamma_5)_{ik} (\gamma^\mu \gamma_5)_{\ell j}$, respectively.

With this Fierz rearrangement, we may write the integrand of the amplitude (omitting the index β for simplicity) as follows:

$$\begin{aligned}
 \bar{u}_i O_{ik} T_{k\ell} O_{\ell j} u_j &= \sum_\alpha C_\alpha (\bar{u}_i \Omega_{ij}^\alpha u_j) (\Omega_{\ell k}^\alpha T_{k\ell}) \\
 &= \sum_\alpha (\bar{u} \Omega^\alpha u) C_\alpha \text{Tr}[\Omega^\alpha T], \quad (2)
 \end{aligned}$$

where the external nucleon current (or biprodut $\bar{u} \Omega^\alpha u$) part is now factorized from the internal scattering part given by the trace of the quark loop ($\text{Tr}[\Omega^\alpha T]$) as depicted in the right portion of Fig. 2. With this rearrangement of the same amplitude, one may get the general structure of the target hadron's current more immediately and factorize the details of the internal probing mechanism just due to the relevant constituents for the current of the target hadron. It provides an efficient and unified way to analyze the general structure of the amplitudes sharing the commonality of the same type of diagram for the process.

III. SIMPLEST ILLUSTRATION

For an illustration of the basic idea, we start from the simple example of a fermion self-energy amplitude, which

¹Although our definition of the axial-vector operator, i.e., $O_A = \gamma_\mu \gamma_5$, differs from the corresponding operator $\gamma_5 \gamma_\mu$ used in Ref. [6], Table I is identical to the Fierz coefficients given in the same reference [6] [see, e.g., Eq. (A5)] because the swap of γ_5 and γ_μ does not matter on the level of the four-fermion operator $(\hat{O}_A)_{ik} (\hat{O}^A)_{\ell j}$. See more details in Appendix A.

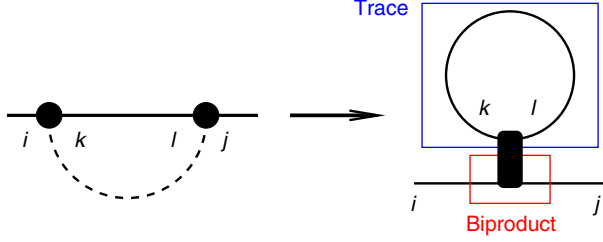


FIG. 3 (color online). Self-energy amplitude and the corresponding Fierz rearrangement.

does not have any external photons but just has one loop due to an exchanged boson as shown in Fig. 3. Such a process may occur in chiral perturbation theory to yield the self-energy of the nucleon due to the surrounding pion cloud [8]. Also, in the Yukawa model with a scalar coupling, the fermion self-energy due to a scalar boson has been investigated [9].

For the purpose of simple illustration, we consider here only scalar and pseudoscalar couplings (rather than the pseudovector coupling in the chiral perturbation theory) and write the self-energy amplitude for a nucleon of mass M , four-momentum p , and spin s in a unified formula both for scalar and pseudoscalar couplings:

$$\begin{aligned} \hat{\Sigma}(p, s) &= \bar{u}_i(p, s) \hat{\Sigma}_{ij} u_j(p, s) \\ &= \Sigma_S \bar{u}(p, s) u(p, s) + \Sigma_V^\mu \bar{u}(p, s) \gamma_\mu u(p, s), \end{aligned} \quad (3)$$

where, modulo the appropriate normalization factor, the self-energy operator $\hat{\Sigma}_{ij}$ is given by

$$\hat{\Sigma}_{ij} = \int \frac{d^4 k}{(2\pi)^4} \frac{O_{ik}(\not{p} - \not{k} + M)_{kl} O_{lj}}{D_k D_N}, \quad (4)$$

with $D_k = k^2 - m_X^2 + i\epsilon$ (m_X is the intermediate meson mass) and $D_N = (p - k)^2 - M^2 + i\epsilon$. The four-fermion operator $O_{ik} O_{lj}$ becomes $\mathbf{I}_{ik} \mathbf{I}_{lj} = \delta_{ik} \delta_{lj}$ for the scalar coupling theory, while it becomes $(\gamma_5)_{ik} (\gamma_5)_{lj}$ for the pseudoscalar coupling theory. From Table I, we get

$$\begin{aligned} \delta_{ik} \delta_{lj} &= \frac{1}{4} \delta_{ij} \delta_{lk} + \frac{1}{4} (\gamma_\mu)_{ij} (\gamma^\mu)_{lk} + \frac{1}{8} (\sigma_{\mu\nu})_{ij} (\sigma^{\mu\nu})_{lk} \\ &\quad - \frac{1}{4} (\gamma_\mu \gamma_5)_{ij} (\gamma^\mu \gamma_5)_{lk} + \frac{1}{4} (\gamma_5)_{ij} (\gamma_5)_{lk} \end{aligned} \quad (5)$$

and

$$\begin{aligned} (\gamma_5)_{ik} (\gamma_5)_{lj} &= \frac{1}{4} \delta_{ij} \delta_{lk} - \frac{1}{4} (\gamma_\mu)_{ij} (\gamma^\mu)_{lk} \\ &\quad + \frac{1}{8} (\sigma_{\mu\nu})_{ij} (\sigma^{\mu\nu})_{lk} + \frac{1}{4} (\gamma_\mu \gamma_5)_{ij} (\gamma^\mu \gamma_5)_{lk} \\ &\quad + \frac{1}{4} (\gamma_5)_{ij} (\gamma_5)_{lk}, \end{aligned} \quad (6)$$

using Eq. (1) and Table I. The structure given by $\sum_\alpha \mathbf{C}_\alpha^\beta (\Omega_\alpha)_{ij} (\Omega^\alpha)_{lk}$ in Eq. (1) is manifest both in Eqs. (5) and (6). Now, using the Fierz rearrangement given by Eq. (1), one may replace $O_{ik} O_{lj}$ with $\sum_\alpha \mathbf{C}_\alpha^\beta (\Omega_\alpha)_{ij} \times (\Omega^\alpha)_{lk}$. Then, the multiplication of the operator factor $(\Omega^\alpha)_{lk}$ with the factor $(\not{p} - \not{k} + M)_{kl}$ in Eq. (4) yields the trace $\text{Tr}[\Omega^\alpha (\not{p} - \not{k} + M)]$ corresponding to the fermion loop shown in the right side of Fig. 3.

Computing the trace $\text{Tr}[\Omega^\alpha (\not{p} - \not{k} + M)]$, one can easily see that only $\alpha = S$ and V survive while $\alpha = P$, A and T vanish as expected from the structure of the fermion self-energy given by Eq. (3). Since $\text{Tr}[\not{p} - \not{k} + M] = 4M$ and $\text{Tr}[\gamma^\mu (\not{p} - \not{k} + M)] = 4(p - k)^\mu$, we get Σ_S and Σ_V^μ in Eq. (3) as

$$\begin{aligned} \Sigma_S &= 4\mathbf{C}_S M \int \frac{d^4 k}{(2\pi)^4} \frac{1}{D_k D_N}, \\ \Sigma_V^\mu &= 4\mathbf{C}_V \int \frac{d^4 k}{(2\pi)^4} \frac{(p - k)^\mu}{D_k D_N}, \end{aligned} \quad (7)$$

where $\mathbf{C}_S = 1/4(1/4)$ and $\mathbf{C}_V = 1/4(-1/4)$ for the scalar (pseudoscalar) coupling case from Table I. This shows that both scalar and pseudoscalar coupling theories share the same expressions given by Eq. (7). From this unified formula, one can rather easily find a relationship between the two results, one from the scalar coupling theory and the other from the pseudoscalar theory, i.e.,

$$(\Sigma_S)^S = (\Sigma_S)^P \quad \text{and} \quad (\Sigma_V^\mu)^S = -(\Sigma_V^\mu)^P. \quad (8)$$

The usual dimensional regularization method can be applied to obtain explicit results for Σ_S and Σ_V^μ after the four-dimensional integration over the internal four-momentum k^μ in the fermion loop. They are found to be identical to the previous results [8,9] obtained by the direct calculation without using the Fierz rearrangement. It is amusing to notice that the results for the scalar coupling theory [9] and the pseudoscalar coupling theory [8] indeed satisfy the relationship given by Eq. (8). Using the Fierz rearrangement, we now understand explicitly how and why they are related to each other.

IV. APPLICATION TO FORM FACTORS

We now apply the idea of the four-fermion operator and Fierz rearrangement to the form factors shown in Fig. 1 and present the result that covers both the nucleon form factors in a quark-diquark model and the electron form factors in QED. Using the four-fermion method illustrated in Sec. II, the current operator J^μ in the amplitude $\bar{u}(p') J^\mu u(p)$ from the triangle diagram with the external fermion mass M , the internal fermion mass m , and the intermediate boson mass m_X can be given by (modulo normalization)

$$J^\mu = ig^2 \int \frac{d^4 k}{(2\pi)^4} \frac{\mathbf{N}^\mu}{D_{p_1} D_{p_2} D_k}, \quad (9)$$

where $p_{1(2)} = p(p') - k$, $D_{p_i} = p_i^2 - m^2 + i\varepsilon$, $D_k = k^2 - m_X^2 + i\varepsilon$, and \mathbf{N}^μ is the numerator of the amplitude corresponding to the triangle diagram (e.g., Fig. 1). Using the Fierz rearrangement given by Eq. (1), the numerator \mathbf{N}^μ can be written as

$$\mathbf{N}^\mu = \sum_{\alpha} \mathbf{C}_{\alpha} \text{Tr}[(\not{p}_2 + m)\gamma^{\mu}(\not{p}_1 + m)\Omega^{\alpha}]\Omega_{\alpha}, \quad (10)$$

where $\alpha = S, V, T, A, P$ and the corresponding $\Omega^{\alpha} = \mathbf{I}, \gamma^{\nu}, \sigma^{\nu\delta}, \gamma^{\nu}\gamma_5, \gamma_5$ with the dummy Lorentz indices ν and δ . As expected from the parity conservation in the electromagnetic current, $\alpha = P$ never contributes to \mathbf{N}^μ , i.e., $\text{Tr}[(\not{p}_2 + m)\gamma^{\mu}(\not{p}_1 + m)\gamma_5] = 0$. Thus, after the trace calculation, we get

$$\begin{aligned} \mathbf{N}^\mu = & 4[\mathbf{C}_S m(p_1 + p_2)^{\mu} + \mathbf{C}_V \{(m^2 - p_1 \cdot p_2)\gamma^{\mu} \\ & + p_2^{\mu}(p_1 \cdot \gamma) + (p_2 \cdot \gamma)p_1^{\mu}\} + i\mathbf{C}_T m(g^{\mu\alpha}q^{\beta} \\ & - g^{\mu\beta}q^{\alpha})\sigma_{\alpha\beta} - i\mathbf{C}_A \epsilon^{\mu\nu\alpha\beta}(p_2)_{\nu}\gamma_{\alpha}(p_1)_{\beta}\gamma_5], \quad (11) \end{aligned}$$

where $q = p' - p$.

Now, using the usual Feynman parametrization for the loop integration, the denominator of the integrand in Eq. (9) yields

$$\begin{aligned} & \frac{1}{D_{p_1}D_{p_2}D_k} \\ & = 2 \int_0^1 dx \int_0^{1-x} \frac{dy}{[D_k + x(D_{p_1} - D_k) + y(D_{p_2} - D_k)]^3} \\ & = 2 \int_0^1 dx \int_0^{1-x} \frac{dy}{(k'^2 - M_{\text{cov}}^2)^3}, \quad (12) \end{aligned}$$

where in the second line we used the shifted momentum $k' = k - xp - yp'$ and defined

$$\begin{aligned} M_{\text{cov}}^2 = & (x + y)m^2 + (1 - x - y)m_X^2 - xyq^2 \\ & - (x + y)(1 - x - y)M^2 \quad (13) \end{aligned}$$

with the on-shell condition $p^2 = p'^2 = M^2$. Then, the current operator J^μ given by Eq. (9) becomes

$$J^\mu = 2ig^2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^4k'}{(2\pi)^4} \frac{\tilde{\mathbf{N}}^\mu}{(k'^2 - M_{\text{cov}}^2)^3}, \quad (14)$$

where the numerator with shifted momentum is now given by

$$\begin{aligned} \tilde{\mathbf{N}}^\mu = & 4\left[\mathbf{C}_S(1 - x - y)m(p' + p)^\mu + \mathbf{C}_V\left\{\left(m^2 - k'^2 - (1 - x - y)^2M^2 + (1 - x - y + 2xy)\frac{q^2}{2}\right)\gamma^\mu + 2k'^{\mu}\not{k}' \right. \right. \\ & \left. \left. + \frac{(1 - x - y)^2}{2}(p' + p)^\mu(\not{p} + \not{p}') - \frac{1 - (x - y)^2}{2}q^\mu\not{q}\right\} + 2i\mathbf{C}_T mg^{\mu\alpha}q^\beta\sigma_{\alpha\beta} + i\mathbf{C}_A(1 - x - y)\epsilon^{\mu\nu\alpha\beta}p_\nu\gamma_\alpha p'_\beta\gamma_5\right]. \quad (15) \end{aligned}$$

Although one expects to get $J^\mu = \gamma^\mu F_1(q^2) + i\frac{\sigma^{\mu\nu}q^\nu}{2M}F_2(q^2)$, our result for J^μ appears to exhibit not only the vector and tensor currents but also the scalar and axial vector currents. This issue can be resolved by the Gordon decomposition and a similar extension, namely,

$$(p' + p)^\mu \rightarrow 2M\gamma^\mu - i\sigma^{\mu\nu}q_\nu, \quad i\epsilon^{\mu\nu\alpha\beta}\gamma_5 p_\nu\gamma_\alpha p'_\beta \rightarrow \frac{q^2}{2}\gamma^\mu - iM\sigma^{\mu\nu}q_\nu. \quad (16)$$

Using Eq. (16), we get the expected decomposition of J^μ in terms of just vector and tensor currents and find the form factors ($i = 1, 2$) as follows:

$$F_i(q^2) = 8ig^2 \int_0^1 dx \int_0^{1-x} dy \int d^4k' \frac{\tilde{\mathbf{N}}_i}{(k'^2 - M_{\text{cov}}^2)^3}, \quad (17)$$

where

$$\begin{aligned} \tilde{\mathbf{N}}_1 = & 2mM(1 - x - y)\mathbf{C}_S + (1 - x - y)\frac{q^2}{2}\mathbf{C}_A + \left[m^2 + (1 - x - y)^2M^2 + (1 - x - y + 2xy)\frac{q^2}{2} - \frac{k^2}{2}\right]\mathbf{C}_V, \\ \tilde{\mathbf{N}}_2 = & -2mM(1 - x - y)\mathbf{C}_S - 2(1 - x - y)M^2\mathbf{C}_A - 2(1 - x - y)^2M^2\mathbf{C}_V + 4mM\mathbf{C}_T. \quad (18) \end{aligned}$$

Apparently, F_1 is UV divergent and requires a regularization along with the renormalization set by the normalization condition $F_1(0) = 1$. More explicit expressions for $F_1(q^2)$ and $F_2(q^2)$ are derived in Appendix B using dimensional regularization with Wick rotation.

We may point out that the results in Eq. (17) can cover not only the nucleon form factors in a quark-diquark model, whether the diquark is scalar or axial-vector, but also the electron form factors in QED taking the corresponding Fierz coefficients and masses. For example, from

Table I, $C_S = C_V = 2C_T = -C_A = C_P = 1/4$ if the diquark is taken as a scalar boson, while $C_S = -1$, $C_V = -1/2$, $C_T = 0$, $C_A = -1/2$, and $C_P = 1$ if the diquark is taken as an axial vector boson in a quark-diquark model for the nucleon form factors. For the electron form factors in QED, one should take of course $C_S = 1$, $C_V = -1/2$, $C_T = 0$, $C_A = -1/2$, and $C_P = -1$. Also, M , m , and m_X are the nucleon, quark, and diquark masses for the nucleon form factors, while $M = m$ is the electron mass in the QED calculation of the electron form factors. It is interesting to see that the Fierz coefficient C_P appears neither in the nucleon form factors nor in the electron form factors reflecting the parity conservation both in the strong and electromagnetic interactions. We should note,

$$\begin{aligned}
 J^\mu &= \gamma^\mu F_1 + i \frac{\sigma^{\mu\nu} q_\nu}{2M} F_2 = \gamma^\mu (F_1 + F_2) + \frac{(p + p')^\mu}{2M} F_2 = \frac{(p + p')^\mu}{2M} \frac{4M^2 F_1 + q^2 F_2}{4M^2 - q^2} - i \epsilon^{\mu\nu\alpha\beta} \gamma_5 \gamma_\nu p_\alpha p'_\beta \frac{2(F_1 + F_2)}{4M^2 - q^2} \\
 &= \frac{(p + p')^\mu}{2M} F_1 + i \frac{\sigma^{\mu\nu} q_\nu}{2M} (F_1 + F_2) = \gamma^\mu \left(F_1 + \frac{q^2}{4M^2} F_2 \right) - i \epsilon^{\mu\nu\alpha\beta} \gamma_5 \gamma_\nu p_\alpha p'_\beta \frac{F_2}{2M^2} \\
 &= i \frac{\sigma^{\mu\nu} q_\nu}{2M} \left(\frac{4M^2}{q^2} F_1 + F_2 \right) + i \epsilon^{\mu\nu\alpha\beta} \gamma_5 \gamma_\nu p_\alpha p'_\beta \frac{2F_1}{q^2}.
 \end{aligned} \tag{19}$$

One should note, however, that the equivalence presented in Eq. (19) meant the equality on the level of matrix elements, e.g., $\bar{u} J^\mu u$, but not on the level of operators themselves. In other words, Eq. (19) is valid only for the spin-1/2 fermion case such as the nucleon. Thus, for the nucleon target, these six different decompositions in Eq. (19) are all equivalent. Any particular choice of decomposition may depend on a matter of convenience and/or effectiveness in the given situation of computation.

V. CONCLUSION

The idea of rearranging four-fermion operators provides an effective way to analyze hadronic processes. It factorizes the details of the internal probing mechanism from the external global structure owing to the target hadrons. In this work, we illustrated the idea of Fierz rearrangement to the fermion self-energy and electromagnetic form factor calculations. Processes sharing a certain commonality (e.g., the same type of diagrams) may be described in a unified way. For instance, whether the mesons surrounding the nucleon are scalar or pseudoscalar bosons, the Fierz rearrangement of the four-fermion operators can be used to yield a unified expression for the nucleon self-energy amplitude and provide a relationship between the two amplitudes, one for the scalar coupling and the other for the pseudoscalar coupling. Likewise, the electromagnetic nucleon form factors in a quark-diquark model and the electron form factors in QED can be given by a unified expression based on the commonality of sharing the same type of diagram, e.g., the triangle diagrams shown in Fig. 1. Moreover, the quark-diquark calculations of baryon form

however, that the disappearance of C_P in Eq. (17) is not coming from the Fierz rearrangement itself but coming from the trace calculation reflecting the conservation of parity in the single photon process; e.g., a pion can never decay into a single photon but can decay into two photons. Thus, we may expect that the contribution from Ω_P would show up in the amplitude defined in a process involving two photons such as the generalized parton distributions in deeply virtual Compton scattering.

Finally, we note that the usual decomposition of $J^\mu = \gamma^\mu F_1(q^2) + i \frac{\sigma^{\mu\nu} q_\nu}{2M} F_2(q^2)$ in terms of vector and tensor currents with the Dirac (F_1) and Pauli (F_2) form factors is just one of six possible decompositions:

factors using the idea of rearranging four-fermion operators proposed in this work may provide a unified expression that can cover all types of diquarks such as scalar, pseudoscalar, vector, axial-vector, and tensor diquarks. With this idea, we can offer a clear understanding of the interrelationships among different calculations sharing a commonality.

While we presented only the basic idea and a few simple examples in this paper, we may foresee a great potential for further application to other hadronic processes. In particular, the application to the two-photon processes would be interesting since the generalized hadronic tensor structure of deeply virtual Compton scattering still needs further investigation [10] in view of forthcoming experiments with the 12 GeV upgrade at JLab. Work along this line is in progress.

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APPENDIX A: CONVENTIONS IN FIERZ IDENTITIES FOR DIRAC MATRICES

A Fierz identity is an identity that allows one to rewrite bilinears of the product of two spinors as a linear combination of products of the bilinears of the individual spinors. In all, 16 bilinear terms can be constructed using bispinors \bar{a} and b . The linear combinations of these terms form five

different types of Lorentz-covariant quantities, $\bar{a}b$, $\bar{a}\gamma^\mu b$, $\bar{a}\sigma^{\mu\nu}b$, $\bar{a}\gamma^5\gamma^\mu b$, $\bar{a}\gamma^5b$, where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$. These covariants are normalized as follows:

$$\begin{aligned} \mathbf{I} \cdot \mathbf{I} &= 1, & \gamma_\mu \gamma^\mu &= 4, & \sigma_{\mu\nu} \sigma^{\mu\nu} &= 12, \\ (\gamma_5 \gamma_\mu)(\gamma^5 \gamma^\mu) &= -4, & \gamma_5 \gamma^5 &= 1. \end{aligned} \quad (\text{A1})$$

There are several ways of constructing the Lorentz scalar out of four bispinors \bar{a} , b , \bar{c} , d [6,7,11]. According to the convention by Weber [6], the five Lorentz scalars can be constructed out of four bispinors \bar{u}_1 , u_2 , \bar{u}_3 , u_4 as follows:

$$\begin{aligned} S\text{-variant: } S^W(4, 2; 3, 1) &= (\bar{u}_4 u_2)(\bar{u}_3 d_1), \\ V\text{-variant: } V^W(4, 2; 3, 1) &= (\bar{u}_4 \gamma^\mu u_2)(\bar{u}_3 \gamma_\mu u_1), \\ T\text{-variant: } T^W(4, 2; 3, 1) &= (\bar{u}_4 \sigma^{\mu\nu} u_2)(\bar{u}_3 \sigma_{\mu\nu} u_1), \\ A\text{-variant: } A^W(4, 2; 3, 1) &= (\bar{u}_4 \gamma^5 \gamma^\mu u_2)(\bar{u}_3 \gamma_5 \gamma_\mu u_1), \\ P\text{-variant: } P^W(4, 2; 3, 1) &= (\bar{u}_4 \gamma^5 u_2)(\bar{u}_3 \gamma_5 u_1). \end{aligned} \quad (\text{A2})$$

Counting only the independent tensors, $\sigma^{\mu\nu}$ with $\mu < \nu$, the 16 matrices (\mathbf{I} , γ^μ , $\sigma^{\mu\nu}$, $\gamma^5 \gamma^\mu$, γ_5) form a complete set so that any one of the above variants can be expressed as a linear combination of variants with a changed sequence of spinors:

$$(\bar{u}_4 W^i u_2)(\bar{u}_3 W_i u_1) = \sum_k C_k^i (\bar{u}_4 W^k u_1)(\bar{u}_3 W_k u_2), \quad (\text{A3})$$

where

$$W_S = \mathbf{I}, \quad W_V = \gamma_\mu, \quad W_T = \sigma_{\mu\nu}, \quad W_A = \gamma_5 \gamma_\mu, \quad W_P = \gamma_5. \quad (\text{A4})$$

Our vertex operators denoted by O_β as well as the rearranged operators denoted by Ω_α are the same as Weber's operators W_i except for the swap of γ_5 and γ_μ in the axial vector operator, i.e., $O_S = W_S$, $O_V = W_V$, $O_T = W_T$, $O_A = -W_A$, $O_P = W_P$. Thus, the coefficients C_k^i in Eq. (A3) are identical to the Fierz coefficients given by Table I. More explicitly, one may write Eq. (A3) as a matrix equation, i.e.,

$$\begin{aligned} \begin{bmatrix} S^W \\ V^W \\ T^W \\ A^W \\ P^W \end{bmatrix} (4, 2; 3, 1) &= \frac{1}{4} \begin{bmatrix} 1 & 1 & \frac{1}{2} & -1 & 1 \\ 4 & -2 & 0 & -2 & -4 \\ 12 & 0 & -2 & 0 & 12 \\ -4 & -2 & 0 & -2 & 4 \\ 1 & -1 & \frac{1}{2} & 1 & 1 \end{bmatrix} \\ &\times \begin{bmatrix} S^W \\ V^W \\ T^W \\ A^W \\ P^W \end{bmatrix} (4, 1; 3, 2). \end{aligned} \quad (\text{A5})$$

For example, either from Table I or Eq. (A5), one can read off the following coefficients: ($C_S^T = 3$, $C_V^T = 0$,

$C_T^T = -1/2$, $C_A^T = 0$, $C_P^T = 3$) for the tensor product or T variant $(\bar{u}_4 \sigma^{\mu\nu} u_2)(\bar{u}_3 \sigma_{\mu\nu} u_1)$ and ($C_S^A = -1$, $C_V^A = -1/2$, $C_T^A = 0$, $C_A^A = -1/2$, $C_P^A = 1$) for the axial-vector product or A variant $(\bar{u}_4 \gamma^5 \gamma^\mu u_2)(\bar{u}_3 \gamma_5 \gamma_\mu u_1)$.

On the other hand, according to the convention by Itzykson and Zuber (IZ) [7], the five different Lorentz-covariant quantities are taken as $\{\bar{a}b, \bar{a}\gamma^\mu b, \bar{a}\sigma^{\mu\nu}b, \bar{a}\gamma^5\gamma^\mu b, \bar{a}(i\gamma^5)b\}$. Note here that the factor i in front of γ^5 makes the pseudoscalar operator $i\gamma^5$ Hermitian. These five Lorentz-covariant quantities are paired with their partners $\{\bar{a}b, \bar{a}\gamma_\mu b, \bar{a}\sigma_{\mu\nu}b, \bar{a}\gamma_\mu \gamma_5 b, \bar{a}(-i\gamma_5)b\}$ to construct the corresponding five Lorentz scalars. Using the four bispinors \bar{u}_1 , u_2 , \bar{u}_3 , u_4 , we may write those Lorentz scalars as follows:

$$\begin{aligned} S^{IZ}(4, 2; 3, 1) &= (\bar{u}_4 u_2)(\bar{u}_3 d_1), \\ V^{IZ}(4, 2; 3, 1) &= (\bar{u}_4 \gamma^\mu u_2)(\bar{u}_3 \gamma_\mu u_1), \\ T^{IZ}(4, 2; 3, 1) &= \frac{1}{2}(\bar{u}_4 \sigma^{\mu\nu} u_2)(\bar{u}_3 \sigma_{\mu\nu} u_1), \\ A^{IZ}(4, 2; 3, 1) &= (\bar{u}_4 \gamma^5 \gamma^\mu u_2)(\bar{u}_3 \gamma_\mu \gamma_5 u_1), \\ P^{IZ}(4, 2; 3, 1) &= (\bar{u}_4 \gamma^5 u_2)(\bar{u}_3 \gamma_5 u_1). \end{aligned} \quad (\text{A6})$$

Note here that the factors i in $i\gamma^5$ and $-i$ in $-i\gamma_5$ are not written explicitly in P^{IZ} because they cancel out. Also, the usual summation convention is used in T^{IZ} to sum over all twelve tensor operators. Because only six independent tensor operators exist, the factor of $\frac{1}{2}$ is introduced in T^{IZ} to compensate for this double counting. Finally, we note that the T and A variants defined in Eq. (A6) are different from those in Eq. (A2), i.e., $T^{IZ} = \frac{1}{2}T^W$ and $A^{IZ} = -A^W$. Accordingly, the coefficients C_k^i in the IZ convention of Lorentz scalars are given by

$$\begin{aligned} \begin{bmatrix} S^{IZ} \\ V^{IZ} \\ T^{IZ} \\ A^{IZ} \\ P^{IZ} \end{bmatrix} (4, 2; 3, 1) &= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & -2 & 0 & 2 & -4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & 2 & 0 & -2 & -4 \\ 1 & -1 & 1 & -1 & 1 \end{bmatrix} \\ &\times \begin{bmatrix} S^{IZ} \\ V^{IZ} \\ T^{IZ} \\ A^{IZ} \\ P^{IZ} \end{bmatrix} (4, 1; 3, 2). \end{aligned} \quad (\text{A7})$$

APPENDIX B: EXPLICIT RESULTS OF FORM FACTORS IN EQ. (17)

Using the four-fermion method illustrated in Sec. II and the usual Feynman parametrization for the loop integration, we computed the triangle diagrams shown

in Fig. 1 for the electromagnetic form factors of the spin-1/2 target particle and obtained Eq. (17) as presented in Sec. IV. Since the momentum integral in Eq. (17) diverges for the k'^2 and $k'^\mu k'^\alpha$ terms in the ultraviolet region, we need to regularize it. In this appendix, we perform the four-dimensional k' integration using the Wick rotation to Euclidean space, $k' = i\kappa_E$, and the dimensional regularization to find the more explicit expressions for $F_1(q^2)$ and $F_2(q^2)$. Then, Eq. (14) is rewritten as

$$J^\mu = 2\mu^{4-d} g^2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^d \kappa_E}{(2\pi)^d} \frac{\tilde{N}_E^\mu}{(\kappa_E^2 + M_{\text{cov}}^2)^3}, \quad (\text{B1})$$

where μ^{d-4} is the usual mass factor that comes in to compensate the change in the dimensionality of the momentum integration. Using the property of symmetric integration for $\kappa_E^\mu \kappa_E^\alpha = \kappa_E^2 g^{\mu\alpha}/d$, we have now the following numerator, which corresponds to Eq. (15):

$$\begin{aligned} \tilde{N}_E^\mu = d \left[\mathbf{C}_S (1-x-y) m (p' + p)^\mu + \mathbf{C}_V \left\{ \left(m^2 + \left(1 - \frac{2}{d} \right) \kappa_E^2 - (1-x-y)^2 M^2 + (1-x-y+2xy) \frac{q^2}{2} \right) \gamma^\mu \right. \right. \\ \left. \left. + (1-x-y)^2 M (p' + p)^\mu \right\} + 2i \mathbf{C}_T m \sigma^{\mu\nu} q_\nu + i \mathbf{C}_A (1-x-y) \epsilon^{\mu\nu\alpha\beta} p_\nu \gamma_\alpha \gamma_5 p'_\beta \right]. \end{aligned} \quad (\text{B2})$$

Now, using Eq. (16), we can decompose J^μ in terms of vector and tensor currents and find the form factors $F_i (i = 1, 2)$:

$$F_i(q^2) = 2d\mu^{4-d} g^2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^d \kappa_E}{(2\pi)^d} \frac{N_i}{(\kappa_E^2 + M_{\text{cov}}^2)^3}, \quad (\text{B3})$$

where

$$\begin{aligned} N_1 &= 2mM(1-x-y)\mathbf{C}_S + (1-x-y)\frac{q^2}{2}\mathbf{C}_A + \left[m^2 + (1-x-y)^2 M^2 + (1-x-y+2xy)\frac{q^2}{2} + \left(1 - \frac{2}{d} \right) k \right] \mathbf{C}_V, \\ N_2 &= -2mM(1-x-y)\mathbf{C}_S - 2(1-x-y)^2 M^2 \mathbf{C}_V - 2(1-x-y)M^2 \mathbf{C}_A + 4mM\mathbf{C}_T. \end{aligned} \quad (\text{B4})$$

The momentum integration can be performed in Eq. (B3) using the following standard results:

$$d\mu^{4-d} \int d^d \kappa_E \frac{1}{(\kappa_E^2 + M_{\text{cov}}^2)^3} = \frac{2\pi^2}{M_{\text{cov}}^2} + \mathcal{O}(\epsilon), \quad d\mu^{4-d} \int d^d \kappa_E \frac{\kappa_E^2}{(\kappa_E^2 + M_{\text{cov}}^2)^3} = \pi^2 (2 - \epsilon)^2 \left(\frac{\mu^2}{\pi M_{\text{cov}}^2} \right)^\epsilon \Gamma(\epsilon), \quad (\text{B5})$$

where on the right-hand side we used the definition $2\epsilon = 4 - d$. Expanding the second result above for small ϵ , we have

$$\left(1 - \frac{2}{d} \right) d\mu^{4-d} \int d^d \kappa_E \frac{\kappa_E^2}{(\kappa_E^2 + M_{\text{cov}}^2)^3} = 2\pi^2 \left\{ \frac{1}{\epsilon} - \gamma - \frac{3}{2} + \ln \left(\frac{\mu^2}{\pi M_{\text{cov}}^2} \right) + \mathcal{O}(\epsilon) \right\}. \quad (\text{B6})$$

We finally get

$$\begin{aligned} F_1(q^2) &= \frac{g^2}{4\pi^2} \int_0^1 dx \int_0^{1-x} dy \left\{ \left[\frac{1}{\epsilon} - \gamma - \frac{3}{2} + \ln \left(\frac{\mu^2}{\pi M_{\text{cov}}^2} \right) \right] \mathbf{C}_V + \frac{2Mm(1-x-y)\mathbf{C}_S + (1-x-y)\frac{q^2}{2}\mathbf{C}_A}{M_{\text{cov}}^2} \right. \\ &\quad \left. + \frac{[m^2 + (1-x-y)^2 M^2 + (1-x-y+2xy)\frac{q^2}{2}] \mathbf{C}_V}{M_{\text{cov}}^2} \right\} \end{aligned} \quad (\text{B7})$$

and

$$F_2(q^2) = \frac{g^2}{2\pi^2} \int_0^1 dx \int_0^{1-x} dy \left\{ \frac{2Mm\mathbf{C}_T - Mm(1-x-y)\mathbf{C}_S}{M_{\text{cov}}^2} - \frac{(1-x-y)^2 M^2 \mathbf{C}_V + (1-x-y)M^2 \mathbf{C}_A}{M_{\text{cov}}^2} \right\}. \quad (\text{B8})$$

We now check whether the form factors given by Eqs. (B7) and (B8) are consistent with specific spectator particles, such as scalar meson exchange, vector meson/photon exchange, etc. To do that, we need to consider how the different coefficients are expressed, and this is achieved by using appropriate Fierz rearrangements in $O_{ik} O_{\ell j}$ given by Eq. (1) and Table I. For example, if we want the coefficients for the scalar meson exchange (e.g., the Yukawa model), the proper coefficients are $\mathbf{C}_S = \mathbf{C}_V = 2\mathbf{C}_T = -\mathbf{C}_A = \mathbf{C}_P = \frac{1}{4}$. Substituting these values in Eqs. (B7) and (B8), we get

$$F_1^{\text{scalar}}(q^2) = \frac{g^2}{16\pi^2} \int_0^1 dx \int_0^{1-x} dy \left\{ \frac{1}{\epsilon} - \gamma - \frac{3}{2} + \ln\left(\frac{\mu^2}{\pi M_{\text{cov}}^2}\right) + \frac{[m + (1-x-y)M]^2 + xyq^2}{M_{\text{cov}}^2} \right\},$$

$$F_2^{\text{scalar}}(q^2) = \frac{g^2}{8\pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{(x+y)[m + (1-x-y)M]M}{M_{\text{cov}}^2}.$$
(B9)

This is exactly what we get from the standard calculation, i.e., $N^\mu = (\not{p}_2 + m)\gamma^\mu(\not{p}_1 + m)$ in Eq. (9).

Another example is the calculation of the electron vertex correction in four-dimensional QED, where the exchanged particle is a vector photon (i.e., $m_X = 0$). In this case, the external fermion lines have the same mass as the internal ones, i.e., $M = m$. From the Fierz transformation relations in Table I for this case, we have $C_S = 1$, $C_V = -\frac{1}{2}$, $C_T = 0$, and $C_A = -\frac{1}{2}$, $C_P = -1$. Substituting these values in Eqs. (B7) and (B8), we get

$$F_1^{\text{QED}}(q^2) = -\frac{g^2}{8\pi^2} \int_0^1 dx \int_0^{1-x} dy \left\{ \frac{1}{\epsilon} - \gamma - \frac{3}{2} + \ln\left(\frac{\mu^2}{\pi M_{\text{QED}}^2}\right) + \frac{[(x+y)^2 + 2(x+y-1)]m^2 + (1-x)(1-y)q^2}{M_{\text{QED}}^2} \right\},$$

$$F_2^{\text{QED}}(q^2) = -\frac{g^2}{4\pi^2} \int_0^1 dx \int_0^{1-x} dy \frac{(x+y)(1-x-y)m^2}{M_{\text{QED}}^2},$$
(B10)

where $M_{\text{QED}}^2 = M_{\text{cov}}^2(M \rightarrow m, m_X \rightarrow 0)$. Again, this is exactly the result we get from the standard calculation in Feynman gauge, i.e., $N^\mu = \gamma^\nu(\not{p}_2 + m)\gamma^\mu(\not{p}_1 + m)\gamma_\nu$ in Eq. (9).

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