Critical behavior of the $\mathcal{PT}$-symmetric $i\phi^3$ quantum field theory

Carl M. Bender,[1,*] V. Branchina,[2,†] and Emanuele Messina[2,‡]

[1]*Department of Physics, Washington University, St. Louis, Missouri 63130, USA*
[2]*Department of Physics, University of Catania and INFN, Sezione di Catania, Via Santa Sofia 64, I-95123 Catania, Italy*

It was shown recently that a $\mathcal{PT}$-symmetric $i\phi^3$ quantum field theory in $6 - \epsilon$ dimensions possesses a nontrivial fixed point. The critical behavior of this theory around the fixed point is examined and it is shown that the corresponding phase transition is related to the existence of a nontrivial solution of the gap equation. The theory is studied first in the mean-field approximation and the critical exponents are calculated. Then, it is examined beyond the mean-field approximation by using renormalization-group techniques, and the critical exponents for $6 - \epsilon$ dimensions are calculated to order ϵ. It is shown that because of its stability the $\mathcal{PT}$-symmetric $i\phi^3$ theory has a higher predictive power than the conventional ϕ^3 theory. A comparison of the $i\phi^3$ model with the Lee-Yang model is given.

I. INTRODUCTION

The study of $\mathcal{PT}$-symmetric quantum theory [1] was originally motivated by the discovery that the eigenvalues of the quantum-mechanical Hamiltonian $H = p^2 + ix^3$ are real, positive, and discrete [2]. This work naturally led to a number of studies of the properties of a scalar quantum field theory having an imaginary cubic self-interaction term [3]. A very recent study of the renormalization-group (RG) equations for the $\mathcal{PT}$-symmetric $ig\phi^3$ quantum field theory in $d = 6 - \epsilon$ dimensions shows the existence of a nontrivial fixed point [4]. This allows for a nonperturbative renormalization of the theory and suggests that the theory undergoes a continuous phase transition.

In the present paper we study this transition in detail; that is, we examine the critical behavior of the theory. We will see that such a transition is associated with the existence of a nontrivial solution to the gap equation at a critical value m_c^2 of the bare mass m^2. The correlation length (the inverse of the renormalized scalar mass) diverges at $m^2 = m_c^2$.

In Sec. II we study this transition within the framework of the mean-field approximation and in Sec. III we obtain the critical behavior of the theory by calculating the critical exponents. Next, in Sec. IV we compare the conventional and the $\mathcal{PT}$-symmetric ϕ^3 theories and analyze the relation between their renormalization and stability properties. We show that compared with the conventional ϕ^3 model, the $\mathcal{PT}$-symmetric $i\phi^3$ theory exhibits new and interesting features. Remarkably, the $\mathcal{PT}$-symmetric theory has a higher predictive power than the ϕ^3 theory. This is because its critical behavior (and therefore its renormalization properties) is governed by one parameter less than the conventional theory. We will show that this property is

related to the different stability properties of the two theories.

Section V goes beyond mean-field analysis. We study the theory near $d = 6$ dimensions and calculate the critical exponents up to O(ϵ). For $d < 6$, the fluctuations around the mean-field configuration become important and the analysis of the critical behavior requires the use of RG techniques. With the help of the hyperscaling relations, the critical exponents are calculated for $d = 6 - \epsilon$. Some conclusions are given in Sec. VI.

II. MEAN-FIELD ANALYSIS

In Ref. [4] a nontrivial fixed point of the RG equations of the $\mathcal{PT}$-symmetric $ig\phi^3$ quantum field theory in $d = 6 - \epsilon$ dimensions was found. The existence of such a fixed point suggested the onset of a continuous transition. In this section we study this transition by performing a mean-field analysis.

The partition function $Z[h]$ of the quantum field theory in d dimensions is given by

$$Z[h] = \int \mathcal{D}\phi\, e^{-S[\phi] - i\int d^d x\, h\phi}, \tag{1}$$

where h is an external field. The action is $S[\phi] = \int d^d x[(\partial_\mu \phi)^2/2 + V(\phi)]$ and the potential is $V(\phi) = m^2\phi^2/2 + ig\phi^3/6$.

We consider the mean-field approximation to $Z[h]$, first searching for a constant-field saddle point and then performing a semiclassical expansion around this configuration. [The exponential in (1) contains an implicit factor of $1/\hbar$, and in the semiclassical approximation $\hbar$ is treated as small, which justifies the use of steepest-descent asymptotic techniques [5].] Without loss of generality, we assume that $g > 0$.

The constant-field saddle points are given by the gap equation

$$m^2\bar{\phi} + ig\bar{\phi}^2/2 = -ih. \tag{2}$$

*cmb@wustl.edu
†branchina@ct.infn.it
‡emanuele.messina@ct.infn.it

When $h = 0$, (2) becomes

$$\bar{\phi}(m^2 + ig\bar{\phi}/2) = 0, \qquad (3)$$

which has the two solutions $\bar{\phi}_1 = 0$ and $\bar{\phi}_2 = 2im^2/g$. We must determine which of the saddle points, $\bar{\phi}_1$ or $\bar{\phi}_2$, contributes to the small-$\hbar$ asymptotic behavior of the path integral (1).

A. Asymptotic analysis for $d = 0$

Let us first examine the problem for the simple case $d = 0$. The second derivative of the exponential in (1) at $h = 0$ is $-m^2 - ig\phi$. Thus, at $\phi = \bar{\phi}_1$ the second derivative is $-m^2$, and at $\phi = \bar{\phi}_2$ the second derivative is m^2. The sign of the second derivative of the potential determines the directions of the steepest-descent paths (constant-phase contours) in the neighborhood of the saddle points. Thus, if m^2 is positive, the *down* directions from $\bar{\phi}_1$ are horizontal (parallel to the Reϕ axis) and the *down* directions from $\bar{\phi}_2$ are vertical (parallel to the Imϕ axis). However, if m^2 is negative, the directions are reversed; the *down* directions from $\bar{\phi}_2$ are horizontal and the *down* directions from $\bar{\phi}_1$ are vertical.

It is necessary to deform the original integration path in (1), which lies on the real axis, into a constant-phase contour that passes through the appropriate saddle point. If we let $\phi = u + iv$ and note that the phase (the imaginary part of the exponent) at the saddle points vanishes, we obtain a cubic algebraic equation for the constant-phase contour: $u(gv^2/2 - m^2v - gu^6/6) = 0$. Thus, near ∞ the steepest-descent contours asymptote to the angles $\pi/2$, $-\pi/6$, and $-5\pi/6$ and the steepest-ascent paths asymptote to the angles $-\pi/2$, $\pi/6$, and $5\pi/6$. Thus, the *uniquely determined* steepest-descent contour is a $\mathcal{PT}$-symmetric (left-right-symmetric) path that terminates at the angles $-\pi/6$ and $-5\pi/6$.

Hence, the crucial observation is this: When $m^2 > 0$, $\bar{\phi}_2$ lies above $\bar{\phi}_1$ on the Imϕ axis and the steepest-descent path passes through $\bar{\phi}_1$ (and not $\bar{\phi}_2$). However, when $m^2 < 0$, $\bar{\phi}_2$ lies below $\bar{\phi}_1$ on the Imϕ axis and the steepest-descent path passes through $\bar{\phi}_2$ (and not $\bar{\phi}_1$). A phase transition occurs at $m^2 = 0$ where the saddle points coincide.

B. Asymptotic analysis for $d > 0$

When $d > 0$, we consider the Hessian matrix $D^{-1}(q, p) = \Delta^{-1}(q^2)\delta^d(p + q)$ in Fourier space, where

$$\Delta^{-1}(q^2) = \int d^d x d^d y e^{-iq(x-y)} \frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)} \bigg|_{\bar{\phi}_i}$$

$$= q^2 + m^2 + ig\bar{\phi}_i.$$

This is just the inverse tree-level correlator when $\bar{\phi}_i$ is the vacuum. For $m^2 \geq 0$, $D^{-1}(q, p)$ is positive definite when evaluated at $\bar{\phi}_1 = 0$. In this case we find that

$$\Delta^{-1}(q^2) = q^2 + m^2. \qquad (4)$$

On the other hand, for $m^2 < 0$, $D^{-1}(q, p)$ is positive definite when calculated at $\bar{\phi}_2 = 2im^2/g$, and we get

$$\Delta^{-1}(q^2) = q^2 + m^2 - 2m^2 = q^2 - m^2. \qquad (5)$$

These results imply that there is a phase transition in the Euclidean $\mathcal{PT}$-symmetric $ig\phi^3$ theory, with the two phases being determined by the order parameter $\bar{\phi}$, where $\bar{\phi}_1 = 0$ and $\bar{\phi}_2 = 2im^2/g$.

We proceed to identify the *relevant couplings*; in general, these are the couplings that control the phase transition. In the ferromagnetic case, the relevant couplings are the *temperature* and the *external magnetic field* [6]. Thus, we look for the two couplings that in our case play the role that the temperature and the external magnetic field play in the ferromagnetic case. We begin by defining the *temperature T* as

$$T \equiv 2m^2/g, \qquad (6)$$

where the coefficient $2/g$ in (6) is chosen for later convenience [7]. The critical temperature T_c is obtained with the help of (3) for the nontrivial saddle point (that is, the solution to $m^2 + ig\bar{\phi}/2 = 0$). We seek the limiting value of T for which $\bar{\phi}$ vanishes, and in our case we have $T_c = 0$. Therefore, the reduced temperature is given by

$$\tau = T - T_c = 2m^2/g. \qquad (7)$$

The *external magnetic field* (the conjugate field) is just the parameter h.

From (4) and (5) we see that the excitations above the trivial vacuum $\bar{\phi}_1 = 0$ and those above the nontrivial one $\bar{\phi}_2 = 2im^2/g$ have the same mass, $M = |m^2|^{1/2}$, which goes to zero as we approach the critical point. Moreover, as we move continuously from positive to negative values of m^2, $\bar{\phi} = 2im^2/g$ moves continuously down the negative-imaginary axis starting from $\bar{\phi} = 0$ (when $m^2 \geq 0$). The presence of such a divergent correlation length (the inverse of the scalar mass $M = 0$) and of a continuously varying order parameter $\bar{\phi}$ are the indications of a continuous (second-order) phase transition. In RG language, this is due to the existence of the nontrivial fixed point found in Ref. [4].

We have shown that at least within the framework of the mean-field approximation used in this section, the fixed point found in Ref. [4] governs the transition from the $\bar{\phi}_1 = 0$ phase to the $\bar{\phi}_2 = 2im^2/g$ phase. In Sec. III we calculate the critical exponents that govern the behavior of the theory in the critical region in terms of the parameters h and τ.

III. CRITICAL EXPONENTS IN THE MEAN-FIELD APPROXIMATION

In this section we give a quantitative description of the critical behavior of the model for $d > 6$ dimensions. We define and calculate the critical exponents in the mean-field approximation by following the standard terminology used for Ising-like models. With the help of (7) we write the saddle-point equation (2) for $h \neq 0$ as

$$\tau + i\bar{\phi} = -2ih/(g\bar{\phi}), \qquad (8)$$

which has the typical form of a gap equation. From (8) we can immediately deduce the mean-field exponents β and δ. The exponent β is determined by $\bar{\phi}|_{h=0} \sim \tau^{\beta}$, which gives $\beta = 1$. The quantity δ is defined by the relation $\bar{\phi}|_{\tau=0} \sim h^{1/\delta}$, which implies that $\delta = 2$.

To evaluate the exponents η, ν, and γ, we need the two-point inverse correlator given in (4) and (5) for the $\bar{\phi} = 0$ and the $\bar{\phi} = 2im^2/g$ phases, respectively. The exponent η is defined by $\Delta(q^2) \sim q^{\eta-2}(q^2 \to \infty)$, which implies that $\eta = 0$ in both phases.

To calculate ν we need the correlation length. The latter is proportional to the inverse of the renormalized mass M (the pole of the propagator). In the mean-field approximation discussed in the present section, $M^2 = |m^2|$. For the phase with nonzero $\bar{\phi}$ ($\bar{\phi} = \bar{\phi}_2$) we find that $\xi^{-2} = -m^2 \sim \bar{\phi}_2 \sim \tau$. This implies that the mean-field exponent ν, which is defined by $\xi \sim \tau^{-\nu}$, is $\nu = 1/2$.

Finally, the exponent γ is obtained as follows. The susceptivity χ is given by $\chi \equiv i\delta\bar{\phi}/\delta h|_{h=0}$. By differentiating (2) with respect to h we have $m^2\delta\bar{\phi}/\delta h + ig\bar{\phi}\delta\bar{\phi}/\delta h = -i$, which we evaluate at $h = 0$ to get $(m^2 + ig\bar{\phi})\chi = 1$. We thus obtain

$$\chi = (m^2)^{-1} \sim \tau^{-1} \quad \text{for } T > T_c (m^2 > 0),$$
$$\chi = (-m^2)^{-1} \sim \tau^{-1} \quad \text{for } T < T_c (m^2 < 0).$$

Therefore, γ, which is defined by the equation $\chi \sim \tau^{-\gamma}$, turns out to be $\gamma = 1$.

In this and in the previous section we have studied the features of the continuous transition of the $\mathcal{PT}$-symmetric $ig\phi^3$ theory in the mean-field approximation by calculating the critical exponents. In the following section we deepen our understanding of the behavior of the theory in the critical region. To this end, we compare the conventional and the $\mathcal{PT}$-symmetric theories by relating their critical behaviors to their stability properties.

IV. RENORMALIZATION AND STABILITY

In Ref. [4] we compared the conventional ϕ^3 theory with the corresponding $i\phi^3$ $\mathcal{PT}$-symmetric theory near and at $d = 6$ dimensions, and the instability of the former was contrasted with the stability of the latter. Moreover, for $d = 6 - \epsilon$ dimensions, the perturbative renormalizability of the ϕ^3 theory (obtained by taking the bare parameters

around the Gaussian fixed point) was compared with the renormalizability of the $\mathcal{PT}$-symmetric theory. The latter is *nonperturbative* because it is realized around a non-Gaussian fixed point (which collapses onto the Gaussian fixed point at $d = 6$).

As shown in Ref. [4], there is a crucial difference between the RG properties of the conventional and the $\mathcal{PT}$-symmetric theories. In the former both m^2 and g are relevant directions, but in the latter m^2 is the only relevant direction (g being irrelevant). Accordingly, in Sec. III we found that to make the nontrivial solution $\bar{\phi}_2$ vanish we only need to tune the temperature $T = 2m^2/g$ towards its critical value $T_c = 0$. For generic values of the bare coupling g (which is kept fixed), we must tune only m^2 towards zero. This is similar to what happens in conventional ϕ^4 theory, where the only relevant direction is m^2 and the quartic coupling plays no role in determining the phase transition. It appears that the instability of the ϕ^3 theory gives rise to an additional condition to reach the critical region.

In this section we examine the origin of the profound difference between the two theories. To this end, we study the mean-field (classical) potential of the conventional theory

$$V(\phi) = m^2\phi^2/2 + g\phi^3/6. \qquad (9)$$

As in the previous section, without loss of generality we consider the case $g > 0$.

This potential is unstable for both the $m^2 > 0$ and $m^2 < 0$ cases (see Fig. 1). However, if the false vacuum is sufficiently long lived (that is, if the tunneling time is
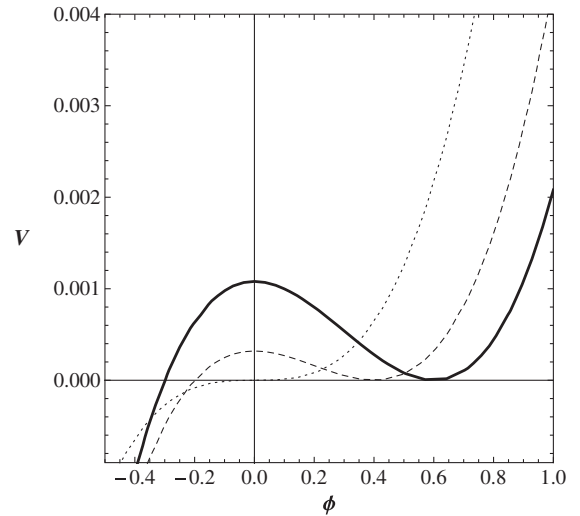

FIG. 1. The potential $V(\phi) = m^2\phi^2/2 + g\phi^3/6 - 2m^6/(3g^2)$ for $g = 6 \times 10^{-2}$ and $m^2 = -1.8 \times 10^{-2}$ (solid line), $m^2 = -1.2 \times 10^{-2}$ (dashed line) and $m^2 = 0$ (dotted line). For the solid and the dashed lines, the vacuum is at $\bar{\phi} = -2m^2/g$ while the intersection with the negative ϕ axis is at $\phi = m^2/g$. The area $\mathcal{A}$ of the surface included between the ϕ axis and $V(\phi)$ is proportional to the lifetime of the false vacuum.

sufficiently long), the theory can be consistently defined around this vacuum. For $h = 0$ the gap equation for the potential in (9) is $\phi(m^2 + g\phi/2) = 0$. For $m^2 > 0$, the potential has a minimum at the origin $\bar{\phi} = 0$ and a maximum at $\bar{\phi} = -2m^2/g$; for $m^2 < 0$, there is a minimum at $\bar{\phi} = -2m^2/g$ and the maximum is at $\bar{\phi} = 0$. Defining the reduced temperature τ as

$$\tau \equiv m^2/g, \tag{10}$$

we study the continuous phase transition from $\bar{\phi} = -2m^2/g$ to $\bar{\phi} = 0$. We stress that in the $\mathcal{PT}$-symmetric $ig\phi^3$ theory (as well as in the conventional $g\phi^4$ theory) the scaling region $\tau \to 0$ is reached by keeping g finite and taking the limit $m^2 \to 0$. This is not true for the ordinary ϕ^3 theory.

The potential in (9) for $m^2 < 0$ is given by the solid line in Fig. 1 [apart from a trivial constant shift $V(\phi) \to V(\phi) - 2m^6/(3g^2)$]. In the limit $m^2 \to 0$ with $g =$ finite the potential becomes the dotted line (cubic parabola), which has an inflection point at $\bar{\phi} = 0$. In this limit the vacuum disappears.

Let us define the area $\mathcal{A}$:

$$\mathcal{A} \equiv \int_{m^2/g}^{-2m^2/g} d\phi\, V(\phi) = \frac{9m^8}{8g^3} \tag{11}$$

of the surface under the potential function between the points $\bar{\phi} = -2m^2/g$ and $\phi = m^2/g$ (see Fig. 1). Note that in the limit $m^2 \to 0$ with $g =$ finite the nontrivial minimum $\bar{\phi} = -2m^2/g$ moves toward $\bar{\phi} = 0$, so $\mathcal{A}$ decreases. Therefore, the lifetime of the false vacuum, which is proportional to $\mathcal{A}$, becomes smaller, and the theory is destabilized. As already mentioned, from the RG viewpoint this happens because in addition to m^2, g is a relevant parameter. To reach the continuous phase transition ($\tau \to 0$) while keeping the theory stable ($\mathcal{A} =$ fixed), we must tune not only m^2 but also g. From (10) and (11) we get

$$m^2 \sim \mathcal{A}/\tau^3 \quad \text{and} \quad g \sim \mathcal{A}/\tau^4. \tag{12}$$

This means that in order to reach the continuum limit, $1/m^2$ and $1/g$ must be tuned separately to zero with $\tau \to 0$ according to (12) while keeping $\mathcal{A}$ (the vacuum lifetime) fixed. It is clear that the conventional ϕ^3 theory has an additional relevant direction as compared to its $\mathcal{PT}$-symmetric counterpart because of the intrinsic instability of the theory. The $\mathcal{PT}$-symmetric model, being energetically stable, has only one relevant direction, thus showing a higher degree of predictive power.

V. CRITICAL EXPONENTS NEAR $d = 6$

In this section we study the critical behavior of the theory for $d < 6$. (The mean-field results provide a good description for $d > 6$.) We turn our attention to the calculation of the critical exponents of the $\mathcal{PT}$-symmetric theory beyond the mean-field approximation considered

in Sec. III. [According to the mean-field analysis of Sec. III, in the presence of the external source h the two relevant parameters are $\tau = T - T_c = 2m^2/g$ (the *reduced temperature*) and h itself (the *external field*).] As is known from the general theory of critical phenomena, below the upper critical dimension ($d = 6$ in this case), the fluctuations around the mean-field configuration became important. RG techniques provide an essential tool for calculating the scaling behavior of the theory. (Note that although the exponents η and δ appear in the work of Fisher [8] on the Yang-Lee zero problem for the first time, the evaluation of the other exponents is an original achievement of this paper.)

Let us consider our theory in $d = 6 - \epsilon$ dimensions. The RG equations imply the existence of two nontrivial fixed points [4]:

$$h^* = 0, \qquad m^{2*} = 0, \qquad g^* = \pm\sqrt{128\pi^3\epsilon/3}.$$

To each of these points is associated the phase transition that we have just described in the mean-field approximation. (The above analysis was done for $g^* > 0$ but for $g^* < 0$ the results are analogous.)

According to Ref. [4] the scalings of h, m^2, and g with the running scale $t = \ln(\mu/\mu_0)$ are given by $h(t) = c_1 e^{g_1 t}$, $m^2(t) = c_2 e^{g_2 t}$, and $g(t) = g^* + c_3 e^{g_3 t}$, respectively, where

$$g_1 = -4 + 4\epsilon/9, \quad g_2 = -2 + 5\epsilon/9, \quad g_3 = \epsilon. \tag{13}$$

From the above equations we see that the two relevant parameters are h and $\tau = 2m^2/g$.

With the help of the hyperscaling relations

$$\beta = -\frac{d + g_1}{g_2}, \qquad \delta = -\frac{g_1}{d + g_1}, \qquad \gamma = \frac{2g_1 + d}{g_2},$$

$$\nu = -\frac{1}{g_2}, \qquad \eta = 2 + d + 2g_1, \tag{14}$$

we can calculate from (13) the critical exponents, which turn out to be

$$\beta = 1, \tag{15}$$

$$\gamma = 1 + \epsilon/3, \tag{16}$$

$$\delta = 2 + \epsilon/3, \tag{17}$$

$$\nu = 1/2 + 5\epsilon/36, \tag{18}$$

$$\eta = -\epsilon/9. \tag{19}$$

From (19), the scaling dimension of the scalar field is $[\phi] = (d - 2 + \eta)/2 = 2 - 5\epsilon/9$. Note that for $\epsilon = 0$ these exponents coincide with the mean-field values calculated in Sec. III.

It is interesting to compare the above RG expressions with known exact results in $d = 1$ dimensions (although one must recognize that the RG analysis is performed in the framework of the ϵ expansion with ϵ being treated as a small parameter). To this end we consider the exponents δ and η. From the hyperscaling relations (14) we see that g_1 can be eliminated, yielding

$$\delta = \frac{d + 2 - \eta}{d - 2 + \eta}. \tag{20}$$

Inserting into this relation $\eta = -\frac{\epsilon}{9} = -\frac{5}{9}$ from (19) and setting $d = 1$, we obtain $\delta = -2.29$. This compares well with the exact value $\delta = -2$. While the exponent δ is reasonably well approximated by the RG result, the same is not true for η ($\eta_{\text{exact}} = -1$).

VI. CONCLUSIONS

The $i\phi^3$ model was introduced in Ref. [8] to study the density of the Lee-Yang zeros of the partition function $Z[H]$ on the imaginary axis of H (H being the magnetic field). The "critical exponents" in Ref. [8] are not critical exponents in the physical sense, but rather they are parameters governing the mathematical behavior of the function that gives the asymptotic density of zeros on the H imaginary axis near the branch point $H = 0$. If we set $m^2 = 0$ in our theory, we obtain the Lee-Yang model studied in Ref. [8], where the exponents η and δ already appear; β, γ, and ν are given for the first time in (15), (16), and (18).

In contrast to the Lee-Yang model, our $i g \phi^3$ theory has a physical interpretation as a $\mathcal{PT}$-symmetric Euclidean quantum field theory. Moreover, to show that our theory undergoes a second-order phase transition at $m^2 = 0$ (that is, to show the renormalizability of the theory) we have had to investigate its behavior in the neighborhood of $m^2 = 0$; it was not sufficient to study the theory at $m^2 = 0$. We emphasize that the physical symmetry of the theory (that is, the $\mathcal{PT}$ symmetry) allows for the presence of the ϕ^2 operator in addition to the linear and cubic terms.

Using RG techniques, we have studied the theory beyond the mean-field approximation and have calculated the critical exponents for $d = 6 - \epsilon$ dimensions up to O(ϵ). In studying the critical behavior of the $\mathcal{PT}$-symmetric $i g \phi^3$ quantum field theory, we have shown that the phase transition is associated with the existence of a nontrivial solution of the gap equation at a critical value m_c^2 of m^2. We conclude that one can view the Lee-Yang model considered in Ref. [8] as the critical theory of the $\mathcal{PT}$-symmetric $i g \phi^3$ model.

Compared with the conventional ϕ^3 model, the $\mathcal{PT}$-symmetric theory exhibits new and interesting properties. In particular, it has a higher degree of predictive power because its critical behavior is governed by one parameter less than in the ϕ^3 theory. We have shown that this crucial difference is related to the different stability properties of the two theories. Thus, from our work it appears that the renormalization properties of the $\mathcal{PT}$-symmetric $i g \phi^3$ model, when compared with those of the conventional ϕ^3 theory, are quite remarkable and encouraging for further studies and future applications.

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[6] The relevance of these terms is due to the fact that they are the only parameters governing the behavior of the different physical quantities near the phase transition. This point is further clarified in Sec. III with the computation of the critical exponents.

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