Shifted orbifold models with magnetic flux

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We propose a mechanism to obtain the generation of matter in the standard model. We start from the analysis of the T^2/Z_N shifted orbifold with magnetic flux, which imposes a Z_N symmetry on torus. We also consider several orbifolds such as $(T^2 \times T^2)/Z_N$, $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'})$ and $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'} \times Z_{N''})$. On such orbifolds, we study the behavior of fermions in two different means—the operator formalism and the explicit analysis of wave functions. For an interesting result, it is found that the number of zero-mode fermions is related to N of the Z_N symmetry. In other words, the generation of matter relates to the type of orbifolds. Moreover, we find that shifted orbifold models are severely restricted from realizing three generations. For example, the three-generation model on the type of $M^4 \times (T^2 \times T^2)/Z_N$ is unique. One can also construct other types of three-generation orbifold models with rich flavor structure. Those results may bring us a realistic model with desired Yukawa structure.

I. INTRODUCTION

Extra-dimensional field theories play an important role in particle physics and cosmology. In particular, string-derived higher-dimensional models would be interesting.

A key issue is how to realize a four-dimensional chiral theory through a certain type of compactification, when we start with a higher-dimensional theory. For example, the toroidal compactification is simple, but it does not lead to a four-dimensional chiral theory without any gauge background. More complicated geometrical backgrounds such as Calabi-Yau compactifications can lead to a four-dimensional chiral theory. In general, it is difficult to solve zero-mode equations analytically on Calabi-Yau manifolds, although topological discussions are applicable.

Torus compactifications with magnetic fluxes are quite interesting extra-dimensional backgrounds [1,2].[1] Chiral massless spectra in a four-dimensional theory can be realized. One can solve zero-mode equations analytically and their zero-mode profiles are nontrivially quasilocalized. Furthermore, such supersymmetric Yang-Mills models on the torus with magnetic fluxes can be derived from the superstring theory with D-branes [4]. Also, the general form of wave functions and the spectra have been derived in arbitrary n-dimensional torus with magnetic flux [5].

The number of zero modes is determined by the magnitude of the magnetic flux. Thus, one can realize the three generations of chiral matter fields by choosing a proper pattern of magnetic fluxes.[2] In addition, nontrivial profiles of quasilocalized zero modes can lead to hierarchically small couplings when they are localized far away from each other, although their couplings would be of $\mathcal{O}(1)$ for zero modes localized near places. Thus, these models are phenomenologically quite interesting. Indeed, several studies have been carried out for various phenomenological aspects such as explicit model building and computations of four-dimensional low-energy effective field theories, e.g., Yukawa couplings [2], realization of quark/lepton masses and their mixing angles [8], higher order couplings [9], flavor symmetries [10–12], massive modes [13], etc. (See also Refs. [14–17].)

The orbifold compactifiction with magnetic flux is also interesting [18,19].[3] The (twisted) orbifold is constructed by dividing the torus by a discrete rotation [21]. On the Z_2 orbifold with magnetic flux, zero-mode wave functions are classified into even and odd functions under the Z_2 reflection. Either Z_2 even or odd zero modes are projected out exclusively by the orbifold projection. Then, we can obtain that the number of generations differs from one in the simple toroidal compactification with the same magnetic flux. Hence, the flavor structure becomes rich.

In this paper, we study another type of orbifolds with magnetic flux. Instead of the discrete rotation, we divide

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[1]See for string magnetized D-brane models [3] and references therein.

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[2]On the other hand, there have been some bottom-up approaches realizing the three generations of chiral matter fields, which are called the extra-dimensional models. For example, an attempt is to derive the three generations of chiral matter fields on the basis of the relation between gauge symmetry and boundary conditions on four-dimensional spacetime added to S^1/Z_2 twisted orbifold [6]. Another one is to derive the three generations of chiral matter fields through the geometry of extra dimension with point interactions instead of magnetic flux [7].

[3]See also for heterotic models on magnetized orbifolds [20].

the simple toroidal compactification with magnetic flux by some discrete shift symmetries such as Z_N, i.e., the shifted orbifold.[4] We can classify zero-mode wave functions by their behaviors under the shift symmetry and project out some of them. Then, we derive a new type of models and flavor structures. In particular, we consider T^2/Z_N, $(T^2 \times T^2)/Z_N$, $(T^2 \times T^2)/Z_N \times T^2$, $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'})$, and $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'} \times Z_{N''})$ shifted orbifolds. Then, we study the number of zero modes and their wave functions on the above shifted orbifolds with magnetic fluxes. For an interesting result, it is found that the number of zero-mode fermions is related to N of the Z_N symmetry. In other words, the generation of matter relates to the type of orbifolds. Moreover, we find that shifted orbifold models are, in general, severely restricted from realizing three generations. For example, the three-generation model on the type of $M^4 \times (T^2 \times T^2)/Z_N$ is unique. One can also construct other types of three-generation orbifold models with rich flavor structure. Those results may bring us a realistic model with desired Yukawa structure.

This paper is organized as follows. In Sec. II, we study the shifted orbifold with magnetic fluxes in the operator formalism. In Sec. III, we study zero-mode wave functions of spinor fields and flavor structure. Section IV is devoted to the conclusion and discussion. In Appendix A, we discuss the general form of Z_N shifted orbifold and basis transformation. In Appendix B, we discuss the degeneracy of spectrum on $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'} \times Z_{N''})$. In Appendix C, we discuss Wilson line phases and the redefinition of fields under the existence of magnetic flux.

II. OPERATOR FORMALISM ON SHIFTED ORBIFOLD WITH MAGNETIC FLUX

First of all, we introduce a homogeneous magnetic flux along a $U(1)$ gauge group, which may be a subgroup in some non-Abelian gauge group, and concentrate only on the $U(1)$ gauge theory. In this section, we consider the quantum mechanical system of the $U(1)$ gauge theory on some shifted orbifold with homogeneous magnetic flux. The shifted orbifold is defined by orbifolding with some discrete symmetries on the torus, as we will see below. For the analysis, we make use of a technique of operator formalism, in which it is easy to understand the number of degeneracy compared with the analysis with the wave functions. We first investigate the quantum mechanical system with a homogeneous magnetic field on T^2/Z_N and show that the energy spectrum of this system has degeneracy due to the magnetic flux, which is well known as the Landau levels. In this case, however, there is no reason to restrict the number of degeneracy. After that, we

also investigate the system on $(T^2 \times T^2)/Z_N$. Differently from T^2/Z_N, we obtain the remarkable result that the number of degeneracy is restricted to a multiple of N. This result implies that the three-generation structure could be derived from some higher-dimensional gauge theory. We also discuss the extension of this idea to $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'})$ as well as $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'} \times Z_{N''})$ in the latter half of this section.

A. T^2/Z_N shifted orbifold
1. Operator formalism on T^2

Let us start from an analysis of a quantum mechanical system in a homogeneous magnetic field on T^2. We define the vector notation on T^2 as $\boldsymbol{y} \equiv (y_1, y_2)^{\mathrm{T}}$. The Schrödinger equation and the Hamiltonian we consider are written as

$$H\psi(\boldsymbol{y}) = E\psi(\boldsymbol{y}), \qquad H = (-i\boldsymbol{\nabla} + q\boldsymbol{A}(\boldsymbol{y}))^2, \quad (1)$$

where $\boldsymbol{\nabla} \equiv (\partial_{y_1}, \partial_{y_2})^{\mathrm{T}}$, q is a $U(1)$ charge, and $\boldsymbol{A}(\boldsymbol{y})$ provides a homogeneous magnetic field on T^2 as

$$\boldsymbol{A}(\boldsymbol{y}) = -\frac{1}{2}\Omega\boldsymbol{y} + \frac{2\pi}{q}\boldsymbol{a} \qquad (2)$$

or in the component

$$A_i(\boldsymbol{y}) = -\frac{1}{2}\sum_{j=1}^{2}\Omega_{ij}y_j + \frac{2\pi}{q}a_i, \qquad (i = 1, 2). \quad (3)$$

Here, $\boldsymbol{a}$ is a Wilson line phase which is composed of real constants. We emphasize that only the antisymmetric part of Ω is physical, and the symmetric part depends on a choice of gauge. We will take a suitable gauge as we will see later. Introducing a two-dimensional lattice,

$$\Lambda = \left\{ \sum_{a=1}^{2} n_a \boldsymbol{u}_a \,\middle|\, n_a \in \mathbb{Z} \right\}, \qquad (4)$$

we define the two-dimensional torus as $T^2 = \mathbb{R}^2/\Lambda$. Here $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are the basis vectors of torus. In other words, T^2 is defined with the identification

$$\boldsymbol{y} \sim \boldsymbol{y} + \sum_{a=1}^{2} n_a \boldsymbol{u}_a. \qquad (5)$$

For the Schrödinger equation to be compatible with this condition, the wave function $\psi(\boldsymbol{y})$ has to satisfy the pseudoperiodic boundary conditions,

$$\psi(\boldsymbol{y} + \boldsymbol{u}_a) = e^{iq\boldsymbol{y}^{\mathrm{T}}\Omega\boldsymbol{u}_a/2}\psi(\boldsymbol{y}) \quad \text{for } a = 1, 2. \quad (6)$$

Here we made a constant phase appearing at the right-hand side of this equation absorbed into $\boldsymbol{a}$. Now, we require that the wave function must be single-valued on the torus. The requirement leads to the magnetic flux quantization condition,

$$q\boldsymbol{u}_a^{\mathrm{T}}B\boldsymbol{u}_b = 2\pi Q_{ab}, \qquad (7)$$

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where $B \equiv \frac{1}{2}(\Omega - \Omega^{\mathrm{T}})$, which corresponds to a homogeneous magnetic field, and $Q_{ab} = -Q_{ba} \in \mathbb{Z}$.

To move on the operator formalism, we introduce a momentum operator as $\boldsymbol{p} \equiv -i\boldsymbol{\nabla}$, which is canonically conjugate to $\boldsymbol{y}$. This operator satisfies the canonical commutation relations,

$$[y_j, p_k] = i\delta_{jk}, \quad \text{the others} = 0, \qquad (j, k = 1, 2). \quad (8)$$

We rewrite the wave function $\psi(\boldsymbol{y})$ in the language of the operator formalism as $\psi(\boldsymbol{y}) = \langle \boldsymbol{y} | \psi \rangle$ and then the system is rewritten by

$$\hat{H}|\psi\rangle = E|\psi\rangle, \qquad \hat{H} = \left(\hat{\boldsymbol{p}} - \frac{q}{2}\Omega\hat{\boldsymbol{y}} + 2\pi\boldsymbol{a}\right)^2 \equiv \hat{\boldsymbol{p}}'^2. \quad (9)$$

This is restricted by the constraint conditions,

$$e^{i\hat{T}_a}|\psi\rangle = |\psi\rangle, \quad \hat{T}_a = \boldsymbol{u}_a^{\mathrm{T}}\left(\hat{\boldsymbol{p}} - \frac{q}{2}\Omega^{\mathrm{T}}\hat{\boldsymbol{y}}\right), \quad (a = 1, 2). \quad (10)$$

These conditions come from the pseudoperiodic boundary conditions (6), and we have taken the suitable gauge in which the additional term in $\hat{T}_a$ vanishes, i.e., $\boldsymbol{u}_a^{\mathrm{T}}\Omega\boldsymbol{u}_a = 0$. To investigate the eigenstates of this system and their energy eigenvalues, it is convenient to define new variables under a canonical transformation. Then we introduce the new variables,

$$\hat{Y} \equiv \frac{\sqrt{2}}{\omega}\hat{p}'_2, \quad \hat{P} \equiv \sqrt{2}\hat{p}'_1, \quad \hat{\tilde{Y}} \equiv -\frac{1}{2\pi M}\hat{T}_2, \quad \hat{\tilde{P}} \equiv \hat{T}_1, \quad (11)$$

where $\omega \equiv 2qB_{12} = 2qb/|\boldsymbol{u}_1 \times \boldsymbol{u}_2|$[5] and $M \equiv Q_{12}$. They satisfy the canonical commutation relations, i.e.,

$$[\hat{Y}, \hat{P}] = i, \qquad [\hat{\tilde{Y}}, \hat{\tilde{P}}] = i, \quad \text{the others} = 0. \quad (12)$$

The transformation $\{y_i, p_i; i = 1, 2\} \mapsto \{Y, P, \tilde{Y}, \tilde{P}\}$ is a canonical one since it preserves the canonical commutation relations. Moreover, under the new variables, the system can be reformulated into an effectively one-dimensional harmonic oscillator,

$$\hat{H} = \frac{1}{2}\hat{P}^2 + \frac{\omega^2}{2}\hat{Y}^2, \quad (13)$$

with the two constraint conditions

$$e^{i\hat{\tilde{P}}}|\psi\rangle = |\psi\rangle, \qquad e^{-i2\pi M\hat{\tilde{Y}}}|\psi\rangle = |\psi\rangle. \quad (14)$$

These two constraint conditions are, however, independent of the energy spectrum of the harmonic oscillator because $\hat{H}$ is constructed only from $\hat{Y}$ and $\hat{P}$ but not from $\hat{\tilde{Y}}$ and $\hat{\tilde{P}}$. The constraint conditions (14) lead to the coordinate quantization as below. We take the coordinate representation $|\tilde{y}\rangle$ which diagonalizes the operator $\hat{\tilde{Y}}$ as
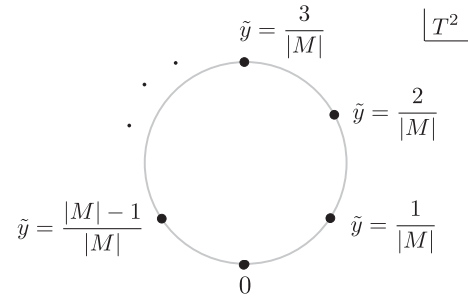

FIG. 1.   Schematic figure of the quantization of $\tilde{y} = j/|M|$.

$$e^{-2\pi i M\hat{\tilde{Y}}}|\tilde{y}\rangle = e^{-2\pi i M\tilde{y}}|\tilde{y}\rangle. \quad (15)$$

Making $e^{-i2\pi M\hat{\tilde{Y}}}$ operate on $e^{ia\hat{\tilde{P}}}|\tilde{y}\rangle (a \in \mathbb{R})$ and using Eq. (12), we can obtain the condition

$$e^{ia\hat{\tilde{P}}}|\tilde{y}\rangle = |\tilde{y} - a\rangle. \quad (16)$$

From Eqs. (14) and (16), we can obtain the periodic condition,

$$|\tilde{y} - 1\rangle = |\tilde{y}\rangle, \quad (17)$$

with the coordinate quantization condition,

$$\tilde{y} = \frac{j}{M}, \qquad (j = 0, 1, 2, \ldots, |M| - 1). \quad (18)$$

A schematic figure of the quantization condition is represented in Fig. 1. These results imply that the eigenstates of $\hat{H}$ and their energy eigenvalues are given by

$$\hat{H}\left|n, \frac{j}{M}\right\rangle_{T^2} = E_n\left|n, \frac{j}{M}\right\rangle_{T^2}, \qquad E_n = \omega\left(n + \frac{1}{2}\right), \quad (19)$$

where $n = 0, 1, 2, \ldots$ and $j = 0, 1, 2, \ldots, |M| - 1$. Thus, there is $|M|$-fold degeneracy at each energy level in this system, i.e.,

$$\text{the number of degeneracy} = |M|. \quad (20)$$

Note that E_n corresponds to eigenvalues of the two-dimensional Laplace operator with magnetic flux (1), i.e., the mass squares of scalar fields $m_n^2 = \omega(n + 1/2)$. The spinor and two-dimensional vector have mass spectra as $m_n^2 = \omega n$ and $m_n^2 = \omega(n - 1/2)$, respectively [2,13].

2. Z_N shifted orbifolding

From here, we investigate the quantum system on T^2/Z_N shifted orbifold, which is defined as the identification

$$Z_N: \boldsymbol{y} \sim \boldsymbol{y} + \frac{1}{N}\boldsymbol{u}_1, \quad (21)$$

where N is some positive integer. We should comment here that Eq. (21) is not the general form in the Z_N shifted

orbifolding, but we can always transform the general form into a special one such as Eq. (21) without any loss of generality. We show the details in Appendix A. This identification can be translated into the requirement for physical states on T^2/Z_N in the operator formalism as[6]

$$\hat{U}_{Z_N}|\psi\rangle_{T^2/Z_N} = |\psi\rangle_{T^2/Z_N}, \qquad \hat{U}_{Z_N} \equiv e^{i\hat{T}_1/N}, \quad (22)$$

where $\hat{U}_{Z_N}$ is the operator that shifts the coordinate y by $\frac{1}{N}u_1$, as in Eq. (21). The Z_N shift operator has to be consistent with the torus identification (10) and leads to the consistency conditions

$$\hat{U}_{Z_N}e^{i\hat{T}_a} = e^{i\hat{T}_a}\hat{U}_{Z_N}, \qquad (a = 1, 2). \quad (23)$$

From these conditions, we can obtain

$$\frac{M}{N} \in \mathbb{Z}. \quad (24)$$

In other words, we can write

$$M = tN, \quad (25)$$

where t is some integer. We can understand briefly what this result means physically. Since the Z_N shifted orbifolding reduces the fundamental region $\mathcal{A}$ of T^2 to $\mathcal{A}/N$, the magnetic flux quantization requires that M has to be a multiple of N.

Then we define the physical states on T^2/Z_N as

$$\begin{aligned}\left|n, \frac{j}{M}\right\rangle_{T^2/Z_N} &\equiv \frac{1}{\sqrt{N}}\sum_{\ell=0}^{N-1}(\hat{U}_{Z_N})^{-\ell}\left|n, \frac{j}{M}\right\rangle_{T^2} \\ &= \frac{1}{\sqrt{N}}\sum_{\ell=0}^{N-1}\left|n, \frac{j+\ell t}{M}\right\rangle_{T^2},\end{aligned} \quad (26)$$

where $j + \ell t$ is identical with an element of the set of j modulo $|M|$. Here we used the fact that $\hat{U}_{Z_N}$ operate on the states on T^2, i.e.,

$$(\hat{U}_{Z_N})^{-1}\left|n, \frac{j}{M}\right\rangle_{T^2} = \left|n, \frac{j}{M} + \frac{1}{N}\right\rangle_{T^2} = \left|n, \frac{j+t}{M}\right\rangle_{T^2}. \quad (27)$$

Finally, we mention the number of degeneracy in this system, which is given by

$$\text{the number of degeneracy} = \frac{|M|}{N} = |t|. \quad (28)$$

We notice that we cannot obtain any constraint for the number of degeneracy on T^2/Z_N.

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[6]We note that there is always a Z_N-phase ambiguity to put $e^{2\pi i\theta/N}(\theta \in \mathbb{Z})$ in the definition of $\hat{U}_{Z_N}$. Here we assume $e^{2\pi i\theta/N} = 1$ because it does not affect the analysis below. For the same reason, we will apply this assumption to the cases of other orbifolds.

B. $(T^2 \times T^2)/Z_N$ shifted orbifold

In this subsection, using the analysis in Sec. II A, we analyze the quantum mechanics on $(T^2 \times T^2)/Z_N$ shifted orbifold with a homogeneous magnetic field. The extension of the previous analysis is straightforward, but the result is nontrivial. In this case, the number of degeneracy is restricted to a multiple of N.

1. Operator formalism on $T^2 \times T^2$

Let us start from a Hamiltonian of the quantum system on $T^2 \times T^2$ with a homogeneous magnetic field, which is given by

$$H = \sum_{g=1}^{2}(-i\boldsymbol{\nabla}^{(g)} + q\boldsymbol{A}^{(g)}(\boldsymbol{y}^{(g)}))^2, \quad (29)$$

where

$$\boldsymbol{A}^{(g)}(\boldsymbol{y}^{(g)}) = -\frac{1}{2}\Omega^{(g)}\boldsymbol{y}^{(g)} + \frac{2\pi}{q}\boldsymbol{a}^{(g)}, \quad (g = 1, 2). \quad (30)$$

Here we introduced an index g to be a label of each torus. The two two-dimensional tori $T^2 \times T^2$ are defined as

$$T^2 \times T^2 = \mathbb{R}/\Lambda^{(1)} \times \mathbb{R}/\Lambda^{(2)}, \quad (31)$$

where

$$\Lambda^{(g)} = \left\{\sum_{a=1}^{2}n_a^{(g)}\boldsymbol{u}_a^{(g)}|n_a^{(g)} \in \mathbb{Z}\right\}, \qquad (g = 1, 2). \quad (32)$$

In other words, $T^2 \times T^2$ can be defined with the identifications,

$$\boldsymbol{y}^{(g)} \sim \boldsymbol{y}^{(g)} + \sum_{a=1}^{2}n_a^{(g)}\boldsymbol{u}_a^{(g)}. \quad (33)$$

On each torus, the wave function $\psi(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)})$ has to satisfy the pseudoperiodic boundary conditions,

$$\begin{aligned}\psi(\boldsymbol{y}^{(1)} + \boldsymbol{u}_a^{(1)}, \boldsymbol{y}^{(2)}) &= e^{iq(\boldsymbol{y}^{(1)})^{\mathrm{T}}\Omega^{(1)}\boldsymbol{u}_a^{(1)}/2}\psi(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}), \\ \psi(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)} + \boldsymbol{u}_a^{(2)}) &= e^{iq(\boldsymbol{y}^{(2)})^{\mathrm{T}}\Omega^{(2)}\boldsymbol{u}_a^{(2)}/2}\psi(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}),\end{aligned} \quad (34)$$

for the Schrödinger equation to be compatible with Eq. (33). The requirement that the wave function is single-valued on each torus leads to the magnetic flux quantization conditions,

$$q\boldsymbol{u}_a^{(g)\mathrm{T}}B^{(g)}\boldsymbol{u}_b^{(g)} = 2\pi Q_{ab}^{(g)}, \quad (35)$$

where $B^{(g)} \equiv \frac{1}{2}(\Omega^{(g)} - \Omega^{(g)\mathrm{T}})$ and $Q_{ab}^{(g)} = -Q_{ba}^{(g)} \in \mathbb{Z}$.

Here we introduce momentum operators as $\boldsymbol{p}^{(g)} \equiv -i\boldsymbol{\nabla}^{(g)}$, which are canonically conjugate to $\boldsymbol{y}^{(g)}$. These operators satisfy the canonical commutation relations

$$[y_j^{(g)}, p_k^{(g')}] = i\delta_{gg'}\delta_{jk}, \qquad \text{the others} = 0, \quad (36)$$

where $g, g' = 1, 2$ and $j, k = 1, 2$. In the same way as the previous subsection, the pseudoperiodic boundary conditions (34) can be regarded as the constraint conditions for the physical states in the operator formalism. The system is rewritten by

$$\hat{H}|\psi\rangle = E|\psi\rangle,$$

$$\hat{H} = \sum_{g=1}^{2}\left(\hat{\boldsymbol{p}}^{(g)} - \frac{q}{2}\Omega^{(g)}\hat{\boldsymbol{y}}^{(g)} + 2\pi\boldsymbol{a}^{(g)}\right)^2 \equiv \sum_{g=1}^{2}(\hat{\boldsymbol{p}}'^{(g)})^2, \tag{37}$$

with the constraint conditions

$$e^{i\hat{T}_a^{(g)\mathrm{T}}}|\psi\rangle = |\psi\rangle, \quad \hat{T}_a^{(g)} = \boldsymbol{u}_a^{(g)\mathrm{T}}\left(\hat{\boldsymbol{p}}^{(g)} - \frac{q}{2}\Omega^{(g)\mathrm{T}}\hat{\boldsymbol{y}}^{(g)}\right), \tag{38}$$

where $a = 1, 2$ and $g = 1, 2$. In the above, we chose the gauge as $\boldsymbol{u}_a^{(g)\mathrm{T}}\Omega^{(g)}\boldsymbol{u}_a^{(g)} = 0$. Under the system, we consider the canonical transformation,

$$\hat{Y}^{(g)} \equiv \frac{\sqrt{2}}{\omega^{(g)}}\hat{p}_2'^{(g)}, \qquad \hat{P}^{(g)} \equiv \sqrt{2}\hat{p}_1'^{(g)},$$

$$\hat{\tilde{Y}}^{(g)} \equiv -\frac{1}{2\pi M^{(g)}}\hat{T}_2^{(g)}, \qquad \hat{\tilde{P}}^{(g)} \equiv \hat{T}_1^{(g)}, \tag{39}$$

where $\omega^{(g)} \equiv 2qB_{12}^{(g)} = 2qb^{(g)}/|\boldsymbol{u}_1^{(g)}\times\boldsymbol{u}_2^{(g)}|$ and $M^{(g)} \equiv Q_{12}^{(g)}$. These new variables satisfy the relations

$$[\hat{Y}^{(g)},\hat{P}^{(g')}] = i\delta_{gg'}, \quad [\hat{\tilde{Y}}^{(g)},\hat{\tilde{P}}^{(g')}] = i\delta_{gg'}, \quad \text{the others} = 0. \tag{40}$$

Then, we can rewrite the Hamiltonian as

$$\hat{H} = \sum_{g=1}^{2}\left[\frac{1}{2}(\hat{P}^{(g)})^2 + \frac{(\omega^{(g)})^2}{2}(\hat{Y}^{(g)})^2\right], \tag{41}$$

with the constraints

$$e^{i\hat{\tilde{P}}^{(g)}}|\psi\rangle = |\psi\rangle, \quad e^{-i2\pi M^{(g)}\hat{\tilde{Y}}^{(g)}}|\psi\rangle = |\psi\rangle \quad (g = 1, 2). \tag{42}$$

In a way similar to the previous subsection, $\hat{H}$ is constructed only from $\hat{P}^{(g)}$ and $\hat{Y}^{(g)}$, so that the constraint conditions (42) do not affect the energy spectrum of the system. $\hat{H}$ is just the sum of the two one-dimensional harmonic oscillators, which implies that the energy eigenvalues of $\hat{H}$ are given by

$$E_{n^{(1)}n^{(2)}} = \sum_{g=1}^{2}\omega^{(g)}\left(n^{(g)} + \frac{1}{2}\right), \quad (n^{(g)} = 0, 1, 2, \ldots). \tag{43}$$

Furthermore, the constraint conditions (42) lead to the quantization on the coordinates $\tilde{y}^{(g)}$,

$$\tilde{y}^{(g)} = \frac{j_g}{M^{(g)}}, \qquad (j_g = 0, 1, 2, \ldots, |M^{(g)}| - 1), \tag{44}$$

where $\tilde{y}^{(g)}$ is an eigenvalue of $\hat{\tilde{Y}}^{(g)}$ with the identification $\tilde{y}^{(g)} \sim \tilde{y}^{(g)} + 1$. Thus the eigenstates of this system are described by not only $n^{(1)}$ and $n^{(2)}$ but also j_1 and j_2 as

$$\left|n^{(1)}, n^{(2)}; \frac{j_1}{M^{(1)}}, \frac{j_2}{M^{(2)}}\right\rangle_{T^2\times T^2}. \tag{45}$$

Since the eigenvalues of the Hamiltonian are independent of the value j_g, there is $|M^{(1)}M^{(2)}|$-fold degeneracy at each energy level, i.e.,

$$\text{the number of degeneracy} = |M^{(1)}M^{(2)}|. \tag{46}$$

2. Z_N shifted orbifolding

From here, we investigate the quantum system on $(T^2 \times T^2)/Z_N$ shifted orbifold, which is defined by the identification

$$Z_N: (\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}) \sim \left(\boldsymbol{y}^{(1)} + \frac{d}{N}\boldsymbol{u}_1^{(1)}, \boldsymbol{y}^{(2)} + \frac{1}{N}\boldsymbol{u}_1^{(2)}\right), \tag{47}$$

where each of N and d is some integer and d is relatively prime with N. As we mentioned in the previous subsection, Eq. (47) is not the general form in the Z_N shifted orbifolding. However, we can always transform the general form into a special one such as Eq. (47) without any loss of generality. We discuss it in Appendix A. This identification can be translated into the requirement for physical states in the operator formalism as

$$\hat{U}_{Z_N}|\psi\rangle_{(T^2\times T^2)/Z_N} = |\psi\rangle_{(T^2\times T^2)/Z_N}, \quad \hat{U}_{Z_N} \equiv e^{i(d\hat{T}_1^{(1)} + \hat{T}_1^{(2)})/N}. \tag{48}$$

Since the Z_N shift operator has to be compatible with Eq. (38), we obtain the consistency conditions

$$\hat{U}_{Z_N}e^{i\hat{T}_a^{(g)}} = e^{i\hat{T}_a^{(g)}}\hat{U}_{Z_N}, \qquad (a = 1, 2; g = 1, 2). \tag{49}$$

These conditions lead to

$$\frac{dM^{(1)}}{N}, \qquad \frac{M^{(2)}}{N} \in \mathbb{Z}, \tag{50}$$

which imply that both $M^{(1)}$ and $M^{(2)}$ must be multiples of N, i.e.,

$$M^{(1)} = t_1 N, \qquad M^{(2)} = t_2 N, \tag{51}$$

where each of t_1 and t_2 is some integer. The physical states on $(T^2 \times T^2)/Z_N$, which are nothing but Z_N-invariant states, can be constructed as

$$\left|n^{(1)}, n^{(2)}; \frac{j_1}{M^{(1)}}, \frac{j_2}{M^{(2)}}\right\rangle_{(T^2\times T^2)/Z_N}$$

$$\equiv \frac{1}{\sqrt{N}}\sum_{\ell=0}^{N-1}(\hat{U}_{Z_N})^{-\ell}\left|n^{(1)}, n^{(2)}; \frac{j_1}{M^{(1)}}, \frac{j_2}{M^{(2)}}\right\rangle_{T^2\times T^2}$$

$$= \frac{1}{\sqrt{N}}\sum_{\ell=0}^{N-1}\left|n^{(1)}, n^{(2)}; \frac{j_1+\ell dt_1}{M^{(1)}}, \frac{j_2+\ell t_2}{M^{(2)}}\right\rangle_{T^2\times T^2}, \tag{52}$$

where each of $j_1 + \ell dt_1$ and $j_2 + \ell t_2$ is identical with an element of the set of j_g modulo $|M^{(g)}|$. Moreover, we can obtain the result that the number of degeneracy in this system is a multiple of N, i.e.,

$$\text{the number of degeneracy} = \frac{|M^{(1)}M^{(2)}|}{N} = |t_1 t_2|N. \tag{53}$$

We would like to note that the number of zero-mode fermions for $(T^2 \times T^2)/Z_N$ is given by a multiple of N, while it can be an arbitrary integer for T^2/Z_N. This result leads to an important conclusion—that there is only one

possibility for deriving the three generations of matter, i.e., $(N; M^{(1)}, M^{(2)}) = (3; 3, 3)$ on $(T^2 \times T^2)/Z_N$. Moreover, in a case of $(T^2 \times T^2)/Z_N \times T^2$, we obtain only one condition delivering the three generations of matter such as $(N; M^{(1)}, M^{(2)}, M^{(3)}) = (3; 3, 3, 1)$. We will show that these results coincide with that of the analysis with wave functions in Sec. III.

C. $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'})$ shifted orbifold

We can also apply the previous results to the system on $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'})$ shifted orbifold. As in the previous model, the number of degeneracy is also restricted in this case and can produce threefold degeneracy. We do not present the analysis by the quantum system on $T^2 \times T^2 \times T^2$ in detail because it is just a simple extension of the previous one. The energy spectrum of the system on $T^2 \times T^2 \times T^2$ is labeled by three quantum numbers, $n^{(1)}$, $n^{(2)}$ and $n^{(3)}$, and is given by

$$E_{n^{(1)}n^{(2)}n^{(3)}} = \sum_{g=1}^{3} \omega^{(g)} \left(n^{(g)} + \frac{1}{2} \right), \quad (n^{(g)} = 0, 1, 2, \ldots), \quad (54)$$

which is constructed of the three one-dimensional harmonic oscillators. Then the physical states compatible with Eq. (54) are labeled by not only $n^{(g)}$ but also the quantum numbers j_g, which come from the constraint conditions in each torus, i.e.,

$$\left| n^{(1)}, n^{(2)}, n^{(3)}; \frac{j_1}{M^{(1)}}, \frac{j_2}{M^{(2)}}, \frac{j_3}{M^{(3)}} \right\rangle_{T^2 \times T^2 \times T^2}, \quad (55)$$

where $j_g = 0, 1, 2, \ldots, |M^{(g)}| - 1$.

Next we analyze the quantum system on $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'})$. We define the Z_N and $Z_{N'}$ shifted orbifoldings as the identifications,

$$Z_N: (y^{(1)}, y^{(2)}, y^{(3)}) \sim \left(y^{(1)} + \frac{d}{N} u_1^{(1)}, y^{(2)} + \frac{1}{N} u_1^{(2)}, y^{(3)} \right),$$

$$Z_{N'}: (y^{(1)}, y^{(2)}, y^{(3)}) \sim \left(y^{(1)}, y^{(2)} + \frac{1}{N'} (s_1 u_1^{(2)} + s_2 u_2^{(2)}), \right.$$

$$\left. y^{(3)} + \frac{d'}{N'} u_1^{(3)} \right), \quad (56)$$

where each of N, N', d, and d' is some positive integer and each of s_1 and s_2 is some integer. Here we require that $d(d')$ is relatively prime with $N(N')$, and when we define s' as the greatest common divisor (gcd) of s_1 and s_2, the gcd of s' and d' is relatively prime with N'. As in Appendix A, we can always transform the general forms of the Z_N and $Z_{N'}$ shifted orbifoldings into special ones such as Eq. (56) without any loss of generality. In the operator formalism, we define the operators generating the identifications (56) as

$$\hat{U}_{Z_N} \equiv e^{i(d\hat{T}_1^{(1)} + \hat{T}_1^{(2)})/N}, \quad (57)$$

$$\hat{U}_{Z_{N'}} \equiv e^{i\pi s_1 s_2 M^{(2)}/N'} e^{i(s_1 \hat{T}_1^{(2)} + s_2 \hat{T}_2^{(2)} + d' \hat{T}_1^{(3)})/N'}. \quad (58)$$

We note that the phase factor $e^{i\pi s_1 s_2 M^{(2)}/N'}$ in Eq. (58) is necessary to be consistent with $(\hat{U}_{Z_{N'}})^{N'} = 1$. The identifications (56) lead to the requirement for physical states on $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'})$ as

$$\hat{U}_{Z_N} |\psi\rangle_{(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'})} = |\psi\rangle_{(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'})},$$

$$\hat{U}_{Z_{N'}} |\psi\rangle_{(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'})} = |\psi\rangle_{(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'})}. \quad (59)$$

Since these conditions have to be compatible with the torus conditions,

$$e^{i\hat{T}_a^{(g)}} |\psi\rangle = |\psi\rangle, \quad \hat{T}_a^{(g)} = u_a^{(g)\mathrm{T}} \left(\hat{p}^{(g)} - \frac{q}{2} \Omega^{(g)\mathrm{T}} \hat{y}^{(g)} \right), \quad (60)$$

where $g = 1, 2, 3$ and $a = 1, 2$, we obtain the consistency conditions,

$$\hat{U}_{Z_N} e^{i\hat{T}_a^{(g)}} = e^{i\hat{T}_a^{(g)}} \hat{U}_{Z_N}, \quad \hat{U}_{Z_{N'}} e^{i\hat{T}_a^{(g)}} = e^{i\hat{T}_a^{(g)}} \hat{U}_{Z_{N'}}. \quad (61)$$

Moreover, the compatibility between the Z_N and $Z_{N'}$ shift operators leads to the extra consistency condition,

$$\hat{U}_{Z_N} \hat{U}_{Z_{N'}} = \hat{U}_{Z_{N'}} \hat{U}_{Z_N}. \quad (62)$$

From Eq. (61), we obtain the conditions

$$M^{(1)} = t_1 N, \quad M^{(2)} = t_2 \frac{NN'}{d_2}, \quad M^{(3)} = t'_3 N', \quad (63)$$

where each of t_1, t_2, and t'_3 is some integer and d_2 is the gcd of N and N'. From Eq. (62), we obtain the additional constraint,

$$\frac{s_2 M^{(2)}}{NN'} \in \mathbb{Z}. \quad (64)$$

Using Eq. (63), we can rewrite this constraint as

$$\frac{s_2 t_2}{d_2} \in \mathbb{Z}. \quad (65)$$

When we define the gcd of s_2 and d_2 as γ,[7] we obtain

$$s_2 = \tilde{s}_2 \gamma, \quad d_2 = \tilde{d} \gamma, \quad t_2 = \tilde{t}_2 \tilde{d}, \quad (66)$$

where each of $\tilde{s}_2$ and $\tilde{t}_2$ is some integer and $\tilde{d}$ is some positive integer. In the case of $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'})$, the form of $M^{(2)}$, which is compatible with the consistency conditions, can be rewritten as

$$M^{(2)} = \tilde{t}_2 \frac{NN'}{\gamma}. \quad (67)$$

The physical states on $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'})$, which are nothing but Z_N- and $Z_{N'}$-invariant states, can be constructed as

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[7]In the case of $s_2 = 0$, we define the gcd of 0 and d_2 as d_2.

$$\left| n^{(1)}, n^{(2)}, n^{(3)}; \frac{j_1}{M^{(1)}}, \frac{j_2}{M^{(2)}}, \frac{j_3}{M^{(3)}} \right\rangle_{(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'})}$$

$$\equiv \mathcal{N} \sum_{\ell=0}^{N-1} (\hat{U}_{Z_N})^{-\ell} \sum_{\ell'=0}^{N'-1} (U_{Z_{N'}})^{-\ell'} \left| n^{(1)}, n^{(2)}, n^{(3)}; \frac{j_1}{M^{(1)}}, \frac{j_2}{M^{(2)}}, \frac{j_3}{M^{(3)}} \right\rangle_{T^2 \times T^2 \times T2}$$

$$= \mathcal{N} \sum_{\ell=0}^{N-1} \sum_{\ell'=0}^{N'-1} e^{i\pi\ell' s_2(2j_2 - (N'-\ell')s_1 t_2')/N'} \left| n^{(1)}, n^{(2)}, n^{(3)}; \frac{j_1 + \ell d t_1}{M^{(1)}}, \frac{j_2 + \ell t_2' + \ell' s_1 t_2''}{M^{(2)}}, \frac{j_3 + \ell' d' t_3'}{M^{(3)}} \right\rangle_{T^2 \times T^2 \times T2}, \quad (68)$$

where $\mathcal{N}$ is the normalization factor, $t_2' \equiv \tilde{t}_2 N'/\gamma$ and $t_2'' \equiv \tilde{t}_2 N/\gamma$.

Then we obtain the result that the number of degeneracy in this system is given by

the number of degeneracy

$$= \frac{|M^{(1)}M^{(2)}M^{(3)}|}{NN'} = |t_1 \tilde{t}_2 t_3'| \frac{NN'}{\gamma}, \quad (69)$$

which is a multiple of NN'/γ. Hence, as a conclusion similar to $(T^2 \times T^2)/Z_N$, we would like to note that the number of zero-mode fermions for $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'})$ is given by a multiple of N and N'. This result leads to an important conclusion—that there is only one possibility to derive the three generations of matter, i.e., $(N, N'; M^{(1)}, M^{(2)}, M^{(3)}) = (3, 3; 3, 3, 3)$ on $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'})$. We will show that this result coincides with that of the analysis with wave functions.

In a similar way, we can consider the case of $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'} \times Z_{N''})$. For example, we assume three discrete symmetries as

$$Z_N: (\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}, \boldsymbol{y}^{(3)}) \sim \left(\boldsymbol{y}^{(1)} + \frac{1}{N}, \boldsymbol{y}^{(2)} + \frac{1}{N}, \boldsymbol{y}^{(3)} \right),$$

$$Z_{N'}: (\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}, \boldsymbol{y}^{(3)}) \sim \left(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)} + \frac{1}{N'}, \boldsymbol{y}^{(3)} + \frac{1}{N'} \right), \quad (70)$$

$$Z_{N''}: (\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}, \boldsymbol{y}^{(3)}) \sim \left(\boldsymbol{y}^{(1)} + \frac{1}{N''}, \boldsymbol{y}^{(2)}, \boldsymbol{y}^{(3)} + \frac{1}{N''} \right),$$

where each of N, N', and N'' is some integer. Then we obtain the result that the number of degeneracy in this system is given by (see Appendix B)

the number of degeneracy $= \dfrac{|M^{(1)}M^{(2)}M^{(3)}|}{NN'N''}$

$$= |t_1 t_2 t_3| \frac{NN'N''}{d_1 d_2 d_3}, \quad (71)$$

where d_1 is the gcd of N and N', d_2 is the gcd of N' and N'', and d_3 is the gcd of N'' and N. We would also like to note that there are only two possibilities to derive the three generations of matter, i.e., $(N, N', N''; M^{(1)}, M^{(2)}, M^{(3)}) = (3, 9, 3; 3, 9, 9), (3, 9, 9; 9, 9, 9)$, up to the permutation of parameters for the magnitude of fluxes and the shift symmetries. We will also show that this result coincides with that of the analysis with wave functions.

III. WAVE FUNCTION ON SHIFTED ORBIFOLD WITH MAGNETIC FLUX

In Sec. II, we considered the fields on shifted orbifolds with magnetic flux by the operator formalism. Here we reconsider it by analyzing wave functions explicitly because it is necessary to calculate some physical quantities, e.g., Yukawa couplings and higher-order couplings [2,9]. We will see below that the results of the analysis with wave functions coincide with results by the operator formalism.

A. Review of the $U(1)$ gauge theory on T^2

1. Magnetic flux quantization

First, we review the $U(1)$ gauge theory on the two-dimensional torus with magnetic flux.[8] Here, it is convenient to use the complex coordinate $z = y_1 + iy_2$, $\bar{z} = y_1 - iy_2$ instead of the vector notation $\boldsymbol{y} = (y_1, y_2)^{\mathrm{T}}$ in order to write wave functions explicitly. They satisfy the identification $z \sim z + 1 \sim z + \tau (\tau \in \mathbb{C}, \mathrm{Im}\tau > 0)$ on T^2.[9] Similarly, we make use of the complex basis for the vector potential as

$$A_z = \frac{1}{2}(A_{y_1} - iA_{y_2}), \qquad A_{\bar{z}} = \frac{1}{2}(A_{y_1} + iA_{y_2}). \quad (72)$$

For the nonzero magnetic flux b on T^2, we can write that $b = \int_{T^2} F$ by the field strength

$$F = \frac{ib}{2\mathrm{Im}\tau} dz \wedge d\bar{z}. \quad (73)$$

For $F = dA$, the vector potential A can be written as

$$A(z, \bar{z}) = \frac{b}{2\mathrm{Im}\tau} \mathrm{Im}[(\bar{z} + \bar{a})dz] \equiv A_z(z, \bar{z})dz + A_{\bar{z}}(z, \bar{z})d\bar{z}, \quad (74)$$

where $a(\in \mathbb{C})$ is a Wilson line phase. From Eq. (74), we obtain

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[8]This subsection is based on Ref. [2].
[9]For convenience, we choose $(1, \tau)$ as two circumferences of the two-dimensional torus.

$$A(z + 1, \bar{z} + 1) = A(z, \bar{z}) + \frac{b}{2\mathrm{Im}\tau}\mathrm{Im}dz$$
$$\equiv A(z, \bar{z}) + d\chi_1(z, \bar{z}),$$

$$A(z + \tau, \bar{z} + \bar{\tau}) = A(z, \bar{z}) + \frac{b}{2\mathrm{Im}\tau}\mathrm{Im}(\bar{\tau}dz)$$
$$\equiv A(z, \bar{z}) + d\chi_2(z, \bar{z}), \qquad (75)$$

where[10]

$$\chi_1(z, \bar{z}) = \frac{b}{2\mathrm{Im}\tau}\mathrm{Im}(z + a),$$
$$\chi_2(z, \bar{z}) = \frac{b}{2\mathrm{Im}\tau}\mathrm{Im}(\bar{\tau}(z + a)). \qquad (76)$$

Moreover, let us consider a field $\Phi(z, \bar{z})$ with the $U(1)$ charge q on T^2. We require the Lagrangian density $\mathcal{L}$ to be single-valued as

$$\mathcal{L}(A(z, \bar{z}), \Phi(z, \bar{z})) = \mathcal{L}(A(z + 1, \bar{z} + 1), \Phi(z + 1, \bar{z} + 1))$$
$$= \mathcal{L}(A(z + \tau, \bar{z} + \bar{\tau}), \Phi(z + \tau, \bar{z} + \bar{\tau})). \qquad (77)$$

Then this field $\Phi(z, \bar{z})$ should satisfy the pseudoperiodic boundary conditions,

$$\Phi(z + 1, \bar{z} + 1) = e^{iq\chi_1(z,\bar{z})}\Phi(z, \bar{z}),$$
$$\Phi(z + \tau, \bar{z} + \bar{\tau}) = e^{iq\chi_2(z,\bar{z})}\Phi(z, \bar{z}). \qquad (78)$$

From these, the consistency of the contractible loops, e.g., $z \to z + 1 \to z + 1 + \tau \to z + \tau \to z$, requires the magnetic flux quantization condition,

$$\frac{qb}{2\pi} \equiv M \in \mathbb{Z}. \qquad (79)$$

2. Zero-mode solutions of a fermion

Here we consider zero-mode solutions of a fermion $\psi(z, \bar{z})$ on T^2 with magnetic flux, which satisfy the equation

$$\sum_{a=z,\bar{z}} \Gamma^a(\partial_a - iqA_a)\psi(z, \bar{z}) = 0, \qquad (80)$$

where

$$\partial_z = \frac{1}{2}(\partial_{y_1} - i\partial_{y_2}), \qquad \partial_{\bar{z}} = \frac{1}{2}(\partial_{y_1} + i\partial_{y_2}),$$

$$\Gamma^z = \Gamma^1 + i\Gamma^2 = \sigma^1 + i\sigma^2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \qquad (81)$$

$$\Gamma^{\bar{z}} = \Gamma^1 - i\Gamma^2 = \sigma^1 - i\sigma^2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}.$$

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[10]For $b \neq 0$, the general functions of $\chi_1(z, \bar{z})$ and $\chi_2(z, \bar{z})$ can be written as $\chi_1(z, \bar{z}) = \frac{b}{2\mathrm{Im}\tau}\mathrm{Im}(z + a) + \frac{\pi\alpha_1}{q}$, $\chi_2(z, \bar{z}) = \frac{b}{2\mathrm{Im}\tau}\mathrm{Im}(\bar{\tau}(z + a)) + \frac{\pi\alpha_2}{q}$, where α_1 and α_2 are real numbers. We can always make α_i absorbed into the Wilson line phase a by the redefinition of fields. (See Appendix C.)

Then we can write $\psi(z, \bar{z})$ as a two-component spinor,

$$\psi(z, \bar{z}) = \begin{pmatrix} \psi_+(z, \bar{z}) \\ \psi_-(z, \bar{z}) \end{pmatrix}, \qquad (82)$$

and Eq. (80) can be decomposed as

$$\left(\partial_{\bar{z}} + \frac{\pi M}{2\mathrm{Im}\tau}(z + a)\right)\psi_+(z, \bar{z}) = 0,$$
$$\left(\partial_z - \frac{\pi M}{2\mathrm{Im}\tau}(\bar{z} + \bar{a})\right)\psi_-(z, \bar{z}) = 0. \qquad (83)$$

The fields $\psi_\pm(z, \bar{z})$ should obey the conditions (78), i.e.,

$$\psi_\pm(z + 1, \bar{z} + 1) = e^{iq\chi_1(z,\bar{z})}\psi_\pm(z, \bar{z}),$$
$$\psi_\pm(z + \tau, \bar{z} + \bar{\tau}) = e^{iq\chi_2(z,\bar{z})}\psi_\pm(z, \bar{z}). \qquad (84)$$

From Eqs. (83) and (84), we find that for $M > 0 (M < 0)$ only $\psi_+(\psi_-)$ has solutions. Their zero-mode wave functions are given by

$$\psi_+^j(z, \bar{z}) = \mathcal{N}e^{i\pi M(z+a)\frac{\mathrm{Im}(z+a)}{\mathrm{Im}\tau}}$$
$$\cdot \vartheta\begin{bmatrix} \frac{j}{M} \\ 0 \end{bmatrix}(M(z + a), M\tau) \quad \text{for } M > 0, \qquad (85)$$

$$\psi_-^j(z, \bar{z}) = \mathcal{N}e^{i\pi M(\bar{z}+\bar{a})\frac{\mathrm{Im}(\bar{z}+\bar{a})}{\mathrm{Im}\bar{\tau}}}$$
$$\cdot \vartheta\begin{bmatrix} \frac{j}{M} \\ 0 \end{bmatrix}(M(\bar{z} + \bar{a}), M\bar{\tau}) \quad \text{for } M < 0, \qquad (86)$$

where $j = 0, 1, \ldots, |M| - 1$ and $\mathcal{N}$ is the normalization factor. Here the ϑ function is defined by

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\nu, \tau) = \sum_{l\in\mathbb{Z}} e^{i\pi(a+l)^2\tau}e^{2\pi i(a+l)(\nu+b)}, \qquad (87)$$

with the properties

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\nu + n, \tau) = e^{2\pi ian}\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\nu, \tau),$$
$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\nu + n\tau, \tau) = e^{-i\pi n^2\tau - 2\pi in(\nu+b)}\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\nu, \tau), \qquad (88)$$

where a and b are real numbers, ν and τ are complex numbers, and $\mathrm{Im}\tau > 0$.

B. $U(1)$ gauge theory on T^2/Z_N

Next, we investigate the $U(1)$ gauge theory on T^2/Z_N with magnetic flux, in which the Z_N shifted orbifolding satisfies the identification $z \sim z + e_N^{mn}(e_N^{mn} \equiv (m + n\tau)/N; m, n \in \mathbb{Z})$. Here we consider a general Z_N shift e_N^{mn} for convenience of practical computations, although we could take, say, $(m, n) = (1, 0)$ without any loss of generality as in Eq. (21). From Eq. (74), we obtain

$$A(z + e_N^{mn}, \bar{z} + \bar{e}_N^{mn}) = A(z, \bar{z}) + \frac{\pi M}{q \mathrm{Im}\tau} \mathrm{Im}(\bar{e}_N^{mn} dz)$$
$$\equiv A(z, \bar{z}) + d\chi_N(z, \bar{z}). \qquad (89)$$

Then we define the physical state $\Phi(z, \bar{z})$ which is consistent with Eqs. (78) and (89) as

$$\Phi(z + e_N^{mn}, \bar{z} + \bar{e}_N^{mn}) = e^{iq\chi_N(z,\bar{z})}\Phi(z, \bar{z}), \qquad (90)$$

$$\chi_N(z, \bar{z}) = \frac{m}{N}\chi_1(z, \bar{z}) + \frac{n}{N}\chi_2(z, \bar{z}) + \frac{\pi \alpha_N}{q}, \qquad (91)$$

where α_N is some real number and is determined below. For Eq. (90) to be consistent with Eq. (78), we find the relation

$$e^{iqN\chi_N(z,\bar{z})} = e^{iq(m\chi_1(z,\bar{z})+n\chi_2(z,\bar{z}))}e^{i\pi mnM}. \qquad (92)$$

It follows that α_N can be determined as $\alpha_N = mnM/N$. Then the consistency of the contractible loops, e.g., $z \to z + 1 \to z + 1 + e_N^{mn} \to z + 1 + \tau + e_N^{mn} \to z + \tau + e_N^{mn} \to z + e_N^{mn} \to z$, requires additional conditions such as[11]

$$\frac{mM}{N}, \qquad \frac{nM}{N} \in \mathbb{Z}. \qquad (93)$$

Since N has to be relatively prime with the gcd of m and n for e_N^{mn} to represent a Z_N shift symmetry, the conditions (93) lead to the magnetic flux quantization condition,

$$M = tN, \qquad (94)$$

where t is some integer. We note that this result agrees with Eq. (25), even though in Sec. II A 2, the magnetic flux quantization condition was derived for a special case of the identification (21), which may correspond to $(m, n) = (1, 0)$.

Furthermore, we consider zero-mode fermions $\psi_\pm^j(z, \bar{z})$. It follows from Eqs. (85) and (86) that $\psi_\pm^j(z, \bar{z})$ satisfy the equations,

$$\psi_\pm^j(z + \ell e_N^{mn}, \bar{z} + \ell \bar{e}_N^{mn})$$
$$= e^{iq\ell\chi_N(z,\bar{z})}e^{i\pi\ell m(2j-(N-\ell)nt)/N}\psi_\pm^{j+\ell nt}(z, \bar{z}), \qquad (95)$$

where ℓ is any integer and $\chi_N(z + e_N^{mn}, \bar{z} + \bar{e}_N^{mn}) = \chi_N(z, \bar{z})$. Since $\psi_\pm^j(z, \bar{z})$ do not, in general, satisfy the physical state condition (90) on T^2/Z_N, we may need to take appropriate linear combinations of them. For example, when $(m, n) = (0, 1)$, we obtain

$$\psi_\pm^j\left(z + \frac{\tau\ell}{N}, \bar{z} + \frac{\bar{\tau}\ell}{N}\right) = e^{iq\ell\chi_N(z,\bar{z})}\psi_\pm^{j+\ell t}(z, \bar{z}), \qquad (96)$$

so the physical states $\Psi_\pm^j(z, \bar{z})$ are given by

[11]Actually, we find the consistency conditions $e^{i\pi M(\frac{m}{N}+\frac{n}{N})} = e^{-i\pi M(\frac{m}{N}+\frac{n}{N})} = e^{i\pi M(\frac{m}{N}-\frac{n}{N})} = e^{-i\pi M(\frac{m}{N}-\frac{n}{N})}$, which lead to Eq. (93).

$$\Psi_\pm^j(z, \bar{z}) = \frac{1}{\sqrt{N}}\sum_{\ell=0}^{N-1} e^{-iq\ell\chi_N(z)}\psi_\pm^j\left(z + \frac{\tau\ell}{N}, \bar{z} + \frac{\bar{\tau}\ell}{N}\right)$$
$$= \frac{1}{\sqrt{N}}\sum_{\ell=0}^{N-1}\psi_\pm^{j+\ell t}(z, \bar{z}), \qquad (97)$$

where $j = 0, 1, \ldots, |t| - 1$. We can check that these $\Psi_\pm^j(z, \bar{z})$ indeed satisfy Eq. (90). When $(m, n) = (1, 0)$, we obtain

$$\psi_\pm^j\left(z + \frac{\ell}{N}, \bar{z} + \frac{\ell}{N}\right) = e^{iq\ell\chi_N(z,\bar{z})}e^{2\pi ij\ell/N}\psi_\pm^j(z, \bar{z}), \qquad (98)$$

so the physical states $\Psi_\pm^j(z, \bar{z})$ are given by

$$\Psi_\pm^j(z, \bar{z}) = \frac{1}{N}\sum_{\ell=0}^{N-1} e^{-iq\ell\chi_N(z,\bar{z})}\psi_\pm^j\left(z + \frac{\ell}{N}, \bar{z} + \frac{\ell}{N}\right)$$
$$= \frac{1}{N}\sum_{\ell=0}^{N-1} e^{2\pi ij\ell/N}\psi_\pm^j(z, \bar{z})$$
$$= \begin{cases} \psi_\pm^j(z, \bar{z}) & (j \equiv 0 \bmod N) \\ 0 & (j \not\equiv 0 \bmod N). \end{cases} \qquad (99)$$

We can check that these $\Psi_\pm^j(z, \bar{z})$ satisfy Eq. (90).

For a general Z_N shift e_N^{mn}, we can obtain the physical states $\Psi_\pm^j(z, \bar{z})$, which satisfy Eq. (90), as

$$\Psi_\pm^j(z, \bar{z}) = \mathcal{N}'\sum_{\ell=0}^{N-1} e^{-iq\ell\chi_N(z,\bar{z})}\psi_\pm^j(z + \ell e_N^{mn}, \bar{z} + \ell \bar{e}_N^{mn})$$
$$= \mathcal{N}'\sum_{\ell=0}^{N-1} e^{i\pi\ell m(2j-(N-\ell)nt)/N}\psi_\pm^{j+\ell nt}(z, \bar{z}), \qquad (100)$$

where $\mathcal{N}'$ is the normalization factor. We note that we cannot obtain any constraint on the number of zero-mode fermions, i.e., the generation of matter in the standard model, because the number of degeneracy t is a free parameter.

Here, we comment briefly on couplings on T^2/Z_N. On the two-dimensional torus, a generic L-point coupling $C^{j_{(1)}\cdots j_{(L)}}$ in a four-dimensional low-energy effective field theory is given by the overlap integral of zero-mode functions [2,9],

$$C^{j_{(1)}\cdots j_{(L)}} = c^{j_{(1)}\cdots j_{(L)}}\int_{T^2} d^2z\, \psi^{j_{(1)}} \ldots \psi^{j_{(L)}}, \qquad (101)$$

where $c^{j_{(1)}\cdots j_{(L)}}$ denotes the coupling in a higher-dimensional field theory. Similarly, on the orbifold T^2/Z_N, a generic L-point coupling $C_{\mathrm{orbifold}}^{j_{(1)}\cdots j_{(L)}}$ is given as

$$C_{\mathrm{orbifold}}^{j_{(1)}\cdots j_{(L)}} = c^{j_{(1)}\cdots j_{(L)}}\int_{T^2/Z_N} d^2z\, \Psi^{j_{(1)}} \ldots \Psi^{j_{(L)}}. \qquad (102)$$

Since the zero-mode wave functions Ψ^j are written by linear combinations of ψ^j, as shown in Eq. (100), the coupling $C_{\mathrm{orbifold}}^{j_{(1)}\cdots j_{(L)}}$ is written by a proper linear combination

of $C^{j_{(1)}\cdots j_{(L)}}$. Its extension to other orbifolds such as $(T^2 \times T^2)/Z_N$, $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'})$, and $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'} \times Z_{N''})$ is also straightforward.

Although we have constructed the Z_N-invariant wave functions $\Psi_\pm^j(z, \bar{z})$ as in Eq. (100), it is also worthwhile to consider wave functions $\Phi_\kappa(z, \bar{z})$ with a Z_N charge κ, which is defined as

$$\Phi_\kappa(z + e_N^{mn}, \bar{z} + \bar{e}_N^{mn}) = \omega^\kappa e^{iq\chi_N(z,\bar{z})}\Phi_\kappa(z, \bar{z}), \quad (103)$$

where κ is some integer and $\omega \equiv e^{2\pi i/N}$. Then, in a way similar to Eq. (100), the wave functions $\Psi_{\pm,\kappa}^j(z, \bar{z})$ satisfying Eq. (103) can be constructed from $\psi_\pm^j(z, \bar{z})$ and are given by

$$\Psi_{\pm,\kappa}^j(z, \bar{z})$$
$$= \mathcal{N}' \sum_{\ell=0}^{N-1} \omega^{-\ell\kappa} e^{-iq\ell\chi_N(z,\bar{z})} \psi_\pm^j(z + \ell e_N^{mn}, \bar{z} + \ell \bar{e}_N^{mn})$$
$$= \mathcal{N}' \sum_{\ell=0}^{N-1} e^{i\pi\ell m(2j-(N-\ell)nt)/N} e^{-2\pi i\ell\kappa/N} \psi_\pm^{j+\ell nt}(z, \bar{z}).$$
$$(104)$$

C. $U(1)$ gauge theory on $(T^2 \times T^2)/Z_N$

In a similar way, we consider the $U(1)$ gauge theory on $(T^2 \times T^2)/Z_N$ with magnetic flux. Let us define $z^{(1)}$ and $z^{(2)}$ as the complex coordinates for each torus with the identifications $z^{(g)} \sim z^{(g)} + 1 \sim z^{(g)} + \tau^{(g)}$. When there are nonzero magnetic fluxes $b^{(1)}$ and $b^{(2)}$ on $T^2 \times T^2$, we can write that $b^{(g)} = \int_{T^{2(g)}} F^{(g)}$ by the field strengths,

$$F^{(g)} = \frac{ib^{(g)}}{2\mathrm{Im}\tau^{(g)}} dz^{(g)} \wedge d\bar{z}^{(g)}, \quad (g = 1, 2). \quad (105)$$

For $F^{(g)} = dA^{(g)}$, the vector potentials $A^{(g)}$ can be written as[12]

$$A^{(g)}(z^{(g)}) = \frac{b^{(g)}}{2\mathrm{Im}\tau^{(g)}} \mathrm{Im}[(\bar{z}^{(g)} + \bar{a}^{(g)})dz^{(g)}]$$
$$\equiv A_{z^{(g)}}^{(g)}(z^{(g)})dz^{(g)} + A_{\bar{z}^{(g)}}^{(g)}(z^{(g)})d\bar{z}^{(g)}, \quad (106)$$

where $a^{(g)}(\in \mathbb{C})$ are Wilson line phases. From Eq. (106), we obtain

$$A^{(g)}(z^{(g)} + 1) = A^{(g)}(z^{(g)}) + \frac{b^{(g)}}{2\mathrm{Im}\tau^{(g)}} \mathrm{Im}\,dz^{(g)}$$
$$\equiv A^{(g)}(z^{(g)}) + d\chi_1^{(g)}(z^{(g)}),$$

$$A^{(g)}(z^{(g)} + \tau^{(g)}) = A^{(g)}(z^{(g)}) + \frac{b^{(g)}}{2\mathrm{Im}\tau^{(g)}} \mathrm{Im}(\bar{\tau}^{(g)}dz^{(g)})$$
$$\equiv A^{(g)}(z^{(g)}) + d\chi_2^{(g)}(z^{(g)}), \quad (107)$$

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[12]From here we omit $\bar{z}^{(g)}$ such as $A^{(g)}(z^{(g)})$, $\Phi^{(g)}(z^{(g)})$, and $\psi(z^{(g)})$.

where

$$\chi_1^{(g)}(z^{(g)}) = \frac{b^{(g)}}{2\mathrm{Im}\tau^{(g)}} \mathrm{Im}(z^{(g)} + a^{(g)}),$$
$$\chi_2^{(g)}(z^{(g)}) = \frac{b^{(g)}}{2\mathrm{Im}\tau^{(g)}} \mathrm{Im}(\bar{\tau}^{(g)}(z^{(g)} + a^{(g)})). \quad (108)$$

Let us next consider a field $\Phi(z^{(1)}, z^{(2)})$ with the $U(1)$ charge q on $T^2 \times T^2$. For the Lagrangian density to be single-valued on $T^2 \times T^2$, we require that the field $\Phi(z^{(1)}, z^{(2)})$ satisfies the pseudoperiodic boundary conditions,

$$\Phi(z^{(1)} + 1, z^{(2)}) = e^{iq\chi_1^{(1)}(z^{(1)})}\Phi(z^{(1)}, z^{(2)}),$$
$$\Phi(z^{(1)} + \tau^{(1)}, z^{(2)}) = e^{iq\chi_2^{(1)}(z^{(1)})}\Phi(z^{(1)}, z^{(2)}),$$
$$\Phi(z^{(1)}, z^{(2)} + 1) = e^{iq\chi_1^{(2)}(z^{(2)})}\Phi(z^{(1)}, z^{(2)}),$$
$$\Phi(z^{(1)}, z^{(2)} + \tau^{(2)}) = e^{iq\chi_2^{(2)}(z^{(2)})}\Phi(z^{(1)}, z^{(2)}). \quad (109)$$

The compatibility of the conditions (109) with any contractible loops requires the magnetic flux quantization conditions,

$$\frac{qb^{(1)}}{2\pi} \equiv M^{(1)}, \qquad \frac{qb^{(2)}}{2\pi} \equiv M^{(2)} \in \mathbb{Z}. \quad (110)$$

Moreover, we consider $(T^2 \times T^2)/Z_N$ to impose a Z_N shift symmetry on $T^2 \times T^2$ with the additional identification $(z^{(1)}, z^{(2)}) \sim (z^{(1)} + e_N^{m_1n_1}, z^{(2)} + e_N^{m_2n_2})$. For the Lagrangian density to be single-valued on $(T^2 \times T^2)/Z_N$, we define $\Phi(z^{(1)}, z^{(2)})$ following Eq. (109) as

$$\Phi(z^{(1)} + e_N^{m_1n_1}, z^{(2)} + e_N^{m_2n_2})$$
$$= e^{iq(\chi_N^{(1)}(z^{(1)}) + \chi_N^{(2)}(z^{(2)}))}\Phi(z^{(1)}, z^{(2)}), \quad (111)$$

$$\chi_N^{(g)}(z^{(g)}) = \frac{m_g}{N}\chi_1^{(g)}(z^{(g)}) + \frac{n_g}{N}\chi_2^{(g)}(z^{(g)}) + \frac{\pi\alpha_N^{(g)}}{q}, \quad (112)$$

where $\alpha_N^{(g)}$ are some real numbers. For Eq. (111) to be consistent with Eq. (109), we find the relation

$$e^{iqN(\chi_N^{(1)}(z^{(1)}) + \chi_N^{(2)}(z^{(2)}))} = \prod_{g=1,2} e^{iq(m_g\chi_1^{(g)}(z^{(g)}) + n_g\chi_2^{(g)}(z^{(g)}))}$$
$$\times e^{i\pi m_g n_g M^{(g)}}, \quad (113)$$

which determines the values of $\alpha_N^{(g)}$ to be $\alpha_N^{(g)} = m_g n_g M^{(g)}/N$. Then the consistency of the contractible loops requires the additional magnetic flux quantization conditions,

$$\frac{m_1 M^{(1)}}{N}, \qquad \frac{n_1 M^{(1)}}{N}, \qquad \frac{m_2 M^{(2)}}{N}, \qquad \frac{n_2 M^{(2)}}{N} \in \mathbb{Z}. \quad (114)$$

In the same way as T^2/Z_N, each of $M^{(g)}$ turns out to be a multiple of N, i.e.,

$$M^{(1)} = t_1 N, \qquad M^{(2)} = t_2 N, \qquad (115)$$

where each of t_1 and t_2 is some integer. We note that this result also agrees with Eq. (51) and the number of degeneracy is given by Eq. (53).

Next, we consider zero-mode solutions of a fermion $\psi(z^{(1)}, z^{(2)})$ on $(T^2 \times T^2)/Z_N$, which satisfies the equation

$$\sum_{g=1}^{2} \sum_{a=z^{(g)}, \bar{z}^{(g)}} \Gamma^a (\partial_a - iqA_a^{(g)}) \psi(z^{(1)}, z^{(2)}) = 0, \qquad (116)$$

where

$$\Gamma^{z^{(1)}} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \otimes \sigma^0, \qquad \Gamma^{\bar{z}^{(1)}} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \otimes \sigma^0,$$

$$\Gamma^{z^{(2)}} = \sigma^3 \otimes \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \qquad \Gamma^{\bar{z}^{(2)}} = \sigma^3 \otimes \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}. \qquad (117)$$

Then we can write $\psi(z^{(1)}, z^{(2)})$ as a four-component spinor,

$$\psi(z^{(1)}, z^{(2)}) = \begin{pmatrix} \psi_+^{j_1}(z^{(1)}) \\ \psi_-^{j_1}(z^{(1)}) \end{pmatrix} \otimes \begin{pmatrix} \psi_+^{j_2}(z^{(2)}) \\ \psi_-^{j_2}(z^{(2)}) \end{pmatrix}$$

$$\equiv (\psi_{\mathcal{P}}^J(z^{(1)}, z^{(2)})), \qquad (118)$$

where $\mathcal{P} \equiv (\pm, \pm), (\pm, \mp)$, and $J \equiv (j_1, j_2)$. For the same reason as T^2, depending on $M^{(1)} \lessgtr 0$ and $M^{(2)} \lessgtr 0$, only one of $\psi_{\mathcal{P}}$ is well defined, while the others cannot be normalizable.

Furthermore, the zero-mode fermions $\psi_{\mathcal{P}}^J(z^{(1)}, z^{(2)})$ are constructed of $\psi_\pm^{j_g}(z^{(g)})$ on each torus, and from Eqs. (85) and (86), $\psi_\pm^{j_g}(z^{(g)})$ satisfy the equations

$$\psi_\pm^{j_g}(z^{(g)} + \ell e_N^{m_g n_g})$$
$$= e^{iq\ell\chi_N^{(g)}(z^{(g)})} e^{i\pi\ell m_g(2j_g - (N-\ell)n_g t_g)/N} \psi_\pm^{j_g + \ell n_g t_g}(z^{(g)}), \quad (119)$$

where ℓ is any integer and $\chi_N^{(g)}(z^{(g)} + e_N^{m_g n_g}) = \chi_N^{(g)}(z^{(g)})$. Then, since $\psi_{\mathcal{P}}^J(z^{(1)}, z^{(2)})$ do not, in general, satisfy the physical state condition (111) on $(T^2 \times T^2)/Z_N$, we may need to take appropriate linear combinations of them in order to obtain the physical states $\Psi_{\mathcal{P}}^J(z^{(1)}, z^{(2)})$. For example, when $(m_1, n_1, m_2, n_2) = (0, 1, 0, 1)$, the physical states $\Psi_{\mathcal{P}}^J(z^{(1)}, z^{(2)})$ are given by

$$\Psi_{\mathcal{P}}^J(z^{(1)}, z^{(2)}) = \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} \psi_{\mathcal{P}}^{J+\ell T}(z^{(1)}, z^{(2)}), \qquad (120)$$

where $J + \ell T \equiv (j_1 + \ell t_1, j_2 + \ell t_2)$ and the number of degeneracy of $\Psi_{\mathcal{P}}^J(z^{(1)}, z^{(2)})$ is $|t_1 t_2| N$. When $(m_1, n_1, m_2, n_2) = (1, 0, 1, 0)$, the physical states $\Psi_{\mathcal{P}}^J(z^{(1)}, z^{(2)})$ are given by

$$\Psi_{\mathcal{P}}^J(z^{(1)}, z^{(2)}) = \frac{1}{N} \sum_{\ell=0}^{N-1} e^{2\pi i \ell(j_1 + j_2)/N} \psi_{\mathcal{P}}^J(z^{(1)}, z^{(2)})$$

$$= \begin{cases} \psi_{\mathcal{P}}^J(z^{(1)}, z^{(2)}) & ((j_1 + j_2) \equiv 0 \bmod N) \\ 0 & ((j_1 + j_2) \not\equiv 0 \bmod N), \end{cases} \qquad (121)$$

where the number of degeneracy of $\Psi_{\mathcal{P}}^J(z^{(1)}, z^{(2)})$ is $|t_1 t_2| N$.

In the same way, for a general Z_N shift $e_N^{m_g n_g}$, the physical states $\Psi_{\mathcal{P}}^J(z^{(1)}, z^{(2)})$, which satisfy Eq. (111), are given by

$$\Psi_{\mathcal{P}}^J(z^{(1)}, z^{(2)})$$

$$= \mathcal{N}' \sum_{\ell=0}^{N-1} \prod_{g=1,2} e^{i\pi\ell m_g(2j_g - (N-\ell)n_g t_g)/N} \psi_{\mathcal{P}}^{J+\ell T_{n_1 n_2}}(z^{(1)}, z^{(2)}), \qquad (122)$$

where $\mathcal{N}'$ is the normalization factor and $J + \ell T_{n_1 n_2} \equiv (j_1 + \ell n_1 t_1, j_2 + \ell n_2 t_2)$.

We would like to notice that the number of zero-mode fermions for $(T^2 \times T^2)/Z_N$ is given by a multiple of N, while it can be an arbitrary integer for T^2/Z_N. This result coincides with that of the operator formalism and leads to an important conclusion that there is only one possibility to derive the three generations of matter, i.e., $(N; M^{(1)}, M^{(2)}) = (3; 3, 3)$ on $(T^2 \times T^2)/Z_N$. Moreover, in a case of $(T^2 \times T^2)/Z_N \times T^2$, we obtain only one condition delivering the three generations of matter such as $(N; M^{(1)}, M^{(2)}, M^{(3)}) = (3; 3, 3, 1)$.

Furthermore, it is also worthwhile to consider wave functions $\Phi_\kappa(z^{(1)}, z^{(2)})$ with a Z_N charge κ, which is defined by

$$\Phi_\kappa(z^{(1)} + e_N^{m_1 n_1}, z^{(2)} + e_N^{m_2 n_2})$$
$$= \omega^\kappa e^{iq(\chi_N^{(1)}(z^{(1)}) + \chi_N^{(2)}(z^{(2)}))} \Phi_\kappa(z^{(1)}, z^{(2)}), \qquad (123)$$

where κ is some integer and $\omega \equiv e^{2\pi i/N}$. Then the wave functions $\Psi_{\mathcal{P}, \kappa}^J(z^{(1)}, z^{(2)})$ satisfying Eq. (123) can be constructed from $\psi_\pm^{j_g}(z^{(g)})$ and are given by

$$\Psi_{\mathcal{P}, \kappa}^J(z^{(1)}, z^{(2)})$$

$$= \mathcal{N}' \sum_{\ell=0}^{N-1} \prod_{g=1,2} e^{i\pi\ell m_g(2j_g - (N-\ell)n_g t_g)/N} e^{-2\pi i \ell \kappa/N}$$

$$\times \psi_{\mathcal{P}}^{J+\ell T_{n_1 n_2}}(z^{(1)}, z^{(2)}). \qquad (124)$$

D. $U(1)$ gauge theory on $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'})$

In a way similar to the case of $(T^2 \times T^2)/Z_N$, we consider the $U(1)$ gauge theory on $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'})$ with magnetic flux. Here we leave out the full analysis and discuss some points. We impose the Z_N shift symmetry on the first and second tori, which relates the first torus with

the second one as $(z^{(1)}, z^{(2)}, z^{(3)}) \sim (z^{(1)} + e_N^{m_1 n_1}, z^{(2)} + e_N^{m_2 n_2}, z^{(3)})$. On the other hand, we impose the $Z_{N'}$ shift symmetry on the second and third tori, which relates the second torus with the third one as $(z^{(1)}, z^{(2)}, z^{(3)}) \sim (z^{(1)}, z^{(2)} + e_{N'}^{m'_2 n'_2}, z^{(3)} + e_{N'}^{m'_3 n'_3})$. In this case, for the Lagrangian density to be single-valued, the pseudoperiodic boundary conditions of the physical states $\Phi(z^{(1)}, z^{(2)}, z^{(3)})$ with the $U(1)$ charge q are given by

$$\Phi(z^{(1)} + e_N^{m_1 n_1}, z^{(2)} + e_N^{m_2 n_2}, z^{(3)})$$
$$= e^{iq(\chi_N^{(1)}(z^{(1)}) + \chi_N^{(2)}(z^{(2)}))} \Phi(z^{(1)}, z^{(2)}, z^{(3)}), \quad (125)$$

$$\chi_N^{(g)}(z^{(g)}) = \frac{m_g}{N} \chi_1^{(g)}(z^{(g)}) + \frac{n_g}{N} \chi_2^{(g)}(z^{(g)}) + \frac{\pi \alpha_N^{(g)}}{q}, \quad (126)$$

$$\Phi(z^{(1)}, z^{(2)} + e_{N'}^{m'_2 n'_2}, z^{(3)} + e_{N'}^{m'_3 n'_3})$$
$$= e^{iq(\chi_{N'}^{(2)}(z^{(2)}) + \chi_{N'}^{(3)}(z^{(3)}))} \Phi(z^{(1)}, z^{(2)}, z^{(3)}), \quad (127)$$

$$\chi_{N'}^{(g')}(z^{(g')}) = \frac{m'_{g'}}{N'} \chi_1^{'(g')}(z^{(g')}) + \frac{n'_{g'}}{N'} \chi_2^{'(g')}(z^{(g')}) + \frac{\pi \alpha_{N'}^{'(g')}}{q}, \quad (128)$$

with the relations

$$e^{iqN(\chi_N^{(1)}(z^{(1)}) + \chi_N^{(2)}(z^{(2)}))} = \prod_{g=1,2} e^{iq(m_g \chi_1^{(g)}(z^{(g)}) + n_g \chi_2^{(g)}(z^{(g)}))}$$
$$\times e^{i\pi m_g n_g M^{(g)}}, \quad (129)$$

$$e^{iqN'(\chi_{N'}^{(2)}(z^{(2)}) + \chi_{N'}^{(3)}(z^{(3)}))} = \prod_{g'=2,3} e^{iq(m'_{g'} \chi_1^{'(g')}(z^{(g')}) + n'_{g'} \chi_2^{'(g')}(z^{(g')}))}$$
$$\times e^{i\pi m'_{g'} n'_{g'} M^{(g')}}, \quad (130)$$

where $g = 1, 2$, $g' = 2, 3$, $\alpha_N^{(g)} = m_g n_g M^{(g)}/N$, and $\alpha_{N'}^{'(g')} = m'_{g'} n'_{g'} M^{(g')}/N'$. Then the consistency of the contractible loops requires the magnetic flux quantization conditions,

$$\frac{m_g M^{(g)}}{N}, \quad \frac{n_g M^{(g)}}{N}, \quad \frac{m'_{g'} M^{(g')}}{N'}, \quad \frac{n'_{g'} M^{(g')}}{N'} \in \mathbb{Z}. \quad (131)$$

From these conditions, each of $M^{(1)}$, $M^{(2)}$, and $M^{(3)}$ turns out to be a multiple of N and/or N', i.e.,

$$M^{(1)} = t_1 N, \qquad M^{(2)} = t_2 N = t'_2 N', \qquad M^{(3)} = t'_3 N', \quad (132)$$

where each of t_g and $t'_{g'}$ is some integer. Defining d as the gcd of N and N', we obtain $N \equiv \tilde{n} d$ and $N' \equiv \tilde{n}' d$, where each of $\tilde{n}$ and $\tilde{n}'$ is some positive integer and $\tilde{n}$ is relatively prime with $\tilde{n}'$. Since $t_2 N = t'_2 N'$, we obtain the relation

$t_2 \tilde{n} = t'_2 \tilde{n}'$. When $|\tilde{t}|$ is defined as the gcd of t_2 and t'_2, we can rewrite t_1 and t_2 as $t_2 = \tilde{n}' \tilde{t}$ and $t'_2 = \tilde{n} \tilde{t}$, respectively. Namely, we obtain

$$M^{(2)} = \tilde{t} \frac{NN'}{d}. \quad (133)$$

If $\tilde{d}$ is defined as the gcd of $\tilde{t}$ and d, we obtain the same equation as Eq. (67),

$$M^{(2)} = \tilde{t}_2 \frac{NN'}{\gamma}, \quad (134)$$

where each of $\tilde{t}_2$ and γ is some positive integer, $\tilde{t} = \tilde{t}_2 \tilde{d}$ and $d = \gamma \tilde{d}$. We note that this result also agrees with Eq. (67) and the number of degeneracy is given by Eq. (69).

In a way similar to the case of $(T^2 \times T^2)/Z_N$, we also consider zero-mode solutions of a fermion $\psi(z^{(1)}, z^{(2)}, z^{(3)})$ on $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'})$, which satisfies the equation

$$\sum_{g=1}^{3} \sum_{a=z^{(g)}, \bar{z}^{(g)}} \Gamma^a (\partial_a - iq A_a^{(g)}) \psi(z^{(1)}, z^{(2)}, z^{(3)}) = 0, \quad (135)$$

where

$$\begin{aligned} \Gamma^{z^{(1)}} &= \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \otimes \sigma^0 \otimes \sigma^0, \\ \Gamma^{\bar{z}^{(1)}} &= \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \otimes \sigma^0 \otimes \sigma^0, \\ \Gamma^{z^{(2)}} &= \sigma^3 \otimes \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \otimes \sigma^0, \\ \Gamma^{\bar{z}^{(2)}} &= \sigma^3 \otimes \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \otimes \sigma^0, \\ \Gamma^{z^{(3)}} &= \sigma^3 \otimes \sigma^3 \otimes \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \\ \Gamma^{\bar{z}^{(3)}} &= \sigma^3 \otimes \sigma^3 \otimes \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}. \end{aligned} \quad (136)$$

Then we can write $\psi(z_1, z_2, z_3)$ as an eight-component spinor,

$$\psi(z^{(1)}, z^{(2)}, z^{(3)}) = \begin{pmatrix} \psi_+^{j_1}(z^{(1)}) \\ \psi_-^{j_1}(z^{(1)}) \end{pmatrix} \otimes \begin{pmatrix} \psi_+^{j_2}(z^{(2)}) \\ \psi_-^{j_2}(z^{(2)}) \end{pmatrix} \otimes \begin{pmatrix} \psi_+^{j_3}(z^{(3)}) \\ \psi_-^{j_3}(z^{(3)}) \end{pmatrix}$$
$$\equiv (\psi_{\mathcal{P}}^J(z^{(1)}, z^{(2)}, z^{(3)})), \quad (137)$$

where $\mathcal{P} \equiv (\pm, \pm, \pm), (\pm, \pm, \mp), (\pm, \mp, \pm), (\mp, \pm, \pm)$, and $J \equiv (j_1, j_2, j_3)$. For the same reason as T^2, depending on $M^{(1)} \lessgtr 0$, $M^{(2)} \lessgtr 0$ and $M^{(3)} \lessgtr 0$, only one of eight fields of $\psi_{\mathcal{P}}$ is well defined, while the others cannot be normalizable.

Moreover, the zero-mode fermions $\psi_{\mathcal{P}}^{J}(z^{(1)}, z^{(2)}, z^{(2)})$ are also constructed of $\psi_{\pm}^{j_g}(z^{(g)})$ on each torus, and from Eqs. (85) and (86), $\psi_{\pm}^{j_g}(z^{(g)})$ satisfy Eq. (119). Then, since $\psi_{\mathcal{P}}^{J}(z^{(1)}, z^{(2)}, z^{(3)})$ do not, in general, satisfy the physical state conditions (125) and (127) on $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'})$, we may need to take appropriate linear combinations of them in order to obtain the physical states $\Psi_{\mathcal{P}}^{J}(z^{(1)}, z^{(2)}, z^{(3)})$. Thus, the physical states are given by

$$\Psi_{\mathcal{P}}^{J}(z^{(1)}, z^{(2)}, z^{(3)}) = \mathcal{N}' \sum_{\ell=0}^{N-1} \prod_{g=1,2} e^{i\pi\ell m_g (2j_g - (N-\ell)n_g t_g)/N} \sum_{\ell'=0}^{N'-1} \prod_{g'=2,3} e^{i\pi\ell' m'_{g'} (2j_{g'} - (N'-\ell')n'_{g'}t'_{g'})/N'} \psi_{\mathcal{P}}^{J+\ell T_{n_1 n_2} + \ell' T'_{n'_2 n'_3}}(z^{(1)}, z^{(2)}, z^{(3)}),$$

(138)

where $\mathcal{N}'$ is the normalization factor, $T_{n_1 n_2} \equiv (n_1 t_1, n_2 t_2, 0)$ and $T'_{n'_2 n'_3} \equiv (0, n'_2 t'_2, n'_3 t'_3)$.

As a conclusion similar to $(T^2 \times T^2)/Z_N$, we would like to note that the number of zero-mode fermions for $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'})$ is given by a multiple of N and N'. This result leads to an important conclusion—that there is only one possibility to derive the three generations of matter, i.e., $(N, N'; M^{(1)}, M^{(2)}, M^{(3)}) = (3, 3; 3, 3, 3)$ on $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'})$. Furthermore, we consider the case of $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'} \times Z_{N''})$, which also imposes the additional $Z_{N''}$ shift symmetry on the first and third tori, which relates the first torus with the third one as $(z^{(1)}, z^{(2)}, z^{(3)}) \sim (z^{(1)} + e_{N''}^{m''_1 n''_1}, z^{(2)}, z^{(3)} + e_{N''}^{m''_3 n''_3})$. Then, we obtain an important conclusion that there are only two possibilities for deriving the three generations of matter, i.e., $(N, N', N''; M^{(1)}, M^{(2)}, M^{(3)}) = (3, 9, 3; 3, 9, 9)$, $(3, 9, 9; 9, 9, 9)$, up to the permutation of parameters for the magnitude of fluxes and the shift symmetries. (See Appendix B.) These results coincide with that of the operator formalism.

In the same way, the wave functions with Z_N and $Z_{N'}$ charges, κ and κ', are given by

$$\Psi_{\mathcal{P}\kappa\kappa'}^{J}(z^{(1)}, z^{(2)}, z^{(3)}) = \mathcal{N}' \sum_{\ell=0}^{N-1} \prod_{g=1,2} e^{i\pi\ell m_g (2j_g - (N-\ell)n_g t_g)/N} e^{-2\pi i\ell\kappa/N}$$

$$\times \sum_{\ell'=0}^{N'-1} \prod_{g'=2,3} e^{i\pi\ell' m'_{g'} (2j_b - (N'-\ell')n'_{g'}t'_{g'})/N'} e^{-2\pi i\ell'\kappa'/N'} \psi_{\mathcal{P}}^{J+\ell T_{n_1 n_2} + \ell' T'_{n'_2 n'_3}}(z^{(1)}, z^{(2)}, z^{(3)}), \quad (139)$$

where each of κ and κ' is some integer.

E. Flavor structure

Here we study the flavor structure in shifted orbifold models with magnetic fluxes. First we give a brief review on the torus models [10]. As seen in Sec. III A, if $M > 0$, the number of zero modes is equal to M on T^2, and those wave functions are written by ψ_+^j ($j = 0, \ldots, M-1$) in Eq. (85). Each mode has the Z_M charge j, which corresponds to the quantized coordinate or momentum in terms of $\hat{\tilde{Y}}$ and $\hat{\tilde{P}}$. Such Z_M transformation is represented on

$$(\psi^j) = \begin{pmatrix} \psi^0 \\ \psi^1 \\ \vdots \\ \psi^{M-1} \end{pmatrix} \qquad (140)$$

by

$$Z = \begin{pmatrix} 1 & & & & \\ & \rho & & & \\ & & \rho^2 & & \\ & & & \ddots & \\ & & & & \rho^{M-1} \end{pmatrix}, \qquad (141)$$

where $\rho = e^{2\pi i/M}$. They also have another symmetry under the cyclic permutation,

$$\psi^j \rightarrow \psi^{j+1}, \tag{142}$$

where $\psi^M = \psi^0$. This cyclic permutation $Z_M^{(C)}$ is represented on the multiplet (140) by

$$C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{143}$$

These generators, Z and C, are not commutable but satisfy the following algebraic relation,[13]

$$CZ = \rho ZC. \tag{144}$$

Hence, the flavor symmetry including Z and C is a non-Abelian symmetry. Its diagonal elements are written by $Z^m(Z')^n$ with $m, n = 0, \ldots, M - 1$, where

$$Z' = \begin{pmatrix} \rho & & \\ & \ddots & \\ & & \rho \end{pmatrix} \tag{145}$$

on the multiplet (140). Then, this flavor symmetry would correspond to $(Z_M \times Z'_M) \rtimes Z_M^{(C)}$ on T^2 (see for review on non-Abelian discrete flavor symmetries [28]).[14]

Suppose that each of three tori has the magnetic flux corresponding to $M^{(g)}$ ($g = 1, 2, 3$), where $M^{(g)} > 0$. Then, there are $M^{(g)}$ zero modes on the gth torus. Their symmetry is the direct product of $\prod_{g=1}^3 (Z_{M^{(g)}} \times Z'_{M^{(g)}}) \rtimes Z_{M^{(g)}}^{(C)}$. In order to realize the three-generation models, we choose the magnetic fluxes as $(M^{(1)}, M^{(2)}, M^{(3)}) = (3, 1, 1)$ or their permutations. In this model, only one torus, e.g., the first torus for $(M^{(1)}, M^{(2)}, M^{(3)}) = (3, 1, 1)$ is important to the flavor structure. That is, the three generations of fermions are quasilocalized at places different from each other on the first torus, while those sit at the same places on the other tori. Thus, the zero-mode profiles on the first torus are important to realize the mass ratios between three generations, while the zero-mode profiles on the other tori are relevant to the overall strength of Yukawa couplings. This model has the flavor symmetry $(Z_3 \times Z'_3) \rtimes Z_3^{(C)}$ isomorphic to $\Delta(27)$.

It would be obvious that the T^2/Z_N model has a flavor structure similar to the above. However, the $(T^2 \times T^2)/Z_N$ model as well as the $(T^2 \times T^2)/Z_N \times T^2$ model has a different flavor structure. As an illustrating model, we consider the $(T^2 \times T^2)/Z_3$ model, which leads to the three generations by choosing $M^{(1)} = M^{(2)} = 3$. Before the

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[13]The symmetry, which is called the magnetic translational group, has been discussed in Refs. [5,27].
[14]Similar flavor symmetries are obtained, e.g., within the framework of heterotic orbifold models [29].

Z_3 shifted orbifolding, there appear the three zero modes $\psi^{j_g}(z^{(g)})$ with $g = 1, 2$ and $j_g = 0, 1, 2$ on each torus, and a total of nine zero modes. They have the flavor symmetry $(Z_3 \times Z'_3) \rtimes Z_3^{(C)}$ on each torus, and the total symmetry is their direct product.

By Z_3 shift orbifolding with $(m_1, n_1, m_2, n_2) = (1, 0, 1, 0)$ corresponding to Eq. (121), only the three zero modes $\psi^j(z^{(1)}) \otimes \psi^{3-j}(z^{(2)})$ with $j = 0, 1, 2$ remain, but the others $\psi^{j_1}(z^{(1)}) \otimes \psi^{j_2}(z^{(2)})$ with $j_1 + j_2 \neq 0 \pmod{3}$ are projected out. Through orbifolding, the $Z_{M^{(1)}=3}$ and $Z_{M^{(2)}=3}$ symmetries (141) on the first and second tori, respectively, are broken into the diagonal Z_3 one. The other symmetries such as $Z'_{M^{(g)}}$ and $Z_{M^{(g)}}^{(C)}$ are also broken into the diagonal ones. Then, the flavor symmetry $\Delta(27) \times \Delta(27)$ is broken into the diagonal one $\Delta(27)$. The flavor symmetry itself is the same as one in the three-generation model on $T^2 \times T^2 \times T^2$ without shifted orbifolding. However, in the three-generation model on $(T^2 \times T^2)/Z_3$, the zero-mode profiles of the three generations are localized at places different from each other on both the first and second tori. That is, both tori are relevant to the flavor structure and mass ratios depend on geometrical aspects of both tori such as complex structure moduli.

We have studied quite simple models so far. Furthermore, the flavor structure of shifted orbifold models can become richer in slightly extended models. Suppose that there is an additional $U(1)$ gauge symmetry. We do not introduce the magnetic flux background for the additional $U(1)$, but we embed Z_3 shift orbifolding into this additional $U(1)$. That is, the fermion with the additional $U(1)$ charge q', which is an integer, has the phase $e^{2\pi i q'/3}$. In this case, the zero modes, $\psi^j(z_1) \otimes \psi^{3-j}(z_2)$ with $j = 0, 1, 2$, do not survive in the above model, but the zero modes, $\psi^j(z_1) \otimes \psi^{3-j+k}(z_2)$ with $j = 0, 1, 2$ and $k = -q'$, survive through the Z_3 shift orbifolding. The surviving number, i.e., 3, does not change, but the combinations of surviving wave functions depend on the $U(1)$ charge q'. Hence, the flavor structure becomes rich. For example, when this charge q' corresponds to the hypercharge, the three generations of quarks and leptons have quite an interesting flavor structure. We will study such model building and its flavor structure elsewhere.

IV. CONCLUSIONS

We have studied the $U(1)$ gauge theory on some shifted orbifolds with magnetic flux and proposed a mechanism to obtain the generation of matter in the standard model. On the space, we consider the behavior of fermions in two different means. One is the operator formalism for the quantum mechanical system and the other is the wave functions for the field theory. The operator formalism turns out to be useful to analyze the general structure of the spectrum. On the other hand, the wave function approach becomes important for computing the four-dimensional

Yukawa coupling of phenomenological models on the shifted orbifolds that we consider in this paper. We investigated the relations between the magnetic fluxes and the number of degeneracy of zero-mode fermions in both approaches and showed the results to be consistent with each other.

Then we found that the number of degeneracy of zero-mode fermions is related to N of Z_N, that is, the geometry of space such as $(T^2 \times T^2)/Z_N$. Actually, while there existed no constraint for the degeneracy of zero-mode fermions on T^2/Z_N, we obtained the constraint on the degeneracy of zero-mode fermions on $(T^2 \times T^2)/Z_N$; that is to say, the number of degeneracy of zero-mode fermions is always a multiple of N.

This result is phenomenologically very important, because we have a unique choice of $(N; M_1, M_2) = (3; 3, 3)$ if we want to construct a model deriving the three generations of matter on $M^4 \times (T^2 \times T^2)/Z_N$ with the magnetic fluxes $(M^{(1)}, M^{(2)})$. In a similar way, we found some candidates for the models to derive the three generations of matter in cases of $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'})$ and $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'} \times Z_{N''})$. In the case of $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'})$, the candidate to derive the three generations of matter is that $(N, N'; M^{(1)}, M^{(2)}, M^{(3)}) = (3, 3; 3, 3, 3)$. In the case of $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'} \times Z_{N''})$, the candidates to derive the three generations of matter are that $(N, N', N''; M^{(1)}, M^{(2)}, M^{(3)}) = (3, 9, 3; 3, 9, 9), (3, 9, 9; 9, 9, 9)$ up to the permutation of parameters for the magnitude of fluxes and the shift symmetries. Thus, we may conclude that a very restricted class of shifted orbifold models can produce the three generations of matter, in general.

We comment on the difference between the shifted orbifold and the twisted orbifold. On the twisted orbifold, there are fixed points. Thus, there is the degree of freedom to put localized matter fields on the fixed points in the twisted orbifold models with magnetic fluxes. There is no such a degree of freedom in the shifted orbifold models, because there is no fixed point. Hence, the spectrum of the shifted orbifold models is completely determined by the shift and magnetic fluxes.

For the three-generation models, the torus models without shifted orbifolding and the shifted orbifold models would lead to the same flavor symmetry, i.e., $\Delta(27)$. However, while only one of the tori is relevant to the flavor structure in the former, two or three tori are important to the flavor structure in the latter. These behaviors would lead to phenomenologically interesting aspects. We would study realistic model building and its phenomenological aspects elsewhere.

Our approach is a field-theoretical one. However, our model building, shifted orbifold with magnetic fluxes may be embedded into heterotic string or type II string theory. Such a study would be quite important, but it is beyond our scope.

APPENDIX A: Z_N SHIFTED ORBIFOLDING AND BASIS TRANSFORMATION

In this Appendix, we discuss the general form of Z_N shifted orbifold and its transformation into the simple form using a basis transformation of the torus bases. We can define the T^2/Z_N shifted orbifold as the general identification,

$$Z_N: \boldsymbol{y} \sim \boldsymbol{y} + \frac{1}{N}(r_1 \boldsymbol{u}_1 + r_2 \boldsymbol{u}_2), \tag{A1}$$

where N is some positive integer, and the gcd of the integers r_1 and r_2, say r, is relatively prime with N. However, a choice of lattice bases $(\boldsymbol{u}_1, \boldsymbol{u}_2)$ is not unique, and we can take another lattice base $(\boldsymbol{u}'_1, \boldsymbol{u}'_2)$ using a matrix $U \in SL(2, \mathbb{Z})$ as

$$\begin{pmatrix} \boldsymbol{u}'_1 \\ \boldsymbol{u}'_2 \end{pmatrix} = U \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{pmatrix}, \tag{A2}$$

and a suitable choice of a new basis leads to the simple shifted orbifold form like Eq. (21). Since we assumed that r is the gcd of r_1 and r_2, both of them can be expressed as

$$r_1 = \alpha r, \qquad r_2 = \beta r, \tag{A3}$$

where each of α and β is some integer. Since α and β are relatively prime with each other, there exist some integers γ and δ such that

$$\alpha\delta - \beta\gamma = 1 \quad \text{for } ^\exists\gamma, {}^\exists\delta \in \mathbb{Z}. \tag{A4}$$

Constructing the $SL(2, \mathbb{Z})$ matrix

$$U = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \tag{A5}$$

we can take a new basis,

$$\begin{pmatrix} \boldsymbol{u}'_1 \\ \boldsymbol{u}'_2 \end{pmatrix} = U \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{pmatrix}, \tag{A6}$$

in which the shifted orbifold (A1) is written as

$$Z_N: \boldsymbol{y} \sim \boldsymbol{y} + \frac{r}{N} \boldsymbol{u}'_1. \tag{A7}$$

Since r and N are relatively prime with each other, they satisfy

$$pr - qN = 1 \quad \text{for } {}^\exists p, {}^\exists q \in \mathbb{Z}. \tag{A8}$$

Using the above integer p, we can define the new identification as

$$Z'_N: \boldsymbol{y} \sim \boldsymbol{y} + \frac{pr}{N}\boldsymbol{u}_1 \sim \boldsymbol{y} + \frac{1}{N}\boldsymbol{u}_1, \tag{A9}$$

up to the torus identification.

In a similar way, we can define the $(T^2 \times T^2)/Z_N$ shifted orbifold as the general identification

$$Z_N: (\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}) \sim \left(\boldsymbol{y}^{(1)} + \frac{1}{N}(r_{11}\boldsymbol{u}_1^{(1)} + r_{12}\boldsymbol{u}_2^{(1)}), \boldsymbol{y}^{(2)} \right.$$
$$\left. + \frac{1}{N}(r_{21}\boldsymbol{u}_1^{(2)} + r_{22}\boldsymbol{u}_2^{(2)}) \right), \tag{A10}$$

where N is some positive integer, each of $r_{gj}(g, j = 1, 2)$ is some integer, and $r^{(g)}$, which are defined as the gcd of r_{g1} and r_{g2}, are relatively prime with N. Changing the lattice bases like Eq. (A7), we can rewrite this identification as

$$Z_N: (\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}) \sim \left(\boldsymbol{y}^{(1)} + \frac{r^{(1)}}{N}\boldsymbol{u}_1^{(1)}, \boldsymbol{y}^{(2)} + \frac{r^{(2)}}{N}\boldsymbol{u}_1^{(2)} \right). \tag{A11}$$

We have to note that each of $r^{(g)}$ is relatively prime with N, i.e.,

$$p_g r^{(g)} - q_g N = 1 \quad \text{for } {}^\exists p_g, {}^\exists q_g \in \mathbb{Z}. \tag{A12}$$

Using p_2, we can put $r^{(2)}$ to 1 and define the new identification as

$$Z'_N: (\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}) \sim \left(\boldsymbol{y}^{(1)} + \frac{d}{N}\boldsymbol{u}_1^{(1)}, \boldsymbol{y}^{(2)} + \frac{1}{N}\boldsymbol{u}_1^{(2)} \right), \tag{A13}$$

where

$$d \equiv p_2 r^{(1)} > 0. \tag{A14}$$

The above Z'_N shifted orbifolding is nothing but Eq. (47).

In the case of $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'})$, the situation is a little bit different. For the Z_N shifted orbifold, we can apply the same argument which leads us to the simple shifted orbifold,

$$Z_N: (\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}, \boldsymbol{y}^{(3)}) \sim \left(\boldsymbol{y}^{(1)} + \frac{d}{N}\boldsymbol{u}_1^{(1)}, \boldsymbol{y}^{(2)} + \frac{1}{N}\boldsymbol{u}_1^{(2)}, \boldsymbol{y}^{(3)} \right), \tag{A15}$$

where each of N and d is some positive integer and d is relatively prime with N. However, for the $Z_{N'}$ shifted orbifold, we cannot apply the same argument at the same time since we are unable to execute a basis transformation for the second torus anymore. Because we already executed a basis transformation for the second torus to simplify the Z_N shifted orbifold, we thus have to assume the general form when we consider the $Z_{N'}$ shifted orbifold. However, for the third torus, we can still execute a basis transformation which means that the $Z_{N'}$ shifted orbifold is given by

$$Z_{N'}: (\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)}, \boldsymbol{y}^{(3)}) \sim \left(\boldsymbol{y}^{(1)}, \boldsymbol{y}^{(2)} + \frac{1}{N'}(s_1\boldsymbol{u}_1^{(2)} + s_2\boldsymbol{u}_2^{(2)}), \boldsymbol{y}^{(3)} \right.$$
$$\left. + \frac{d'}{N'}\boldsymbol{u}_1^{(3)} \right), \tag{A16}$$

where each of N' and d' is some positive integer and each of s_1 and s_2 is some integer. When we define s' as the gcd of s_1 and s_2, the gcd of s' and d' is relatively prime with N'.

APPENDIX B: THE DEGENERACY OF SPECTRUM ON $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'} \times Z_{N''})$

We here discuss the degeneracy of the spectrum on $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'} \times Z_{N''})$ shifted orbifold with the identifications (70). We define N, N' and N'' as

$$N \equiv l_1 d_{12} d_{13} d_{123}, \qquad N' \equiv l_2 d_{12} d_{23} d_{123},$$
$$N'' \equiv l_3 d_{13} d_{23} d_{123}, \tag{B1}$$

where any pair of l_1, l_2, and l_3 are relatively prime with each other, d_{123} is the gcd of N, N', and N'', d_{12} is relatively prime with each of l_1, l_2, and d_{123}, d_{23} is relatively prime with each of l_2, l_3, and d_{123}, and d_{13} is relatively prime with each of l_1, l_3, and d_{123}. Since d_1 is the gcd of N and N', d_2 is the gcd of N' and N'', and d_3 is the gcd of N'' and N, we can rewrite d_1, d_2, and d_3 as $d_1 = d_{13}d_{123}$, $d_2 = d_{12}d_{123}$, and $d_3 = d_{23}d_{123}$, respectively. The magnitude of flux on each torus turns out to be of the form

$$M^{(1)} = t_1 l_1 l_3 d_{13} d_{12} d_{23} d_{123},$$
$$M^{(2)} = t_2 l_1 l_2 d_{13} d_{12} d_{23} d_{123}, \tag{B2}$$
$$M^{(3)} = t_3 l_2 l_3 d_{13} d_{12} d_{23} d_{123},$$

where each of t_1, t_2, and t_3 is some integer. Then it follows that the number of degeneracy is given by

$$\frac{|M^{(1)} M^{(2)} M^{(3)}|}{NN'N''} = |t_1 t_2 t_3| \frac{NN'N''}{d_1 d_2 d_3}, \tag{B3}$$

which is the result of Eq. (71). When we want to construct $(T^2 \times T^2 \times T^2)/(Z_N \times Z_{N'} \times Z_{N''})$ shifted orbifold models with the three generations, there are only two possibilities up to the permutation of parameters for the magnitude of fluxes and the shift symmetries. One is $(N, N', N''; M^{(1)}, M^{(2)}, M^{(3)}) = (3, 9, 3; 3, 9, 9)$ on $t_1 = t_2 = t_3 = 1$, $l_1 = l_3 = d_{12} = d_{13} = d_{23} = 1$, and $l_2 = d_{123} = 3$. The other is $(N, N', N''; M^{(1)}, M^{(2)}, M^{(3)}) = (3, 9, 9; 9, 9, 9)$ on $t_1 = t_2 = t_3 = 1$, $l_1 = l_2 = l_3 = d_{12} = d_{13} = 1$, and $d_{23} = d_{123} = 3$.

APPENDIX C: REDEFINITION OF FIELDS AND α_i PARAMETERS

We consider the relation between the redefinition of fields $A(z, \bar{z})$ and $\Phi(z, \bar{z})$ and $\alpha_i(\in \mathbb{R})$ parameters in $\chi_i(z, \bar{z})$, which are given by

$$\chi_1(z, \bar{z}) = \frac{b}{2\text{Im}\tau}\,\text{Im}(z + a) + \frac{\pi\alpha_1}{q},$$

$$\chi_2(z, \bar{z}) = \frac{b}{2\text{Im}\tau}\,\text{Im}(\bar{\tau}(z + a)) + \frac{\pi\alpha_2}{q}. \tag{C1}$$

Let us redefine $\Phi(z, \bar{z})$ in Eq. (78), which has the $U(1)$ charge q, by

$$\Phi(z, \bar{z}) \equiv e^{iq\text{Re}(\bar{\gamma}z)}\tilde{\Phi}(z, \bar{z}), \tag{C2}$$

where γ is any complex number. With this the redefinition, the covariant derivatives for Φ can be written by

$$(\partial_z - iqA_z)\Phi(z, \bar{z}) = e^{iq\text{Re}(\bar{\gamma}z)}(\partial_z - iq\tilde{A}_z)\tilde{\Phi}(z, \bar{z}),$$

$$(\partial_{\bar{z}} - iqA_{\bar{z}})\Phi(z, \bar{z}) = e^{iq\text{Re}(\bar{\gamma}z)}(\partial_{\bar{z}} - iq\tilde{A}_{\bar{z}})\tilde{\Phi}(z, \bar{z}). \tag{C3}$$

Here we defined $\tilde{A}_z$ and $\tilde{A}_{\bar{z}}$ as $\tilde{A}_z \equiv A_z - \bar{\gamma}/2$ and $\tilde{A}_{\bar{z}} \equiv A_{\bar{z}} - \gamma/2$, respectively. Then the Wilson line phases of $\tilde{A}$ are given by $\tilde{a} \equiv a - \gamma\text{Im}\tau/b$.

We notice that under the transformation $\Phi \to \tilde{\Phi}$ and $A \to \tilde{A}$, the Lagrangian density $\mathcal{L}$ is invariant, i.e., $\mathcal{L}(A, \Phi) = \mathcal{L}(\tilde{A}, \tilde{\Phi})$. Defining $\tilde{\chi}_i$ as

$$\tilde{\chi}_1(z, \bar{z}) \equiv \chi_1(z, \bar{z}) - \text{Re}\gamma,$$

$$\tilde{\chi}_2(z, \bar{z}) \equiv \chi_2(z, \bar{z}) - \text{Re}(\bar{\tau}\gamma), \tag{C4}$$

we can check that $\tilde{A}$ and $\tilde{\Phi}$ with $\tilde{\chi}_i$ satisfy

$$\tilde{A}(z + 1, \bar{z} + 1) = \tilde{A}(z, \bar{z}) + d\tilde{\chi}_1(z, \bar{z}),$$

$$\tilde{A}(z + \tau, \bar{z} + \bar{\tau}) = \tilde{A}(z, \bar{z}) + d\tilde{\chi}_2(z, \bar{z}),$$

$$\tilde{\Phi}(z + 1, \bar{z} + 1) = e^{iq\tilde{\chi}_1(z,\bar{z})}\tilde{\Phi}(z, \bar{z}),$$

$$\tilde{\Phi}(z + \tau, \bar{z} + \bar{\tau}) = e^{iq\tilde{\chi}_2(z,\bar{z})}\tilde{\Phi}(z, \bar{z}). \tag{C5}$$

If we take γ to satisfy $\pi\alpha_1 - q\text{Re}\gamma = 0$ and $\pi\alpha_2 - q\text{Re}(\bar{\tau}\gamma) = 0$, we obtain $\tilde{a} = a - i\pi(\alpha_1\bar{\tau} + \alpha_2)/qb$. Thus, since we can take any γ, we can always make α_i absorbed into the Wilson line phase a by the redefinition of fields. This result can be applied in the multitorus case.

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