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Some exact properties of the gluon propagator

Daniel Zwanziger[1]

[1]*Physics Department, New York University, New York, New York 10003, USA*

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Recent numerical studies of the gluon propagator in the minimal Landau and Coulomb gauges in space-time dimension two, three, and four pose a challenge to the Gribov confinement scenario. In these gauges all configurations are transverse, $\partial \cdot A = 0$, and lie inside the Gribov region Ω, where the Faddeev-Popov operator, $M(A) = -\partial_\mu D_\mu(A)$, is positive, that is, $(\psi, M(A)\psi) \geq 0$ for all ψ. We prove, without approximation, that for these gauges the continuum gluon propagator $D(k)$ in SU(N) gauge theory satisfies the bound $\frac{d-1}{d} \frac{1}{(2\pi)^d} \int d^d k \frac{D(k)}{k^2} \leq N$. This holds for the Landau gauge, in which case d is the dimension of space-time, and for the Coulomb gauge, in which case d is the dimension of ordinary space and $D(k)$ is the instantaneous spatial gluon propagator. This bound implies that $\lim_{k \to 0} k^{d-2} D(k) = 0$, where $D(k)$ is the gluon propagator at momentum k, and consequently $D(0) = 0$ in the Landau gauge in space-time $d = 2$ and in the Coulomb gauge in space dimension $d = 2$, but $D(0)$ may be finite in higher dimensions. These results are compatible with numerical studies of the Landau-and Coulomb-gauge propagator. In four-dimensional space-time a regularization is required, and we also prove an analogous bound on the lattice gluon propagator, $\frac{1}{d(2\pi)^d} \int_{-\pi}^{\pi} d^d k \frac{\sum_\mu \cos^2(k_\mu/2) D_{\mu\mu}(k)}{4 \sum_\lambda \sin^2(k_\lambda/2)} \leq N$. Here we have taken the infinite-volume limit of lattice gauge theory at fixed lattice spacing, and the lattice momentum componant k_μ is a continuous angle, $-\pi \leq k_\mu \leq \pi$. Unexpectedly, this implies a bound on a renormalization-group invariant that governs the overall normalization of the continuum gluon propagator in the minimum Landau and Coulomb gauges in four space-time dimensions, which, moreover, is compatible with the perturbative renormalization group when the theory is asymptotically free.

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I. INTRODUCTION

The successes of perturbative calculations at high energy and of numerical studies in lattice gauge theory provide strong evidence that the interactions of quarks and gluons are correctly described by the non-Abelian gauge theory known as QCD. However, we lack a satisfactory understanding of the mechanism by which quarks and gluons are confined, comparable to that provided by the Higgs model of electroweak interactions. There are several suggestive scenarios which involve the dual Meissner effect with condensation of magnetic monopoles, the maximal Abelian gauge, the maximal center gauge, or the light-cone gauge.

There is also a scenario in the Landau and Coulomb gauges that originated with Gribov [1] that is based on the insight that there exist Gribov copies—that is to say gauge-equivalent configurations that satisfy the gauge condition—and moreover that the dynamics is strongly affected if one cuts off the integral over configurations A to avoid over counting these copies. According to this scenario, the cutoff is nearby in infrared directions (in A space), which suppresses the gluon propagator $D(k)$ at small k, so that would-be massless gluons exit the physical spectrum and are said to be confined. However, recent numerical studies—which we shall review shortly—have revealed that the behavior of the gluon propagator is more complicated than expected, and present a challenge to this scenario. In the present article we present exact bounds on the gluon propagator—which result from the cutoff in A space—that are consistent with and clarify the results found in numerical studies of the gluon propagator.

There had been various early conjectures about the behavior of the gluon propagator $D(k)$. Gribov in particular obtained, by an approximate calculation in the Landau gauge [1], $D(k) = \frac{k^2}{k^4 + \gamma}$ in all space-time dimensions. This has the notable property that $D(0) = 0$, in striking contrast to the tree-level gluon propagator, $D(k) = \frac{1}{k^2}$, which has a pole at $k = 0$. This same propagator is also the zeroth-order gluon propagator in a perturbative expansion based on a local, renormalizable action that includes a cutoff at the Gribov horizon [2]. However, according to numerical studies in the Landau gauge it appears that $D(0)$ does not vanish in dimension $2 + 1$ and $3 + 1$. To address this problem, a dynamically refined action for dimension $2 + 1$ and $3 + 1$ has been proposed and studied [3–7].

The gluon propagator has been much studied by Dyson-Schwinger (DS) equations and related methods. The general consensus at present is that there are two types of solutions: a scaling solution, with $D(0) = 0$ [8,9], and a decoupling solution, with $D(0) > 0$, as discussed in Ref. [10], where further references may be found. The scaling solution accords with the Gribov scenario. This may seem paradoxical because the Gribov horizon does not appear in the DS equations. However, the explanation was given some time ago [11]. The functional DS equation is derived by doing a partial integration in A space while neglecting the boundary term. This boundary is commonly thought to be at infinity. However, because the integrand contains the Faddeev-Popov determinant $\det[M(A)]$ as a factor, where $M(A) = -D(A) \cdot \partial$, the integrand in fact also vanishes on every Gribov horizon [because these are the surfaces on which an eigenvalue $\lambda_n(A)$ of the Faddeev-Popov operator $M(A)$ vanishes, $\lambda_n(A) = 0$]. Therefore, a cutoff may be made at the first (or the nth) Gribov horizon without introducing a boundary term, that is, *without changing the form of the DS equation.* Each of these different cutoffs in A space corresponds to a different solution of the same functional DS equation. One of these solutions corresponds to a cutoff at the first Gribov horizon; this is presumably the scaling solution.[1]

The gluon propagator in the Landau gauge has also been the subject of numerical studies in lattice gauge theory. From recent studies, a somewhat puzzling picture has emerged. It appears that in dimension $1 + 1$ the gluon propagator in the Landau gauge does in fact vanish at $k = 0$, $D(0) = 0$, in accordance with Gribov's result [12–14]. In dimension $2 + 1$ it was found that the gluon propagator has a turnover below which $D(k)$ *decreases* with decreasing k, and there is no explanation for this nonintuitive behavior besides the proximity of the Gribov horizon in infrared directions. However, from studies on huge lattices, it appears that in dimension $2 + 1$, $D(k)$ approaches a *finite* value at $k = 0$ [13,15,16]. In dimension $3 + 1$ there may be a kind of shoulder at low momentum and, as in dimension $2 + 1$, it appears that $D(0)$ is finite [13,15,17–19].

Most of the numerical studies of the Coulomb gauge [20–24][2] are for SU(2) gauge theory in dimension $3 + 1$, with results for dimension $2 + 1$ in Refs. [22,24] and for SU(3) gauge theory in dimension $3 + 1$ in Ref. [23]. From these numerical studies it appears that in the Coulomb gauge in $3 + 1$ dimensions, the static gluon propagator vanishes at $\mathbf{k} = 0$, $D(0) = 0$, in accordance with the Gribov scenario (see particularly Ref. [21]). In $2 + 1$ space-time dimensions, the static Coulomb propagator $D(0)$ must vanish at $\mathbf{k} = 0$, as follows from the bound obtained here [Eq. (40)].

To summarize, we are faced with the puzzle that numerical studies in the Landau gauge indicate that $D(0) = 0$ in dimension $1 + 1$—which accords with the Gribov scenario—but $D(0)$ is positive, $D(0) > 0$, in dimension $2 + 1$ and $3 + 1$. The puzzle deepens in view of an argument that led to the conclusion that $D(0) = 0$ in any number of dimensions [25]. This argument involves the free energy, $W(J) \equiv \ln\langle e^{(J,A)}\rangle$, where J is an external source. The input for this argument is (i) a bound on $W(J)$, and (ii) the hypothesis that $W(J)$ is analytic in J. The bound on $W(J)$ appears unassailable, and so if the numerical results are accepted, it must be that $W(J)$ is nonanalytic in J, which is the signal for a change of phase. The nonanalyticity of $W(J)$ will be discussed elsewhere [26].

In the present article we take up another bound—which is found in Appendix B of Ref. [25]—that does not involve the free energy $W(J)$. It is an ellipsoidal bound, satisfied by all configurations A in the Gribov region Ω. From this bound on configurations A, we will obtain a bound on the gluon propagator $D(k)$ by taking expectation values.

In Sec. II A we recall some known results, in Sec. II B we explain the simple idea on which the derivation of the bound is based, and in Sec. II C we note that the bound applies to other gauge bosons. In Sec. III we present exact bounds on the gluon propagator that hold in continuum and in lattice gauge theory, and we also present the renormalized continuum bound that holds in dimension $3 + 1$. In Sec. IV we discuss the implications of these bounds for the Landau- and Coulomb-gauge propagator in dimension $1 + 1$, $2 + 1$, and $3 + 1$. In addition to infrared bounds that are stronger in lower space-time dimensions, we shall find, unexpectedly, that the cutoff at the Gribov horizon implies a bound on the *high*-momentum behavior of the Landau- and Colulomb-gauge propagator in dimension $3 + 1$ given in Eqs. (35) and (50), respectively. Some concluding remarks may be found in Sec. V. In Appendix A we derive an ellipsoidal bound on continuum configurations that lie inside the Gribov horizon. In Appendix B we convert this into a bound on the continuum gluon propagator.[3] In Appendix C we exhibit a simpler ellipsoidal bound on continuum configurations in the infinite-volume limit. In Appendix D we derive the bound on the lattice gluon propagator.

II. SETUP AND BASIC IDEA

A. Elementary properties

We deal with Euclidean QCD in its continuum and lattice formulations. Numerical gauge fixing on the lattice is done by gauge transforming to a local minimum of a lattice analog (specified in Appendix D) of the continuum minimizing functional [27–29]

$$F_A(g) = \| {}^gA \|^2 = \int d^dx |{}^gA(x)|^2. \tag{1}$$

[1]Because there are Gribov copies inside the Gribov region, a correct quantization would be an integral of the Faddeev-Popov weight over a subset of the Gribov region, for example the set of absolute minima on each gauge orbit. In this case, strictly speaking, the standard DS equations would not be satisfied because they should be corrected by a boundary term.

[2]Reference [24] also presents results for gauges that interpolate between Coulomb and Landau gauges, with the gauge condition $\lambda\partial_0 A_0 + \partial_i A_i = 0$.

[3]The same continuum bound is also obtained in Eq. (15) as a limit of the lattice bound derived in Appendix D, but we have provided an independent derivation of the continuum bound because it is simpler.

This is the Hilbert square norm of the configuration ${}^gA_\mu = g^{-1}A_\mu g + g^{-1}\partial_\mu g$, which is the gauge transform of the gauge field $A_\mu(x) = t^b A^b_\mu(x)$ by the local gauge transformation $g(x) \in SU(N)$, where $\mu = 1, \ldots d$. Here the t^a are an anti-Hermitian basis, $(t^a)^\dagger = -t^a$, of the Lie algebra of the $SU(N)$ group, $[t^a, t^b] = f^{abc}t^c$, normalized to $\mathrm{tr}(t^a t^b) = -\delta^{ab}/2$, where $a = 1, \ldots N^2 - 1$. If d is taken to be the dimension of Euclidean space-time, then this gauge fixing produces a gauge in the class of minimal Landau gauges, whereas if d is the dimension of ordinary space then the gauge is in the class of minimal Coulomb gauges. (For the Coulomb gauge this minimization is done at every Euclidean time t). The minimization produces a local minimum of the minimizing functional. Any local minimum will do. In principle it could be the absolute minimum, but this is not necessary for our purposes, nor is it achievable in practice numerically.

At a local minimum (i) the functional $F_A(g)$ is stationary, and (ii) the matrix of its second derivatives is positive. Property (i) gives the transversality condition

$$\partial_\mu A^a_\mu = 0, \tag{2}$$

which is characteristic of the Landau gauge. Property (ii) is the positivity of the Faddeev-Popov operator,

$$(\psi^a, M^{ac}(A)\psi^c) = (\partial_\mu \psi^a, \partial_\mu \psi^a) - (\psi^a f^{abc}, A^b_\mu \partial_\mu \psi^c) \geq 0, \tag{3}$$

for any wave function $\psi^a(x)$. These two properties define the (first) Gribov region Ω, and gauge fixing by this minimization produces configurations A that all lie inside Ω.

B. Basic idea of the bound

It is very easy to establish bounds on configurations A that are in the Gribov region Ω. Take any trial wave function $\psi(A)$ that may depend on A. Then, from Eq. (3), it follows that every A in Ω satisfies the bound

$$(\psi^a(A) f^{abc}, A^b_\mu \partial_\mu \psi^c(A)) \leq (\partial_\mu \psi^a(A), \partial_\mu \psi^a(A)). \tag{4}$$

For an appropriately chosen trial wave function $\psi(A)$, an ellipsoidal bound on A of the form

$$\sum_k C^{bc}_{k,\mu\nu} a^{b*}_{k,\mu} a^c_{k,\nu} \leq 1 \tag{5}$$

is obtained, as shown in Appendix A. Here $a^b_{k,\mu}$ is the component of $A^b_\mu(x)$ in the Fourier expansion,

$$A^b_\mu(x) = \sum_k a^b_{k,\mu} e^{ik\cdot x}, \tag{6}$$

on a finite periodic Euclidean volume $V = L^d$, where $k_\mu = 2\pi n_\mu/L$, and n_μ runs over all integers. Such a bound for a finite lattice was established in Appendix B of Ref. [25], and a stronger ellipsoidal bound is derived in the present article for continuum and lattice gauge fields in Appendices A and D, respectively. Upon taking expectation values, we obtain the bound on the gluon propagator $D_{\mu\nu}(k)$,

$$V^{-1} \sum_k C^{bb}_{\mu\nu}(k) D_{\mu\nu}(k) \leq 1, \tag{7}$$

where we have used $\langle a^{b*}_{k,\mu} a^c_{k,\nu} \rangle = V^{-1}\delta^{bc} D_{\mu\nu}(k)$. In dimension $3+1$, the continuum theory must be regularized, and in Appendix D a bound on the lattice gluon propagator is derived from the positivity of the lattice Faddeev-Popov operator. The limit in which the ultraviolet regulator is removed, $\Lambda \to \infty$, is discussed in Sec. IV.

C. Other gauge bosons

The only input to the bounds obtained here is the restriction of the functional integral to the interior of the Gribov region. For this reason, the bound is the same whether or not the gluons are coupled to quarks or not, although the bound becomes inconsistent in dimension $3+1$ if the theory is not asymptotically free. The bounds obtained here also apply to the propagator of other gauge bosons that belong to an $SU(N)$ gauge group, including those with a Higgs coupling. In the present article we are concerned with QCD gauge theory only. However, it should be noted that the Landau gauge is a special case of the R_ξ gauge, with $\xi = 0$, that is used when the gauge field is coupled to a Higgs boson. This gauge may be given a nonperturbative meaning by the minimizing gauge fixing described above. This is straightforward in dimensions $1+1$ and $2+1$, and our results hold in these dimensions. In dimension $3+1$, a lattice regularization of ultraviolet divergences would be required to give the theory a nonperturbative meaning, but we have not considered other gauge bosons in dimension $3+1$.

III. BOUNDS ON THE GLUON PROPAGATOR

A. Bound on the continuum gluon propagator

The continuum propagator is defined by

$$\langle A^b_\mu(x) A^d_\nu(0) \rangle = V^{-1} \sum_k \delta^{bd} D_{\mu\nu}(k) e^{ik\cdot x}, \tag{8}$$

where the (hyper)cubic periodic volume $V = L^d$ is sufficiently large that (hyper)spherical symmetry holds, and the propagator has the tensor structure

$$D_{\mu\nu}(k) = D(k)\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \tag{9}$$

by transversality. Note that a factor of the coupling constant g_0 has been absorbed into the gauge field A, so $D = g_0^2 D_0$, where D_0 is the unrenormalized, canonical propagator.

Statement: For gauge fixing to the interior of the Gribov region Ω, as in the minimization procedure described above, the gluon propagator $D(k)$ in $SU(N)$ gauge theory satisfies the bound

$$J \equiv \frac{d-1}{d} \frac{1}{(2\pi)^d} \int d^d k \frac{D(k)}{k^2} \leq N. \tag{10}$$

This holds for the Landau gauge, in which case d is the dimension of space-time, and for the Coulomb gauge, in which case d is the dimension of ordinary space and $D(k)$ is the instantaneous spatial gluon propagator. (Because of renormalization, dimension $3+1$ requires a special discussion, which will be given shortly.) This is proven in Appendices A and B.

Lest it be thought that this bound is trivial, note that for the free propagator in the Landau gauge, $D(k) = 1/k^2$, the bound is violated because of an infrared divergence of the k integration in dimensions $1+1$, $2+1$, and $3+1$, and by an ultraviolet divergence of the k integration in dimension $3+1$.

If the angular integration is performed, the bound reads

$$J \equiv \frac{d-1}{d} \frac{S_{d-1}}{(2\pi)^d} \int_0^\infty dk k^{d-3} D(k) \leq N, \tag{11}$$

where $S_1 = 2\pi$, $S_2 = 4\pi$, and $S_3 = 2\pi^2$. Note that J has engineering dimension 0 in all dimensions d.

B. Bound on the lattice gluon propagator

To discuss the case of dimension $3+1$, we must regularize and renormalize. The lattice provides a convenient regularization, and a lattice analog of the bound (10) holds,

$$J \equiv \frac{1}{d(2\pi)^d} \int_{-\pi}^{\pi} d^d k \frac{\sum_\mu \cos^2(k_\mu/2) D_{\mu\mu}(k)}{4\sum_\lambda \sin^2(k_\lambda/2)} \leq N, \tag{12}$$

as is shown in Appendix D. (We have used the same symbols J, $D_{\mu\nu}\ldots$ for continuum quantities and their lattice analogs.) Here we have taken the lattice volume to infinity while keeping the lattice spacing finite, and the lattice momentum k_μ is a continuous angle $-\pi \leq k_\mu \leq \pi$. The lattice propagator is given by $D_{\mu\nu}(k) = \sum_x \langle A^b_{x,\mu} A^c_{0,\nu} \rangle e^{-ik\cdot[x+(e_\mu/2)-(e_\nu/2)]}$, where the lattice variable $A^b_{x,\mu}$ is defined in Eq. (D3).

C. Renormalized form of the continuum bound

The lattice variable A goes over in the continuum limit to

$$A \to g_0(\Lambda) A_0 = g_0(\Lambda) Z_3^{1/2}(\Lambda) A_r, \tag{13}$$

where $g_0 = g_0(\Lambda)$ is the bare coupling constant that depends on the ultraviolet cutoff Λ, A_0 is the unrenormalized or canonical continuum gauge field, and A_r is the renormalized continuum gauge field. Consequently, the lattice gluon propagator is related to the unrenormalized and renormalized gluon propagators by

$$D \to g_0^2(\Lambda) D_0 = g_0^2(\Lambda) Z_3(\Lambda) D_r, \tag{14}$$

where $D_r(k)$ is the finite, renormalized, continuum propagator. The lattice momentum goes over to the continuum momentum by $k \to ak$, where a is the lattice spacing, so the lattice integral $\int_{-\pi}^{\pi} d^d k$ goes over to the continuum integral with a cutoff $\int_{|k|\leq C\Lambda} d^d k$, where $\Lambda = 1/a$, and C is a constant of order 1. Consequently, for large Λ the lattice bound (12) goes over to the bound,

$$\begin{aligned} J &\equiv \frac{d-1}{d} \frac{S_{d-1}}{(2\pi)^d} \int_0^{C\Lambda} dk k^{d-3} D(k) \\ &= g_0^2(\Lambda) Z_3(\Lambda) \frac{d-1}{d} \frac{S_{d-1}}{(2\pi)^d} \int_0^{C\Lambda} dk k^{d-3} D_r(k) \leq N. \end{aligned} \tag{15}$$

This is the renormalized form of the continuum bound (11) that holds in dimension $3+1$, where $d = 4$ for the Landau gauge and $d = 3$ for the Coulomb gauge. We will find that the limit $\Lambda \to \infty$ is independent of C.

IV. DISCUSSION

A. Infrared bound in the Landau gauge

We first discuss the Landau-gauge case. The bound (11) is more stringent in the infrared in lower dimensions because of the factor k^{d-3} in the integrand. Since the bound is finite and the integrand is positive, it follows that in dimension $d = 1+1$ the propagator $D(k)$ must vanish at $k = 0$,

$$\lim_{k\to 0} D(k) = 0 \quad \text{for } d = 1+1. \tag{16}$$

However, in dimension $2+1$, the bound (11) is compatible with a finite value for $D(0)$. $D(k)$ may even be singular at $k = 0$ for $d = 2+1$, provided that the strength of the singularity remains less than $1/k$,

$$\lim_{k\to 0} kD(k) = 0 \quad \text{for } d = 2+1. \tag{17}$$

This condition forbids the existence of gluons of mass zero for $d = 3$. Numerical studies of the gluon propagator in the Landau gauge in dimension $2+1$ indicate a finite value for $D(0)$, as discussed in the Introduction.

Numerical studies also indicate that in dimension $d = 2+1$ the propagator in the Landau gauge is suppressed in the infrared, although not as severely as in dimension $d = 1+1$, with $D(k)$ *decreasing* with k as k decreases to 0, but approaching a finite value, $D(0) > 0$. There is no other explanation for this otherwise counterintuitive decrease besides the proximity of the Gribov horizon in infrared directions. The bound (11) for $d = 2+1$ does not require such a decrease, but only that $D(k)$ not diverge as strongly as $1/k$. Thus it appears that this bound by itself does not fully express the strength of the dynamical consequences of the cutoff of the functional integral at the Gribov horizon in dimension $2+1$.

According to the lattice bound (12), the lattice Landau propagator $D_{\mu\nu}(k)$ in dimension $3+1$ cannot have a singularity as strong as $\frac{1}{\sum_\mu \sin^2(k_\mu/2)}$ at $k = 0$ [where $\cos(k_\mu/2) = 1$],

$$\lim_{k\to 0}\sum_{\mu}\sin^2(k_\mu/2)\sum_{\nu}D_{\nu\nu}(k)=0. \tag{18}$$

Thus the bound on the lattice gluon propagator in the Landau gauge in four Euclidean dimensions does not tolerate a $1/\sin^2(k/2)$ singularity, that is to say, a massless lattice gluon, for any finite value of the lattice spacing a or, in other words, for any finite value of the cutoff $\Lambda = 1/a$. Moreover, the finiteness of the renormalized continuum bound (15) for large but finite Λ implies

$$\lim_{k\to 0}k^2 D_r(k)=0, \quad \text{for } d=3+1. \tag{19}$$

According to numerical studies discussed in the Introduction, in dimension $3+1$ there appears to be a kind of shoulder in $D(k)$ at low k, with a finite value of $D(0)>0$. This is entirely consistent with the infrared bound obtained here, but again, as in dimension $2+1$, the gluon propagator $D(k)$ is apparently more strongly suppressed in the infrared than required by the bound we have obtained. Thus it appears that also in dimension $3+1$, the bound obtained here does not by itself fully express the strength of the dynamical consequences of the cutoff of the functional integral at the Gribov horizon.

B. Bound on the renormalization-group invariant in the Landau gauge

The coefficient $g_0^2(\Lambda)Z_3(\Lambda)$, which appears in the renormalzied continnum bound (15), is either zero or infinite as $\Lambda\to\infty$, so it might be thought that if $g_0^2(\Lambda)Z_3(\Lambda)$ is zero then the bound (15) is trivially satisfied by any finite renormalized propagator D_r, and if $g_0^2(\Lambda)Z_3(\Lambda)$ is infinite then the bound implies that the renormalized propagator $D_r(k)$ vanishes identically for all k, in which case the theory is inconsistent. However, before coming to this conclusion, we must also consider the Λ dependence introduced by the ultraviolet cutoff of the integral at $k=C\Lambda$.

In dimension $d=3+1$, the bound (15) reads

$$J\equiv g_0^2(\Lambda)Z_3(\Lambda)\frac{3}{32\pi^2}\int_0^{C\Lambda}dkkD_r(k)\le N. \tag{20}$$

We evaluate J using the perturbative renormalization group according to which, asymptotically at large Λ,

$$g_0^2(\Lambda)\approx\frac{1}{2b\ln\Lambda}, \tag{21}$$

where b is the leading coefficient of the β function $dg_0/d\ln\Lambda=-bg^3+O(g^5)$. It is gauge independent and has the value ([30], p. 653)

$$b=\frac{1}{(4\pi)^2}\left(\frac{11N}{3}-\frac{2n_f}{3}\right), \tag{22}$$

where n_f is the number of quarks in the fundamental representation. The dependence of $Z_3(\Lambda)$ on Λ is found from the perturbative renormalization group. We have

$$Z_3=1+cg_r^2\ln\Lambda+O(g_r^4), \tag{23}$$

where, in the Landau gauge for SU(N) gauge theory ([30], p. 589),

$$c=\frac{1}{(4\pi)^2}\left(\frac{13N}{3}-\frac{4n_f}{3}\right), \tag{24}$$

and we have added the quark contribution. According to the renormalization group we have

$$\left.\frac{d\ln Z_3}{d\ln\Lambda}\right|_{g_r}=cg_r^2+O\left(g_r^4\right)=cg_0^2+O\left(g_0^4\right)=\frac{c}{2b\ln\Lambda}+\ldots, \tag{25}$$

which is solved by

$$Z_3(\Lambda)=z_3(\ln\Lambda)^p, \tag{26}$$

where

$$p=\frac{c}{2b}=\frac{13N-4n_f}{22N-4n_f}, \tag{27}$$

and z_3 is a finite constant of integration. Thus the renormalized bound at large Λ reads

$$J\equiv\frac{3z_3}{64\pi^2 b}\ln^{p-1}\Lambda\int_0^{C\Lambda}dkkD_r(k)\le N. \tag{28}$$

If $p>1$, the coefficient of the integral, $\ln^{p-1}\Lambda$, diverges, so the bound $J\le N$ requires that the renormalized propagator, $D_r(k)$, *vanish* for all k, in which case the theory is inconsistent.

To see if this happens, we note from Eq. (27) that

$$p<1, \tag{29}$$

provided that the denominator is positive, that is, provided that $11N>2n_f$. This is the restriction on the number of quarks for the theory to be asymptotically free, $b>0$. Thus the inconsistency is avoided provided that the theory is asymptotically free, as we now assume.

Because we are in the case $p<1$, the coefficient of the integral in Eq. (28) vanishes in the limit $\Lambda\to\infty$,

$$\lim_{\Lambda\to\infty}\ln^{p-1}\Lambda=0. \tag{30}$$

There are three possibilities. (i) If the integral in Eq. (28) is finite, the bound is trivially satisfied—and vacuous. This also happens if the integral diverges at large Λ, but too weakly to compensate the vanishing of the coefficient. (ii) If the integral diverges sufficiently strongly that J diverges for $\Lambda\to\infty$, then the theory is inconsistent. (iii) If the integral has a divergence that precisely compensates for the vanishing of the coefficient at large Λ, then a finite bound results.

To find out which possibility is realized, we first note that any finite contribution to the integral is annihilated by the coefficient, and we may write the bound as

$$J \equiv \frac{3z_3}{64\pi^2 b} \ln^{p-1}\Lambda \int_{\mu}^{C\Lambda} dkkD_r(k) \leq N, \tag{31}$$

where μ is an arbitrary mass. Only the asymptotic form of $D_r(k)$ at large k concerns us.

According to the Callan-Symanzik equation, the corrections to scaling in the gluon propagator are logarithmic at high momentum, and the renormalized gluon propagator has the asymptotic behavior

$$D_r(k) \approx \frac{r}{k^2 \ln^p k}, \tag{32}$$

where the same power p appears here as in Z_3, and r is a finite constant that depends on the normalization condition. Since any finite contribution to the integral gets annihilated by the vanishing of the coefficient, $\ln^{p-1}\Lambda$ for $\Lambda \to \infty$, we may extend this asymptotic expression down to finite k, and the quantity J in Eq. (31) is thus—asymptotically at large Λ—given by

$$\begin{aligned} J &= \frac{3z_3 r}{64\pi^2 b} \ln^{p-1}\Lambda \int_{\mu}^{C\Lambda} \frac{dk/k}{\ln^p k} \\ &= \frac{3z_3 r}{64\pi^2 b} \ln^{p-1}\Lambda \left[\frac{\ln^{1-p}(C\Lambda) - \ln^{1-p}\mu}{1-p}\right], \end{aligned} \tag{33}$$

and, with $p < 1$, we obtain the limit

$$\lim_{\Lambda\to\infty} J = \frac{3z_3 r}{64\pi^2 b}\frac{1}{1-p} = \frac{3z_3 r}{32\pi^2}\frac{1}{2b-c} = \frac{z_3 r}{2N}. \tag{34}$$

The result is independent of μ and C, and finite. From $\lim_{\Lambda\to\infty} J \leq N$ we obtain the nontrivial bound

$$z_3 r \leq 2N^2. \tag{35}$$

Four comments are in order.

(i) The trivial and inconsistent possibilities are avoided because the divergence of the integral compensates for the vanishing of the coefficient in the limit $\Lambda \to \infty$.

(ii) This bound governs the overall normalization of the gluon propagator. It is derived from the asymptotic high-momentum gluon propagator that is provided by the perturbative renormalization group (RG). To our knowledge the restriction to the interior of the Gribov horizon has heretofore been used to bound only the infrared behavior of the gluon propagator. Note however that this bound occurs only in dimension 3 + 1.

(iii) The renormalized bound (20) may be expressed in terms of unrenormalized quantities,

$$g_0^2(\Lambda)\frac{3}{32\pi^2}\int_0^{C\Lambda} dkkD_0(k) = \frac{z_3 r}{2N} \leq N. \tag{36}$$

Since the left-hand side is constructed out of unrenormalized quantities, it is independent of any renormalization scheme. On the other hand, it is finite and independent of Λ at large Λ. As such, it is a renormalization-group invariant. Thus the finite quantity $z_3 r$, for which we have just established the bound $z_3 r \leq 2N^2$, is in fact a renormalization-group invariant, although this was not apparent from the way z_3 and r were introduced.

(iv) The number n_f of quark flavors has dropped out of the inequality (35).

C. Infrared bound in the Coulomb gauge

The minimal Coulomb gauge is obtained by minimizing the Hilbert square norm of the space components $A_i^b(\mathbf{x}, t)$ on each time slice t,

$$F_{\mathbf{A}}(g, t) = \|{}^g A_i(t)\|^2 = \int d^d x |{}^g A_i(\mathbf{x}, t)|^2, \tag{37}$$

where $i = 1, \ldots d$, and the dimension of space-time is $d + 1$. Consequently, the equal-time propagator

$$D(\mathbf{k})(\delta_{ij} - k_i k_j/\mathbf{k}^2)\delta^{bc} = \int d^d x e^{-ik\cdot x}\langle A_i^b(\mathbf{x}, t) A_j^c(\mathbf{0}, t)\rangle \tag{38}$$

satisfies in the space dimension d the bounds we have derived in the Landau gauge in space-time dimension d. The expectation value is independent of t by time-translation invariance. For orientation purposes we note that in zeroth-order perturbation theory the equal-time Coulomb-gauge propagator is given by

$$D^{(0)}(\mathbf{k}) = \int \frac{dk_0}{2\pi}\frac{1}{k_0^2 + \mathbf{k}^2} = \frac{1}{2|\mathbf{k}|}. \tag{39}$$

For space dimension $d = 2$ we obtain from Eq. (11)

$$\lim_{\mathbf{k}\to 0} D(\mathbf{k}) = 0, \quad \text{for } d = 2, \tag{40}$$

so the equal-time propagator vanishes at $\mathbf{k} = 0$, $D(\mathbf{0}) = 0$. This states that a gluon of zero momentum cannot be created by applying the field $A_i^b(\mathbf{x}, t)$ to the vacuum, and thus the would-be physical gluons exit the spectrum. From Eq. (15), we obtain, as in Eq. (19),

$$\lim_{\mathbf{k}\to 0} |\mathbf{k}| D_r(\mathbf{k}) = 0, \quad \text{for } d = 3, \tag{41}$$

where $D_r(\mathbf{k})$ is the renormalized, equal-time, space-space Coulomb-gauge propagator.

D. Bound on the RG invariant in the Coulomb gauge

For space dimension $d = 3$, we must consider regularization and renormalization, as in the Landau gauge. Although renormalization in the Coulomb gauge has not been established to all orders, we suppose that it is renormalizable, and that the perturbative renormalization group holds.

We proceed exactly as in the Landau-gauge case, but with different values for the constants. For space dimension $d = 3$, Eq. (15) reads

$$J \equiv g_0^2(\Lambda) Z_3(\Lambda) \frac{1}{3\pi^2} \int_0^{C\Lambda} dk D_{\rm r}(k) \leq N, \tag{42}$$

where $g_0(\Lambda)$ is the unrenormalized coupling constant, and $Z_3(\Lambda)$ is the renormalization constant for the space components, $A_{i,0} = Z_3^{1/2} A_{i,{\rm r}}$, in the Coulomb gauge. (The space and time components of A_μ renormalize differently in the Coulomb gauge).

The renormalization constant Z_3 of the space components A_i of the gauge field is given by Eq. (23) where, by Eq. (B.37) of Ref. [31], with $Z_3 = Z_{\mathbf{A}}^2$, the coefficient c has the value

$$c = \frac{1}{(4\pi)^2}\left(2N - \frac{4n_f}{3}\right) \tag{43}$$

for SU(N), and we have added the quark contribution. As in the Landau gauge, we have

$$Z_3(\Lambda) = z_3 \ln^p \Lambda, \tag{44}$$

where $p = c/2b$. Here c, r, and z_3 have values appropriate for the Coulomb gauge, and we obtain

$$J = \frac{1}{2b\ln\Lambda} z_3 \ln^p \Lambda \frac{1}{3\pi^2} \int_0^{C\Lambda} dk D_r(k) \leq N, \tag{45}$$

where the gauge-independent quantity b is given in Eq. (22). As in the Landau gauge, the theory is consistent only if $p < 1$. We find

$$p = \frac{c}{2b} = \frac{3N - 2n_f}{11N - 2n_f}, \tag{46}$$

and, as in the Landau gauge, we have $p < 1$, provided that the denominator is positive, $11N > 2n_f$. This is, again, the condition on the number of quarks for the theory to be asymptotically free, as we now assume. As before, the coefficient of the integral in Eq. (45) vanishes for $\Lambda \to \infty$, and any finite contribution to the integral is annihilated in this limit.

According to the Callan-Symanzik equation, the renormalized equal-time propagator has logarithmic corrections asymptotically at large $\mathbf{k}$,

$$D_{\rm r}(\mathbf{k}) \approx \frac{r}{2|\mathbf{k}|\ln^p|\mathbf{k}|}, \tag{47}$$

where p is given in Eq. (46), and, as in Eq. (33), we have

$$J = \frac{z_3}{6\pi^2 b} \ln^{p-1}\Lambda \int_\mu^{C\Lambda} dk \frac{r}{2k\ln^p k} \leq N. \tag{48}$$

As in the Landau gauge, this gives (asymptotically at large Λ)

$$J = \frac{z_3 r}{12\pi^2 b(1-p)} = \frac{z_3 r}{6\pi^2(2b - c)} = \frac{z_3 r}{2N}. \tag{49}$$

We thus obtain

$$z_3 r \leq 2N^2, \tag{50}$$

which is the same bound as in the Landau gauge and, again, the number of quark flavors has dropped out.

As in the Landau gauge, the quantity $z_3 r$ is a renormalization-group invariant. In the Coulomb gauge the bound on this quantity governs the overall normalization of the space components of the equal-time gluon propagator $D(\mathbf{k})$, whereas in the Landau gauge it governs the overall normalization of the Lorentz-invariant propagator $D(k)$.

V. CONCLUDING REMARKS

We have obtained the continuum and lattice bounds (11) and (12) on the gluon propagator, and the bound (15) on the renormalized gluon propagator, which hold in the Landau gauge, where d is the dimension of space-time, and in the Coulomb gauge, where d is the dimension of space and $D(k)$ is the instantaneous spatial gluon propagator. In space-time dimensions two, three, and four these bounds imply restrictions on the infrared behavior of the continuum gluon propagator in the Landau and Coulomb gauges that are more severe in lower dimensions, and in space-time dimension four there is, unexpectedly, a restriction on the high-momentum behavior of the continuum gluon propagator in the Landau and Coulomb gauges.

It would be of interest to test the lattice and continuum bounds using numerical lattice data for the gluon propagator in two, three, and four space-time dimensions in the Landau and Coulomb gauges. It is possible that the bounds are not close to being saturated.

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APPENDIX A: ELLIPSOIDAL BOUND ON CONTINUUM CONFIGURATIONS

We shall establish an ellipsoidal bound on continuum configurations A that lie inside the Gribov region Ω. We consider the SU(N) gauge group on a periodic Euclidean volume $V = L^d$. To start, we substitute into the inequality (4) the trial wave function

$$\psi(A) = \psi_0 - \alpha \frac{P_q}{(M_0 - p^2)} M_1(A)\psi_0, \tag{A1}$$

where $M = M_0 + M_1$ is the Faddeev-Popov operator, with $M_0 = -\partial^2$, $M_1^{ac}(A) = -f^{abc} A_\mu^b \partial_\mu$. This wave function is inspired by first-order perturbation theory, according to which the first-order change in the zeroth-order wave function ψ_0 is given by a similar expression, but we shall of course obtain an exact bound. The plane-wave state

$$\psi_0 = V^{-1/2} \exp(ip \cdot x)\eta \tag{A2}$$

is an eigenvector of M_0, $M_0\psi_0 = p^2\psi_0$, η is an x-independent, normalized color vector, and p_μ is an allowed momentum vector on the periodic Euclidean volume $V = L^d$, $p_\mu = 2\pi n_\mu/L$, where n_μ is an integer. The operator P_q is the projector defined by the kernel

$$P_q(x, y) = V^{-1} \sum_{k;|k|\geq|q|} e^{ik\cdot(x-y)}, \tag{A3}$$

which projects onto the direct sum of eigenspaces of M_0 belonging to all eigenvalues k^2 of M_0 that are greater than or equal to some fixed eigenvalue q^2. We stipulate that the inequalities

$$k^2 \geq q^2 > p^2 \geq (2\pi/L)^2 \tag{A4}$$

are satisfied so that the denominator in Eq. (A1) is always positive, and that k_μ, p_μ, and q_μ are allowed momentum vectors on the periodic volume $V = L^d$. The quantities α, p_μ, and q_μ are at our disposal, and α will be a variational parameter. Note that ψ_0 is independent of A, and $M_1 = M_1(A)$ is linear in A, so the trial wave function $\psi = \psi(A)$ has a piece that is independent of A and a piece that is linear in A.

With this wave function we have, by Eq. (4), for all $A \in \Omega$,

$$(\psi, M\psi) = I(\alpha) = X - 2\alpha Y + \alpha^2 Z \geq 0, \tag{A5}$$

where

$$X = (\psi_0, (M_0 + M_1)\psi_0),$$
$$Y = \left(\psi_0, M_1 \frac{P_q}{(M_0 - p^2)} M_1 \psi_0\right),$$
$$Z = \left(\psi_0, M_1 \frac{P_q}{(M_0 - p^2)} (M_0 + M_1) \frac{P_q}{(M_0 - p^2)} M_1 \psi_0\right), \tag{A6}$$

and we have simplified Y using $P_q M_0 \psi_0 = 0$. The positivity of $I(\alpha)$ for all α implies

$$X \geq 0, \qquad Z \geq 0. \tag{A7}$$

Moreover, $I(\alpha)$ has a minimum at $\alpha = Y/Z$, from which we obtain the bound

$$Y^2 \leq XZ. \tag{A8}$$

Because A appears only in $M_1(A)$, which is linear in A, there is a term in Z which is cubic in A, whereas X and Y are at most quadratic in A. We shall bound the cubic term by means of the following lemma.

Lemma: For the SU(N) group the bound

$$(\omega, [M_0 + M_1(A)]\omega) \leq N^2(\omega, M_0\omega) \tag{A9}$$

holds for any A in Ω and any wave function $\omega(A)$.

This lemma is derived in Appendix B of Ref. [25], but we present the derivation here for completeness.

Proof: Consider first the SU(2) group. We decompose M_1 into the sum of three operators, $M_1 = M_1^{(1)} + M_1^{(2)} + M_1^{(3)}$, that are each given by

$$M_1^{(b)}(A) = S^b H^b(A) \tag{A10}$$

(no sum on b), where $(S^b)_{ac} \equiv i\epsilon^{abc}$ is an angular momentum matrix in the spin-one representation that acts on color variables, and $H^b = iA^b_\mu \partial_\mu$ is a Hermitian operator that acts only on space variables. We first bound the operator $M_1^{(3)} = S^3 H^3$. Let $\omega^a_\pm \equiv e^a_\pm \phi(x)$, where $\phi(x)$ is any function of x, and $e_\pm$ and e_0 are x-independent, normalized eigenvectors of S_3, $S_3 e_m = m e_m$. We have

$$0 \leq (\omega_\pm, [M_0 + M_1(A)]\omega_\pm)$$
$$= (\phi, M_0\phi) \pm (\phi, H^3(A)\phi), \tag{A11}$$

where the inequality holds for all $A \in \Omega$, and we have used the fact that $(e_\pm, S^1 e_\pm) = (e_\pm, S^2 e_\pm) = 0$ because S_1 and S_2 are off-diagonal in the S_3 basis. It follows that the inequality

$$|(\phi, H^3(A)\phi)| \leq (\phi, M_0\phi) \tag{A12}$$

holds for any $\phi(x)$ and all $A \in \Omega$. We now decompose any wave function ω according to $\omega = e_+\phi_+ + e_0\phi_0 + e_-\phi_-$, and we have for all ω and all $A \in \Omega$

$$|(\omega, M_1^{(3)}(A)\omega)| = |(\omega, S^3 H^3 \omega)|$$
$$= |(\phi_+, H^3\phi_+) - (\phi_-, H^3\phi_-)|$$
$$\leq |(\phi_+, H^3\phi_+)| + |(\phi_-, H^3\phi_-)|$$
$$\leq (\phi_+, M_0\phi_+) + (\phi_-, M_0\phi_-)$$
$$\leq (\omega, M_0\omega). \tag{A13}$$

The same inequality holds for $M_1^{(1)}$ and $M_1^{(2)}$, which gives

$$|(\omega, M_1(A)\omega)| \leq 3(\omega, M_0\omega). \tag{A14}$$

For the SU(N) group, the proof is identical except that there are $N^2 - 1$ terms in M_1, which gives

$$|(\omega, M_1(A)\omega)| \leq (N^2 - 1)(\omega, M_0\omega), \tag{A15}$$

and Eq. (A9) follows □.

We apply the lemma to Z, which is of the form $Z = (\omega, M\omega)$, where $\omega = \frac{P_q}{(M_0 - p^2)} M_1 \psi_0$. The lemma yields

$$Z \leq N^2 Z_0, \tag{A16}$$

where

$$Z_0 \equiv \left(\psi_0, M_1(A) \frac{P_q M_0}{(M_0 - p^2).^2} M_1(A)\psi_0\right), \tag{A17}$$

and we obtain the bound

$$Y^2 \leq N^2 X Z_0. \tag{A18}$$

The gain here is that Z_0 is only quadratic in A, whereas Z contains a term that is cubic in A.

We further simplify the bound by comparing Y and Z_0. We insert a complete set of eigenstates,

$$\psi_{k,a} = V^{-1/2} e^{ik\cdot x} e_a, \tag{A19}$$

where e_a are a basis of color vectors, and obtain

$$Y = \sum_{k,a;|k|\geq|q|} \frac{1}{k^2 - p^2} |(\psi_{k,a}, M_1 \psi_0)|^2, \tag{A20}$$

$$Z_0 = \sum_{k,a;|k|\geq|q|} \frac{k^2}{(k^2 - p^2)^2} |(\psi_{k,a}, M_1 \psi_0)|^2. \tag{A21}$$

From the restriction $k^2 \geq q^2 > p^2$ it follows that

$$\frac{k^2}{k^2 - p^2} \leq \frac{q^2}{q^2 - p^2}, \tag{A22}$$

and consequently

$$Z_0 \leq \frac{q^2}{q^2 - p^2} Y. \tag{A23}$$

This gives the bound

$$Z_0 Y \leq \frac{q^2}{q^2 - p^2} Y^2 \leq \frac{N^2 q^2}{q^2 - p^2} X Z_0, \tag{A24}$$

and so

$$Y \leq \frac{N^2 q^2}{q^2 - p^2} X. \tag{A25}$$

We next consider

$$X = (\psi_0, (M_0 + M_1)\psi_0) = p^2 - i p_\mu f^{abc} \eta^{a*} a^b_{k=0,\mu} \eta^c, \tag{A26}$$

where we have used the Fourier expansion (6), and $a^b_{k=0,\mu}$ is the 0-momentum component of $A^b_\mu(x)$. We next show that

$$|f^{abc} \eta^{a*} a^b_{k=0,\mu} \eta^c| \leq \frac{2\pi}{L}, \tag{A27}$$

where μ is a fixed Lorentz index. To do so, we use $(\omega, M(A)\omega) \geq 0$ for $A \in \Omega$, where $\omega = V^{-1/2} \exp(\pm i 2\pi x_\mu / L)\eta$. We have

$$(\omega, M\omega) = \left(\frac{2\pi}{L}\right)^2 \mp i\left(\frac{2\pi}{L}\right) f^{abc} \eta^{a*} a^b_{k=0,\mu} \eta^c \geq 0, \tag{A28}$$

from which Eq. (A27) follows. This gives

$$X \leq p^2 + (2\pi/L) \sum_\mu |p_\mu|, \tag{A29}$$

and we obtain the bound on Y,

$$Y \leq \frac{N^2 q^2}{q^2 - p^2} \left[p^2 + (2\pi/L) \sum_\mu |p_\mu| \right]. \tag{A30}$$

Note that Y is quadratic in A, while the right-hand side is independent of A, so this is an ellipsoidal bound on A, as advertised.

We next evaluate Y, Eq. (A20). We have

$$\begin{aligned}(\psi_{k,a}, M_1 \psi_0) &= -i p_\mu f^{abc} V^{-1} \int d^d x A^b_\mu(x) e^{i(p-k)\cdot x} \eta^c \\ &= -i p_\mu f^{abc} a^b_{k-p,\mu} \eta^c.\end{aligned} \tag{A31}$$

This gives

$$\begin{aligned} Y &= \sum_{k,a;|k|\geq|q|} \frac{|p_\mu f^{abc} a^b_{k-p,\mu} \eta^c|^2}{k^2 - p^2} \\ &= \sum_{k,a;|k+p|\geq|q|} \frac{|p_\mu f^{abc} a^b_{k,\mu} \eta^c|^2}{(k+p)^2 - p^2}, \end{aligned} \tag{A32}$$

and we have the bound

$$\begin{aligned} &f^{abc} f^{ade} \eta^{c*} \eta^e p_\mu p_\nu \sum_{k;|k+p|\geq|q|} \frac{a^{b*}_{k,\mu} a^d_{k,\nu}}{(k+p)^2 - p^2} \\ &\quad \leq \frac{N^2 q^2}{q^2 - p^2} \left[p^2 + (2\pi/L) \sum_\mu |p_\mu| \right]. \end{aligned} \tag{A33}$$

For each p and q satisfying $q^2 > p^2 \geq (2\pi/L)^2$, and for each color vector η, this is an ellipsoidal bound on the Fourier components $a^b_{k,\nu}$ that holds at finite Euclidean volume V for all configurations $A \in \Omega$. Geometrically speaking, the configurations A that satisfy the bound (A33) define an ellipsoid E in configuration space (that depends on p and q). The Gribov region Ω is contained in E, and we have the inclusions

$$\Lambda \subset \Omega \subset E. \tag{A34}$$

Here Λ is the fundamental modular region, which consists of the absolute minimum of the minimizing functional on each gauge orbit.

APPENDIX B: BOUND ON THE CONTINUUM GLUON PROPAGATOR

We convert the ellipsoidal bound on configurations (just obtained) to a bound on the gluon propagator by taking expectation values,

$$\begin{aligned} &f^{abc} f^{ade} \eta^{c*} \eta^e p_\mu p_\nu \sum_{k;|k+p|\geq|q|} \frac{\langle a^{b*}_{k,\mu} a^d_{k,\nu} \rangle}{(k+p)^2 - p^2} \\ &\quad \leq \frac{N^2 q^2}{q^2 - p^2} \left[p^2 + (2\pi/L) \sum_\mu |p_\mu| \right]. \end{aligned} \tag{B1}$$

From the Fourier expansions

$$\langle A^b_\mu(x) A^d_\nu(0)\rangle = V^{-1}\sum_k \delta^{bd} D_{\mu\nu}(k) e^{ik\cdot x} = \sum_k \langle a^{b*}_{k,\mu} a^d_{k,\nu}\rangle e^{ik\cdot x}, \tag{B2}$$

where $D_{\mu\nu}(k)$ is the gluon propagator on the finite periodic volume V, we obtain

$$\langle a^{b*}_{k,\mu} a^d_{k,\nu}\rangle = V^{-1}\delta^{bd} D_{\mu\nu}(k), \tag{B3}$$

and the last bound becomes

$$V^{-1}\sum_{k;|k+p|\geq|q|} \frac{p_\mu D_{\mu\nu}(k) p_\nu}{(k+p)^2 - p^2} \leq \frac{Nq^2}{q^2 - p^2}\left[p^2 + (2\pi/L)\sum_\mu |p_\mu|\right], \tag{B4}$$

where we have used $f^{abc} f^{abe} = N\delta^{ce}$. This is an exact bound on the gluon propagator on a finite periodic Euclidean volume $V = L^d$.

We take the infinite-volume limit, $L \to \infty$, keeping k, q, and p finite, and obtain

$$V^{-1}\sum_{k;|k+p|\geq|q|} \frac{p_\mu D_{\mu\nu}(k) p_\nu}{(k+p)^2 - p^2} \leq \frac{Nq^2}{q^2 - p^2} p^2. \tag{B5}$$

We divide out a factor of p^2,

$$V^{-1}\sum_{k;|k+p|\geq|q|} \frac{\hat{p}_\mu D_{\mu\nu}(k) \hat{p}_\nu}{(k+p)^2 - p^2} \leq \frac{Nq^2}{q^2 - p^2}, \tag{B6}$$

where $\hat{p}_\mu = p_\mu/|p|$ is a unit Lorentz vector. Recall that $|p|$, $\hat{p}_\mu$, and q_μ are quantities at our disposal. We take the limit $|p| \to 0$, keeping $\hat{p}_\mu$ and q fixed, which gives

$$V^{-1}\sum_{k;|k|\geq|q|} \frac{\hat{p}_\mu D_{\mu\nu}(k) \hat{p}_\nu}{k^2} \leq N. \tag{B7}$$

We now take the limit $q \to 0$, and convert the sum to an integral, since we are in the infinite-volume limit, and obtain the bound

$$\frac{1}{(2\pi)^d}\int d^d k \frac{\hat{p}_\mu D_{\mu\nu}(k) \hat{p}_\nu}{k^2} \leq N. \tag{B8}$$

Spherical symmetry is regained in the infinite-volume limit so, by transversality, $D_{\mu\nu}(k) = D(k)(\delta_{\mu\nu} - k_\mu k_\nu/k^2)$, and the angular average over k yields the bound

$$\frac{d-1}{d}\frac{1}{(2\pi)^d}\int d^d k \frac{D(k)}{k^2} \leq N. \tag{B9}$$

APPENDIX C: SIMPLE ELLIPSOIDAL BOUND ON CONTINUUM CONFIGURATIONS AT INFINITE VOLUME

We note parenthetically that we could have taken the limit of large volume V, without taking expectation values. In Eq. (A33), we take the limit $L \to \infty$ keeping k, q, and p finite, divide by p^2, take the limit $p \to 0$, followed by the limit $q \to 0$, and we obtain the ellipsoidal bound

$$f^{abc} f^{ade} \eta^{c*} \eta^e \hat{p}_\mu \hat{p}_\nu \sum_k \frac{a^{b*}_{k,\mu} a^d_{k,\nu}}{k^2} \leq N^2. \tag{C1}$$

We obtain a simpler ellipsoidal bound by summing over a complete basis $\sum_{b=1}^{N^2-1} \eta^{c*}_b \eta^e_b = \delta^{ce}$, and $\sum_{\lambda=1}^d \hat{p}^\lambda_\mu \hat{p}^\lambda_\nu = \delta_{\mu\nu}$, which gives

$$\sum_k \frac{a^{b*}_{k,\mu} a^b_{k,\mu}}{k^2} \leq N(N^2-1)d. \tag{C2}$$

In position space this bound reads, by Eq. (6),

$$\int_V d^d x A^b_\mu(x)[(-\partial^2)^{-1} A]^b_\mu(x) \leq N(N^2-1)Vd. \tag{C3}$$

For a configuration $A^b_\mu(x)$ that has compact support, this bound becomes vacuous in the limit $V \to \infty$ because the right-hand side diverges with V, while the left-hand side remains finite. However, for a typical gauge-fixed configuration (and here we may suppose that a lattice regularization is in place), the integral on the left is a bulk quantity of order V, and the bound is meaningful. Bounds on lattice configurations are given in Appendix D.

APPENDIX D: BOUND ON THE LATTICE GLUON PROPAGATOR

1. Notation for lattice quantities

Lattice configurations are defined by link variables $U_{x,\mu} \in SU(N)$ that live on the link $\langle x, x+e_\mu\rangle$, where sites of the lattice are labeled (in lattice units) by integers x_ν, and e_μ is a unit Lorentz vector in the positive μ direction. Numerical gauge fixing is done by minimizing the function

$$F_U(g) = \sum_{x,\mu} \mathrm{Re}\,\mathrm{tr}(1 - {}^g U_{x,\mu}) \tag{D1}$$

with respect to local gauge transformations g_x, where ${}^g U_{x,\mu} \equiv g_x^{-1} U_{x,\mu} g_{x+e_\mu}$ is the gauge transform of the configuration $U_{x,\mu}$ by g_x. In practice there are many local minima, and the particular minimum chosen is algorithm dependent. For our purposes, any minimum will do: the absolute minimum plays no special role. The only properties we shall use are that at any local minimum (i) the functional $F_U(g)$ is stationary, and (ii) the matrix of its second derivatives is positive. Property (i) gives the lattice transversality condition,

$$\sum_\mu (A^a_{x,\mu} - A^a_{x-e_\mu,\mu}) = 0, \tag{D2}$$

where we have introduced the lattice variables

$$A^b_{x,\mu}(U) \equiv -\mathrm{tr}[t^b(U_{x,\mu} - U^\dagger_{x,\mu})], \tag{D3}$$

that are a lattice analog of the continuum variables $A_\mu^b(x)$. Property (ii) is the positivity of the matrix element,

$$(\psi, M(U)\psi) \geq 0, \tag{D4}$$

for all ψ_x^a. Here $M_{xy}^{ab}(U)$ is the lattice Faddeev-Popov matrix. It is a real symmetric matrix that is conveniently expressed as

$$M = M_0 + M_1, \tag{D5}$$

where M_0 and M_1 are defined by the quadratic forms

$$(\psi, M_0\psi) \equiv \sum_{x,\mu} \mathrm{tr}[-(\psi_{x+e_\mu}^* - \psi_x^*)(U_{x,\mu} + U_{x,\mu}^\dagger) \times (\psi_{x+e_\mu} - \psi_x)], \tag{D6}$$

where $\psi_x \equiv t^a\psi_x^a$ and $\psi_x^* \equiv t^a\psi_x^{a*}$, and

$$\begin{aligned}(\psi, M_1(A)\psi) &= \mathrm{tr}\{[\psi_{x+e_\mu}, \psi_x](U_{x,\mu} - U_{x,\mu}^\dagger)\} \\ &= f^{abc}\psi_{x+e_\mu}^a A_{x,\mu}^b \psi_x^c \\ &= -(1/2)f^{abc}(\psi_{x+e_\mu} + \psi_x)^a \\ &\quad \times A_{x,\mu}^b(\psi_{x+e_\mu} - \psi_x)^c.\end{aligned} \tag{D7}$$

The relation to the continuum Faddeev-Popov operator (3) is apparent. The last expression is real when A satisfies the lattice transversality condition (D2). Properties (i) and (ii) define the (first) lattice Gribov region Ω.

We consider a hypercubic periodic lattice of volume $V = \mathcal{N}^d$, where $\mathcal{N}$ is an integer and $x_\mu = 0, 1, \ldots \mathcal{N} - 1$, mod($\mathcal{N}$). The Fourier transformation is given by

$$A_{x,\mu}^b = \sum_k a_{k,\mu}^b \exp[ik \cdot (x + e_\mu/2)], \tag{D8}$$

where $k_\mu = 2\pi n_\mu/\mathcal{N}$, and $n_\mu = 0, 1 \ldots \mathcal{N} - 1$, mod($\mathcal{N}$). The transversality condition (D2) is diagonal in momentum space, where it reads

$$\sum_\mu K_\mu a_{k,\mu}^b = 0, \tag{D9}$$

and we have introduced

$$K_\mu \equiv 2\sin(k_\mu/2), \tag{D10}$$

and similarly $P_\mu \equiv 2\sin(p_\mu/2)$, $Q_\mu \equiv 2\sin(q_\mu/2)$.

2. Ellipsoidal bound on lattice configurations

We proceed as in the continuum case. The lattice Gribov region Ω is defined by the following condition on (transverse) configurations U:

$$-(\psi, M_1(A)\psi) \leq (\psi, M_0(U)\psi), \tag{D11}$$

for all ψ, where $A = A(U)$.

As in the continuum case, the matrix $M_1(A)$ is linear in A. However, in the lattice case $M_0(U)$ is not independent of the configuration U. We nevertheless obtain a simple lattice bound by introducing the matrix $\mathcal{K}_0$ defined by

$$(\psi, \mathcal{K}_0\psi) \equiv \sum_{x,\mu} \mathrm{tr}[-2(\psi_{x+e_\mu} - \psi_x)^*(\psi_{x+e_\mu} - \psi_x)], \tag{D12}$$

which is independent of U [25]. The difference,

$$(\psi, [\mathcal{K}_0 - M_0(U)]\psi) \equiv \sum_{x,\mu} \mathrm{tr}[-(\psi_{x+e_\mu}^* - \psi_x^*)(1 - U_{x,\mu}) \times (1 - U_{x,\mu}^\dagger)(\psi_{x+e_\mu} - \psi_x)], \tag{D13}$$

is manifestly positive for every lattice configuration U and every trial wave function ψ, so we have

$$(\psi, M_0(A)\psi) \leq (\psi, \mathcal{K}_0\psi), \tag{D14}$$

which, by Eq. (D11), yields the inequality

$$-(\psi, M_1(A)\psi) \leq (\psi, \mathcal{K}_0\psi) \tag{D15}$$

for every configuration $U \in \Omega$ and every ψ. Geometrically, it is natural to define a region Θ in configuration space by the condition

$$\Theta \equiv \{U\colon\ -(\psi, M_1(A)\psi) \leq (\psi, \mathcal{K}_0\psi) \quad \text{for all } \psi\}, \tag{D16}$$

where $A = A(U)$ is transverse, and we have the inclusions

$$\Lambda \subset \Omega \subset \Theta. \tag{D17}$$

Here Λ is the fundamental modular region, which consists of every configuration that is the absolute minimum of the minimizing function on its gauge orbit. We shall derive a bound on the lattice gluon propagator that holds for all transverse configurations in Θ, which then holds *a fortiori* for all configurations in the Gribov region Ω. Because $\mathcal{K}_0$ is independent of U and because $M_1(A)$ is linear in $A = A(U)$, the proof goes just as in the continuum case, but with $M_0 \to \mathcal{K}_0$.

We define

$$\psi_0 = V^{-1/2}e^{ip\cdot x}\eta, \tag{D18}$$

where $V = \mathcal{N}^d$ and $p_\mu = 2\pi n_\mu/\mathcal{N}$ is a lattice momentum, and we have

$$\mathcal{K}_0\psi_0 = P^2\psi_0, \tag{D19}$$

where P_μ is defined as in Eq. (D10). The continuum proof goes through, with the substitutions $p^2 \to P^2$, $k^2 \to K^2$, $q^2 \to Q^2$, and $M_0 \to \mathcal{K}_0$, and Eq. (A4) becomes

$$K^2 \geq Q^2 > P^2 \geq 4\sin^2(\pi/\mathcal{N}). \tag{D20}$$

From Eq. (D7) one sees that Eq. (A31) may be replaced by

$$(\psi_{k,a}, M_1 \psi_0) = -i f^{abc} \sum_\mu P_\mu \cos(k_\mu/2) a^b_{k-p,\mu} \eta^c, \tag{D21}$$

Eq. (A26) gets replaced by

$$X = P^2 - i \sum_\mu P_\mu \cos(p_\mu/2) f^{abc} \eta^{a*} a^b_{k=0,\mu} \eta^c, \tag{D22}$$

Eq. (A27) by

$$|f^{abc} \eta^{a*} a^b_{k=0,\mu} \eta^c| \le 2\tan(\pi/\mathcal{N}), \tag{D23}$$

Eq. (A29) by

$$X \le P^2 + 2\tan(\pi/\mathcal{N}) \sum_\mu |\sin p_\mu|, \tag{D24}$$

Eq. (A30) by

$$Y \le \frac{N^2 Q^2}{Q^2 - P^2} \Big[P^2 + 2\tan(\pi/\mathcal{N}) \sum_\mu |\sin p_\mu| \Big], \tag{D25}$$

and finally Eq. (A33) by

$$f^{abc} f^{ade} \eta^{c*} \eta^e \sum_k \frac{C_\mu(k', p) a^{b*}_{k,\mu} a^d_{k,\nu} C_\nu(k', p)}{K'^2 - P^2} \le \frac{N^2 Q^2}{Q^2 - P^2} \Big[P^2 + 2\tan(\pi/\mathcal{N}) \sum_\mu |\sin p_\mu| \Big], \tag{D26}$$

where $k'_\mu \equiv (k+p)_\mu$, $K'^2 \equiv 4\sum_\mu \sin^2(k'/2)$, $C_\mu(k', p) \equiv P_\mu \cos(k'_\mu/2)$, and the sum over k is restricted by $K'^2 \ge Q^2$. This is an ellipsoidal bound that holds on a finite lattice for every configuration $a^b_{k,\mu}$ in the lattice Gribov region Ω.

3. Bound on the lattice gluon propagator

Upon taking expectation values, we obtain a bound on the lattice gluon propagator,

$$V^{-1} \sum_k \frac{C_\mu(k', p) D_{\mu\nu}(k) C_\nu(k', p)}{K'^2 - P^2} \le \frac{N Q^2}{Q^2 - P^2} \Big[P^2 + 2\tan(\pi/\mathcal{N}) \sum_\mu |\sin p_\mu| \Big], \tag{D27}$$

where $D_{\mu\nu}(k)$ is the gluon propagator on a finite lattice, and we have used

$$\langle a^{b*}_{k,\mu} a^d_{k,\nu} \rangle = V^{-1} \delta^{bd} D_{\mu\nu}(k). \tag{D28}$$

We now take the lattice volume to infinity, $\mathcal{N} \to \infty$, while keeping the lattice spacing finite. The lattice momentum $k_\mu = 2\pi n_\mu/\mathcal{N}$ becomes a continuous angle $-\pi \le k_\mu \le \pi$. We divide out P^2, take the limit $p \to 0$ and then $q \to 0$, and we obtain the bound on the gluon propagator on an infinite lattice,

$$\hat{P}_\mu T_{\mu\nu} \hat{P}_\nu \le N, \tag{D29}$$

where $\hat{P}$ is an arbitrary unit vector. Here

$$T_{\mu\nu} \equiv \frac{1}{(2\pi)^d} \int_{-\pi}^{\pi} d^d k \frac{\cos(k_\mu/2) D_{\mu\nu}(k) \cos(k_\nu/2)}{4\sum_\lambda \sin^2(k_\lambda/2)} \tag{D30}$$

(no sum on μ or ν) is a tensor that is invariant under the hypercubic symmetries. It is thus of the form $T_{\mu\nu} = J\delta_{\mu\nu}$, where $J = T_{\mu\mu}/d$, and the bound on the lattice gluon propagator reads $J \le N$, or

$$\frac{1}{d(2\pi)^d} \int_{-\pi}^{\pi} d^d k \frac{\sum_\mu \cos^2(k_\mu/2) D_{\mu\mu}(k)}{4\sum_\lambda \sin^2(k_\lambda/2)} \le N. \tag{D31}$$

[1] V. N. Gribov, Nucl. Phys. **B139**, 1 (1978).
[2] D. Zwanziger, Nucl. Phys. **B323**, 513 (1989).
[3] D. Dudal, S. P. Sorella, N. Vandersickel, and H. Verschelde, Phys. Rev. D **77**, 071501 (2008).
[4] D. Dudal, J. A. Gracey, S. P. Sorella, N. Vandersickel, and H. Verschelde, Phys. Rev. D **78**, 125012 (2008).
[5] D. Dudal, J. A. Gracey, S. P. Sorella, N. Vandersickel, and H. Verschelde, Phys. Rev. D **78**, 065047 (2008).
[6] D. Dudal, S. P. Sorella, N. Vandersickel, and H. Verschelde, Phys. Lett. B **680**, 377 (2009).
[7] D. Dudal, S. Sorella, and N. Vandersickel, Phys. Rev. D **84**, 065039 (2011).
[8] C. Lerche and L. von Smekal, Phys. Rev. D **65**, 125006 (2002).
[9] R. Alkofer and L. von Smekal, Phys. Rep. **353**, 281 (2001).
[10] C. S. Fischer, A. Maas, and J. M. Pawlowski, Ann. Phys. (Amsterdam) **324**, 2408 (2009).
[11] D. Zwanziger, Phys. Rev. D **65**, 094039 (2002).
[12] A. Maas, Phys. Rev. D **75**, 116004 (2007).

[13] A. Cucchieri and T. Mendes, Phys. Rev. Lett. **100**, 241601 (2008).
[14] A. Cucchieri and T. Mendes, AIP Conf. Proc. **1343**, 185 (2011).
[15] A. Cucchieri and T. Mendes, Proc. Sci., LAT2007 (2007) 297.
[16] A. Cucchieri and T. Mendes, Proc. Sci., QCD-TNT09 (2009) 026.
[17] I. L. Bogolubsky, E. M. Ilgenfritz, M. Muller-Preussker, and A. Sternbeck, Proc. Sci., LAT2007 (2007) 290.
[18] I. L. Bogolubsky, E. M. Ilgenfritz, M. Muller-Preussker, and A. Sternbeck, Phys. Lett. B **676**, 69 (2009).
[19] V. Bornyakov, V. Mitrjushkin, and M. Muller-Preussker, Phys. Rev. D **81**, 054503 (2010).
[20] A. Cucchieri, A. Maas, and T. Mendes, Mod. Phys. Lett. A **22**, 2429 (2007).
[21] G. Burgio, M. Quandt, and H. Reinhardt, Phys. Rev. Lett. **102**, 032002 (2009).
[22] G. Burgio, M. Quandt, and H. Reinhardt, Phys. Rev. D **81**, 074502 (2010).
[23] Y. Nakagawa, A. Nakamura, T. Saito, and H. Toki, Phys. Rev. D **83**, 114503 (2011).
[24] A. Maas, T. Mendes, and S. Olejnik, Phys. Rev. D **84**, 114501 (2011).
[25] D. Zwanziger, Nucl. Phys. **B364**, 127 (1991).
[26] A. Maas and D. Zwanziger (unpublished).
[27] T. Maskawa and H. Nakajima, Prog. Theor. Phys. **60**, 1526 (1978); **63**, 642 (1980).
[28] M. Semenov-Tyan-Shanskii and V. Franke, *Zap. Nauch. Sem. Leningrad. Otdelleniya Matematicheskogo Instituta*, in V. A. Steklov, AN SSSR Vol. 120 (in English translation Plenum, New York, 1982), p. 159.
[29] D. Zwanziger, Nucl. Phys. **B209**, 336 (1982).
[30] C. Itzykson and J.-B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).
[31] D. Zwanziger, Nucl. Phys. **B518**, 237 (1998).