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Action principle for the connection dynamics of scalar-tensor theories

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A first-order action for scalar-tensor theories of gravity is proposed. The Hamiltonian analysis of the action gives the desired connection dynamical formalism, which was derived from the geometrical dynamics by canonical transformations. It is shown that this connection formalism in the Jordan frame is equivalent to the alternative connection formalism in the Einstein frame. Therefore, the action principle underlying loop quantum scalar-tensor theories is recovered.

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I. INTRODUCTION

Modified gravity theories have recently received increased attention in issues related to the "dark Universe" and nontrivial tests on gravity beyond general relativity (GR). Since 1998, a series of independent astronomic observations implied that our Universe is currently undergoing a period of accelerated expansion [1]. This causes the "dark energy" problem in the framework of GR. It is thus reasonable to consider the possibility that GR is not a valid theory of gravity on a Galactic or cosmological scale. A simple and typical modification of GR is the so-called $f(R)$ theory of gravity [2]. Besides $f(R)$ theories, a well-known competing relativistic theory of gravity was proposed by Brans and Dicke in 1961 [3], which is apparently compatible with Mach's principle. To represent a varying "gravitational constant," a scalar field is nonminimally coupled to the metric in the Brans-Dicke theory. To be compared with the observational results within the framework of a broad class of theories, the Brans-Dicke theory was generalized by Bergmann [4] and Wagoner [5] to general scalar-tensor theories (STT). Scalar-tensor modifications of GR are also popular in unification schemes such as string theory (see, e.g., Refs. [6–8]). Note that the metric $f(R)$ theories and Palatini $f(R)$ theories are equivalent to the special kinds of STT with the coupling parameter $\omega = 0$ and $\omega = -\frac{3}{2}$, respectively [2], while the original Brans-Dicke theory is the particular case of constant ω and vanishing potential of ϕ.

In the past two decades, a nonperturbative quantization of GR, called loop quantum gravity (LQG), has matured [9–12]. It is remarkable that both $f(R)$ theories and STT can be nonperturbatively quantized by extending the LQG techniques [13–15]. Thus, LQG is extended to more general metric theories of gravity [16,17]. The background independent quantization method relies on the key observations that these theories can be cast into the connection dynamical formulations with the structure group $SU(2)$. The connection dynamical formulation of $f(R)$ theories and STT were obtained by canonical transformations from their geometrical dynamics [13–15]. However, the action principle for the above connection dynamics of either $f(R)$ theories or STT is still lacking, although the first-order action for the connection dynamics in the Einstein frame of STT was proposed in Ref. [18]. The purpose of this paper is to fill this gap. We will propose a first-order action for general STT of gravity, which includes $f(R)$ theories as special cases. The connection dynamical formalism will be derived from this action by Hamiltonian analysis. It turns out that this connection dynamics is exactly the same as that derived from the geometrical dynamics by canonical transformations. Moreover, the equivalence between this connection formalism in the Jordan frame and the alternative one in Einstein frame will be proved. Hence, loop quantum STT, as well as loop quantum $f(R)$ theories, have their foundation of action principle.

Throughout the paper, we use the Latin alphabet $a, b, c, \dots$, to represent abstract index notation of spacetime [19], the capital Latin alphabet $I, J, K, \dots$, for internal Lorentzian indices, and $i, j, k, \dots$, for internal $SU(2)$ indics. The other conventions are as follows. The internal Minkowski metric is denoted by $\eta_{IJ} = \mathrm{diag}(-1, 1, 1, 1)$. The Hodge dual of a differential form F_{IJ} is denoted by ${}^\star F_{IJ} = \frac{1}{2}\epsilon_{IJKL}F^{KL}$, where ϵ_{IJKL} is the internal Livi-Civital symbol. The antisymmetry of a tensor A_{IJ} is defined by $A_{[IJ]} = A_{IJ} - A_{JI}$.

II. EQUATIONS OF MOTION

In order to get the Lagrangian formalism of connection dynamics of STT proposed in Ref. [15], let us first consider the following first-order action on a four-dimensional spacetime M,

$$\begin{aligned} S[e, \omega, \phi] &= \int_M \mathcal{L} d^4x \\ &= \int_M \frac{1}{2}\Big(\phi e e_I^a e_J^b \bar{\Omega}_{ab}{}^{IJ} - 2 e e_I^a e_J^b \bar{\omega}_a^{IJ} \bar{\partial}_b \phi \\ &\quad + e e_I^{[a} e_J^{b]} \bar{\partial}_a (e_b^I e^{cJ} \bar{\partial}_c \phi) \\ &\quad + \Big(\frac{3}{2\phi} - K(\phi)\Big) e \bar{\partial}_a \phi \bar{\partial}^a \phi \\ &\quad - 2eV(\phi) + e e_I^a e_J^b \frac{1}{\gamma} {}^\star\bar{\Omega}_{ab}{}^{IJ}\Big) d^4x, \end{aligned} \tag{1}$$

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where $e = \det(e_a^I)$ is the determinant of the right-handed cotetrad e_a^I, $\bar{\Omega}_{ab}{}^{IJ} = \bar{\partial}_{[a}\bar{\omega}_{b]}^{IJ} + \bar{\omega}_{[a}^{IK}\bar{\omega}_{b]K}{}^{J}$ is the curvature of the $SL(2,\mathbb{C})$ spin connection $\bar{\omega}_a^{IJ}$, $V(\phi)$ is the potential of the scalar field ϕ with ϕ satisfying $\phi > 0$, $K(\phi)$ is an arbitrary function of ϕ, and γ is an arbitrary real number. The variation of action (1) with respect to $\bar{\omega}_a^{IJ}$ gives

$$\phi\bar{\mathcal{D}}_a(ee_I^{[a}e_J^{b]}) + \frac{1}{\gamma}{}^{\star}\bar{\mathcal{D}}_a(ee_I^{[a}e_J^{b]}) = 0. \qquad (2)$$

Here the generalized derivative operator $\bar{\mathcal{D}}_a$ is defined as

$$\bar{\mathcal{D}}_a e_b^I = \bar{\partial}_\alpha e_b^I - \bar{\Gamma}_{ab}^c e_c^I + \bar{\omega}_a^{IJ} e_{bJ}, \qquad (3)$$

where $\bar{\Gamma}_{ab}{}^{c}$ is a torsion-free affine connection. From Eq. (2) we have (see Ref. [20] for details)

$$\bar{\mathcal{D}}_{[a}(e_{b]}^I) = 0, \qquad (4)$$

which tells us that the spin connection $\bar{\omega}_a^{IJ}$ is compatible with tetrad e_a^I. On the other hand, taking account of Eq. (4), the variation of action (1) with respect to the tetrad e_a^I gives

$$\begin{aligned}\phi G_{ab} = {}& \left(K - \frac{3}{2\phi}\right)\left((\bar{\partial}_a\phi)\bar{\partial}_b\phi - \frac{1}{2}g_{ab}(\bar{\partial}_c\phi)\bar{\partial}^c\phi\right) \\ & + \bar{\nabla}_a\bar{\nabla}_b\phi - g_{ab}\bar{\nabla}_c\bar{\nabla}^c\phi - g_{ab}V, \end{aligned} \qquad (5)$$

where G_{ab} is the Einstein tensor of e_a^I and $\bar{\nabla}_a$ is the covariant derivative operator compatible with g_{ab}.

Finally, taking account of Eq. (4), the variation of action (1) with respect to the scalar field ϕ gives

$$R + 2\left(K - \frac{3}{2\phi}\right)\bar{\nabla}_a\bar{\nabla}^a\phi - \left(K - \frac{3}{2\phi}\right)'(\bar{\partial}_a\phi)\bar{\partial}^a\phi - 2V' = 0, \qquad (6)$$

where a prime over a function represents a derivative with respect to the argument ϕ. We define a new function

$$\frac{\omega(\phi)}{\phi} := K(\phi) - \frac{3}{2\phi}. \qquad (7)$$

Then it is straightforward to transform Eqs. (5) and (6) into the form in Ref. [15]. Hence, the first-order action (1) gives exactly the equations of motion of STT.

III. HAMILTONIAN ANALYSIS

Let the spacetime M be topologically $\Sigma \times \mathbb{R}$ for some three-manifold Σ. One introduces a foliation of M and a time-evolution vector field t^a in it. t^a can be decomposed with respect to the unit normal vector n^a of Σ as

$$t^a = Nn^a + N^a, \qquad (8)$$

where N and N^a are the lapse function and shift vector, respectively. In the $(3+1)$ decomposition of M, it is convenient to make a gauge fixing $n_I := n^a e_{aI} = (1,0,0,0)$ in the internal space [21]. In a coordinate system adopted to the $(3+1)$ decomposition, the Lagrangian density in Eq. (1) reads

$$\begin{aligned}\mathcal{L} = {}& \frac{1}{\gamma}\tilde{E}_j^b(\gamma\dot{K}_b^j + \dot{\omega}_b^j) - \frac{1}{\phi}\tilde{E}_j^b K_b^j\dot{\phi} + \bar{K}_t^j\left(\mathcal{D}_b\tilde{E}_j^b - \frac{1}{\gamma\phi^2}\epsilon_{jl}{}^{m}K_b^l\tilde{E}_m^b\right) + \frac{1}{\gamma}\bar{\omega}_t^j(\partial_b\tilde{E}_j^b + \epsilon_{jl}{}^{m}(\gamma K_b^l + \omega_b^l)\tilde{E}_m^b) \\ & - N^a\left(\tilde{E}_j^b\mathcal{D}_{[a}K_{b]}^j - \frac{1}{\phi}\tilde{E}_j^b K_b^j\partial_a\phi\right) - N^a\left(\frac{1}{\gamma}\tilde{E}_j^b\Omega_{ab}{}^{j} - \tilde{E}_j^b\frac{1}{\gamma\phi^2}\epsilon^j{}_{lm}K_a^l K_b^m\right) - \frac{\phi}{2}\underline{N}\tilde{E}_i^a\tilde{E}_j^b\epsilon^{ij}{}_k\left(\Omega_{ab}{}^{k} - \frac{1}{\phi^2}\epsilon^k{}_{lm}K_a^l K_b^m\right) \\ & - NEE_j^b(\partial_b(E^{cj}\partial_c\phi) + \omega_b^{jk}E_k^c\partial_c\phi) + \frac{K}{2\underline{N}}(\dot{\phi} - N^a\partial_a\phi)^2 - \frac{1}{2}\left(K - \frac{3}{2\phi}\right)\underline{N}\tilde{E}_i^a\tilde{E}^{bi}(\partial_a\phi)\partial_b\phi \\ & + \frac{1}{\gamma}\underline{N}\tilde{E}_i^a\tilde{E}_j^b\epsilon^{ij}{}_k\mathcal{D}_a\omega_b^{k0} - NEV(\phi), \end{aligned} \qquad (9)$$

where a dot over a letter represents a derivative with respect to the time coordinate, and we have defined

$$\bar{K}_a^i := \phi\bar{\omega}_a^{io} + \frac{1}{2}E_a^i n^c\bar{\partial}_c\phi, \qquad (10)$$

$$\Omega_{ab}{}^{k} := \partial_{[a}\omega_{b]}^k + \epsilon^k{}_{lm}\omega_a^l\omega_b^m, \qquad (11)$$

$$\bar{\omega}_a^i := -\frac{1}{2}\epsilon^i{}_{jk}\bar{\omega}_a^{jk}, \qquad (12)$$

and $\bar{K}_t^i := t^a\bar{K}_a^i$, $\bar{\omega}_t^i := t^a\bar{\omega}_a^i$ are the time component of $\bar{K}_a^i$ and $\bar{\omega}_a^i$, E is the square root of the determinant of the spatial metric $q_{ab} := g_{ab} + n_a n_b$, $E_I^a := q_b^a e_I^b$, $\omega_a^{IJ} := q_a^b\bar{\omega}_b^{IJ}$ $K_a^i := q_a^b\bar{K}_b^i$ are the spatial components of e_I^a, $\bar{\omega}_a^{IJ}$ and $\bar{K}_a^i$, respectively, $\mathcal{D}_a$ is the spatial $SO(1,3)$ generalized covariant derivative operator reduced from $\bar{\mathcal{D}}_a$ and corresponds to a $SO(1,3)$-valued spatial connection one-form ω_a^{ij}, ∂_a is the flat derivative operator on Σ reduced from $\bar{\partial}_a$, $\underline{N} := N/E$ is the densitized lapse scalar of weight -1, and $\tilde{E}_i^a := EE_i^a$ is the densitized spatial triad of weight 1.

Recall that the unique torsion-free $SO(3)$ generalized covariant derivative operator annihilating E_i^a is defined as

$$\nabla_a E_i^b = \partial_a E_i^b + \Gamma_{ac}^b E_i^b + \Gamma_{ai}{}^{j}E_j^b = 0, \qquad (13)$$

where Γ_{ac}^b and $\Gamma_{ai}{}^{j}$ are, respectively, the Levi-Civita connection and the spin connection on Σ. For convenience we define

$$\Gamma_a^i := -\frac{1}{2}\epsilon^i{}_{jk}\Gamma_a^{jk}. \tag{14}$$

Let $C_a^i := \omega_a^i - \Gamma_a^i$. We further define new variables:

$$\gamma M_b^j := \gamma K_b^j + C_b^j, \tag{15}$$

$$Q_b^j := \gamma M_b^j + \Gamma_b^j. \tag{16}$$

Then by using the definitions of Eqs. (10) and (15), the connection components ω_a^{io} can be rewritten as

$$\omega_a^{io} = \frac{1}{\phi}\left(M_a^i - \frac{1}{\gamma}C_a^i - \frac{1}{2}E_a^i n^c \bar{\partial}_c \phi\right). \tag{17}$$

Note that we have the identity

$$E_j^b R_{ab}{}^j = 0, \tag{18}$$

where the curvature $R_{ab}{}^j$ is defined as

$$R_{ab}{}^j := \partial_{[a}\Gamma_{b]}^j + \epsilon^j{}_{lm}\Gamma_a^l\Gamma_b^m. \tag{19}$$

Note also that the two constraint equations with respect to the Lagrangian multipliers $\bar{K}_t^j$ and $\bar{\omega}_t^j$ are equivalent to

$$\epsilon_{jl}{}^m C_b^l \tilde{E}_m^b = 0, \tag{20}$$

$$\epsilon_{jl}{}^m M_b^l \tilde{E}_m^b = 0. \tag{21}$$

We will denote Ω^j, Λ^j as the corresponding Lagrangian multipliers. Then the Lagrangian density (9) can be expressed as

$$\begin{aligned}\mathcal{L} = {} & \frac{1}{\gamma}\tilde{E}_j^b\dot{Q}_b^j - \frac{1}{\phi}\tilde{E}_j^b M_b^j\dot{\phi} + \Lambda^j(\partial_b\tilde{E}_j^b + \epsilon_{jl}{}^m Q_b^l\tilde{E}_m^b) - N^a\left(\tilde{E}_j^b\nabla_{[a}M_{b]}^j - \frac{1}{\phi}\tilde{E}_j^b M_b^j\partial_a\phi\right)\\ & - \frac{\phi}{2}\underline{N}\tilde{E}_i^a\tilde{E}_j^b\epsilon^{ij}{}_k\left(R_{ab}{}^k - \frac{1}{\phi^2}\epsilon^k{}_{lm}M_a^l M_b^m\right) - \underline{N}\tilde{E}_i^a\tilde{E}^{bi}\nabla_a\nabla_b\phi + \frac{K}{2\underline{N}}(\dot{\phi} - N^a\partial_a\phi)^2\\ & - \frac{1}{2}\left(K - \frac{3}{2\phi}\right)\underline{N}\tilde{E}_i^a\tilde{E}^{bi}(\partial_a\phi)\partial_b\phi - \frac{\phi}{2}\underline{N}\left(1 + \frac{1}{\phi^2\gamma^2}\right)(C^2 - C_{ij}C^{ij}) - NEV(\phi),\end{aligned} \tag{22}$$

where $C_{ij} := C_{ai}\tilde{E}_j^a$ and $C := \delta^{ij}C_{ij}$. Since the variation of the action with respect to C_{ij} gives

$$C_{ij} = 0, \tag{23}$$

the Lagrangian density (22) can be reduced to

$$\begin{aligned}\mathcal{L} = {} & \frac{1}{\gamma}\tilde{E}_j^b\dot{A}_b^j - \frac{1}{\phi}\tilde{E}_j^b K_b^j\dot{\phi} + \Lambda^j(\partial_b\tilde{E}_j^b + \epsilon_{jl}{}^m A_b^l\tilde{E}_m^b)\\ & - N^a\left(\tilde{E}_j^b\nabla_{[a}K_{b]}^j - \frac{1}{\phi}\tilde{E}_j^b K_b^j\partial_a\phi\right)\\ & - \frac{\phi}{2}\underline{N}\tilde{E}_i^a\tilde{E}_j^b\epsilon^{ij}{}_k\left(R_{ab}{}^k - \frac{1}{\phi^2}\epsilon^k{}_{lm}K_a^l K_b^m\right)\\ & + \frac{K}{2\underline{N}}(\dot{\phi} - N^a\partial_a\phi)^2 - \frac{1}{2}\left(K - \frac{3}{2\phi}\right)\underline{N}\tilde{E}_i^a\tilde{E}^{bi}(\partial_a\phi)\partial_b\phi\\ & - \underline{N}\tilde{E}_i^a\tilde{E}^{bi}\nabla_a\nabla_b\phi - NEV(\phi),\end{aligned} \tag{24}$$

where

$$A_b^j := \gamma K_b^j + \Gamma_b^j. \tag{25}$$

By Legendre transformation, the momentum conjugate to the configuration variables A_a^i and ϕ are defined, respectively, as

$$\pi_i^a := \frac{\delta\mathcal{L}}{\delta\dot{A}_a^i} = \frac{1}{\gamma}\tilde{E}_i^a, \tag{26}$$

$$\pi := \frac{\delta\mathcal{L}}{\delta\dot{\phi}} = -\frac{1}{\phi}\tilde{E}_j^b K_b^j + \frac{K}{\underline{N}}(\dot{\phi} - N^a\partial_a\phi). \tag{27}$$

The fundamental Poisson brackets read

$$\{A_a^i(x), \tilde{E}_j^b(y)\} = \gamma\delta_a^b\delta_j^i\delta^3(x-y), \tag{28}$$

$$\{\phi(x), \pi(y)\} = \delta^3(x-y). \tag{29}$$

It should be noted that the second-class constraints that appeared in the Hamiltonian analysis have been solved by the partial gauge fixing. In the case when $K \neq 0$, the corresponding Hamiltonian reads

$$H = \int d^3x(\Lambda^i\mathcal{G}_i + N^a\mathcal{C}_a + \underline{N}\mathcal{C}), \tag{30}$$

where the Gaussian, vector, and scalar constraints read, respectively, as

$$\mathcal{G}_j = \partial_b\tilde{E}_j^b + \epsilon_{jl}{}^m A_b^l\tilde{E}_m^b, \tag{31}$$

$$\mathcal{C}_a = \tilde{E}_j^b\nabla_{[a}K_{b]}^j + \pi\partial_a\phi, \tag{32}$$

$$\begin{aligned}\mathcal{C} = {} & \frac{\phi}{2}\tilde{E}_i^a\tilde{E}_j^b\epsilon^{ij}{}_k\left(R_{ab}{}^k - \frac{1}{\phi^2}\epsilon^k{}_{lm}K_a^l K_b^m\right)\\ & + \tilde{E}_i^a\tilde{E}^{bi}\nabla_a\nabla_b\phi + \frac{1}{2}\left(K - \frac{3}{2\phi}\right)\tilde{E}_i^a\tilde{E}^{bi}(\partial_a\phi)\partial_b\phi\\ & + \frac{1}{2K}\left(\pi + \frac{1}{\phi}\tilde{E}_j^b K_b^j\right)^2 + E^2V(\phi).\end{aligned} \tag{33}$$

In the special case when $K = 0$, it is easy to see from Eq. (27) that there is a primary constraint

$$S = \pi\phi + \tilde{E}^b_j K^j_b, \tag{34}$$

which is called the conformal constraint in Ref. [15]. Thus, the Hamiltonian becomes

$$H = \int d^3x(\Lambda^i \mathcal{G}_i + N^a \mathcal{C}_a + \underline{N}\mathcal{C}_0 + \lambda S), \tag{35}$$

where the scalar constraint reads

$$\begin{aligned}\mathcal{C}_0 = \frac{\phi}{2}\tilde{E}^a_i\tilde{E}^b_j\epsilon^{ij}{}_k\left(R_{ab}{}^k - \frac{1}{\phi^2}\epsilon^k{}_{lm}K^l_aK^m_b\right) \\ + \tilde{E}^a_i\tilde{E}^{bi}\nabla_a\nabla_b\phi - \frac{3}{4\phi}\tilde{E}^a_i\tilde{E}^{bi}(\partial_a\phi)\partial_b\phi + E^2V(\phi).\end{aligned} \tag{36}$$

It is obvious that the above Hamiltonian formulations in both cases coincide with those in Ref. [15].

On the other hand, as pointed out in Ref. [18], the first-order action

$$\begin{aligned}S[e,\omega,\phi] = \int\Bigg[\frac{1}{2}\phi e e^a_I e^b_J\left(\bar{\Omega}_{ab}{}^{IJ} + \frac{1}{\gamma}{}^\star\bar{\Omega}_{ab}{}^{IJ}\right) \\ - \frac{1}{2}K(\phi)ee^{Ia}e^b_I(\bar{\partial}_a\phi)\bar{\partial}_b\phi - eV(\phi)\Bigg]d^4x\end{aligned} \tag{37}$$

can give a connection dynamics of STT in the Einstein frame. We now show that the Hamiltonian formalism of action (37) is equivalent to the one that we just derived from action (1) because they are related to each other by a canonical transformation. In the case when $K \neq 0$, the Hamiltonian corresponding to action (37) is a linear combination of first-class constraints as

$$H = \int d^3x(\Lambda^i\hat{\mathcal{G}}_i + N^a\hat{\mathcal{C}}_a + \underline{N}\,\hat{\mathcal{C}}), \tag{38}$$

where

$$\hat{\mathcal{G}}_i = \gamma^{-1}\hat{\mathcal{D}}_a\hat{E}^a_i, \tag{39}$$

$$\hat{\mathcal{C}}_a = \hat{E}^b_i\hat{F}_{ab}{}^i + \hat{\pi}\partial_a\phi, \tag{40}$$

$$\begin{aligned}\hat{\mathcal{C}} = -\gamma^{-1}\frac{1}{2\phi}\epsilon^{ij}{}_k\hat{E}^a_i\hat{E}^b_j[\hat{F}_{ab}{}^k - (\gamma+\gamma^{-1})\hat{R}_{ab}{}^k] \\ + \frac{K(\phi)}{2\phi^2}\hat{E}^{ai}\hat{E}^b_i(\partial_a\phi)\partial_b\phi + \frac{\hat{\pi}^2}{2K(\phi)} + V\sqrt{\det(\tilde{E}^{ai}\tilde{E}^b_i)},\end{aligned} \tag{41}$$

with

$$\hat{\mathcal{D}}_a\hat{E}^a_i := \partial_a\hat{E}^a_i + \gamma\epsilon_{ij}{}^k\hat{A}^j_a\hat{E}^a_k, \tag{42}$$

and $\hat{F}_{ab}{}^i$ and $\hat{R}_{ab}{}^i$ standing for the curvature of $\hat{A}^i_a$ and $\hat{\Gamma}^i_a$, respectively, i.e.,

$$\hat{F}_{ab}{}^i = \partial_{[a}\hat{A}^i_{b]} + \gamma\epsilon^i{}_{jk}\hat{A}^j_a\hat{A}^k_b, \tag{43}$$

$$\hat{R}_{ab}{}^i = \partial_{[a}\hat{\Gamma}^i_{b]} + \epsilon^i{}_{jk}\hat{\Gamma}^j_a\hat{\Gamma}^k_b. \tag{44}$$

Here $\hat{\Gamma}^i_a$ is the $SU(2)$ spin connection satisfying

$$\hat{D}_a\hat{E}^b_i = \partial_a\hat{E}^b_i + \hat{\Gamma}_{ac}{}^b\hat{E}^c_i - \hat{\Gamma}_{ca}{}^c\hat{E}^b_i + \epsilon_{ij}{}^k\hat{\Gamma}^j_a\hat{E}^b_k = 0, \tag{45}$$

where $\hat{\Gamma}_{ab}{}^c$ is the Christoffel connection determined by the spatial metric

$$\hat{q}^{ab} = \hat{E}\hat{E}^{ai}\hat{E}^{bi}, \tag{46}$$

with $\hat{E} := 1/\det(\hat{E}^a_i)$. The fundamental Poisson brackets are

$$\{\hat{A}^i_a(x), \hat{E}^b_j(y)\} = \delta^b_a\delta^i_j\delta^3(x-y), \tag{47}$$

$$\{\phi(x), \hat{\pi}(y)\} = \delta^3(x-y). \tag{48}$$

To do the canonical transformation, we first define

$$K^i_a := \phi(\hat{A}^i_a - \gamma^{-1}\hat{\Gamma}^i_a), \tag{49}$$

$$\tilde{E}^a_i := \phi^{-1}\hat{E}^a_i. \tag{50}$$

Then we further define

$$\pi := \hat{\pi} - \frac{1}{\phi}K^i_a\tilde{E}^a_i, \tag{51}$$

$$A^i_a := \Gamma^i_a + \gamma K^i_a. \tag{52}$$

Using Eqs. (47) and (48), we can get the Poisson brackets between new variables as

$$\{A^i_a(x), \tilde{E}^b_j(y)\} = \gamma\delta^b_a\delta^i_j\delta^3(x-y), \tag{53}$$

$$\{\phi(x), \pi(y)\} = \delta^3(x-y), \tag{54}$$

$$\{A^i_a(x), A^j_b(y)\} = 0 = \{\tilde{E}^a_i(x), \tilde{E}^b_j(y)\}, \tag{55}$$

$$\{\phi(x), \phi(y)\} = 0 = \{\pi(x), \pi(y)\}. \tag{56}$$

Taking account of Eq. (7), the constraints (39)–(41) can be written in terms of new variables, up to Gaussian constraint, as

$$\hat{\mathcal{G}}_i = \gamma(\partial_a\tilde{E}^a_i + \epsilon_{ij}{}^kA^j_a\tilde{E}^a_k), \tag{57}$$

$$\hat{\mathcal{C}}_a = \gamma^{-1}\tilde{E}^b_iF_{ab}{}^i + \pi\partial_a\phi, \tag{58}$$

$$\begin{aligned}\hat{\mathcal{C}} = {}& \frac{\phi}{2}\epsilon_i{}^{lm}\tilde{E}^a_l\tilde{E}^b_m\left[F_{ab}{}^i - \left(\gamma^2 + \frac{1}{\phi^2}\right)\epsilon^i{}_{jk}K^j_aK^k_b\right] \\ & + \frac{1}{2\omega(\phi)+3}\left(\frac{1}{\phi}(K^i_a\tilde{E}^a_i)^2 + 2\tilde{\pi}K^i_a\tilde{E}^a_i + \pi^2\phi\right) \\ & + \frac{\omega(\phi)}{2\phi}\tilde{E}^{ai}\tilde{E}^b_i(\partial_a\phi)\partial_b\phi + \tilde{E}^{ai}\tilde{E}^b_i(\partial_a\partial_b\phi - \Gamma_{ab}{}^c\partial_c\phi) \\ & + V\sqrt{\det(\tilde{E}^{ai}\tilde{E}^b_i)}, \end{aligned} \tag{59}$$

where $F_{ab}{}^i := \partial_{[a}A^i_{b]} + \epsilon^i{}_{jk}A^j_aA^k_b$. It is obvious that these constraints coincide with our results as well as those in Ref. [15]. Similarly, it is easy to get the same conclusion in the special case when $K = 0$.

IV. CONCLUDING REMARKS

As candidate modified gravity theories, STT provide the great possibility to account for the dark Universe and some fundamental issues in physics. The nonperturbative loop quantization of STT is based on their connection dynamical formalism obtained in Hamiltonian formulation in Ref. [15]. The achievement in this paper is to set up an action principle for the connection dynamics of STT in the Jordan frame. Since $f(R)$ theories of gravity can be regarded as the special kinds of STT, our action principle is also valid for the connection dynamics of $f(R)$ theories. To get the action principle, we first show that the first-order action (1) gives the right equations of motion for general STT. Then a detailed Hamiltonian analysis is done to this action. By a partial gauge fixing, the internal $SL(2,\mathbb{C})$ group of the theory is reduced to $SU(2)$, and the second-class constraints are solved. Thus, we obtain a first-class Hamiltonian system with a $SU(2)$ connection as a configuration variable. This Hamiltonian formalism is exactly the same as the one in Ref. [15] derived from the geometrical dynamics by canonical transformations.

On the other hand, the directly corresponding Hamiltonian connection formulation of action (37) is in the Einstein frame, while as shown in Ref. [15], the natural connection formulation obtained by canonical transformations in Hamiltonian framework is in the Jordan frame. However, we have shown that they are equivalent to each other at the classical level. Nevertheless, the ambiguity, whether one should start with the Jordan frame or Einstein frame to quantize STT, still exits. Besides providing the action principle for connection dynamics of STT, actions (1) and (37) also lay the foundation of spinfoam path-integral quantization of STT. We leave this issue for future study.

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