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Scattering in plane-wave backgrounds: Infrared effects and pole structure

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We consider two aspects of scattering in strong plane-wave backgrounds. First, we show that the infrared divergences in elastic scattering depend on the structure of the background, but can be removed using the usual Bloch-Nordsieck approach. Second, we analyze the infinite series of shifted-mass-shell poles in the particle (Volkov) propagator using lightfront quantization. The complete series of poles is shown to describe a single, on-shell, propagating particle.

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I. INTRODUCTION

Strong external fields impact many aspects of gauge theories. In QCD, magnetic fields affect the vacuum [1], phase diagram [2], electric dipole moments [3] and the quark-gluon plasma [4]. In QED, scattering processes in the fields of intense lasers currently attract quite some interest [5,6], with the aim of investigating both nonperturbative effects [7] and beyond-standard-model physics [8,9]. Modeling the laser as a plane wave allows scattering amplitudes to be calculated for arbitrarily strong fields because the fermion propagator in a plane wave is known exactly; this is the Volkov propagator [10].

In this paper we will consider the propagation of quantum particles in strong plane waves. While the basic results for scattering in plane waves were given in the 1960s [11–13], those calculations assumed either monochromatic waves or crossed fields (constant plane waves). Both of these fields are of infinite extent and are therefore rather special cases; two statements related to them will be examined below.

We begin by considering the infrared (IR) structure of processes in strong plane waves, focusing on soft corrections to elastic scattering. In a crossed field, the differential probability of photon emission scales like $1/\omega^{2/3}$ rather than as $1/\omega$ as in bremsstrahlung. Thus the logarithmic divergence of the IR becomes an integrable singularity. That this happens in a field which never vanishes is potentially interesting because IR problems originate in the (incorrect) assumption that the QED coupling switches off at large distances [14,15]. This weakening of the divergence actually comes at the expense of admitting unphysical large distance behavior (see Ref. [16]), but it has raised the question of whether a partially nonperturbative treatment of plane-wave backgrounds can offer insight into the IR problem [17–19], which is still under active investigation [20–25]. It may also be that background fields lead to new IR problems; it has been suggested, for example, that IR divergences do not factorize in pair-creating backgrounds [26]. These are all motivations for the first part of this paper, in which we investigate how the structure of the background field impacts the structure of IR divergences.

In the second part of the paper, we turn to the basic building block of elastic scattering, the electron propagator. Our focus is on the poles of the propagator, particularly in monochromatic fields. These are still used as an intuitive basis for more general calculations in finite pulses, and this has led to some debate concerning the "intensity-dependent mass shift" in Refs. [27–30]. The mass shift (or rather, its effect) can be seen in the spectrum of undulator radiation [31], but its theoretical definition is more elusive. When it appears, the mass shift leads to poles in the propagator away from $p^2 = m^2$, suggesting the presence of heavy states. We investigate this by directly constructing the quantum states of a particle in a plane wave and also by resumming the pole contributions to scattering amplitudes.

The paper is organized as follows. We begin by briefly reviewing necessary previous results. In Sec. II we discuss loop and soft emission corrections to elastic scattering in plane-wave backgrounds, and the cancellation of IR divergences following [32,33]. In Sec. III we discuss the propagator and the quantum states using lightfront quantization, and explicitly relate the poles of the propagator to the ordinary mass-shell condition. We conclude in Sec. IV.

Much of the groundwork for this paper was laid in Ref. [16]. However, this paper can be read independently. The reader who is interested in the details behind our results is referred to the appendixes. Appendix A collects useful results on the propagator, asymptotic states and S-matrix normalization in external plane waves. Appendix B contains full details for our IR calculations.

A. Conventions and review

Consider a classical particle in a plane wave $F_{\mu\nu}(\phi)$ depending on $\phi = k.x$ with $k^2 = 0$, lightlike. We take $k.x = \omega x^+$, lightfront time. (As usual, $x^\pm = x^0 \pm x^3$, $x^\perp = \{x^1, x^2\}$, and $x^\pm = 2x_\mp$.) We consider finite duration fields, for which $F_{\mu\nu}$ vanishes before some ϕ_i and after

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some ϕ_f. The lightfront structure of the plane wave means that *all* initially present particles enter (leave) the wave at the same lightfront time ϕ_i (ϕ_f). A particle entering the wave with momentum p_μ has a subsequent kinematic momentum π_μ given by

$$\pi_\mu(p;\phi) = p_\mu - eC_\mu(\phi) + k_\mu \frac{2ep.C(\phi) - e^2C^2(\phi)}{2k.p}, \tag{1}$$

in which $C_\mu(\phi)$ is the integral of the electric field strength from the initial to the elapsed lightfront time:

$$C_j(\phi) = \frac{1}{\omega}\int_{\phi_i}^{\phi} \mathrm{d}\varphi E^j(\varphi), \qquad j \in \{1,2\}, \tag{2}$$

and $C_\pm = 0$. No gauge potential is used or needed here. (1) follows directly from solving the classical Lorentz equation, which depends only on the field strength [34]. (It is easily checked, though, that the vector C_μ, as we defined it, is a gauge potential for $F_{\mu\nu}$.) We are always free to choose ϕ_i. Note that $C(\infty)$ is proportional to the (trivially gauge invariant) Fourier zero mode of the electric field strength [16]:

$$C(\infty) = \frac{1}{\omega}\int \mathrm{d}\varphi E(\varphi) = \frac{1}{\omega}\tilde{E}(0). \tag{3}$$

From here on we write $C_\infty \equiv C(\infty)$ and $\pi_\infty \equiv \pi(p;\infty)$. We are interested in "unipolar" pulses for which the integral over the entire electric field is nonzero [35], hence $C_\infty \neq 0$, and which can be taken as crude models for fields which provide vacuum acceleration [36–38]. Our interest here is not in the phenomenology of vacuum acceleration, but rather the effect of such accelerating "structures" on the IR sector of scattering amplitudes. After $F_{\mu\nu}$ has switched off, C^μ becomes constant, and we can identify it with its value at $\phi = \infty$, so we write $C^\mu(\phi) = C^\mu_\infty$ for $\phi > \phi_f$ throughout. C_∞ is, as we will shortly see, central to the IR in plane-wave backgrounds. Using (1), the difference in momentum for an electron passing through the wave obeys $(\pi(p;\infty) - p)^2 = e^2C_\infty^2 \leq 0$, which has the right sign for scattering. When $C_\infty = 0$, there is no net acceleration and $\pi_\infty = p$.

II. THE INFRARED SECTOR

We begin by recalling some results on the IR sector of classical and quantum electrodynamics. See [Ref. [39], §6] for a clear introduction to the IR, and our Appendix B for full details of our IR calculations. Beginning classically, the spectral density of radiation emitted from a classical particle in any background field is proportional, at low frequency, to 1/frequency, which signals an IR divergence in the integrated photon number. In a plane wave, the constant of proportionality is nonzero when the wave is unipolar, and zero for "ordinary" waves. In other words, there is an IR divergence when the background is capable of giving net acceleration to a particle passing through it, i.e., when $C_\infty \neq 0$.

We now turn to QED, where the same divergences can be seen in the IR behavior of scattering processes in plane-wave backgrounds. Consider first single photon emission from an electron in a plane wave, called "nonlinear Compton scattering" [12,13,40]. As an introduction, it is straightforward to calculate this process in ordinary perturbation theory. (As a caveat, see Ref. [41] for the radius of convergence of perturbative expansions in background field problems such as that considered here.) Assuming that the initial electron is at rest here (and only here) for simplicity of presentation, the lowest order expression for the probability of nonlinear Compton is[1] [16]

$$\mathbb{P}_{\text{pert}} = \int_0^\infty \frac{\mathrm{d}s}{2\pi}\left(\frac{e\tilde{E}^j(s)}{m\omega}\right)^2 \frac{1}{s} \times \frac{\alpha}{2}\int_{-1}^{1}\mathrm{d}(\cos\theta)\left(\frac{\omega_s'}{s\omega}\right)^2\left[\frac{\omega_s'}{s\omega} + \frac{s\omega}{\omega_s'} - \sin^2\theta\right], \tag{4}$$

which is a sum over Klein-Nishina probabilities for ordinary Compton scattering of incoming photons of all frequencies $s\omega$ (second line), modulated by the strength of the electromagnetic fields (first line). Gauge invariance is manifest. The photon frequencies ω_s' which can be produced by each Fourier mode s of the background fields E^j is

$$\omega_s' = \frac{s\omega}{1 + \frac{s\omega}{m}(1 - \cos\theta)}. \tag{5}$$

The lower limit of the frequency integral in (4), $s = 0$, contains a logarithmic IR divergence, corresponding to the emitted photons becoming arbitrarily soft. Expanding (4) for small s, we have

$$\mathbb{P}_{\text{pert}} = \frac{4\alpha}{3}\int_0 \frac{\mathrm{d}s}{2\pi}\frac{|eC_\infty|^2}{m^2}\frac{1}{s} + \mathcal{O}(s). \tag{6}$$

A standard perturbative calculation therefore reveals that when $C_\infty \neq 0$, the probability is IR (log) divergent at $s = 0$. The IR divergence is present when the Fourier zero frequency mode of the background field strength is nonzero. This coincides with the ability of the field to transfer net acceleration to a particle; see above.

However, we do *not* wish to treat the background perturbatively. There are two reasons for this. First, we have a phenomenological interest in high intensity laser matter interactions [6], for which the background is strong and not amenable to perturbation theory. (The convergence of perturbation theory in the background intensity is discussed in Ref. [41].) Second, working perturbatively can generate divergent diagrams which require careful treatment in

[1] To convert from the notation of Ref. [16], use that f' in that paper is $f' = eE/(m\omega a_0)$.

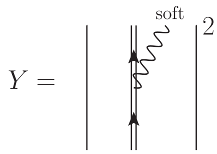

FIG. 1. Nonlinear Compton scattering of a soft photon, at tree level. Double lines indicate the background-dressed Volkov propagator.

order to obtain the correct result [42], a result which is obtained directly by treating the background exactly. We therefore work in the Furry picture, the application of which to QED in strong background fields was pioneered many years ago in Refs. [11–13]. In this approach, interactions between *quantized* fields are treated in perturbation theory, just as in ordinary QED, but the background field are treated *exactly* i.e., without recourse to perturbation theory.

If one calculates exactly in the background, the *soft contribution* to the probability of nonlinear Compton is, using dim reg in $d > 3$ dimensions,

$$Y := -e^2 \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{2l_0} \left(\frac{\pi_\infty}{l.\pi_\infty} - \frac{p}{l.p} \right)^2, \tag{7}$$

with a cutoff corresponding to, say, detector resolution. See Fig. 1. This result is logarithmically divergent in $d = 3$ when $C_\infty \neq 0$, but *vanishes* when $C_\infty = 0$. Hence, this IR divergence depends on the properties of the background; we have a divergence in a unipolar wave, and no divergence in an "ordinary" wave, which agrees precisely with the classical theory.

IR divergences typically arise when virtual particles come close to the mass shell. The field-dependent IR divergences in plane-wave backgrounds arise as follows. At each vertex, the structure of the background allows the x^- and $x^\perp$ integrals to be performed immediately. The p_+ integral can be performed using the residue theorem, which restricts the remaining x^+ integral by introducing a lightfront time ordering. One then sees that it is the large lightfront time parts of these integrals which yield singularities. In other words, our IR divergences essentially arise from the background-free regions of spacetime, before and after the pulse. See Appendix B.

A. Double Compton scattering: Hard-soft factorization

Consider now the emission of two photons from an electron in a plane wave, i.e., the process

$$e^-(p) \overset{\text{inlaser}}{\longrightarrow} e^-(p') + \gamma(k') + \gamma(l), \tag{8}$$

as recently investigated in Refs. [43–45]. Assume that the photon with momentum l_μ is soft. It can be emitted from either the incoming or outgoing leg, and the S-matrix element takes the form

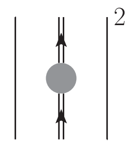

FIG. 2. The probability for elastic scattering. The grey dot denotes all loop corrections.

$$S_{fi} = e\epsilon_{\text{soft}} \cdot \left(\frac{p}{l.p} - \frac{p'}{l.p'} \right) S_{NLC}(p \to p', k'), \tag{9}$$

in which S_{NLC} is the S-matrix element for nonlinear Compton. This is the expected form of a soft correction to a hard scattering process; two-photon emission becomes degenerate with nonlinear Compton when one of the emitted photons is soft. The soft divergence implied by (9) is *independent* of the structure of the background. This is an example of a general result (see the Appendix B): the structure of the plane wave has no impact on the hard-soft factorization of IR divergences, or the severity of those divergences, which give the usual $1/\varepsilon_{\text{IR}}$ poles in $4 + 2\varepsilon_{\text{IR}}$ dimensions. (See Refs. [46,47] for examples in crossed fields.) Essentially, taking the soft limit removes the background-field dressing from emission vertices, and factorization proceeds as in QED without background.

However, the soft divergence implied by (9) is not the highest order divergence in two-photon emission. Rather, this comes from the case in which both photons are soft; two-photon emission then becomes degenerate with elastic scattering; see Fig. 2. The IR divergent part of the emission probability in this case is

$$\mathbb{P} \overset{\text{IR}}{=} \frac{1}{2} \left[e^2 \int \frac{\mathrm{d}^d l}{(2\pi)^d 2l_0} \left(\frac{\pi_\infty}{l.\pi_\infty} - \frac{p}{l.p} \right)^2 \right]^2 = \frac{1}{2} Y^2. \tag{10}$$

Unlike in (9), this divergence does depend on the structure of the background. For $C_\infty \neq 0$ each integral contributes the same divergent term; the leading singularity is therefore $1/\varepsilon_{\text{IR}}^2$.

B. Elastic scattering

As shown in Appendix B [leading to (B21)], the elastic scattering probability including the soft contribution from all loop orders (see Fig. 2) is

$$\mathbb{P} \overset{\text{IR}}{=} \exp \left[e^2 \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{2l_0} \left(\frac{\pi_\infty}{l.\pi_\infty} - \frac{p}{l.p} \right)^2 \right] =: e^X, \tag{11}$$

which defines X. This expression has two parts. The exponential is the all-orders soft loop contribution which is log divergent in $d = 3$ when $C_\infty \neq 0$. This multiplies a "1" which is (see the Appendix B), the exact tree-level probability of elastic scattering. (That $\mathbb{P} = 1$ at tree level is already a sign that something is wrong).

These problems are related and their resolution is clear: it is not possible to observe "elastic scattering" alone, due to the potential emission of soft, unobservable photons.

When we calculate the inclusive probability, IR divergences coming from soft emission should cancel those coming from the loops. We turn to this now.

C. IR cancellation

Generalizing (10), the tree-level probability of emitting n soft photons is $Y^n/n!$. The all-orders soft loop contribution to each of these processes is e^X, as in (11). What can be observed in an experiment is the sum of (1) the probability for elastic scattering and (2) the probabilities for emission of any number of undetected soft photons. This sum (see Fig. 3) is the measurable probability of observing scattering of the electron without photon emission, and its IR part is

$$\mathbb{P}(e^- \to e^-) \overset{\text{IR}}{=} e^X \cdot 1 + e^X \cdot \left(Y + \frac{1}{2}Y^2 + \cdots \right) \overset{\text{IR}}{=} e^{X+Y}. \tag{12}$$

We see that the IR contributions factorize. Comparing (7) and (11), we see also that $X = -Y$, so that

$$\mathbb{P}(e^- \to e^-) \overset{\text{IR}}{=} 1, \tag{13}$$

and the leading field-dependent soft divergences cancel to all orders. Two remarks are now in order.

First, we have followed Ref. [33] and considered only the divergent IR contributions to amplitudes, showing that these cancel. In order to extend our results to the complete amplitudes, i.e., in order to include the IR finite parts, one can instead follow the method of Ref. [32]. (See Refs. [46–49] for various loop calculations).

Second, there are no purely soft divergences in a crossed field [12]. In such a field, all particles are accelerated to the speed of light. Further, the literature results assume that the particle is also *initially* moving at the speed of light. This leads to the replacements $\{\pi, p\} \to k$ in (10) and (11), so that X and Y vanish individually. If a field which is nonzero and constant for a long but finite time is used instead, the C_∞-dependent logarithmic divergence reemerges [16]. Thus, the "improved" IR behavior in crossed fields comes at the expense of introducing unphysical large distance behavior. It does not seem to say anything about the large distance structure of QED.

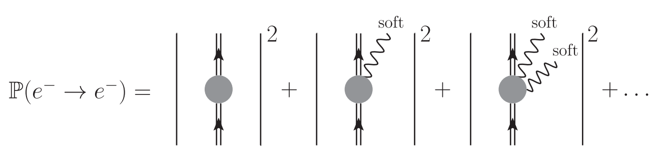

FIG. 3. The measurable probability of "scattering without emission" is the sum of the probabilities for elastic scattering, and the probabilities for the emission of arbitrary numbers of soft (unobserved) photons.

D. Indistinguishable processes

Suppose now that $C_\infty = 0$, i.e., there is no vacuum acceleration, and consider again nonlinear Compton scattering. The final state integrals are now IR finite, and one can integrate over all photon momenta to calculate the "full probability." This number, though, is not measurable. Even when processes are IR finite, one must still account for indistinguishable processes.

Physically, this issue originates in the nonzero frequency range of a pulsed field. If we parametrize this frequency range as $s\omega$ where s is real, then each s can produce photons with a frequency ω'_s, where (taking the incoming electron to be at rest for simplicity) [16]

$$\frac{1}{\omega'_s} = \frac{1}{s\omega} + \frac{1}{m}(1 - \cos\theta). \tag{14}$$

Each ω'_s is bounded below, $\omega'_s > \omega s$, but this lower bound can extend down to zero frequency in a pulse, even if there is no support at zero frequency itself, i.e., if $C_\infty = 0$ [16]. Suppose, then, that an experiment can detect only photons of frequency higher than ω_0. The measurable probability for one-photon emission is then the sum of that for nonlinear Compton scattering with $\omega' > \omega_0$, together with the sum of all n-photon emission probabilities in which $n - 1$ photons are soft, $\omega' < \omega_0$.

For a discussion of indistinguishable processes and the structure of the Volkov propagator, see Ref. [50]. The essential part of the amplitudes calculated above is in fact just this propagator: elastic scattering at tree level is given by the double amputation of the propagator (see Appendix A), and this is multiplied at higher orders by soft corrections. We therefore turn now to the propagator itself.

III. THE POLES OF THE PROPAGATOR

The Källén-Lehmann representation of the two-point function provides a natural way to identify the mass of single-particle states, as the location of the poles in momentum space [51]. One of the most discussed properties of the Volkov propagator is its infinite series of poles at

$$(p - lk)^2 = m_*^2, \qquad l \in \mathbb{Z}, \tag{15}$$

where $m_*^2 = m^2(1 + a_0^2)$ is the shifted mass of a particle in a plane wave of intensity a_0 [34,52,53]. There is no pole at the ordinary mass shell. The $l = 0$ pole, $p^2 = m_*^2$, might suggest a change in the rest mass, while the other poles describe such a heavy particle absorbing "multiple photons" from the background. Our focus in this section is on the pole structure. It is important to note that the poles "are there." Signals of the mass shift are seen daily[2] in, for example, the spectra of undulator radiation [31]. The mass

[2] Observation of the mass shift in the famous SLAC E-144 experiment [54,55] is apparently less certain [56].

shift is therefore a real effect, but its interpretation, and also that of the poles, is debated. It was originally thought that the poles were divergences [57], and that they would be regulated (given a finite height and width) by loop corrections [58]. The fact that the poles are discrete has lead to the claim that the spectrum of a particle in a plane wave is discrete [57,59]. See Ref. [60] for an interpretation of the mass-shift m_* as a finite mass renormalization, and Ref. [61] for an interpretation in terms of effective potentials.

At this point we recall the debate in Refs. [27,28] on the existence of the mass shift. Using exact treatments of the background field, the mass shift was shown to play an important role in scattering amplitudes, beginning with the work of Ref. [11] and being pursued in Refs. [12,13]. It can be observed through, for example, an associated frequency shift in the photon spectrum emitted by electrons in a plane wave [28]. However, in the perturbative treatment of Ref. [27], it was claimed that no mass shift existed. This was resolved in Ref. [42]. The authors of Ref. [27] had discarded a set of diagrams as being divergent and therefore unphysical, whereas Ref. [42] showed that the contributing sum of all such diagrams was actually finite, and that the finite term was precisely the missing mass shift. The lesson here is that it is important to treat the background field exactly, without perturbation, in order to keep track of the mass shift, which is of central interest. That is what we will do here (and throughout this paper).

Since a plane wave cannot spontaneously produce pairs [62], particle number is conserved and our theory is simple. However, progress with the equal-time quantization of this theory has only very recently been made [50]. The principle difficulty lies in proving certain orthogonality relations of the Volkov solutions [63,64], this difficulty being due to the fact that the background singles out preferred *lightlike* directions. It is therefore natural to approach problems in plane waves using lightfront quantization, as suggested in Ref. [65]. We will show here that the theory is trivially quantized on the lightfront. We will then recover the Volkov propagator as a lightfront time-ordered product and investigate the poles.

Spin effects do not impact our discussion so we restrict to a scalar particle. Propagator poles are usually tied to the optical theorem and intermediate states, so we will begin by constructing the quantum states of a particle in a plane wave explicitly. Loop corrections are not necessary here, so we turn off the QED interaction. We therefore consider only a scalar particle in a plane wave.

A. Particle states in plane waves

In lightfront field theory, the dynamical (nonconstrained) gauge fields are those transverse to the lightfront and, therefore, to the propagation of our background laser [66,67]. If we take a transverse potential $A_\mu \equiv A_\mu(k\cdot x)$ depending only on $k\cdot x$, then the physical external field is

$$F^{\text{ext}}_{\mu\nu}(k\cdot x) = k_\mu A'_\nu(k\cdot x) - A'_\mu(k\cdot x)k_\nu, \tag{16}$$

which is a plane wave with transverse electric fields $E_j = \omega A'_j$. It can then be easily checked that a transverse potential for our plane wave background with fields E_j is the vector C_μ described in Sec. I A. With this, we turn to the equations of motion for a complex scalar in a plane wave, which are $(D^2+m^2)\varphi = 0$. The general solution is

$$\varphi(x) = \int \frac{\mathrm{d}^3\mathsf{p}\,\theta(p_-)}{(2\pi)^3 2p_-} a_{\mathsf{p}}\varphi_{\mathsf{p}}(x) + b^\dagger_{\mathsf{p}}\varphi_{-\mathsf{p}}(x), \tag{17}$$

where $\mathsf{p} := \{p_\perp, p_-\}$. The functions φ_{p} are scalar Volkov solutions obeying the initial condition $\varphi_{\mathsf{p}}(\phi=0) = e^{-i\mathsf{p}.\mathsf{x}}$ [10]. The set $\{a_{\mathsf{p}}, b_{\mathsf{p}}\}$ is therefore the initial data at $\phi = 0$, and the Volkov solutions recover the classical kinematic momenta of a particle in a plane wave via

$$iD_\mu(\phi)\varphi_{\mathsf{p}}(x) = \pi_\mu(p;\phi)\varphi_{\mathsf{p}}(x). \tag{18}$$

To quantize, we impose the lightfront commutation relation (LCR) [67],

$$[\varphi(x), 2\partial_-\varphi^\dagger(y)]|_{x^+=y^+} = i\delta^{\perp,-}(x-y). \tag{19}$$

Using (17) to calculate the left-hand side of (19), the complicated exponentials in the Volkov solutions cancel immediately, and the commutator reduces to that of the free lightfront theory. The LCR is obeyed if

$$[a_{\mathsf{p}}, a^\dagger_{\mathsf{q}}] = [b_{\mathsf{p}}, b^\dagger_{\mathsf{q}}] = (2\pi)^3 2p_-\delta^3(\mathsf{p}-\mathsf{q}), \tag{20}$$

which are the usual free-field commutators on the lightfront. The particle interpretation of our theory is as follows. The vacuum $|0\rangle$ is, as in the free theory,[3] the state annihilated by the a_{p} and b_{p}, while the first excited states are

$$|p\rangle := a^\dagger_{\mathsf{p}}|0\rangle, \quad \text{and} \quad |\bar{p}\rangle := b^\dagger_{\mathsf{p}}|0\rangle, \tag{21}$$

which we will now see are one-particle/antiparticle states respectively. Using the energy momentum tensor, we combine the normal ordered Hamiltonian and kinematic momenta into

$$\begin{aligned}\Pi_\mu(\phi) = T_{-\mu} &= \int \mathrm{d}^3\mathsf{x}\,\partial_-\varphi^\dagger D_\mu\varphi(x) + \text{c.c.},\\ &= \int \frac{\mathrm{d}^3\mathsf{p}\,\theta(p_-)}{(2\pi)^3 2p_-}\pi_\mu(p;\phi)a^\dagger_{\mathsf{p}}a_{\mathsf{p}} + \bar{\pi}_\mu(p;\phi)b^\dagger_{\mathsf{p}}b_{\mathsf{p}},\end{aligned} \tag{22}$$

where $\bar{\pi}_\mu \equiv \pi_\mu|_{e\to -e}$. The vacuum is annihilated by the Hamiltonian, and is therefore stable; this is just the long-known statement that a single plane wave is incapable of

[3]This provides a very simple example of the triviality of the lightfront vacuum [67].

spontaneous pair production [62]. The states (21) then have time-dependent energies and momenta given by

$$\begin{aligned}\Pi_\mu(\phi)|p\rangle &= \pi_\mu(p;\phi)|p\rangle,\\ \Pi_\mu(\phi)|\bar{p}\rangle &= \bar{\pi}_\mu(p;\phi)|\bar{p}\rangle.\end{aligned} \tag{23}$$

The excited states therefore carry the time-dependent momenta of the classical theory. So, $a^\dagger$ and $b^\dagger$ create particles which, on the initial surface $\phi = 0$, have on-shell kinetic momenta p_μ. This labels the *continuous* spectrum of states. At all subsequent times, the states carry momentum $\pi_\mu(p;\phi)$ with $\pi^2 = m^2$. Thus, we have the same particle content as a free theory.

B. The propagator

We have seen that the one-particle states have mass m^2, and that this spectrum of states is continuous. The background-dependent structure in the field operators (essentially the Volkov solutions) simply describes the action of the Lorentz force on a particle in a background field. How should this be reconciled with (15) which might suggest that additional "heavy" states appear? To answer this we turn to the propagator, which is the lightfront time-ordered product of the fields (17):

$$\begin{aligned}G(x,y) &= \langle 0|\mathcal{T}_+\varphi(x)\varphi^\dagger(y)|0\rangle\\ &= \int\frac{\mathrm{d}^3\mathrm{p}\,\theta(p_-)}{(2\pi)^3 2p_-}\theta(x^+-y^+)\varphi_\mathrm{p}(x)\varphi^\dagger_\mathrm{p}(y)\\ &\quad + \theta(y^+-x^+)\varphi_{-\mathrm{p}}(x)\varphi^\dagger_{-\mathrm{p}}(y).\end{aligned} \tag{24}$$

The lightfront time ordering may be made covariant by using an $i\epsilon$ prescription to yield

$$\begin{aligned}G(x,y) &= i\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{e^{-ip.(x-y)-\frac{i}{2k.p}\int_{k.y}^{k.x}2eA.p-e^2A^2}}{p^2-m^2+i\epsilon}\\ &= i\int\frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\varphi_p(x)\varphi_p^\dagger(y)}{p^2-m^2+i\epsilon}.\end{aligned} \tag{25}$$

This is the Volkov propagator [68,69]. In the second line, we have written the Volkov solutions $\varphi_p(x)$ with a script label to indicate that p_μ is arbitrary, i.e., $p^2 \neq m^2$ in general. Performing the integral over p_+ (on which φ_p depends trivially), the simple pole returns us to (24), putting the initial momentum p_μ, and therefore the kinetic momentum π_μ, onto the mass shell.

Since we can write $p^2 - m^2 \equiv \pi^2(p;\phi) - m^2$, we see that (25) is a spectral representation of the explicitly time-dependent operator $D^2 + m^2$. It is not, in contrast to the free propagator, a Fourier representation, which is available only for simple fields such as monochromatic waves. Fortunately, this is just the case of interest. So, consider a circularly polarized, monochromatic field

$$C_\mu(\phi) = l^1_\mu\sin\phi + l^2_\mu\cos\phi, \tag{26}$$

with $e^2 l^i.l^j = -m^2a_0^2\delta^{ij}$, $l^j.k = 0$. The intensity a_0 appears here and is equal to $eE_{rms}/m\omega$. We define

$$re^{i\theta} = i\frac{l^1.p}{k.p} + \frac{l^2.p}{k.p} \quad\text{and}\quad q_\mu = p_\mu + \frac{a_0^2}{2k.p}k_\mu. \tag{27}$$

The well-known quasimomentum q_μ obeys $q^2 = m^2(1+a_0^2) \equiv m_*^2$, yielding the shifted mass. The Volkov solutions in this field are

$$\varphi_p(x) = e^{-iq.x-ir\sin(\phi-\theta)-ir\sin(\theta)}. \tag{28}$$

(The final term in the exponent follows from initial conditions, but is usually dropped in the literature. Our results hold in either case.) The Fourier transformed propagator can be constructed directly from (24) or (25) and coincides with the known result [60,68,69]

$$\begin{aligned}\tilde{G}_{\mathrm{mon}}(p',p) &= \sum_{n,l\in\mathbb{Z}} J_{n+l}(r)J_l(r)e^{in\theta}(2\pi)^4\\ &\quad\times\delta^4(p'-p-nk)\frac{i}{(p-lk)^2-m_*^2+i\epsilon}.\end{aligned} \tag{29}$$

The delta-comb structure is due to the periodicity of the background [60,70]. Since the Bessel functions are everywhere regular, we see that when p_μ and p'_μ are such that the nth delta function has support, we recover the infinite series of poles (15).

C. Pole resummation

Writing a Feynman amplitude in momentum space, internal lines become the Fourier transform of the propagator. From (29), we see that one then hits a "resonance" for particular values of l and n, each of which corresponds to taking particular values of energy-momentum from the background.[4] In general, all the poles' terms contribute to a given amplitude, and there is no obvious way to single out a particular pole over any other. We therefore consider the sum of contributions from all the poles. This can be extracted by taking real and imaginary parts, i.e., applying the standard result

$$\frac{1}{x+i\epsilon} = -i\pi\delta(x) + \frac{\mathcal{P}}{x}, \tag{30}$$

to (29) and retaining only the delta function terms; call this part of the propagator R. In position space, one has

[4]Related structures are seen in the lower order three-point processes of strong-field QED; due to the periodicity of the background, the emission rates take the form of a diffraction pattern. When the momentum transfer over a cycle is a multiple of the laser frequency, there is a peak in the emission rate which is analogous to a patch of constructive interference [65,71].

$$R(x,y) = \int \frac{\mathrm{d}^4(p,p')}{(2\pi)^8} e^{ip.y+ip'.x} \sum_{n,l\in\mathbb{Z}} J_{n+l}(r) J_l(r) e^{in\varphi}(2\pi)^4 \times \delta(p'-p-nk)\pi\delta((p-lk)^2 - m_*^2). \quad (31)$$

Performing the p' integrals, changing variables $n \to s = n + l$ and $p \to p - lk + \frac{a^2}{2kp}k$, and summing over the Bessel functions using

$$\sum_{s\in\mathbb{Z}} J_s(r) e^{-is(k.x-\varphi)} = \exp(-ir\sin(kx-\varphi)) \quad (32)$$

gives the final result

$$R(x,y) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \pi\delta(p^2 - m^2)\varphi_p(x)\varphi_p^*(y), \quad (33)$$

which is a sum over on-shell Volkov wave functions (28). The same result is obtained directly from (25) by sending

$$\frac{i}{p^2 - m^2 + i\epsilon} \to \pi\delta(p^2 - m^2). \quad (34)$$

Hence, the total contribution of the infinite series of poles is to replace the propagator by an integral over all real, on-shell intermediate states. In other words, the poles contribute the "imaginary parts" one obtains from cutting the propagator, in the sense of the optical theorem. This confirms that the off-shell poles do not describe heavy states. The reason that the usual Källén-Lehmann interpretation does not go through directly is that its derivation assumes Poincaré invariance of the theory. This is explicitly broken by the presence of background fields (though covariance is not). Related to this, the Fourier transform cannot be interpreted in the same way as in a free theory since canonical momentum (the Fourier variable) and kinematic momentum are not equal. See Ref. [72] for analogous statements and an investigation of the Källén-Lehmann representation in AdS space.

Our treatment of the poles has been formal: the poles do play a role in the detailed structure of emission rates [44], and they collectively describe the regime of momentum exchange in which sufficient energy is taken from the background to put normally virtual (intermediate) particles onto the mass shell. Associating individual poles to physical states is misleading, though. The confusion arises because in monochromatic (periodic) waves, the one-particle states are also eigenstates of cycle-averaged momentum operators, with constant eigenvalues equal to the quasimomenta, which square to the shifted mass. One then speaks of the quasimomenta as the "good quantum numbers" of the system [73]. However, such nonlocal operators do not tell us much about the states. A particle in a background field represents a time-dependent problem and so "eigenvalues" are in general time dependent, as in (23).

The results above extend to general plane waves as follows. Given Volkov solutions $\varphi_p(x)$ in a particular plane wave, we define $R(x,y)$ as in (33) and introduce the Fourier transform Γ via

$$\varphi_p(x) = e^{-ip.x} \int \frac{\mathrm{d}s}{2\pi} e^{-isk.x}\Gamma_s(p), \quad (35)$$

from which we obtain the implicit Fourier transform

$$\tilde{G}(p',p) = \int \frac{\mathrm{d}l\mathrm{d}n}{(2\pi)^2} \Gamma_{n+l}(p)\Gamma_l^*(p)(2\pi)^4\delta^4(p'-p-nk) \times \frac{i}{(p-lk)^2 - m^2 + i\epsilon}. \quad (36)$$

It is then trivial to check that

$$\tilde{R}(p',p) = \int \frac{\mathrm{d}l\mathrm{d}n}{(2\pi)^2} \Gamma_{n+l}(p)\Gamma_l^*(p)(2\pi)^4 \times \delta^4(p'-p-nk)\mathrm{Re}\Big[\frac{i}{(p-lk)^2 - m^2 + i\epsilon}\Big]. \quad (37)$$

The sum over on-shell intermediate states, R, is therefore given by taking the real part of the free propagator buried inside G. In this context the contributions from the background act as a "dressing" which encodes the time dependence of the system, but the plane wave does not change the fundamental particle content of the theory.

IV. CONCLUSIONS

Hard-soft factorization of scattering processes in QED goes ahead in the presence of a plane-wave background field of arbitrary strength and shape. The factorization is not sensitive to the structure of the background. The implication is that, just as in ordinary QED, IR divergences in plane-wave backgrounds exponentiate and cancel from measurable processes. A rough explanation of why is as follows. A scattering process in a fixed background obeying Maxwell's equations can be rewritten as scattering between asymptotic, coherent, photon states. As these are free-theory states, the scattering process is equivalent to a sum over ordinary QED process with all numbers of photons. Hence, if the IR divergences cancel in QED, they should also cancel here. (Compare Ref. [26], which suggests that pair-creating backgrounds, which do not obey Maxwell's equations in vacuum, may lead to nonfactorizable divergences.)

However, in those processes which are entirely soft, e.g., elastic scattering, the IR divergences depend on the Fourier zero mode of the field strength (if this is nonzero, the pulse can transfer net energy to a classical particle passing through it). Nevertheless, we have shown that the soft IR divergences in loop corrections to elastic scattering are canceled by divergences coming from multiple soft photon emissions, as normal.

The structure of the plane wave leads to the appearance of lightfront time ordering in scattering amplitudes, and IR divergences then arise from the large lightfront time regions before and after the pulse. This suggests that the natural setting for strong field QED is lightfront quantization, as employed in Ref. [65]. We used lightfront quantization to explicitly construct the quantum states of a particle in a plane wave (which are continuous, not discrete as previously claimed), and recover the Volkov propagator as a lightfront-time-ordered product. We showed that the the sum of shifted mass-shell poles in the Fourier representation of the propagator actually correspond to the ordinary mass shell. The reason that the poles do not correspond to particle masses is due to the explicit breaking of Lorentz invariance induced by the background; the presence of the laser means that the Fourier variable does not coincide with physical momentum, and hence the Källén-Lehmann interpretation of the poles does not apply to the Fourier representation of the propagator. It seems more natural, therefore, to talk of the mass shift in terms of its observable effects, namely the spectral properties of photons emitted in nonlinear Compton scattering [28]. See Ref. [31] for a discussion of such effects beyond the monochromatic approximation.

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APPENDIX A: THE VOLKOV PROPAGATOR AND WAVE FUNCTIONS

1. LSZ reduction

We write $eC(\phi) = a(\phi)$ from here on to compactify notation. We work in the Furry picture, treating the coupling to the background exactly and the interactions between the quantized fields in perturbation theory as normal. Feynman diagrams are therefore built from ordinary QED vertices and the spinor Volkov propagator S, which is the inverse of $\epsilon - i[i\not{D} - m]$:

$$S(x,y) = i\int \frac{\mathrm{d}^4 q}{(2\pi)^4} K_{qx} \frac{e^{-iq.(x-y)}}{\not{q} - m + i\epsilon} \bar{K}_{qy} e^{-i\int_{k.y}^{k.x} V_q}, \qquad \text{(A1)}$$

where we have defined

$$K_{px} := \mathbb{1} + \frac{\not{k}\not{a}}{2k.p}, \qquad V_p = \frac{2a.p - a^2}{2k.p}, \qquad \text{(A2)}$$

and $\bar{K} = \gamma^0 K^\dagger \gamma^0$. For S-matrix elements we also need Lehmann-Symanzik-Zimmermann (LSZ) reduction, which, in the absence of background fields, tells us to replace external legs with free particle wave functions. In a background, LSZ reduction transforms external propagators into incoming and outgoing fermion wave functions; the following short calculation shows this explicitly in plane waves (although the result holds more generally). According to LSZ reduction, the incoming electron wave function is given by

$$\Psi_p^{\mathrm{in}}(x) = -i\int \mathrm{d}^4 y S(x,y)[-i\overleftarrow{\not{\partial}}_y - m]e^{-ip.y}u_p. \qquad \text{(A3)}$$

Since $\bar{K}\overleftarrow{\not{\partial}} = 0$, we find $-i\overleftarrow{\not{\partial}} \to \not{q} + V_q\not{k}$ in (A1). The $y^\perp$, y^- integrals set $q_{-,\perp} = p_{-,\perp}$. Writing $q = p + tk$ we get, after simplifying the spin term,

$$\Psi_p^{\mathrm{in}}(x) = K_{px}u_p e^{-ip.x}\int \mathrm{d}\phi_y \int \frac{\mathrm{d}t}{2\pi}\left(1 + \frac{V_p}{t + i\epsilon}\right) \times e^{-it(\phi_x - \phi_y) - i\int_{\phi_y}^{\phi_x} V_p}. \qquad \text{(A4)}$$

Now we perform the t integral. The first term in the round brackets gives a delta function which sets $\phi_y = \phi_x$. The second term yields a step function, and the resulting ϕ_y integral is exact. Performing this integral we obtain

$$\Psi_p^{\mathrm{in}}(x) = K_{px}u_p^\sigma \exp\left(-ip.x - i\int_0^{\phi_x} V_p\right), \qquad \text{(A5)}$$

which is the Volkov electron wave function (the solution to the Dirac equation in a plane wave) with kinetic momentum p^μ and spin σ in the far past. Its current is

$$\frac{1}{2m}\bar{\Psi}_p^{\mathrm{in}}\gamma_\mu\Psi_p^{\mathrm{in}} = \pi_\mu(p;\phi), \qquad \text{(A6)}$$

which corresponds to a particle with kinematic momenta (1). We now turn to outgoing particles. The asymptotic theory in the past is the usual free theory, while in the future the theory contains an additional constant gauge field $a_\infty \neq 0$ (but with zero field strength). The effect of a nonzero a_∞ is seen when one constructs the LSZ reduction formulas; these differ by phase factors depending on a_∞, which leads to incoming and outgoing particles having different wave functions, as is well known [28]. The outgoing electron wave function which carries kinetic momentum p^μ and spin σ in the far *future* is given by [16,28]

$$\bar{\Psi}_p^{\mathrm{out}}(x) = \bar{u}_p^\sigma \delta\bar{K}_{px}\exp\left(i(p + a_\infty).x - i\int_\phi^{\phi_f}\delta V_p\right), \qquad \text{(A7)}$$

in which, and from here on, δ means $a \to a - a_\infty$. This wave function obeys the same Dirac equation as the incoming wave function, which in the asymptotic future becomes the Dirac equation with a constant gauge potential a_∞. Positron solutions are obtained by sending $u \to v$ and $a \to -a$. In summary, LSZ transforms external lines into Volkov wave functions, which describe (on mass shell) particles in a background plane wave.

One can of course use other gauge potentials for the background [74]; the consequence of doing so would be that *both* the incoming and outgoing wave functions would have to be adjusted [each picking up additional terms, similarly to (A7) relative to (A5)], in order for them to obey the Dirac equation and have the correct boundary conditions in the asymptotic regimes. In this case, one would always find that physical, gauge invariant results depended on a combination of the chosen potential which reduced to C_∞ as in (2), thus reproducing our results. The whole calculation is simplest with our chosen potential, which is almost universal when working with plane waves [6,11–13].

2. Normalization

The use of the Furry picture Feynman rules and the LSZ formulas above correspond to calculating S-matrix elements using equal-time quantization as normal, with asymptotic momentum states $|p\rangle$ obeying

$$\langle q|p\rangle = 2p_0(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q}). \tag{A8}$$

The free states evolve in time to become Volkov wave functions. The structure of the plane wave means firstly that these wave functions are naturally normalized on the lightfront [not as in (A8)], and secondly that S-matrix elements of such wave functions conserve overall $\mathsf{p} := \{p_-, p_\perp\}$, not three-vector $\mathbf{p}$. We give here a clear prescription for dealing with normalizations which eliminates the need for volume factors or trying to compare the infinite volumes $\delta^3(\mathbf{p})$ and $\delta^3_{\perp,-}(\mathsf{p})$; see also Refs. [75,76].

An S-matrix element calculated using the Volkov solutions (A5) and (A7) for an incoming electron, momentum p_μ, and a set of outgoing particles with momenta $\{p_f\}$ takes the form (sums/products of p_f are implicit)

$$S_{fi} = (2\pi)^3\delta^3_{-,\perp}(\mathsf{p}_f-\mathsf{p})M(p\to p_f), \tag{A9}$$

which defines M. Now, we should really consider scattering between properly normalized wave packets rather than momentum states. Final states will always be integrated out to obtain the full probability with Lorentz invariant measure

$$\sum_{p_f} = \prod_f \int \frac{\mathrm{d}^3 p_f}{(2\pi)^3 2p_{f0}}, \tag{A10}$$

and it is enough to consider only the incoming electron wave packet. This corresponds to multiplying (A9) by the factor

$$\int \frac{\mathrm{d}^3\mathbf{p}}{\sqrt{(2\pi)^3 2p_0}}\psi(p) \quad \text{with} \quad \int \mathrm{d}^3\mathbf{p}|\psi(p)|^2 = 1. \tag{A11}$$

The S-matrix element mod-squared then becomes

$$|S_{fi}|^2 = \int \mathrm{d}^3\mathbf{p}|\psi(p)|^2 \times (2\pi)^3\delta^3_{-,\perp}(\mathsf{p}_f-\mathsf{p})|M(p\to p_f)|^2\frac{1}{2p_-}. \tag{A12}$$

Making the usual assumption that the wave packet is sharply peaked corresponds to calculating (A9) and then dropping the first line in (A12) for $|S_{fi}|^2$. In short, the incoming electron should carry a normalization factor of $1/2p_-$, rather than the usual $1/2p_0$, at the level of the probability, the final expression for which is summed (averaged) over final (initial) polarization and spin:

$$\mathbb{P} = \frac{1}{2}\sum_{p_f,\sigma,\epsilon}\frac{1}{2p_-}(2\pi)^3\delta^3_{\perp,-}(\mathsf{p}_f-\mathsf{p})|M(p\to p_f)|^2. \tag{A13}$$

3. Example: Elastic scattering

As an example, consider the matrix element for elastic scattering at tree level. According to the LSZ reduction formulas this is obtained by amputating the Volkov wave function for the incoming electron:

$$\begin{aligned} S^{(0)} &= -i\int \mathrm{d}^4x e^{i(p'+a_\infty).x}\bar{u}_{p'}(i\not{D}-m)\Psi^{\mathrm{in}}_p(x) \\ &= (2\pi)^3\delta^3_{\perp,-}(p'+a_\infty-p)\bar{u}_{p'}\frac{\not{k}}{2k_+}u_p \\ &\quad \times i\int_{-\infty}^{f}\mathrm{d}\phi(V_\infty - V)\exp\Big(i\int_0^\phi V_\infty - V\Big). \end{aligned} \tag{A14}$$

The integral over ϕ needs to be regulated. Inserting a small convergence factor we find, for $a \neq 0$,

$$S^{(0)} = (2\pi)^3\delta^3_{\perp,-}(p'+a_\infty-p)\bar{u}_{p'}\frac{\not{k}}{2k_+}u_p e^{i\theta}, \tag{A15}$$

where

$$\theta := \int_0^{\phi_f}(V_\infty - V). \tag{A16}$$

We now apply (A13). The spin sum and average in our case gives

$$\frac{1}{2}\cdot\frac{1}{4k_+^2}\mathrm{Tr}[(\not{p}+m)\not{k}(\not{\pi}+m)\not{k}] = 4p_-^2, \tag{A17}$$

and it follows that the total probability is

$$\begin{aligned} \mathbb{P} &= \int\frac{\mathrm{d}^3\mathbf{p}'}{(2\pi)^3 2p_0'}\frac{1}{2p_-}\cdot(2\pi)^3\delta^3_{\perp,-}(\mathsf{p}'-\mathsf{p})\cdot 4p_-^2 \\ &= \int\frac{\mathrm{d}^3\mathsf{p}'\theta(p_-)}{2p_-'}2p_-\delta^3_{\perp,-}(\mathsf{p}'-\mathsf{p}) = 1, \end{aligned} \tag{A18}$$

where we used the Lorentz invariance of the measure in the second line to change variables. The reason why the probability is unity is discussed in Appendix B; it is a manifestation of the IR problem.

APPENDIX B: IR STRUCTURE

Here we establish in general how soft photons affect a given Feynman diagram in the Furry picture. We focus on the leading order IR divergences, following Weinberg's treatment [33].

We use dimensional regularization to take care of the IR divergences, working in $1+d$ dimensions with $d>3$. Noting that our background field singles out a lightlike direction $\phi = k.x \sim x^+$, we follow Ref. [77] and place the extra dimensions into the $d-1$ directions transverse to the background. This means in particular that all the structure of the plane wave is preserved by the regularization. The measures in position and momentum space are then, in lightfront coordinates,

$$\begin{aligned} \mathrm{d}x &:= \frac{1}{2}\,\mathrm{d}x^+\mathrm{d}x^-\mathrm{d}^{d-1}x^\perp \equiv \frac{\mathrm{d}\phi_x}{2k_+}\,\mathrm{d}x^-\mathrm{d}^{d-1}x^\perp, \\ \mathrm{d}q &:= \frac{\mathrm{d}^{d+1}q}{(2\pi)^{d+1}} = \frac{\mathrm{d}q_+}{\pi}\,\frac{\mathrm{d}q_-\mathrm{d}q_\perp^{d-1}}{(2\pi)^d}. \end{aligned} \tag{B1}$$

As we are only interested in the IR, we assume there is a cutoff in place to take care of the UV divergences, regarding which we note the following. By employing a proper time representation, all propagators can be expressed in terms of heat kernels, and these are easily continued to $d \neq 3$ following Ref. [78]. The short-time expansion of the heat kernel then gives a very convenient method for identifying UV divergences even when background fields are present. See Ref. [69] for an example.

1. Soft photon correction to an external line

We concentrate on incoming electron lines; other external lines can be treated similarly. A Feynman diagram in which the incoming electron emits n *soft* photons (which may be real or virtual, we consider both cases below) with small momenta l_j contains the term

$$\begin{aligned} E_n(x_{n+1}) := (-ie)^n \int \mathrm{d}x_n \ldots \mathrm{d}x_1 G(x_{n+1}, x_n)\gamma^{\mu_n} \\ \times e^{il_n.x_n} \ldots G(x_2, x_1)\gamma^{\mu_1} e^{il_1.x_1}\Psi_p^{\mathrm{in}}(x_1). \end{aligned} \tag{B2}$$

We will first factor out the infrared divergent part. To begin, consider $n=1$,

$$\begin{aligned} E_1(x_2) = e\int \mathrm{d}x_1 \int \mathrm{d}q K_{q2}\frac{\not{q}+m}{q^2-m^2+i\epsilon}\bar{K}_{q1}\gamma^{\mu_1}K_{p1}u_p \\ \times \exp\bigg(-iq.x_2 + i(q+l_1-p).x_1 \\ - i\int_1^2 V_q - i\int_0^1 V_p\bigg). \end{aligned} \tag{B3}$$

The x^- and $x^\perp$ integrals set $\mathsf{q} = \mathsf{p} - \mathsf{l}_1$. (In the absence of the field, we would also be able to perform the x^+ integral to set $q^\mu = p^\mu - l^\mu$.) Since we are only interested in the soft sector, and in particular the divergent terms, we employ the usual eikonal approximation, replacing $k.q \to k.p$ in K. Define now a new variable t by $q^2 - m^2 = 2k.qt \approx 2k.pt$. Changing the integration variable from q_+ to t we find, again to lowest order in l,

$$\begin{aligned} E_1(x_2) = \frac{e}{2k.p}\int \mathrm{d}\phi_1 \int \frac{\mathrm{d}t}{2\pi}K_{p2}\bigg[\frac{\not{p}+m}{t+i\epsilon}+\not{k}\bigg]\bar{K}_{p1}\gamma^{\mu_1}K_{p1}u_p \\ \times \exp\bigg(-it(\phi_2-\phi_1) - i(p-l_1).x_2 \\ -i\int_0^2 V_p - i\int_1^2 \frac{l_1.\pi}{k.p}\bigg). \end{aligned} \tag{B4}$$

We have condensed our notation further: when π appears under a ϕ integral it means $\pi(p;\phi)$, and when it appears outside such integrals it means $\pi(p;\infty)$. Consider the term in square brackets. Performing the t integral, the $\not{k}$ term leads to a delta function setting $\phi_2 = \phi_1$. This is the contribution from the lightfront zero mode [67], which is interesting in itself but does not contribute to the IR divergence and so we drop it. The remaining term in the square brackets gives $\theta(\phi_2 - \phi_1)$, which restricts the ϕ_1 integral. The spin term can be simplified to $(\not{p}+m)\bar{K}\gamma^\mu K u = 2\pi^\mu u$, and we then find

$$\begin{aligned} E_1(x_2) = \Psi_p^{\mathrm{in}}(x_2)e^{il_1.x_2}\bigg(\frac{-ie}{k.p}\bigg) \\ \times \int_{-\infty}^{\phi_2}\mathrm{d}\phi_1 \pi_1^{\mu_1}\exp\bigg(-i\int_{\phi_1}^{\phi_2}\frac{l_1.\pi}{k.p}\bigg). \end{aligned} \tag{B5}$$

The calculation is easily extended, so that we obtain

$$\begin{aligned} E_n(x) = \Psi_p^{\mathrm{in}}(x)\exp\bigg(i\sum_{j=1}^n l_j.x\bigg)\bigg(-\frac{ie}{k.p}\bigg)^n \\ \times \int_{-\infty}^x \mathrm{d}\phi_n \ldots \int_{-\infty}^2 \mathrm{d}\phi_1 \prod_{j=1}^n \pi_j^{\mu_j}\exp\bigg(-i\int_j^x \frac{l_j.\pi}{k.p}\bigg). \end{aligned} \tag{B6}$$

The leading term is the incoming Volkov solution. The evaluation of the remaining integrals, which contains the IR divergence, depends on whether (a) the leg is attached to a "hard" vertex and is therefore part of a multiparticle scattering process or (b) the leg continues to an outgoing line, in which case there are only soft photons in the process. We now consider these two cases, which are illustrated in Fig. 4.

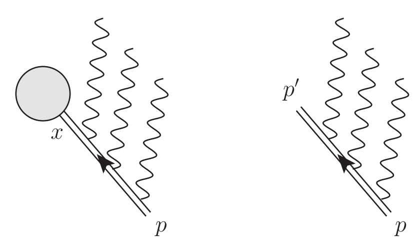

FIG. 4. Left: An external leg emits *soft* photons (or emits and absorbs virtual photons) as part of a scattering process with a hard vertex at x^μ. Right: Emission of many *soft* photons from a single electron. There is no hard scattering part to this diagram.

2. Between a soft and a hard place

We begin with the left-hand diagram in Fig. 4, considering the effect of the soft photon lines in (B6) (which can correspond to real emission or virtual loops, as we will need both) on a hard scattering process. We will see that in this case, the IR divergence does not depend on the properties of our background field.

So, we assume that $E_n(x)$ is connected to a hard vertex. In that case, there is a hard photon momentum in the exponent of (B6) at position x, and relative to this we can neglect the term $\sum l_j$. Since we have reduced the spin terms to products of π's in (B6), the integrand therein is symmetric with respect to the ϕ_j except for the step functions. This simplifies when we sum over all the orders in which the soft photons can be emitted (i.e., when we account properly for all diagrams):

$$\sum_{l\text{-perm}} E_n(x) = \Psi_p^{\text{in}}(x)\Big(-\frac{ie}{k.p}\Big)^n \prod_{j=1}^n \int_{-\infty}^x \mathrm{d}\phi_j \pi_j^{\mu_j} \times \exp\Big(-i\int_j^x \frac{l_j.\pi}{k.p}\Big), \tag{B7}$$

and we see that the step functions drop out. The divergent part of the ϕ integrals comes from the region before the pulse turns on. Since the leg is attached to the hard vertex, the upper limit of the integrals in the exponent is unimportant as long as it is finite, and we can make the following replacement without affecting the leading divergence:

$$\int_{-\infty}^x \phi_j \to \int_{-\infty}^0 \phi_j. \tag{B8}$$

Performing the ϕ integrals with the help of a convergence factor as usual, we find

$$\sum E_n = \Psi_p^{\text{in}}(x)\prod_{j=1}^n \frac{-ep_j^{\mu_j}}{l_j.p - i\epsilon}, \tag{B9}$$

in which the soft contributions factor off, meaning that the Feynman amplitude factorizes into a hard part and a soft correction, as for QED without background fields. Similar expressions hold for other external lines. Hence, we have reduced the IR problem in plane waves to the case with no background, with no surprises thus far, and we may proceed as in [Ref. [33], § 13] to show that the leading infrared divergences cancel to all orders when one sums the probabilities for indistinguishable processes. These statements hold for processes in which a "hard part" can be identified, i.e., assuming that the external lines are connected to hard vertices. We turn now to single electron processes with only soft vertices.

3. Soft processes

The S-matrix element for a diagram with a single electron and n soft vertices (see the right-hand diagram of Fig. 4) contains

$$S_n := -ie\int \mathrm{d}x \bar{\Psi}_{p'}^{\text{out}}(x)\gamma^\mu e^{il.x}E_{n-1}(x). \tag{B10}$$

Since the photons are soft, $k.q \approx k.p$ and $k.p' \approx k.p$. The spin terms can then be simplified and taken outside the integrals, leaving

$$S_n = (2\pi)^d \delta^3_{-,\perp}(p' + a_\infty - p)e^{i\theta}\bar{u}_{p'}\frac{\not{k}}{2k_+}u_p\Big(-\frac{ie}{k.p}\Big)^n \times \int_{-\infty}^{\infty}\mathrm{d}\phi_n \int_{-\infty}^{n}\mathrm{d}\phi_{n-1}\ldots\int_{-\infty}^{2}\mathrm{d}\phi_1 \prod_{j=1}^n \pi_j^{\mu_j} \times \exp\Big(i\int_0^j \frac{l_j.\pi}{k.p}\Big), \tag{B11}$$

with θ as in (A16). (We drop the factor $\sum l_j$ from the delta functions. In a more rigorous treatment the soft photon energies should be restricted so that this sum is less than some given energy; see Ref. [33]. The divergent part is still the same.) As before, S_n simplifies when we sum over the permutations of the l_j,

$$\sum_{l\text{-perm}} S_n = (2\pi)^d \delta^3_{-,\perp}(p' + a_\infty - p)e^{i\theta}\bar{u}'\frac{\not{k}}{2k_+}u\Big(-\frac{ie}{kp}\Big)^n \times \prod_{j=1}^n \int \mathrm{d}\phi_j \pi_j \exp\Big(i\int_0^j \frac{l_j.\pi}{k.p}\Big). \tag{B12}$$

We have derived this formula for $n \geq 1$ (with $E_0 = \Psi^{\text{in}}$), but it also holds for $n = 0$, when it describes elastic scattering at tree level; see Appendix A 3 above. The significant difference compared to the case of hard-soft factorization is that in soft processes, the outgoing electron's momentum is fixed by classical momentum conservation, in other words by the properties of the background field, and in particular a_∞.

The soft photons can be real or virtual. For each real emission we multiply (B12) by a polarization vector ϵ, giving

$$-\frac{ie}{k.p}\int \mathrm{d}\phi\, \epsilon.\pi \exp\Big(i\int^\phi \frac{l.\pi}{k.p}\Big) = e\epsilon\Big(\frac{\pi}{l.\pi} - \frac{p}{l.p}\Big). \tag{B13}$$

At the level of the probability we sum over polarizations, which gives minus the above expression squared, and then integrate over the photon momenta. We get the same factor for each photon, but with a symmetry factor of $1/n!$. The contribution from the emission of n soft photons is therefore

$$\frac{1}{n!}\Big[-e^2\int \frac{\mathrm{d}^d l}{(2\pi)^d}\frac{1}{2l_0}\Big(\frac{\pi}{l.\pi} - \frac{p}{l.p}\Big)^2\Big]^n. \tag{B14}$$

For each virtual photon we choose $l_j = -l_i = l$ and multiply (B12) by

$$\int \mathrm{d}l \frac{-ig_{\mu_i\mu_j}}{l^2+i\epsilon}. \tag{B15}$$

We then have

$$\int \mathrm{d}l \frac{-i}{l^2+i\epsilon}\left(-\frac{ie}{k.p}\right)^2 \int \mathrm{d}\phi_i \mathrm{d}\phi_j \pi_i \pi_j \exp\left(-i\int_j^i \frac{l.\pi}{k.p}\right). \tag{B16}$$

We divide this into two parts, $\phi_i > \phi_j$ and $\phi_i < \phi_j$, and change variable $l \to -l$ in the $<$ part. We can then close the l_0 contour in the lower half plane, and (B16) becomes

$$\int \frac{\mathrm{d}^d l}{(2\pi)^d}\frac{1}{2l_0}\left(\frac{e}{k.p}\right)^2 \int \mathrm{d}\phi_i \mathrm{d}\phi_j \pi_i \pi_j (\theta_> e^- + \theta_< e^+), \tag{B17}$$

with obvious notation. The imaginary part of (B16) diverges like

$$\mathrm{Im} \sim \int \mathrm{d}\phi, \tag{B18}$$

but we will shortly see that it drops out of probabilities. In the real part, the ϕ_i integrals can be performed and are finite, and (B16) becomes

$$e^2 \int \frac{\mathrm{d}^d l}{(2\pi)^d}\frac{1}{2l_0}\left(\frac{\pi}{l.\pi}-\frac{p}{l.p}\right)^2 + i\ldots \tag{B19}$$

We get one such factor for each virtual photon with a factor of $1/2^n n!$, which is the number of identical permutations of the sum over l_j for n virtual photons. Summing over all n we get the all-orders loop contribution to a soft process,

$$\exp\left(\frac{e^2}{2}\int \frac{\mathrm{d}^d l}{(2\pi)^d}\frac{1}{2l_0}\left(\frac{\pi}{l.\pi}-\frac{p}{l.p}\right)^2 + i\ldots\right). \tag{B20}$$

This contribution appears mod-squared at the level of the probability, which removes both the leading factor of one half, and the divergent imaginary term; the latter are the usual phase divergences. Hence, returning to the example of Appendix A 3, the probability for elastic scattering including all soft loop contributions is given by the modulus squared of (B20),

$$\mathbb{P} = \exp\left[e^2 \int \frac{\mathrm{d}^d l}{(2\pi)^d}\frac{1}{2l_0}\left(\frac{\pi}{l.\pi}-\frac{p}{l.p}\right)^2\right]. \tag{B21}$$

When $a_\infty \neq 0$, i.e., when the background field is unipolar, the loops give an IR divergent contribution as $d \to 3$.

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