Exact self-similar solutions in Born–Infeld theory

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We present a new class of exact self-similar solutions possessing cylindrical or spherical symmetry in Born–Infeld theory. A cylindrically symmetric solution describes the propagation of a cylindrical electromagnetic disturbance in a constant background magnetic field in Born–Infeld electrodynamics. We show that this solution corresponds to vacuum breakdown and the subsequent propagation of an electron-positron avalanche. The proposed method of finding exact analytical solutions can be generalized to the model of a spherically symmetric scalar Born–Infeld field in the $(n + 1)$-dimensional Minkowski space-time. As an example, the case $n = 3$ is discussed.

Born–Infeld electrodynamics was proposed in the 1930s as a nonlinear generalization of Maxwell electromagnetism [1]. The central idea of Born and Infeld was to create a relativistic field theory that could admit a finite energy classical solution describing an elementary electric charge. Since the appearance of quantum electrodynamics (QED), the interest in classical field theory has faded considerably. However, in that time, Born–Infeld (BI) theory received a fairly unexpected development. It was related to Heisenberg's suggestion to describe multiple π-meson emission in high-energy hadronic collisions as an expansion of a nonlinear wave packet in the context of the effective scalar BI field [2]. This classical Heisenberg's model has proven to be very fruitful and continues to be developed in the physics of quark-gluon plasma [3,4].

In the past decades, since its rediscovery in a low-energy limit of string theories [5–10], BI electrodynamics has been studied extensively by a large number of workers (see, e.g., [8–16] and references therein). A substantial degree of interest in the subject has also been stimulated by recent advances in laser technology. Modern high-power lasers make it possible to reach an intensity level of 10^{26}–10^{28} W/cm^2. The corresponding field strength is sufficient to make nonlinear electrodynamic effects in a vacuum measurable [17–19]. The experimental verification of BI electrodynamics could serve as an important evidence in favor of string theory, and various ways for laboratory testing of the corresponding nonlinear effects are currently discussed in the literature [19–23].

It is known that BI electrodynamics has unique properties (causal propagation and the absence of birefringence [24–26]) among other relativistic nonlinear theories of the electromagnetic field. Born–Infeld theory possesses an impressive mathematical beauty and has already found numerous applications in different branches of physics.

Because of a rather complicated nonlinearity of the BI field equations, only several exact solutions are known in

this theory. These are the point charge solution found by the creators of the theory [1], 2D electrostatic solutions [27,28], and plane wave solutions [29–32]. The problem of finding new exact solutions in BI electrodynamics is very topical. It is well known that BI equations admit the existence of exact static singular solutions with finite total energy (the so-called BIon solutions). The simplest cases are cylindrically or spherically symmetric solutions with a singularity on the axis or at the origin, respectively. Recently, such solutions have received much attention in string or M theory [7,9]. It should be noted, however, that the properties of nonstationary solutions, which describe the propagation of cylindrical and spherical wave disturbances, remain poorly studied not only in BI theory but also in Maxwell electrodynamics of nonlinear media. Some exact axisymmetric solutions of the Maxwell equations in a nonlinear medium have recently been found in Refs. [33–35]. In this work, we obtain exact self-similar solutions possessing cylindrical or spherical symmetry in BI theory.

The Lagrangian density of BI electrodynamics is given by

$$L = (4\pi)^{-1}b^2\left(1 - \sqrt{1 - b^{-2}I - b^{-4}J^2}\right), \qquad (1)$$

where b is Born's constant [1], $I = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = \mathbf{E}^2 - \mathbf{B}^2$ and $J = -\frac{1}{4}F_{\mu\nu}\mathcal{F}^{\mu\nu} = \mathbf{E} \cdot \mathbf{B}$ are the Poincaré invariants, and $F_{\mu\nu}$ and $\mathcal{F}^{\mu\nu}$ are the electromagnetic field tensor and the dual tensor, respectively. The electromagnetic field equations following from Lagrangian density (1) formally coincide with the Maxwell equations if the constitutive relations have the form $\mathbf{D} = 4\pi\partial L/\partial\mathbf{E}$ and $\mathbf{H} = -4\pi\partial L/\partial\mathbf{B}$. We introduce a cylindrical coordinate system (r, ϕ, z) and assume that the fields are independent of ϕ and z. Then the BI equations admit solutions in which only the E_z and B_ϕ components are nonzero. Denoting these components as E and B, respectively, and using the fact that J vanishes in this case, we can write the field equations in the form

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$$\partial_r H + r^{-1} H = \partial_\tau D, \qquad \partial_r E = \partial_\tau B, \qquad (2)$$

where $D = E(1 - b^{-2}I)^{-1/2}$, $H = B(1 - b^{-2}I)^{-1/2}$, $I = E^2 - B^2$, $\tau = ct$, and c is the speed of light. The second equation in system (2) is satisfied by putting

$$E = -b\partial_\tau \psi, \qquad B = -b\partial_r \psi, \qquad (3)$$

where ψ is the normalized (to b) z component of the vector potential. Assuming that $I \neq b^2$, from the first equation in system (2) we obtain

$$[1 - (\partial_\tau \psi)^2]\partial_r^2 \psi + 2\partial_r \psi \partial_\tau \psi \partial_r(\partial_\tau \psi) - [1 + (\partial_r \psi)^2]\partial_\tau^2 \psi$$
$$+ (n-1)r^{-1}[1 - (\partial_\tau \psi)^2 + (\partial_r \psi)^2]\partial_r \psi = 0, \qquad (4)$$

where $n = 2$ in the case considered. Here, the integer parameter n is introduced for the following reasons. Equation (4) can be considered as the Euler–Lagrange equation for the scalar field $\psi(r, t)$ in the $(n + 1)$-dimensional Minkowski space-time, which follows from the action

$$S = \int [1 - (\partial_\tau \psi)^2 + (\partial_r \psi)^2]^{1/2} r^{n-1} dr d\tau, \qquad (5)$$

where $r = \sqrt{x_1^2 + \cdots + x_n^2}$. Although our main attention will be focused on the cylindrical symmetry in BI electrodynamics ($n = 2$ and $r = \sqrt{x^2 + y^2}$), we will also consider the spherical symmetry that corresponds to the case where $n = 3$ and $r = \sqrt{x^2 + y^2 + z^2}$. The generalization to the higher dimensions is straightforward.

Equation (4) admits self-similar solutions of the form

$$\psi = ru(s), \qquad s = \tau r^{-1}. \qquad (6)$$

Substituting Eq. (6) into Eq. (4) yields the ordinary differential equation

$$(s^2 - u^2 - 1)u'' + (n-1)(u - su')[1 - (u')^2 + (u - su')^2]$$
$$= 0, \qquad (7)$$

where the prime denotes the derivative with respect to s. We will seek an exact solution of Eq. (7) in parametric form:

$$u = \xi^{1/2} \cosh \alpha, \qquad s = \xi^{1/2} \sinh \alpha. \qquad (8)$$

Here, $\alpha = \int \Phi(\xi) d\xi + q$, where q is an arbitrary integration constant. Substituting expressions (8) into Eq. (7) and using the formulas

$$u'_s = u'_\xi / s'_\xi, \qquad u''_{ss} = (s'_\xi u''_{\xi\xi} - u'_\xi s''_{\xi\xi})/(s'_\xi)^3, \qquad (9)$$

we arrive at the Bernoulli equation

$$\frac{d\Phi}{d\xi} = -2\xi[(n-1)\xi - 1]\Phi^3 - \frac{(4-n)\xi + 3}{2\xi(\xi + 1)}\Phi. \qquad (10)$$

Integration of Eq. (10) gives

$$\Phi = \pm \frac{1}{2} \frac{(\xi + 1)^\gamma}{\xi \sqrt{\chi_n(\xi)}}, \qquad (11)$$

where $\gamma = (n - 1)/2$, $\chi_n(\xi) = (\xi + 1)^n - (p + n)\xi$ is a polynomial of order n, and p is an integration constant. Restricting ourselves to consideration only of the simplest cases $n = 2$ and $n = 3$, we write

$$\chi_2(\xi) = \xi^2 - p\xi + 1 \qquad (12)$$

and

$$\chi_3(\xi) = \xi^3 + 3\xi^2 - p\xi + 1. \qquad (13)$$

Thus, Eqs. (6), (8), and (11) give an exact solution of Eq. (4). From these expressions, we have

$$\partial_\tau \psi = \pm \frac{\sqrt{\chi_n(\xi)} \cosh \alpha + (\xi + 1)^\gamma \sinh \alpha}{\sqrt{\chi_n(\xi)} \sinh \alpha + (\xi + 1)^\gamma \cosh \alpha},$$
$$\partial_r \psi = \pm \frac{\sqrt{\xi}(\xi + 1)^\gamma}{\sqrt{\chi_n(\xi)} \sinh \alpha + (\xi + 1)^\gamma \cosh \alpha}. \qquad (14)$$

Now we should examine what physically meaningful solutions can be obtained by appropriately choosing the arbitrary constants p and q and the signs in Eqs. (11) and (14).

Cylindrical symmetry.—Let us consider the following representation of the quantity α in the case $n = 2$:

$$\alpha = \frac{1}{2} \int_{\xi_2}^\xi \frac{\sqrt{\xi + 1} d\xi}{\xi \sqrt{(\xi - \xi_1)(\xi - \xi_2)}}, \qquad (15)$$

where $\xi_1 = p/2 - \sqrt{p^2/4 - 1}$, $\xi_2 = p/2 + \sqrt{p^2/4 - 1}$, $p > 2$, and $\xi > \xi_2$. Note that ξ_1 and ξ_2 are the real-valued roots of the polynomial $\chi_2(\xi)$ in Eq. (12). Reduction of the elliptic integral in Eq. (15) to the standard form gives

$$\alpha = \frac{1}{\sqrt{\xi_2 + 1}}[(\xi_1 - \xi_2)\Pi(\beta, \nu, k) + (\xi_2 + 1)F(\beta, k)], \qquad (16)$$

where F and Π are incomplete elliptic integrals of the first and third kinds, respectively, $\beta = \sqrt{(\xi - \xi_2)/(\xi - \xi_1)}$, $\nu = \xi_1/\xi_2$, and $k = \sqrt{(\xi_1 + 1)/(\xi_2 + 1)}$. We use the following notation for the functions Π and F:

$$\Pi(\beta, \nu, k) = \int_0^\beta \frac{d\zeta}{(1 - \nu\zeta^2)\sqrt{(1 - \zeta^2)(1 - k^2\zeta^2)}} \qquad (17)$$

and $F(\beta, k) = \Pi(\beta, 0, k)$. The components of the axisymmetric electromagnetic field in BI electrodynamics are given by Eqs. (3), (12), (14), and (16). With the identity $ct/r = \xi^{1/2} \sinh \alpha$, these formulas determine E and B as

functions of the radial coordinate and time via the parameter ξ ($\xi_2 < \xi < \infty$).

The energy conservation law $\partial_t W + \nabla \cdot \mathbf{\Sigma} = 0$, with the energy density

$$W = \frac{b^2}{4\pi}\left(\frac{1 + b^{-2}B^2}{\sqrt{1 - b^{-2}I}} - 1\right) \qquad (18)$$

and the Poynting vector

$$\mathbf{\Sigma} = -\hat{\mathbf{e}}_r \frac{c}{4\pi} \frac{EB}{\sqrt{1 - b^{-2}I}}, \qquad (19)$$

can easily be derived directly from the field equations (2).

Figure 1 shows the results of calculations of E and B by Eqs. (3) and (14) for $p = 2.2$ and $p = 6$. The quantities D and H are also presented in this figure for the same values of p. The plots of Fig. 1 can be considered as oscillograms of the field quantities at a fixed point $r = \text{const} \neq 0$. The branch of $\xi^{1/2}$ in Eq. (8) and the signs in Eq. (14) were chosen to provide the continuity of E and B at $\tau = 0$ for any $r \neq 0$ and ensure that the radial component Σ of the Poynting vector is negative for $\tau < 0$ and positive for $\tau > 0$. Figure 2 shows the energy density W, the radial component Σ of the Poynting vector, and the invariant I as functions of τ/r. Figure 3 presents the snapshots of E and B as functions of r for $p = 2.2$ and various values of τ. Since there is no characteristic spatial scale in the problem considered, the radial coordinate is given in arbitrary units. It is seen in Figs. 1 and 3 that the field quantities described by the obtained solution are single-valued continuous functions at any space-time point, except for the points on the symmetry axis. The solution describes the propagation of a
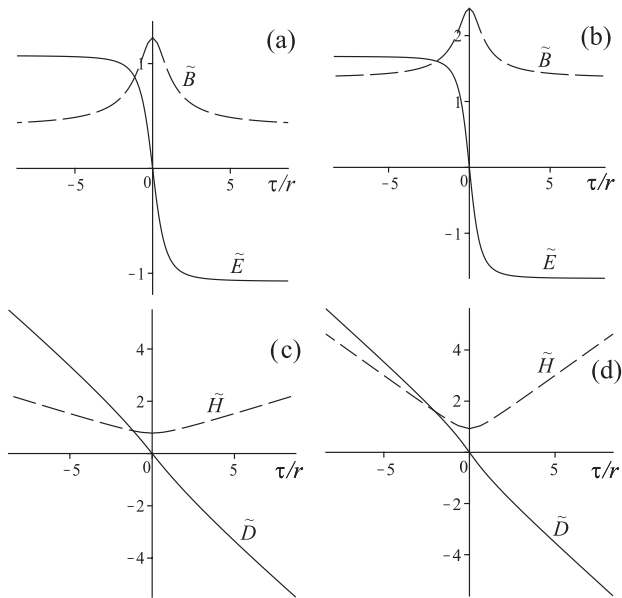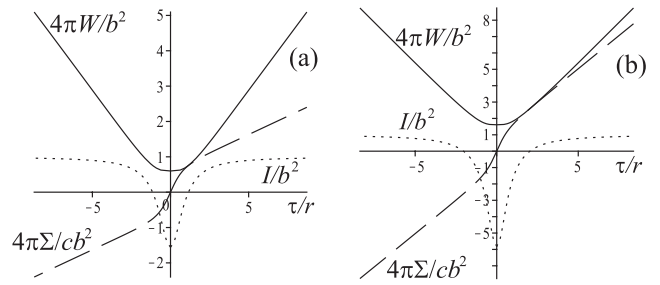

FIG. 2. Normalized energy density, the radial component of the Poynting vector, and the invariant I ($4\pi W/b^2$, $4\pi\Sigma/cb^2$, and I/b^2, respectively) as functions of τ/r for $p = 2.2$ (a) and $p = 6$ (b). It is seen that $I/b^2 \to 1$ for $\tau/r \to \pm\infty$.

cylindrically symmetric disturbance in the azimuthally magnetized vacuum. For $\tau < 0$, the cylindrical electromagnetic wave converges to the axis and the wave profile becomes steeper [see Fig. 3(a)]. As a result, a shock wave forms at $r = 0$ and $\tau = 0$. At this time instant, the electric field and the energy-flow direction reverse their signs, so that for $\tau > 0$ we observe propagation of a divergent cylindrical wave [see Fig. 3(b)]. Although the fields E and B are everywhere finite, it can be shown that D, H, and W behave as $\text{const} \times r^{-1}$ for $r \to 0$ and $\tau \neq 0$. Such a singularity is responsible for a linear increase in the dependences $D(\tau/r)$, $H(\tau/r)$, $W(\tau/r)$, and $\Sigma(\tau/r)$ in Figs. 1(c), 1(d), 2(a), and 2(b). It is also seen in these figures that (i) an increase in W corresponds to the positive values of the invariant I, (ii) the quantities D, H, and W diverge not only at $\tau = \text{const} \neq 0$ and $r \to 0$, but also at $r = \text{const} \neq 0$ and $\tau \to \infty$, and (iii) the local group velocity $v_g = |\Sigma|/W$ does not exceed c. The energy density is everywhere integrable.

The presence of the above singularity allows us to propose a physical interpretation of the obtained exact solution as that due to a distributional source on the axis, which is similar to static BIon solutions [8,9]. The divergent cylindrical wave [see Fig. 3(b)] can be
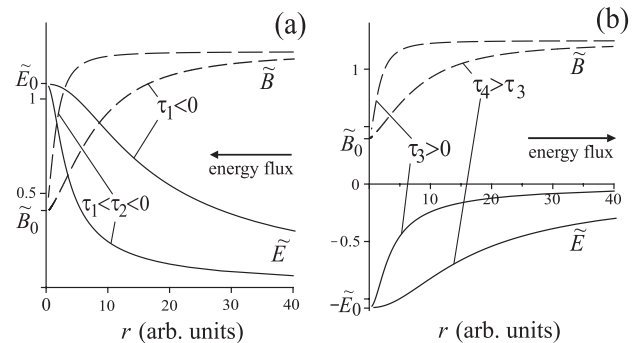

FIG. 1. Normalized field components $\tilde{E} = E/b$, $\tilde{B} = B/b$, $\tilde{D} = D/b$, and $\tilde{H} = H/b$ as functions of τ/r for $p = 2.2$ (a, c) and $p = 6$ (b, d).

FIG. 3. Normalized electric and magnetic fields ($\tilde{E} = E/b$ and $\tilde{B} = B/b$, respectively) as functions of r for $p = 2.2$ and various values of τ: $\tau_1 < 0$ and $\tau_1 < \tau_2 < 0$ (a) and $\tau_3 > 0$ and $\tau_4 > \tau_3$ (b).

excited by the ''switching on'' of a delta-function source on the axis at the time instant $\tau = 0$. Since charges and currents cannot be specified independently of the field in nonlinear BI electrodynamics [1], the existence of such a source is inseparably related to the presence of a constant background azimuthal magnetic field $B(r) \equiv B_m = b\sqrt{\xi_2} > b$ at $\tau = 0$ and $r \neq 0$. On the axis, the chosen source supports constant fields $E(r = 0, \tau > 0) = -E_0$ and $B(r = 0, \tau > 0) = B_0$ such that $I = E_0^2 - B_0^2 = b^2$ and $(D, H) \to \infty$. This limiting field state existing in BI electrodynamics has received an interpretation in string theory as a divergence in the rate of pair production of open strings [9,36]. An analogous effect follows from the QED model of vacuum polarization. In QED, the electric field, the strength of which is close to the Schwinger limit, may cause electron-positron pair creation from a vacuum [37]. The necessary condition for the Schwinger pair creation process is $I > 0$ [17,37]. The interaction of the created electrons and positrons with a sufficiently strong field can lead to production of multiple new particles and avalanchelike vacuum breakdown [38,39]. Thus, we can state that the obtained solution describes radial expansion of the electron-positron plasma bunch. Because of an avalanchelike electromagnetic cascade, we have infinite polarization and magnetization of the vacuum. This process, which is created on the axis, progressively propagates in the whole space. From the classical viewpoint, such an expansion can be explained intuitively as a drift of charged particles in the crossed fields E_z and B_ϕ.

Spherical symmetry.—Since a continuous tangential vector field on a sphere cannot depend only on the radial coordinate, this case obviously has no direct bearing on electrodynamics. However, the model of a spherically symmetric scalar field with BI action (5) is studied intensely in connection with string/M theory [40–43].

In what follows, we consider the solution of Eq. (4) with $n = 3$ in a way similar to that used for the axial symmetry. For $n = 3$, the constants p and q can be chosen so that the quantity α is given by

$$\alpha = \frac{1}{2} \int_{\xi_3}^{\xi} \frac{(\xi + 1)d\xi}{\xi\sqrt{(\xi - \xi_1)(\xi - \xi_2)(\xi - \xi_3)}}, \quad (20)$$

where ξ_1, ξ_2, and ξ_3 are the real-valued roots of the polynomial $\chi_3(\xi)$ in Eq. (13) such that $\xi_3 > \xi_2 > \xi_1$. Since complete analysis of the cubic equation is cumbersome, we consider only a single value $p = 5$ in Eq. (13). In this case, the above-mentioned roots are equal to $\xi_1 = -2 - \sqrt{5}$, $\xi_2 = -2 + \sqrt{5}$, and $\xi_3 = 1$, so that α can be represented as

$$\alpha = \frac{1}{\xi_2\sqrt{1 - \xi_1}}[(\xi_2 - 1)\Pi(\beta, \xi_2, k) + (\xi_2 + 1)F(\beta, k)], \quad (21)$$
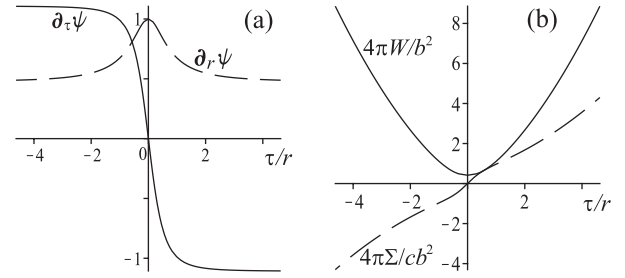

FIG. 4. Quantities $\partial_\tau \psi$ and $\partial_r \psi$ of the spherically symmetric BI field (a) and the energy density and the radial component of the energy flux (b) as functions of τ/r.

where $\beta = \sqrt{(\xi - 1)/(\xi - \xi_2)}$ and $k = \sqrt{(\xi_2 - \xi_1)/(1 - \xi_1)}$. The quantities $\partial_\tau \psi$ and $\partial_r \psi$ are given by Eqs. (13), (14), and (21). The energy density and flux can readily be found from Eqs. (18) and (19) by representing E and B in terms of the derivatives of ψ in accordance with Eq. (3). The results of calculations of $\partial_\tau \psi$, $\partial_r \psi$, W, and Σ are shown in Fig. 4. It is seen in the figure that by analogy with the cylindrical case, the solution describes the propagation of a spherically symmetric disturbance in a constant background field. It can be shown that $W \sim \text{const} \times r^{-2}$ for $r \to 0$ ($\tau \neq 0$) and, hence, the singularity at the origin is integrable. The physical constraint $v_g = |\Sigma|/W \leq c$ is also satisfied.

In conclusion, we emphasize that the obtained exact solutions exist essentially due to BI nonlinearity and spatial symmetry. Maxwell equations ($b \to \infty$) as well as BI equations in flat geometry [$n = 1$ in Eq. (4)] obviously do not allow the existence of such solutions. Although the obtained solutions do not have a finite total energy, their energy density is locally integrable. Because of this, by suitably cutting off, the obtained solutions, as, e.g., plane wave solutions, may provide useful approximations to solutions with finite total energy. Moreover, BI wave propagation in a constant background field represents considerable interest in string theory [10]. Finally, we note that Eq. (4) also admits self-similar solutions $\psi = \tau u(r/\tau)$. The parametrization $u = \tanh \alpha$ and $r\tau^{-1} = (-\xi)^{-1/2} \times (\cosh \alpha)^{-1}$ with $\alpha = \int \Phi(\xi)d\xi + q$ leads again to Eq. (10). Because of discontinuity or ambiguity, we failed to give some of these solutions any physical interpretation. However, it can be assumed that among the whole set of partial solutions, there may exist physically meaningful ones. Since they are determined by three governing parameters (n and two integration constants) of the problem, an appropriate choice of these parameters needs further studies.

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