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Note on an application of the method of uniqueness to reduced quantum electrodynamics

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Using the method of uniqueness a two-loop massless propagator Feynman diagram with a noninteger index on the central line is evaluated in a very transparent way. The result is applied to the computation of the two-loop polarization operator in reduced quantum electrodynamics.

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I. INTRODUCTION

The exact analytical computation of multiloop Feynman diagrams is of crucial importance for the evaluation of renormalization group functions, i.e., β functions and anomalous dimensions of fields. Since the early days of quantum field theory a variety of methods have been developed, and often combined, in order to achieve this task, e.g., the Gegenbauer polynomial technique [1,2], integration by parts [3,4], and the method of uniqueness [3,5–7]. The latter allows, in principle, the computation of complicated Feynman diagrams using sequences of simple transformations. A given diagram is straightforwardly integrated once the appropriate sequence is found. The task of finding such a sequence for a given diagram is, however, nontrivial; see Ref. [8] for a review.

One of the basic building blocks of multiloop calculations is the two-loop massless propagator diagram:

$$J(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5) = \iint \frac{\mathrm{d}^D k_1 \mathrm{d}^D k_2}{k_1^{2\alpha_1} k_2^{2\alpha_2} (k_2-p)^{2\alpha_3} (k_1-p)^{2\alpha_4} (k_2-k_1)^{2\alpha_5}}, \quad (1)$$

with arbitrary indices α_i and external momentum p in a Euclidean space-time of dimensionality D (Fig. 1); see Ref. [9] for a historical review on this diagram. When all indices are integers the diagram of Eq. (1) is well known and easily calculated. Its evaluation for arbitrary indices is however highly nontrivial: the results can be represented [10] as a combination of twofold series. In some particular cases, however, the results can be obtained [2,3,7,11–14] in significantly simpler form. In Ref. [2] a class of complicated diagrams, with two integer indices on adjacent lines and three other arbitrary indices, has been computed exactly on the basis of a new development of the Gegenbauer polynomial technique. The latter approach is technically involved and the result is expressed in terms of a generalized hypergeometric function, ${}_3F_2$ with argument 1. For this class of diagrams, similar results have been found in Ref. [13] using an ansatz to solve the recurrence relations for the two-loop diagram arising from integration by parts. In this brief report we consider the simplest but an important diagram belonging to this class, whose coefficient function reads

$$I(\lambda) = \frac{p^{2(2-\lambda)}}{\pi^D} J(1,1,1,1,\lambda), \qquad \lambda = \frac{D}{2} - 1, \quad (2)$$

where the index α_5 has been restricted to λ, all other indices being 1. This diagram has been already calculated [3] (see also discussions in Ref. [12]) but it seems that our evaluation is simpler and more transparent. As will be seen in the following, the diagram can indeed be straightforwardly integrated using the method of uniqueness in momentum space with the help of a simple but ingenious three-step transformation. As an application, we compute the two-loop polarization operator (Fig. 2) in reduced quantum electrodynamics [15].

II. DEFINITIONS AND NOTATIONS

In what follows we use dimensional regularization. All calculations are performed in a Euclidean space-time of dimensionality $D = 2 + 2\lambda$ which may either be even dimensional ($\lambda \to 1$) or odd dimensional ($\lambda \to 1/2$). Diagrams will be analyzed in momentum space. For a given diagram, the integrations are over loop momenta and the lines are simple power laws of the form: $1/k^{2\alpha}$

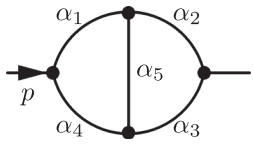
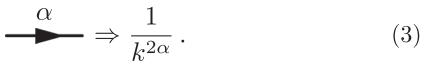


FIG. 1. Two-loop massless propagator diagram.

(a) (b)

FIG. 2. Two-loop vacuum polarization diagrams.

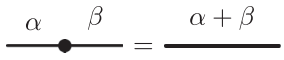
where α is the index of the line. The index of a diagram is defined as the sum of the indices of its constituent lines. A line with an arbitrary index α can be represented graphically as



(3)

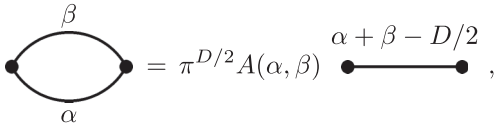
In order to simplify notations, arrows will not be displayed in the following but they are implied, e.g., Fig. 1.

With these definitions and graphical notations, chains reduce to the product of propagators:



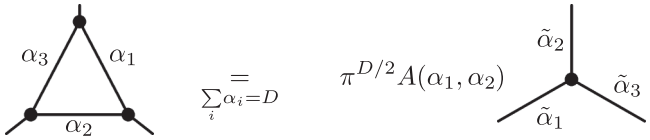
(4)

On the other hand simple loops involve an integration:



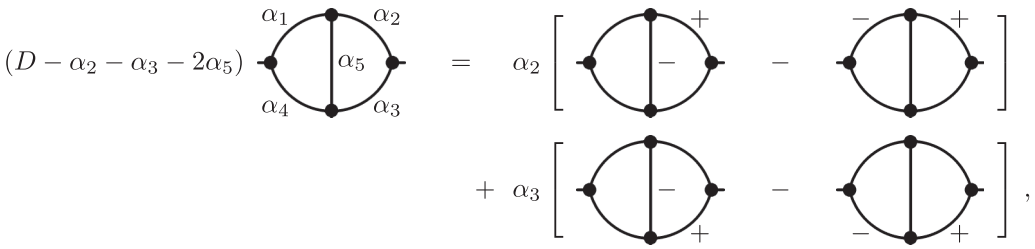
$$A(\alpha,\beta) = \frac{a(\alpha)a(\beta)}{a(\alpha+\beta-D/2)}, \quad a(\alpha) = \frac{\Gamma(D/2-\alpha)}{\Gamma(\alpha)}. \tag{5}$$

In momentum space a vertex and a triangle are said to be unique if their indices are equal to $D/2 = 1 + \lambda$ and $D = 2 + 2\lambda$, respectively. They are related to each other by the uniqueness (or star-triangle) relation:



(6)

where $\tilde{\alpha}_i = D/2 - \alpha_i$ is the index dual to α_i. Finally, for an arbitrary triangle (unique or not) the following recurrence relation is obtained from integration by parts [3,4]:

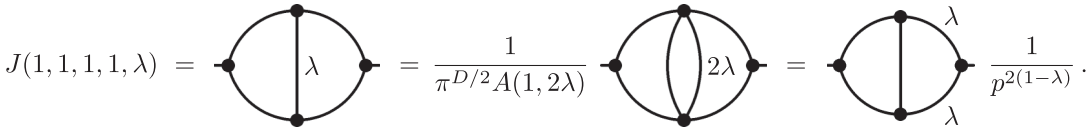


(7)

where $\pm$ on the right-hand side of the equation denotes the increase or decrease of a line index by 1 with respect to its value on the left-hand side. In the following, in order to simply notations, we will assume that lines with no index are ordinary lines. In momentum space ordinary lines have index $\alpha = 1$.

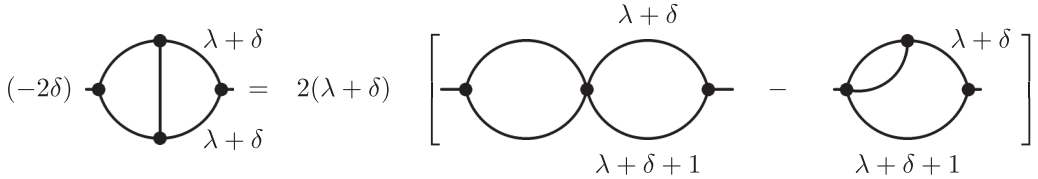
III. CALCULATION OF THE DIAGRAM

With the help of the above notations and identities we proceed on calculating $I(\lambda)$. The first transformation consists in replacing the central line by a loop [16], Eq. (5), in order to make the right triangle unique. The uniqueness relation, Eq. (6), can then be used. In graphical notations this reads



(8)

Finally, using integration by parts, Eq. (7), the last diagram is reduced to sequences of chains and simple loops which can immediately be integrated with the help of Eqs. (4) and (5):



(9a)

$$= \frac{\pi^D 2(\lambda+\delta)}{p^{2(1+2\delta)}} A(1,1)[A(\lambda+\delta+1,\lambda+\delta) - A(\lambda+\delta+1,1+\delta)]_1, \tag{9b}$$

where the parameter δ has been introduced. The function $A(1,1)$ and the bracketed terms in Eq. (9b) read

$$A(1,1) = \frac{\Gamma^2(\lambda)\Gamma(1-\lambda)}{\Gamma(2\lambda)}, \tag{10a}$$

$$(\lambda+\delta)[\bullet]_1 = \frac{\Gamma(-\delta)}{\Gamma(\lambda+\delta)} \frac{\Gamma(\lambda-\delta)\Gamma(1+2\delta)}{\Gamma(1+\delta)\Gamma(\lambda-2\delta)} \left[\frac{\Gamma(1-\delta)\Gamma(1+\delta)\Gamma(\lambda+2\delta)\Gamma(\lambda-2\delta)}{\Gamma(1-2\delta)\Gamma(1+2\delta)\Gamma(\lambda+\delta)\Gamma(\lambda-\delta)} - 1\right]_2. \tag{10b}$$

At this point it is convenient to use the following product expansion of the Gamma function:

$$\Gamma(x+\varepsilon) = \Gamma(x)\exp\left[\sum_{k=1}^{\infty} \psi^{(k-1)}(x)\frac{\varepsilon^k}{k!}\right], \qquad \psi(x) = \psi^{(0)}(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \qquad \psi^{(k)}(x) = \frac{\mathrm{d}^k}{\mathrm{d}x^k}\psi(x), \tag{11}$$

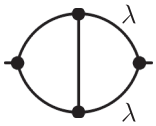
where $\psi^{(k)}$ is the polygamma function of order k. From Eq. (11), the following relation is obtained:

$$\Gamma(x+\varepsilon)\Gamma(x-\varepsilon) = \Gamma^2(x)\exp\left[2\sum_{m=1}^{\infty} \psi^{(2m-1)}(x)\frac{\varepsilon^{2m}}{(2m)!}\right]. \tag{12}$$

Making use of Eq. (12) in the bracket of Eq. (10b) yields

$$[\bullet]_2 = \exp\left[2\sum_{m=1}^{\infty}(2^{2m}-1)[\psi^{(2m-1)}(\lambda) - \psi^{(2m-1)}(1)]\frac{\delta^{2m}}{(2m)!}\right] = 3\delta^2[\psi'(\lambda) - \psi'(1)]_3 + \mathrm{O}(\delta^4). \tag{13}$$

Substituting back Eq. (13) in (10) and performing the remaining δ expansion yields

$$(\lambda+\delta)\left[\bullet\right]_1 = \frac{-3\delta}{\Gamma(\lambda)}\left[\bullet\right]_3 \underset{\text{Eq. (9)}}{\Rightarrow}$$  $$= \frac{\pi^D}{p^2}\, 3\, \frac{\Gamma(\lambda)\Gamma(1-\lambda)}{\Gamma(2\lambda)}\left[\psi'(\lambda) - \psi'(1)\right], \tag{14}$$

where, in the last step, Eq. (9) has been used and δ sent to zero. Substituting the final result of Eq. (14) in Eq. (8) and using Eq. (2), we obtain the advertised result [3,12] for the coefficient function:

$$I(\lambda) = 3\frac{\Gamma(\lambda)\Gamma(1-\lambda)}{\Gamma(2\lambda)}[\psi'(\lambda) - \psi'(1)], \tag{15}$$

where $\psi'(x)$ is the trigamma function. In the even-dimensional case ($\lambda \to 1$ or $D \to 4$) the well-known result, $I(1) = 6\zeta(3)$, is obtained. On the other hand, in the odd-dimensional case ($\lambda \to 1/2$ or $D \to 3$), which is one of the cases of interest to Refs. [12,17], the result reads $I(1/2) = 6\pi\zeta(2)$.

IV. APPLICATION

We now focus on the computation of radiative corrections to the polarization operator $\Pi^{\mu\nu}(q) = \Pi(q^2)(g^{\mu\nu}q^2 - q^\mu q^\nu)$ in reduced quantum electrodynamics [15] (RQED$_{d_\gamma,d_e}$). The latter describes the interaction of a photon field living in d_γ dimensions with a fermion field living in a reduced space-time of d_e dimensions ($d_e \leq d_\gamma$). Within dimensional regularization, the computation of Feynman integrals in such a reduced theory can be carried out by introducing two epsilon parameters, ε_γ and ε_e, such that $d_\gamma = 4 - 2\varepsilon_\gamma$ and $d_e = 4 - 2\varepsilon_e - 2\varepsilon_\gamma$, respectively. In Ref. [17] the corrections up to two loops (see Fig. 2 where the corresponding diagrams were displayed) were computed for an arbitrary RQED$_{d_\gamma,d_e}$ using the general result of Ref. [2] for $I(\lambda)$. The resulting expression is rather cumbersome. Here, we focus on the case of RQED$_{4,d_e}$. In the limit $\varepsilon_\gamma \to 0$ and using Eq. (15) we obtain the following simpler and more explicit formulas (see definitions in Ref. [17]):

$$2\Pi_{2a}(q^2) = d\frac{e^4\Gamma(\lambda)\Gamma^2(1+\varepsilon_\gamma)}{(4\pi)^{3+\lambda-2\varepsilon_\gamma}(q^2)^{1-\lambda+2\varepsilon_\gamma}}\frac{16\Gamma(1+\lambda)\Gamma(1-\lambda)}{\Gamma(3+2\lambda)}\left\{\lambda^2\left(\frac{1}{\varepsilon_\gamma}+\bar{\psi}+\frac{2}{1+2\lambda}\right)+\frac{3\lambda}{2}-2+\frac{2}{1+\lambda}+O(\varepsilon_\gamma)\right\}, \tag{16a}$$

$$\Pi_{2b}(q^2) = d\frac{e^4\Gamma(\lambda)\Gamma^2(1+\varepsilon_\gamma)}{(4\pi)^{3+\lambda-2\varepsilon_\gamma}(q^2)^{1-\lambda+2\varepsilon_\gamma}}\frac{16\Gamma(1+\lambda)\Gamma(1-\lambda)}{\Gamma(3+2\lambda)}\left\{-\lambda^2\left(\frac{1}{\varepsilon_\gamma}+\bar{\psi}+\frac{2}{1+2\lambda}\right)+\frac{\lambda}{2}-\frac{1}{2}-\frac{3}{2\lambda}\right.$$
$$\left.-\frac{1}{1+\lambda}+\frac{3}{2}\lambda(1+\lambda)[\psi'(\lambda)-\psi'(1)]+O(\varepsilon_\gamma)\right\}, \tag{16b}$$

where $\lambda = 1-\varepsilon_e$ and $\bar{\psi} = 3\psi(2\lambda)-2\psi(\lambda)+2\psi(1-\lambda)-3\psi(1)$. The one-loop and total two-loop contributions therefore read

$$\Pi_1(q^2) = -d\frac{e^2\Gamma(\lambda)}{(4\pi)^{1+\lambda}(q^2)^{1-\lambda}}\frac{\Gamma(1+\lambda)\Gamma(1-\lambda)}{(1+2\lambda)\Gamma(2\lambda)}, \tag{17a}$$

$$\Pi_2(q^2) = d\frac{e^4\Gamma(\lambda)}{(4\pi)^{3+\lambda}(q^2)^{1-\lambda}}\frac{16\Gamma(1+\lambda)\Gamma(1-\lambda)}{\Gamma(3+2\lambda)}C_1(\lambda), \tag{17b}$$

$$C_1(\lambda) = 2\lambda-\frac{5}{2}-\frac{3}{2\lambda}+\frac{1}{1+\lambda}+\frac{3}{2}\lambda(1+\lambda)[\psi'(\lambda)-\psi'(1)]. \tag{17c}$$

From Eq. (17) we see that $\Pi_1(q^2)$ and $\Pi_2(q^2)$ are finite as long as $\lambda \neq 1$. We can then replace e^2 by $4\pi\alpha$ in (17) which yields

$$\Pi_2(q^2) = -\frac{\alpha}{\pi\lambda(1+\lambda)}C_1(\lambda)\Pi_1(q^2). \tag{18}$$

It should be noted that these results cannot be used for QED_4 which can be reached from a general $RQED_{d_\gamma,d_e}$ (see Ref. [17]) by first fixing $\varepsilon_e = 0$ and then taking the limit $\varepsilon_\gamma \to 0$. The results (17) are singular in the limit $\lambda \to 1$ but this limit corresponds to $\varepsilon_\gamma = 0$ and $\varepsilon_e \to 0$, which does not lead to QED_4.

The total, up to two loops, gauge-field self-energy in $RQED_{4,d_e}$ ($\varepsilon_\gamma = 0$ and arbitrary ε_e) may then bewritten as

$$\Pi(q^2) = \Pi_1(q^2)(1+\alpha C(\lambda)+O(\alpha^2)),$$
$$C(\lambda) = -\frac{1}{\pi\lambda(1+\lambda)}C_1(\lambda) = -\frac{1}{2\pi}\left(3[\psi'(\lambda+2)-\psi'(1)]+\frac{4}{1+\lambda}+\frac{1}{(1+\lambda)^2}\right). \tag{19}$$

For $\lambda = 1/2$, i.e., in the case of $RQED_{4,3}$ ($\varepsilon_\gamma = 0$ and $\varepsilon_e = 1/2$) which corresponds to an ultrarelativistic model of graphene [18] (a 2-brane), we reproduce the basic result of Refs. [17,19]:

$$C_1(1/2) = \frac{9\pi^2-92}{24}, \qquad C(1/2) = \frac{92-9\pi^2}{18\pi}. \tag{20}$$

This coefficient is small, $C(1/2) \approx 0.056$, in qualitative agreement with some results obtained in the nonrelativistic limit [22] (see however Ref. [23]) as well as experimental results [24] where $C(1/2)$ corresponds to an interaction correction coefficient to the optical conductivity of undoped graphene.

As a next step of our future investigations we would like to evaluate the fermion self-energy of general $RQED_{d_\gamma,d_e}$ in the ultrarelativistic limit.

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