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Kinetic theory with Berry curvature from quantum field theories

Dam Thanh Son[1] and Naoki Yamamoto[2,3]

[1]*Enrico Fermi Institute, University of Chicago, 5640 South Ellis Avenue, Chicago, Illinois 60637, USA*
[2]*Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan*
[3]*Maryland Center for Fundamental Physics, Department of Physics, University of Maryland, College Park, Maryland 20742-4111, USA*

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A kinetic theory can be modified to incorporate triangle anomalies and the chiral magnetic effect by taking into account the Berry curvature flux through the Fermi surface. We show how such a kinetic theory can be derived from underlying quantum field theories. Using the new kinetic theory, we also compute the parity-odd correlation function that is found to be identical to the result in the perturbation theory in the next-to-leading order hard dense loop approximation.

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I. INTRODUCTION

Kinetic theory [1] has wide applications in condensed matter physics, nuclear physics, astrophysics, and cosmology. There is, however, a key deficiency in the conventional relativistic kinetic framework: it misses the effect of triangle anomalies [2,3]—an important feature of relativistic quantum field theories. Recently it has been shown in Ref. [4][1] that a kinetic theory for Fermi liquids can be modified to include such anomalous effects by taking into account the Berry phase and Berry curvature [12]—the notions extensively studied and widely applied in condensed matter physics [13]. It was shown that not only the form of the transport equation but also the definition of the particle number current must be modified when the Berry curvature has a nonzero flux through the Fermi surface. A consequence of this modification is the generation of parity-violating and dissipationless current in the presence of magnetic field called the chiral magnetic effect [14–17]. This had been previously found in the perturbation theory [14,17] and the gauge/gravity duality [18,19] and was incorporated in the framework of hydrodynamics [20] (see also Refs. [21,22] for more recent developments). The chiral magnetic effect may have been experimentally observed in relativistic heavy ion collisions [17,23] and is potentially observable in Weyl semimetals which possess band-touching points [24–26].

On the other hand, one should be able to derive the kinetic theories from the underlying quantum field theories by following the standard procedure: starting from the equations of motion for the two-point function $\langle\psi(x)\psi^\dagger(y)\rangle$ and performing a derivative expansion for its gauge-covariant Wigner transform, one arrives at the Vlasov equation (see, e.g., Ref. [27] for a review). So far, Berry curvature corrections to the relativistic kinetic theory have been ignored in the field theoretic derivation. Also, microscopic origin of the modification to the particle number current is not yet clear.

In this paper, we microscopically derive the kinetic theory with Berry curvature corrections from underlying quantum field theories.[2] For concreteness, we consider the system of relativistic chiral fermions at finite chemical potential μ (which is known to have a nonzero Berry curvature flux). Our starting point is the high density effective theory [30,31] that describes the physics near the Fermi surface of chiral fermions. In this effective theory, one decomposes two-component chiral fermions into single-component particles ψ_+ with positive energy $E = |\mathbf{p}| - \mu$ and antiparticles ψ_- with negative energy $E = -|\mathbf{p}| - \mu$. Then one usually concentrates on the former with $E \sim 0$ for $|\mathbf{p}| \sim \mu$, while neglecting the latter with $E \sim -2\mu$; picking up only ψ_+ degrees of freedom leads to the conventional Vlasov equation. As we shall demonstrate in this paper, however, if one carefully integrates out ψ_- degrees of freedom, Berry curvature corrections emerge in the kinetic theory from the mixing between ψ_+ and ψ_- (or ψ_- and ψ_-). The modification to Liouville's theorem on the phase space known in the condensed matter literature [32,33] and the modification to the current found in Ref. [4] can be naturally understood from this deliberate integrating out procedure [see Eqs. (71) and (76)]. Apparently, the essential ingredient in this field theoretic argument to lead to Berry curvature corrections is a Fermi surface of chiral fermions.

We also compute the parity-violating correlation function using the kinetic theory with Berry curvature corrections. In the case of the conventional Vlasov equation, it is known that the parity-even correlation function computed in the kinetic theory coincides with the one in the perturbation theory under the hard dense loop approximation [27,34]. In this paper, we will see that the parity-odd correlation function derived from the new kinetic theory

[1]See Refs. [5–9] for further investigations and applications. See also Refs. [10,11] for different approaches to derive kinetic equations with triangle anomalies without referring to the Berry curvature.

[2]See Refs. [28,29] for related attempts.

is equivalent to the result in the perturbation theory beyond the leading-order hard dense loop approximation.

The paper is organized as follows. In Sec. II, we review the kinetic theory with Berry curvature corrections. We also derive a new relation for the spin magnetic moment of quasiparticles in Fermi liquids. In Sec. III, we derive the new kinetic theory starting from quantum field theories. In Sec. IV, we compute the parity-violating correlation functions using both the new kinetic theory and perturbation theory and confirm their agreement. Section V is devoted to our conclusions.

Throughout the paper, we consider sufficiently low temperature regime $T \ll \mu$ where the Fermi surface is well defined. We also concentrate on the collisionless limit of the kinetic theory.

II. KINETIC THEORY WITH BERRY CURVATURE

In this section, we review the kinetic theory in the presence of the Berry curvature which exhibits triangle anomalies [4] (see also Refs. [7,9]) and provide the proper definitions of particle number density and current. We also derive the dispersion relation of quasiparticles according to the constraints of Lorentz invariance.

A. Berry curvature and Poisson brackets

We first consider a single chiral fermion expressed by the two-component spinor $u_{\mathbf{p}}$ satisfying the Weyl equation

$$(\boldsymbol{\sigma} \cdot \mathbf{p}) u_{\mathbf{p}} = \pm |\mathbf{p}| u_{\mathbf{p}}, \tag{1}$$

where the signs $+$ and $-$ correspond to right-handed and left-handed fermions, respectively. The two-component spinor described above has a nonzero Berry connection defined by [12]

$$i\boldsymbol{\mathcal{A}}_{\mathbf{p}} \equiv u_{\mathbf{p}}^\dagger \boldsymbol{\nabla}_{\mathbf{p}} u_{\mathbf{p}}, \tag{2}$$

and a nonzero Berry curvature,

$$\boldsymbol{\Omega}_{\mathbf{p}} \equiv \boldsymbol{\nabla}_{\mathbf{p}} \times \boldsymbol{\mathcal{A}}_{\mathbf{p}} = \pm \frac{\hat{\mathbf{p}}}{2|\mathbf{p}|^2}, \tag{3}$$

where $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$ is a unit vector. Equations (2) and (3) can be regarded as the fictitious vector potential and magnetic field in the momentum space. This fictitious magnetic field can be associated with the one from a "magnetic monopole" with the charge $\pm 1/2$ put in the center of the momentum space. As a result, the motion of chiral fermions is affected by the Berry curvature in the momentum space, in addition to the usual electromagnetic fields in the coordinate space. In particular, the effects of the Berry curvature work oppositely between right-handed and left-handed chiral fermions.

Let us now consider the action of a single quasiparticle in the presence of the electromagnetic fields and Berry curvature [32,33],

$$S = \int dt [p^i \dot{x}^i + A^i(x)\dot{x}^i - \mathcal{A}^i(p)\dot{p}^i - \epsilon_{\mathbf{p}}(x) - A^0(x)]. \tag{4}$$

Note that the quasiparticle energy $\epsilon_{\mathbf{p}}$ is a function of x in general; indeed, chiral fermions have the magnetic moment at finite chemical potential μ and their energy depends on the magnetic field $\mathbf{B}(x)$ [see Eq. (43) below]. The action (4) can be summarized in the following form by combining space x and momentum p into a set of variables ξ^a $(a = 1, \ldots, 6)$,

$$S = \int dt [-\omega_a(\xi)\dot{\xi}^a - H(\xi)], \tag{5}$$

where $H(\xi) = \epsilon_{\mathbf{p}} + A_0$ is the Hamiltonian.

The equations of motion of the action (5) read

$$\omega_{ab}\dot{\xi}^b = -\partial_a H, \tag{6}$$

where $\omega_{ab} = \partial_a \omega_b - \partial_b \omega_a$ and $\partial_a \equiv \partial/\partial\xi^a$. This equation can be rewritten as

$$\dot{\xi}^a = -\omega^{ab}\partial_b H, \tag{7}$$

where $\omega^{ab} \equiv (\omega^{-1})^{ab}$ is the inverse matrix of ω_{ab}. Here we assume the existence of the inverse matrix, i.e., $\omega \equiv \det \omega_{ab} \neq 0$. Equation (7) can be interpreted as

$$\dot{\xi}^a = \{H, \xi^a\} = -\{\xi^a, \xi^b\}\frac{\partial H}{\partial \xi^b}, \tag{8}$$

once we define the Poisson brackets as

$$\{\xi_a, \xi_b\} = \omega^{ab}. \tag{9}$$

The explicit forms of the Poisson brackets for the action (5) read [33]

$$\begin{aligned} \{p_i, p_j\} &= -\frac{\epsilon_{ijk}B_k}{1 + \mathbf{B}\cdot\boldsymbol{\Omega}}, \\ \{x_i, x_j\} &= \frac{\epsilon_{ijk}\Omega_k}{1 + \mathbf{B}\cdot\boldsymbol{\Omega}}, \\ \{p_i, x_j\} &= \frac{\delta_{ij} + \Omega_i B_j}{1 + \mathbf{B}\cdot\boldsymbol{\Omega}}, \end{aligned} \tag{10}$$

where $B^i = \epsilon^{ijk}\partial A^k/\partial x^j$.

These Poisson brackets should be compared with the usual ones in the absence of the Berry curvature and electromagnetic fields

$$\{p_i, p_j\} = 0, \qquad \{x_i, x_j\} = 0, \qquad \{p_i, x_j\} = \delta_{ij}, \tag{11}$$

whose invariant phase space is $d\mathbf{p}d\mathbf{x}/(2\pi)^3$. As a consequence of the modifications to the Poisson brackets above, the invariant phase space is modified to [32]

$$d\Gamma = \sqrt{\omega} d\xi = (1 + \mathbf{B}\cdot\boldsymbol{\Omega}_{\mathbf{p}})\frac{d\mathbf{p}d\mathbf{x}}{(2\pi)^3}, \tag{12}$$

where $\omega \equiv \det \omega_{ab}$.

B. Kinetic theory with Berry curvature and triangle anomalies

Let us construct the collisionless kinetic theory incorporating the effects of the Berry curvature. If collisions between particles are negligible, each particle constitutes a closed subsystem. According to Liouville's theorem, which states that a volume element in the phase space does not change during its time evolution, the one-particle distribution function $n(\xi)$ would obey $dn/dt = 0$. However, the invariant phase space is modified as Eq. (12) due to the Berry curvature, and the probability of finding a particle in the phase space is $\sqrt{\omega} n(\xi) d\xi$. As a result, we instead use the modified distribution function $\rho(\xi) = \sqrt{\omega} n(\xi)$ that obeys the equation $d\rho/dt = 0$, or equivalently,

$$\dot{\rho} + \partial_a(\dot{\xi}^a \rho) = 0. \tag{13}$$

Using Eq. (7), this reduces to

$$\dot{n}_{\mathbf{p}} - \omega^{ab}\partial_b H \partial_a n_{\mathbf{p}} = 0. \tag{14}$$

Setting $H = \epsilon_{\mathbf{p}} + A_0$, we can explicitly write down the kinetic equation (see also Refs. [7,9,33])

$$\begin{aligned} \dot{n}_{\mathbf{p}} + \frac{1}{1 + \mathbf{B}\cdot\mathbf{\Omega}_{\mathbf{p}}}\bigg[&(\tilde{\mathbf{E}} + \tilde{\mathbf{v}} \times \mathbf{B} + (\tilde{\mathbf{E}}\cdot\mathbf{B})\mathbf{\Omega}_{\mathbf{p}}) \cdot \frac{\partial n_{\mathbf{p}}}{\partial \mathbf{p}} \\ &+ (\tilde{\mathbf{v}} + \tilde{\mathbf{E}} \times \mathbf{\Omega}_{\mathbf{p}} + (\tilde{\mathbf{v}}\cdot\mathbf{\Omega}_{\mathbf{p}})\mathbf{B}) \cdot \frac{\partial n_{\mathbf{p}}}{\partial \mathbf{x}}\bigg] = 0, \end{aligned} \tag{15}$$

where $\tilde{\mathbf{v}} = \partial\epsilon_{\mathbf{p}}/\partial\mathbf{p}$ and $\tilde{\mathbf{E}} = \mathbf{E} - \partial\epsilon_{\mathbf{p}}/\partial\mathbf{x}$. This is a general low-energy effective theory in the presence of Berry curvature corrections that describes the evolution of $n_{\mathbf{p}}$. Note that $\tilde{\mathbf{v}}$ is different from the unit vector $\hat{\mathbf{p}}$ when the quasiparticle energy $\epsilon_{\mathbf{p}}$ has the contribution from the magnetic moment [see Eq. (43) below]. If we turn off the Berry curvature and ignore the $\mathbf{x}$ dependence of $\epsilon_{\mathbf{p}}$ (i.e., $\mathbf{\Omega}_{\mathbf{p}} = 0$ and $\partial\epsilon_{\mathbf{p}}/\partial\mathbf{x} = 0$), this reduces to the usual Vlasov equation.

We now define the particle number density

$$n = \int \frac{d^3p}{(2\pi)^3}(1 + \mathbf{B}\cdot\mathbf{\Omega}_{\mathbf{p}})n_{\mathbf{p}}, \tag{16}$$

where the invariant phase space is modified according to Eq. (12). Multiplying the kinetic equation (15) by $\sqrt{\omega}$, performing the integral over momentum $\mathbf{p}$, and using Maxwell equations $\nabla\cdot\mathbf{B} = 0$ and $\partial_t\mathbf{B} + \nabla\times\mathbf{E} = 0$, we obtain the following identity [4]:

$$\partial_t n + \nabla\cdot\mathbf{j} = -\int\frac{d^3p}{(2\pi)^3}\left(\mathbf{\Omega}_{\mathbf{p}}\cdot\frac{\partial n_{\mathbf{p}}}{\partial\mathbf{p}}\right)\mathbf{E}\cdot\mathbf{B}, \tag{17}$$

where

$$\begin{aligned} \mathbf{j} = -\int\frac{d^3p}{(2\pi)^3}\bigg[&\epsilon_{\mathbf{p}}\frac{\partial n_{\mathbf{p}}}{\partial\mathbf{p}} + \left(\mathbf{\Omega}_{\mathbf{p}}\cdot\frac{\partial n_{\mathbf{p}}}{\partial\mathbf{p}}\right)\epsilon_{\mathbf{p}}\mathbf{B} \\ &+ \epsilon_{\mathbf{p}}\mathbf{\Omega}_{\mathbf{p}}\times\frac{\partial n_{\mathbf{p}}}{\partial\mathbf{x}}\bigg] + \mathbf{E}\times\boldsymbol{\sigma} \end{aligned} \tag{18}$$

is identified with the current and $\boldsymbol{\sigma}$ is defined as

$$\boldsymbol{\sigma} = \int\frac{d^3p}{(2\pi)^3}\mathbf{\Omega}_{\mathbf{p}}n_{\mathbf{p}}. \tag{19}$$

In Eq. (17), we observe that the particle number of chiral fermions is no longer conserved when we turn on both electric and magnetic fields. By integration by parts and using $\nabla_{\mathbf{p}}\cdot\mathbf{\Omega}_{\mathbf{p}} = 0$ around the Fermi surface, $n_{\mathbf{p}} = 1$ deep inside the Fermi surface and $n_{\mathbf{p}} = 0$ far outside the Fermi surface, it can be evaluated as

$$\partial_t n + \nabla\cdot\mathbf{j} = \pm\frac{1}{4\pi^2}\mathbf{E}\cdot\mathbf{B}, \tag{20}$$

for right-handed and left-handed fermions, respectively. This is exactly the equation of triangle anomalies in relativistic quantum field theories [2,3], which holds independently of interactions.

The first term in Eq. (18) is the usual particle number current of the kinetic theory, while the remaining terms are the Berry curvature corrections. The same form of the current can be obtained in the Hamiltonian formalism using the commutation relations postulated in Ref. [4]. The final term in Eq. (18) is the anomalous Hall current, which vanishes for a spherically symmetric distribution function at rest. In this case, the current is

$$\begin{aligned} \mathbf{j} = -\int\frac{d^3p}{(2\pi)^3}\bigg[&\epsilon_{\mathbf{p}}\frac{\partial n_{\mathbf{p}}}{\partial\mathbf{p}} + \left(\mathbf{\Omega}_{\mathbf{p}}\cdot\frac{\partial n_{\mathbf{p}}}{\partial\mathbf{p}}\right)\epsilon_{\mathbf{p}}\mathbf{B} \\ &+ \epsilon_{\mathbf{p}}\mathbf{\Omega}_{\mathbf{p}}\times\frac{\partial n_{\mathbf{p}}}{\partial\mathbf{x}}\bigg]. \end{aligned} \tag{21}$$

At this moment, microscopic origins of the Berry curvature corrections to the particle number density and current in Eqs. (16) and (21) are not so clear. In Sec. III, microscopic meanings of these corrections will be clarified in the field theoretic language.

It should be remarked that there is an ambiguity to define the number current from the continuity equation because $\tilde{\mathbf{j}} = \mathbf{j} + \nabla\times\mathbf{a}$ with any vector $\mathbf{a}$ is also a solution to the continuity equation. In order to fix this ambiguity, we look at the energy and momentum conservations. We define the energy density and the momentum density,

$$\begin{aligned} \epsilon &= \int\frac{d^3p}{(2\pi)^3}(1 + \mathbf{B}\cdot\mathbf{\Omega}_{\mathbf{p}})\epsilon_{\mathbf{p}}n_{\mathbf{p}}, \\ \pi^i &= \int\frac{d^3p}{(2\pi)^3}(1 + \mathbf{B}\cdot\mathbf{\Omega}_{\mathbf{p}})p^i n_{\mathbf{p}}. \end{aligned} \tag{22}$$

Multiplying Eq. (15) by $\epsilon_{\mathbf{p}}\sqrt{\omega}$ and $p^i\sqrt{\omega}$, and performing the integral over momentum $\mathbf{p}$, we have

$$\partial_t \epsilon + \int \frac{d^3p}{(2\pi)^3} \epsilon_{\mathbf{p}} F = 0, \qquad \partial_t \pi^i + \int \frac{d^3p}{(2\pi)^3} p^i F = 0, \tag{23}$$

where F is the piece in the square brackets of Eq. (15). The above equations can be respectively interpreted as the energy and momentum conservation laws

$$\partial_\mu T^{0\mu} = E^i j^i, \qquad \partial_\mu T^{i\mu} = nE^i + \epsilon^{ijk} j^j B^k, \tag{24}$$

where

$$T^{0i} = -\int \frac{d^3p}{(2\pi)^3} \left[(\delta^{ij} + B^i \Omega^j) \frac{\epsilon_{\mathbf{p}}^2}{2} \frac{\partial n_{\mathbf{p}}}{\partial p^j} + \epsilon^{ijk} \frac{\epsilon_{\mathbf{p}}^2}{2} \Omega^j \frac{\partial n_{\mathbf{p}}}{\partial x^k} \right], \tag{25}$$

$$T^{ij} = -\int \frac{d^3p}{(2\pi)^3} p^i \left[\epsilon_{\mathbf{p}} (\delta^{jk} + B^j \Omega^k) \frac{\partial n_{\mathbf{p}}}{\partial p^k} + \epsilon^{jkl} \Omega^k \left(E^l n_{\mathbf{p}} + \epsilon_{\mathbf{p}} \frac{\partial n_{\mathbf{p}}}{\partial x^l} \right) \right] - \delta^{ij} \epsilon, \tag{26}$$

which indicates that j^i is the genuine current.

Alternatively, this ambiguity is avoided if we use the definition of the current [7,9]

$$\mathbf{j} = \int \frac{d^3p}{(2\pi)^3} \sqrt{\omega}\, \dot{\mathbf{x}}. \tag{27}$$

By using Eq. (7), one can actually check that this current is equal to Eq. (18).

In equilibrium where $n_{\mathbf{p}}$ is homogeneous, the first and third terms in the right-hand side of Eq. (21) vanish, while the second term is nonvanishing. Using $\epsilon_{\mathbf{p}} = \mu$ at the Fermi surface with μ the chemical potential, we find

$$\mathbf{j} = -\int \frac{d^3p}{(2\pi)^3} \left(\mathbf{\Omega}_{\mathbf{p}} \cdot \frac{\partial n_{\mathbf{p}}}{\partial \mathbf{p}} \right) \mu \mathbf{B} = \pm \frac{\mu}{4\pi^2} \mathbf{B}. \tag{28}$$

This is the relation of the chiral magnetic effect [14,15,17]: the equilibrium current induced in the direction of the magnetic field for chiral fermions at finite chemical potential μ.

C. Lorentz invariance in Fermi liquids

Here we consider the consequences of Lorentz invariance in a system described by Landau's Fermi liquid theory [35]. The constraint due to Lorentz invariance is that the energy flux is equal to the momentum density, $T^{0i} = \pi^i$. From Eqs. (22) and (25), this condition in the homogeneous system becomes

$$-\int \frac{d^3p}{(2\pi)^3} (\delta^{ij} + B^i \Omega^j_{\mathbf{p}}) \frac{\epsilon_{\mathbf{p}}^2}{2} \frac{\partial n_{\mathbf{p}}}{\partial p^j} = \int \frac{d^3p}{(2\pi)^3} (1 + \mathbf{B} \cdot \mathbf{\Omega}_{\mathbf{p}}) p^i n_{\mathbf{p}}. \tag{29}$$

We vary both sides of Eq. (29) as $n_{\mathbf{p}} = n^0_{\mathbf{p}} + \delta n_{\mathbf{p}}$ and $\epsilon_{\mathbf{p}} = \epsilon^0_{\mathbf{p}} + \delta\epsilon_{\mathbf{p}}$, where

$$\delta\epsilon_{\mathbf{p}} = \int \frac{d^3q}{(2\pi)^3} (1 + \mathbf{B} \cdot \mathbf{\Omega}_{\mathbf{q}}) f(\mathbf{p}, \mathbf{q}) \delta n_{\mathbf{q}}, \tag{30}$$

with $f(\mathbf{p}, \mathbf{q})$ being some function characterizing the interactions among quasiparticles called the Landau parameters. By integration by parts, we have

$$\int \frac{d^3p}{(2\pi)^3} (\delta^{ij} + B^i \Omega^j_{\mathbf{p}}) \left[\frac{1}{2} \frac{\partial (\epsilon^0_{\mathbf{p}})^2}{\partial p^j} \delta n_{\mathbf{p}} - \epsilon^0_{\mathbf{p}} \delta\epsilon_{\mathbf{p}} \frac{\partial n^0_{\mathbf{p}}}{\partial p^j} \right] = \int \frac{d^3p}{(2\pi)^3} (1 + \mathbf{B} \cdot \mathbf{\Omega}_{\mathbf{p}}) p^i \delta n_{\mathbf{p}}. \tag{31}$$

Using Eq. (30) and renaming variables $\mathbf{q} \leftrightarrow \mathbf{p}$, the second part of the left-hand side reduces to

$$-\int \frac{d^3p}{(2\pi)^3} (1 + \mathbf{B} \cdot \mathbf{\Omega}_{\mathbf{p}}) \times \left(\int \frac{d^3q}{(2\pi)^3} (\delta^{ij} + B^i \Omega^j_{\mathbf{q}}) f(\mathbf{p}, \mathbf{q}) \epsilon^0_{\mathbf{q}} \frac{\partial n^0_{\mathbf{q}}}{\partial q^j} \right) \delta n_{\mathbf{p}}. \tag{32}$$

Because $\delta n_{\mathbf{p}}$ is arbitrary, we have the relation

$$(\delta^{ij} + B^i \Omega^j_{\mathbf{p}}) \epsilon^0_{\mathbf{p}} \frac{\partial \epsilon^0_{\mathbf{p}}}{\partial p^j} - (1 + \mathbf{B} \cdot \mathbf{\Omega}_{\mathbf{p}}) \int \frac{d^3q}{(2\pi)^3} (\delta^{ij} + B^i \Omega^j_{\mathbf{q}}) f(\mathbf{p}, \mathbf{q}) \epsilon^0_{\mathbf{q}} \frac{\partial n^0_{\mathbf{q}}}{\partial q^j} = (1 + \mathbf{B} \cdot \mathbf{\Omega}_{\mathbf{p}}) p^i. \tag{33}$$

To proceed, we take an ansatz

$$\epsilon^0_{\mathbf{p}} = v_f (p - p_f) + \mu + \gamma(p) \mathbf{B} \cdot \mathbf{p} \tag{34}$$

to the linear order in $\mathbf{B}$, with v_f and p_f being some constants and γ a scalar function of $p \equiv |\mathbf{p}|$. Note that the Fermi velocity defined by $\partial\epsilon^0_{\mathbf{p}}/\partial p$ is $\mathbf{B}$ dependent. Note also that Landau parameters are functions of $\mathbf{B}$; from the property $f(\mathbf{p}, \mathbf{q}) = f(\mathbf{q}, \mathbf{p})$, Landau parameters are composed of two parts (to the linear order in $\mathbf{B}$),

$$f(\mathbf{p}, \mathbf{q}) = f^A(\mathbf{p}, \mathbf{q}) + \frac{\mathbf{B} \cdot (\hat{\mathbf{p}} + \hat{\mathbf{q}})}{2p_f^2} f^B(\mathbf{p}, \mathbf{q}), \tag{35}$$

where $f^{A,B}(\mathbf{p}, \mathbf{q})$ are independent of $\mathbf{B}$ and can be expanded by the Legendre functions as

$$f^{A,B}(\mathbf{p}, \mathbf{q}) = \sum_{l=0}^{\infty} f_l^{A,B} P_l(\cos\theta), \tag{36}$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{q}$ both taken on the Fermi surface.

We now evaluate both sides of Eq. (33) to the linear order in $\mathbf{B}$. Substituting Eqs. (34) and (35), replacing p by

p_f, and performing the angular integral (note also that $n^0_{\mathbf{p}}$ has the $\mathbf{B}$ dependence), we have

$$\begin{aligned}\mu v_f\left(1+\frac{1}{3}F_1^A\right)\hat{p}^i &+ \left[p_f\left(v_f\gamma+\mu\gamma'-\frac{1}{3}\mu v_f F_1'^A\right)\right.\\ &\left.+\frac{\mu v_f}{2p_f^2}\left(\frac{1}{3}F_1^A+\frac{1}{3}F_1^B+\frac{1}{5}F_2^B\right)\right](\mathbf{B}\cdot\hat{\mathbf{p}})\hat{p}^i\\ &+\mu\left[\frac{v_f}{2p_f^2}\left(1+F_0^A+\frac{1}{3}F_0^B-\frac{1}{15}F_2^B\right)+\gamma\right]B^i\\ =\left(p_f+\frac{\mathbf{B}\cdot\hat{\mathbf{p}}}{2p_f}\right)\hat{p}^i,\end{aligned} \tag{37}$$

where $\gamma\equiv\gamma(p_f)$, $\gamma'\equiv\frac{\partial}{\partial p}\gamma(p_f)$, and we defined

$$\int d\hat{\mathbf{q}}\frac{\partial f^A(\mathbf{p},\mathbf{q})}{\partial q^i}(\mathbf{B}\cdot\hat{\mathbf{q}})\equiv\frac{1}{3}f_1'^A(\mathbf{B}\cdot\hat{\mathbf{p}})\hat{p}^i. \tag{38}$$

In order to satisfy Eq. (37) for any $\mathbf{B}$ and $\hat{\mathbf{p}}$, we must have

$$v_f\left(1+\frac{1}{3}F_1^A\right)=\frac{p_f}{\mu}, \tag{39}$$

$$\gamma=-\frac{v_f}{2p_f^2}\left(1+F_0^A+\frac{1}{3}F_0^B-\frac{1}{15}F_2^B\right), \tag{40}$$

$$p_f\left(v_f\gamma+\mu\gamma'-\frac{1}{3}\mu v_f F_1'^A\right)+\frac{\mu v_f}{2p_f^2}\left(\frac{1}{3}F_1^A+\frac{1}{3}F_1^B+\frac{1}{5}F_2^B\right)=\frac{1}{2p_f}. \tag{41}$$

While Eq. (39) is the relation obtained by Baym and Chin [36], Eq. (40) is a new relation for the anomalous spin magnetic moment. A constraint from the gauge invariance on the anomalous *angular* magnetic moment in Fermi liquids was originally given by Migdal [37] and was studied in detail in Ref. [38]. Here we have provided the new constraint on the anomalous *spin* magnetic moment from the viewpoint of the Berry curvature together with the Lorentz invariance.

In particular, in the noninteracting limit where $f(\mathbf{p},\mathbf{q})$ is turned off and $p_f=\mu$, we obtain

$$v_f=1,\qquad \gamma(\mu)=-\frac{1}{2\mu^2},\qquad \gamma'(\mu)=\frac{1}{\mu^3}. \tag{42}$$

A solution to satisfy these relations is taken as $\gamma(p)=-1/(2p^2)$. In this case, we have

$$\epsilon^0_{\mathbf{p}}=p-\frac{\mathbf{B}\cdot\hat{\mathbf{p}}}{2p}. \tag{43}$$

We shall see in Sec. III [Eqs. (51) and (65)] based on the microscopic quantum field theories, that this is actually the dispersion relation of chiral fermions near the Fermi surface in a magnetic field.

III. FROM QUANTUM FIELD THEORIES TO KINETIC THEORY WITH BERRY CURVATURE

In this section, we derive the kinetic theory constructed in Sec. II B from the microscopic quantum field theories: the kinetic equation (15) and the modified number density (16) and current (18) are reproduced microscopically. The resultant kinetic theory exhibits triangle anomalies and the chiral magnetic effect.

Our procedure is as follows: we first consider the high density effective theory [30,31], which is an effective field theory valid near the Fermi surface. The expansion parameter of the theory is l/μ where l is the residual momentum measured from the Fermi surface. We then derive the kinetic theory by performing the derivative expansion for the equations of motion of the Wigner function defined in the high density effective theory. The expansion parameter here is the slowly varying disturbances ∂_X taken to be much smaller than the chemical potential μ and the gauge field A_μ. Note that our procedure does not rely on the expansion in terms of the coupling constant, and hence, it is applicable even when the interactions are strong as long as the notion of quasiparticles is well defined.

In this and following sections, we consider the theory with right-handed chiral fermions.

A. High density effective theory

We first review the derivation of the high density effective theory [30,31]. We start with the Lagrangian for right-handed fermions,

$$\mathcal{L}=\psi^\dagger(i\not{D}+\mu)\psi, \tag{44}$$

where $\not{D}=\sigma^\mu D_\mu$ with $D_\mu=\partial_\mu+iA_\mu$ and $\sigma^\mu=(1,\boldsymbol{\sigma})$, and A_μ is the external background field.

In order to focus on the particles near the Fermi surface, we decompose the energy and momentum of a particle (or a hole) near the Fermi surface as $p^0=\mu+l^0$ and $\mathbf{p}=\mu\mathbf{v}+\mathbf{l}$ with $l^0,|\mathbf{l}|\ll\mu$, where $\mathbf{v}$ is a unit vector which specifies a direction to a point on the Fermi surface. The momentum can be shifted by $\mu\mathbf{v}$ by performing a Fourier transformation

$$\psi(x)=\sum_v e^{i\mu\mathbf{v}\cdot\mathbf{x}}\psi_v(x), \tag{45}$$

where summation is taken over $\mathbf{v}$. The matrix $\boldsymbol{\sigma}\cdot\mathbf{v}$ in the momentum space can be diagonalized by using the projectors as

$$\psi_{\pm v}=P_\pm(\mathbf{v})\psi_v,\qquad P_\pm(\mathbf{v})=\frac{1\pm\boldsymbol{\sigma}\cdot\mathbf{v}}{2}. \tag{46}$$

In the absence of the external electromagnetic field, $\psi_\pm$ satisfy the eigenvalue equations, $(\boldsymbol{\sigma}\cdot\mathbf{v})\psi_\pm=\pm\psi_\pm$.

In terms of $\psi_{\pm v}$, Eq. (44) reduces to

$$\psi^\dagger(i\not{D}+\mu)\psi = \sum_v [\psi^\dagger_{+v} iv\cdot D\psi_{+v} + \psi^\dagger_{-v}(2\mu + i\bar{v}\cdot D)\psi_{-v} + (\psi^\dagger_{+v} i\not{D}_\perp \psi_{-v} + \text{H.c.})], \tag{47}$$

where $v^\mu = (1, \mathbf{v})$, $\bar{v}^\mu = (1, -\mathbf{v})$, $\sigma^\mu_\perp = (0, \boldsymbol{\sigma} - \mathbf{v}(\mathbf{v}\cdot\boldsymbol{\sigma}))$, $D^\mu_\perp = (0, \mathbf{D} - \mathbf{v}(\mathbf{v}\cdot\mathbf{D}))$, and $\not{D}_\perp = \sigma^\mu_\perp D_\mu = \sigma^\mu D^\perp_\mu$. By integrating out ψ_{-v} using the equation of motion for ψ_{-v},

$$(2\mu + i\bar{v}\cdot D)\psi_{-v} + i\not{D}_\perp\psi_{+v} = 0, \tag{48}$$

the effective Lagrangian in terms of ψ_+ can be written down order by order in $1/\mu$,

$$\mathcal{L}_{\text{EFT}} = \sum_n \mathcal{L}^{(n)}, \qquad \mathcal{L}^{(n)} = \sum_v \psi^\dagger_{+v}\mathcal{D}^{(n)}\psi_{+v}, \tag{49}$$

where $\mathcal{L}^{(n)}$ denotes the effective Lagrangian of the nth order in $1/\mu$ ($n = 0, 1, 2, \ldots$). The explicit expressions for $\mathcal{D}^{(n)}$ ($n = 0, 1, 2$) are

$$\mathcal{D}^{(0)} = iv\cdot D, \qquad \mathcal{D}^{(1)} = \frac{\not{D}^2_\perp}{2\mu}, \qquad \mathcal{D}^{(2)} = -\frac{i}{4\mu^2}\not{D}_\perp(\bar{v}\cdot D)\not{D}_\perp. \tag{50}$$

Using $\not{D}^2_\perp = D^2_\perp + \mathbf{B}\cdot\boldsymbol{\sigma}$, $\psi^\dagger_{+v}\boldsymbol{\sigma}\psi_{+v} = \psi^\dagger_{+v}\mathbf{v}\psi_{+v}$, and $p = \mu + l_\parallel + l^2_\perp/(2\mu) + O(1/\mu^2)$, the dispersion relation near the Fermi surface reads

$$\epsilon_{\mathbf{p}} = p - \frac{\mathbf{B}\cdot\mathbf{v}}{2\mu} + O\left(\frac{1}{\mu^2}\right). \tag{51}$$

This indeed agrees with Eq. (43) up to $O(1/\mu^2)$. [The agreement will be shown to the order of $O(1/\mu^2)$ in the next subsection.] The second term in Eq. (51) originates from the magnetic moment of chiral fermions at finite μ. This is similar in structure to the Pauli equation that describes the magnetic moment of massive Dirac fermions in the vacuum; the Pauli equation can be obtained by expanding the massive Dirac equation in $1/m$, where m is the mass of Dirac fermions.

B. Kinetic theory via derivative expansion

We construct the kinetic theory based on the effective theory (49) on a patch indicated by a unit vector $\mathbf{v}$. We consider the Dirac operator to the second order in $1/\mu$, $\mathcal{D} = \mathcal{D}^{(0)} + \mathcal{D}^{(1)} + \mathcal{D}^{(2)}$, and introduce a two-point function for ψ_v,

$$G_v(x, y) = \langle\psi_v(x)\psi^\dagger_v(y)\rangle. \tag{52}$$

The function $G_v(x, y)$ satisfies equations of motion together with projection conditions,

$$\mathcal{D}_x G_v(x, y) = 0, \qquad G_v(x, y)\mathcal{D}^\dagger_y = 0, \tag{53}$$

$$P_-(\mathbf{v})G_v(x, y) = 0, \qquad G_v(x, y)P_-(\mathbf{v}) = 0. \tag{54}$$

In thermal equilibrium where the system is homogeneous, G_v depends only on the relative coordinate $s = x - y$. We are interested in the small deviation from the equilibrium where G_v depends both on x and y. It is thus useful to change the coordinates from (x, y) to the center-of-mass and relative coordinates (X, s) defined by

$$x = X + \frac{s}{2}, \qquad y = X - \frac{s}{2}, \tag{55}$$

and consider the derivative expansion with respect to X.

In order to derive a quantum analogue of the classical distribution function, we perform the Wigner transformation

$$G_v(X, l) = \int d^4 s e^{il\cdot s} G_v\left(X + \frac{s}{2}, X - \frac{s}{2}\right), \tag{56}$$

where l^μ is the residual four-momentum. Unlike $G_v(x, y)$, however, $G_v(X, l)$ is not gauge covariant. We will thus use the gauge-covariant definition instead,

$$\tilde{G}_v(X, l) = \int d^4 s e^{il\cdot s} U\left(X, X + \frac{s}{2}\right) \times G_v\left(X + \frac{s}{2}, X - \frac{s}{2}\right) U\left(X - \frac{s}{2}, X\right), \tag{57}$$

where

$$U(x, y) = P\exp\left[-i\int_\gamma dx^\mu A_\mu(x)\right], \tag{58}$$

is the Wilson line. The symbol P is the path ordering along the path γ from x to y. For simplicity, $\tilde{G}_v(X, l)$ is renamed $G_v(X, l)$ in what follows.

In constructing the kinetic theory, we consider the slowly varying disturbances and perform a gradient expansion in terms of ∂_X. To this end, we assume the following counting scheme: $\partial_X = O(\epsilon_1)$, $\partial_s = O(\epsilon_2)$, $A_\mu = O(\epsilon_3)$, and $F_{\mu\nu} = O(\epsilon_1\epsilon_3)$. Here ∂_X and ∂_s are the derivatives with respect to X and s, and ϵ_i ($i = 1, 2, 3$) are independent expansion parameters which satisfy the conditions, $\epsilon_1 \ll \epsilon_{2,3} \ll 1$. The condition $\epsilon_i \ll 1$ ($i = 1, 2, 3$) is necessary for the derivative expansion in the high density effective theory while $\epsilon_1 \ll \epsilon_{2,3}$ is necessary for the derivative expansion in the kinetic theory. In order to take into account triangle anomalies, we consider the kinetic theory to $O(\epsilon_1^2\epsilon_3^2)$.

For simplicity, in this subsection we consider the homogeneous system where $\partial^X_\rho F_{\mu\nu} = 0$ and $\partial^X_i n_{\mathbf{p}} = 0$ ($n_{\mathbf{p}}$ is the distribution function which will be defined below). This is sufficient for our purpose to understand the microscopic origin of the Berry curvature corrections. The generalization to the inhomogeneous case should be straightforward.

Consider the equations, $\mathcal{D}_x G_v(x,y) \pm G_v(x,y)\mathcal{D}_y^\dagger = 0$. We expand them in terms of ∂_X to the second order and perform the Wigner transformation. The Wigner transform of the equations can be written down order by order,

$$I_\pm^{(n)} \equiv \int \frac{d^4 s}{(2\pi)^4} e^{il\cdot s}(\mathcal{D}_x^{(n)} G_v \pm G_v \mathcal{D}_y^{(n)\dagger}), \tag{59}$$

for $n = 0, 1, 2$. The expansion of the gauge field A_μ in ∂_X reads

$$A_\mu(x) \approx A_\mu(X) + \frac{1}{2}(s\cdot\partial^X)A_\mu(X) + \frac{1}{8}(s\cdot\partial^X)^2 A_\mu(X). \tag{60}$$

Combined with the contributions from the Wilson loop in Eq. (57), all the terms involving the gauge field are expressed by the gauge-invariant field strength $F_{\mu\nu}$ at the end. Then the third term in the right-hand side of Eq. (60) will be irrelevant eventually when $\partial_\rho^X F_{\mu\nu} = 0$. Renaming the kinetic residual momentum $\tilde{l}^\mu = l^\mu - A^\mu$ as l^μ, we have

$$I_+^{(0)} = 2(l_0 - l_\parallel)G_v, \qquad I_-^{(0)} = iv^\mu(g_{\mu 0}\partial_t - F_{\mu\nu}\partial_l^\nu)G_v, \tag{61a}$$

$$I_+^{(1)} = \frac{1}{\mu}(-l_\perp^2 + \mathbf{B}\cdot\mathbf{v})G_v, \qquad I_-^{(1)} = \frac{i}{\mu} l_\perp^\mu (g_{\mu 0}\partial_t - F_{\mu\nu}\partial_l^\nu)G_v, \tag{61b}$$

$$I_+^{(2)} = \frac{1}{\mu^2}[l_\parallel(l_\perp^2 - \mathbf{B}\cdot\mathbf{v}) + \mathbf{B}\cdot\mathbf{l}_\perp + (\mathbf{E}\times\mathbf{l})\cdot\mathbf{v}]G_v,$$

$$I_-^{(2)} = -\frac{i}{2\mu^2}\left[2l_\parallel l_\perp^\mu - \frac{1}{2}(l_\perp^2 - \mathbf{B}\cdot\mathbf{v})\bar{v}^\mu - \epsilon^{ijk}v^k\bar{v}_\sigma F^{i\sigma}g^{\mu j}\right](g_{\mu 0}\partial_t - F_{\mu\nu}\partial_l^\nu)G_v, \tag{61c}$$

where ∂_l is the derivative with respect to the residual momentum l, and $l^0 = l_\parallel + O(l^2/\mu)$ is used in Eq. (61c).

From the equation $\mathcal{D}_x G_v(x,y) + G_v(x,y)\mathcal{D}_y^\dagger = 0$, we obtain the on-shell condition. Considering the projection conditions (54), G_v can be written as

$$G_v = 2\pi P_+(\mathbf{v})\delta\left(l_0 - l_\parallel - \frac{l_\perp^2 - \mathbf{B}\cdot\mathbf{v}}{2\mu} + \frac{l_\parallel(l_\perp^2 - \mathbf{B}\cdot\mathbf{v}) + \mathbf{B}\cdot\mathbf{l}_\perp}{2\mu^2}\right)n_l, \tag{62}$$

where $n_l(X)$ is the distribution function expressed by the residual momentum l. Recalling

$$p = \mu + l_\parallel + \frac{l_\perp^2}{2\mu} - \frac{l_\perp^2 l_\parallel}{2\mu^2} + O\left(\frac{1}{\mu^3}\right), \tag{63}$$

$$\frac{\mathbf{B}\cdot\hat{\mathbf{p}}}{2p} = \frac{\mathbf{B}\cdot\mathbf{v}}{2\mu} + \frac{\mathbf{B}\cdot\mathbf{l}_\perp - l_\parallel\mathbf{B}\cdot\mathbf{v}}{2\mu^2} + O\left(\frac{1}{\mu^3}\right), \tag{64}$$

the dispersion relation in the delta function of Eq. (62) is equivalent to the condition $p^0 = \epsilon_\mathbf{p}$ with

$$\epsilon_\mathbf{p} = p - \frac{\mathbf{B}\cdot\hat{\mathbf{p}}}{2p}, \tag{65}$$

which indeed coincides with Eq. (43). Accordingly, the distribution function n_l can be replaced by $n_\mathbf{p}$ in terms of the original momentum $\mathbf{p}$.

On the other hand, from the equation $\mathcal{D}_x G_v(x,y) - G_v(x,y)\mathcal{D}_y^\dagger = 0$, we obtain the transport equation for $n_l(X)$. Using

$$\hat{\mathbf{p}} = \mathbf{v} + \frac{\mathbf{l}_\perp}{\mu} - \frac{l_\perp^2\mathbf{v} + 2l_\parallel\mathbf{l}_\perp}{2\mu^2} + O\left(\frac{l^3}{\mu^3}\right), \tag{66}$$

the transport equation can also be expressed by the original momentum $\mathbf{p}$. We end up with the transport equation

$$\left(1 + \frac{\mathbf{B}\cdot\hat{\mathbf{p}}}{2\mu^2}\right)\dot{n}_\mathbf{p} + \left[(\mathbf{E} + \hat{\mathbf{p}}\times\mathbf{B}) + (\mathbf{E}\cdot\mathbf{B})\frac{\hat{\mathbf{p}}}{2\mu^2}\right]\cdot\frac{\partial n_\mathbf{p}}{\partial\mathbf{p}} = 0. \tag{67}$$

Equation (67) indeed agrees with the homogeneous limit of the kinetic theory (15) to $O(\epsilon_1^2\epsilon_3^2)$ if we identify $\mathbf{\Omega}_\mathbf{p} = \hat{\mathbf{p}}/(2p^2)$ at $p = \mu$. Therefore, we have found that the Berry curvature corrections in the kinetic theory emerge as the higher-order corrections in $1/\mu$ to the usual Vlasov equation. Once the kinetic equation (67) is obtained, the relation of triangle anomalies (20) follows, as we have seen in Sec. II B.

C. Particle number density and current

Here we consider the particle number density and current for right-handed fermions without reference to the kinetic theory derived above. To see the chiral magnetic effect, we need to consider the number current to $O(\epsilon_1\epsilon_3)$ in the high density effective theory. Our discussion in this subsection is applicable to inhomogeneous electromagnetic fields.

By definition, the number density of right-handed fermions consists of four parts,

$$n = \langle\psi_{+v}^\dagger\psi_{+v}\rangle + \langle\psi_{+v}^\dagger\psi_{-v}\rangle + \langle\psi_{-v}^\dagger\psi_{+v}\rangle + \langle\psi_{-v}^\dagger\psi_{-v}\rangle \equiv n_{++} + n_{+-} + n_{-+} + n_{--}, \tag{68}$$

where n_{++} is given by

$$n_{++} = \int \frac{d^4p}{(2\pi)^4} \mathrm{tr} G_v = \int \frac{d^3p}{(2\pi)^3} n_l, \tag{69}$$

and $n_{+-} = n_{-+} = 0$ because of the property of projectors, $P_+(\mathbf{v})P_-(\mathbf{v}) = 0$. In order to express n_{--} in terms of ψ_{+v}, we use Eq. (48) which relates ψ_{-v} to ψ_{+v}. Then n_{--} is given by

$$n_{--} = \frac{1}{4\mu^2} \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}(\not{D}_\perp G_v \not{D}_\perp^\dagger) = \int \frac{d^3p}{(2\pi)^3} \frac{\mathbf{B}\cdot\mathbf{v}}{2\mu^2} n_l. \tag{70}$$

Using the momentum $\mathbf{p}$, we have in total

$$n = \int \frac{d^3p}{(2\pi)^3}\left(1 + \frac{\mathbf{B}\cdot\hat{\mathbf{p}}}{2\mu^2}\right) n_{\mathbf{p}}. \tag{71}$$

This is the number density including the Berry curvature correction, Eq. (16).

Similarly, the number current of right-handed fermions is decomposed as

$$\begin{aligned}\mathbf{j}_R &= \langle \psi_{+v}^\dagger \boldsymbol{\sigma} \psi_{+v}\rangle + \langle \psi_{+v}^\dagger \boldsymbol{\sigma} \psi_{-v}\rangle \\ &\quad + \langle \psi_{-v}^\dagger \boldsymbol{\sigma} \psi_{+v}\rangle + \langle \psi_{-v}^\dagger \boldsymbol{\sigma} \psi_{-v}\rangle \\ &\equiv \mathbf{j}_{++} + \mathbf{j}_{+-} + \mathbf{j}_{-+} + \mathbf{j}_{--},\end{aligned} \tag{72}$$

where $\mathbf{j}_{++}$ is given by

$$\mathbf{j}_{++} = \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}(\boldsymbol{\sigma} G_v) = \int \frac{d^3p}{(2\pi)^3} \mathbf{v} n_l. \tag{73}$$

Using Eq. (48), summation of $\mathbf{j}_{+-}$ and $\mathbf{j}_{-+}$ can be written as

$$\begin{aligned} j^i_{+-} + j^i_{-+} = \frac{i}{2\mu} \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}[(D^i_{x\perp} + i\epsilon^{ijk} v^k D^j_x) G_v \\ - G_v (D^{\dagger i}_{y\perp} - i\epsilon^{ijk} v^k D^{\dagger j}_y)], \end{aligned} \tag{74}$$

while $\mathbf{j}_{--} \sim 1/\mu^2$ is higher order in $1/\mu$ and is negligible to the order under consideration. One can then rewrite Eq. (74) by changing the coordinates from (x, y) to the center-of-mass and relative coordinates (X, s) and performing the Wigner transformation in a gauge-covariant way. One finds

$$\begin{aligned} j^i_{+-} + j^i_{-+} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\mu}\Big[-\epsilon^{ijk} v^j \frac{\partial n_l}{\partial X^k} \\ + (\mathbf{B}\cdot\mathbf{v}) \frac{\partial n_l}{\partial l^i} - B^i \left(\mathbf{v}\cdot\frac{\partial n_l}{\partial \mathbf{l}}\right)\Big]. \end{aligned} \tag{75}$$

Putting them together and writing in terms of the momentum $\mathbf{p}$, we arrive at

$$j^i_R = \int \frac{d^3p}{(2\pi)^3}\left[\frac{\partial \epsilon_{\mathbf{p}}}{\partial p^i} n_{\mathbf{p}} - B^i \left(\frac{\hat{\mathbf{p}}}{2\mu}\cdot\frac{\partial n_{\mathbf{p}}}{\partial \mathbf{p}}\right) - \epsilon^{ijk} \frac{\hat{p}^j}{2\mu}\frac{\partial n_{\mathbf{p}}}{\partial X^k}\right], \tag{76}$$

where we used the dispersion relation (51). This is the same form as the current (21), including the chiral magnetic effect and the inhomogeneous term; the Berry curvature corrections in Eq. (21) microscopically originate from the mixing between ψ_+ and ψ_-.

IV. CORRELATION FUNCTIONS

In this section we compute the one-loop polarization tensor in the presence of chiral fermions at finite chemical potential μ at zero temperature using the perturbation theory and the kinetic theory constructed in Sec. II B. We confirm that both calculations give the same result, not only to the leading order but also to the next-to-leading order in $1/\mu$. In particular, we find the kinetic theory with Berry curvature corrections reproduces the parity-odd polarization tensor beyond the leading-order hard dense loop approximation in the perturbation theory.

A. Perturbation theory

We first compute the parity-even and parity-odd one-loop polarization tensors in the perturbation theory under the hard dense loop approximation.[3] The time-ordered Dirac fermion propagator at finite chemical potential μ and zero temperature is given by

$$\begin{aligned} S(x,y) &= \langle T\psi(x)\bar{\psi}(y)\rangle \\ &= \int \frac{d^3p}{(2\pi)^3}\frac{\gamma\cdot p}{2p_0}[\theta(x^0-y^0)(\alpha_p e^{-ip(x-y)} + \bar{\beta}_p e^{ip(x-y)}) \\ &\quad - \theta(y^0-x^0)(\beta_p e^{-ip(x-y)} + \bar{\alpha}_p e^{ip(x-y)})], \end{aligned} \tag{77}$$

where $\alpha_p = \theta(p_0 - \mu)$, $\beta_p = \theta(\mu - p_0)$, $\bar{\alpha}_p = 1$, and $\bar{\beta}_p = 0$.

The one-loop polarization tensor in the presence of chiral fermions is then

$$\Pi^{\mu\nu}(x-y) = \frac{1}{2}\mathrm{tr}[(1+\gamma_5)\gamma^\mu S(x,y)\gamma^\nu S(y,x)]. \tag{78}$$

In the momentum space, it is given by (see Ref. [40] for the case of Dirac fermions)

[3] The parity-odd hard dense loop action was previously derived in Ref. [39].

$$\begin{aligned}\Pi^{\mu\nu}(k) = &\frac{1}{2}\int\frac{d^3q}{(2\pi)^3}\frac{1}{2p_0}\frac{1}{2q_0}\bigg[T^{\mu\nu}(p,q)\bigg(\frac{\alpha_p\beta_q}{p_0-q_0-k_0-i\epsilon}-\frac{\alpha_q\beta_p}{p_0-q_0-k_0+i\eta}\bigg)\\ &+T^{\mu\nu}(p,\bar{q})\bigg(\frac{\alpha_p\bar{\alpha}_q}{p_0+q_0-k_0-i\epsilon}-\frac{\beta_p\bar{\beta}_q}{p_0+q_0-k_0+i\eta}\bigg)+T^{\mu\nu}(\bar{p},q)\bigg(\frac{\bar{\alpha}_p\alpha_q}{p_0+q_0+k_0-i\eta}-\frac{\bar{\beta}_p\beta_q}{p_0+q_0+k_0+i\epsilon}\bigg)\\ &+T^{\mu\nu}(\bar{p},\bar{q})\bigg(\frac{\bar{\alpha}_p\bar{\beta}_q}{p_0-q_0+k_0-i\eta}-\frac{\bar{\beta}_p\bar{\alpha}_q}{p_0-q_0+k_0+i\epsilon}\bigg)\bigg],\end{aligned}\tag{79}$$

where $p=(p_0,\mathbf{p})$, $\bar{p}=(p_0,-\mathbf{p})$, $q=(q_0,\mathbf{q})$, $\bar{q}=(q_0,-\mathbf{p})$, $p_0=|\mathbf{p}|$, $q_0=|\mathbf{q}|$, $\mathbf{p}=\mathbf{q}+\mathbf{k}$, and $T^{\mu\nu}(p,q)=\mathrm{tr}[(1+\gamma_5)\gamma^\mu\not{p}\gamma^\nu\not{q}]$. An infinitesimal quantity η takes $\eta=\epsilon$ for the time-ordered function $\Pi_T^{\mu\nu}$, and $\eta=-\epsilon$ for the retarded function $\Pi_R^{\mu\nu}$.

Substituting explicit expressions of the distribution functions, we find the μ-dependent part of the retarded function (hereafter we suppress the index "R"),

$$\begin{aligned}\Pi_\pm^{\mu\nu}(k) = &\frac{1}{2}\int\frac{d^3q}{(2\pi)^3}\frac{1}{2p_0}\frac{1}{2q_0}\\ &\times\bigg[[\theta(\mu-q_0)-\theta(\mu-p_0)]\frac{T_\pm^{\mu\nu}(p,q)}{p_0-q_0-k_0-i\epsilon}\\ &-\theta(\mu-p_0)\frac{T_\pm^{\mu\nu}(p,\bar{q})}{p_0+q_0-k_0-i\epsilon}\\ &-\theta(\mu-q_0)\frac{T_\pm^{\mu\nu}(\bar{p},q)}{p_0+q_0+k_0+i\epsilon}\bigg],\end{aligned}\tag{80}$$

where $\Pi_\pm$ and $T_\pm^{\mu\nu}$ denote parity-even and -odd parts, $T_+^{\mu\nu}(p,q)=\mathrm{tr}(\gamma^\mu\not{p}\gamma^\nu\not{q})$ and $T_-^{\mu\nu}(p,q)=\mathrm{tr}(\gamma_5\gamma^\mu\not{p}\gamma^\nu\not{q})$. The parity-even part $\Pi_+^{\mu\nu}$ is the leading contribution in the hard dense loop approximation, while the parity-odd part $\Pi_-^{\mu\nu}$ is suppressed compared with $\Pi_+^{\mu\nu}$ by a factor of $|\mathbf{k}|/\mu$.

For completeness, let us first recall the computation of $\Pi_+^{\mu\nu}$. Under the hard dense loop approximation (where $k_0, |\mathbf{k}|\ll\mu$), we end up with [34]

$$\begin{aligned}\Pi_+^{\mu\nu}(k) &= -\frac{1}{2}\int\frac{d^3q}{(2\pi)^3}\delta(\mu-|\mathbf{q}|)\\ &\quad\times\bigg(\bar{v}^\mu v^\nu+v^\mu v^\nu-2\omega\frac{v^\mu v^\nu}{v\cdot k+i\epsilon}\bigg)\\ &= -\frac{\mu^2}{2\pi^2}\bigg[\delta^{\mu 0}\delta^{\nu 0}-\omega\int\frac{d\mathbf{v}}{4\pi}\frac{v^\mu v^\nu}{v\cdot k+i\epsilon}\bigg],\end{aligned}\tag{81}$$

where $v=(1,\mathbf{v})$, $\bar{v}=(1,-\mathbf{v})$ with $\mathbf{v}=\mathbf{q}/|\mathbf{q}|$.

Let us now turn to the parity-odd part $\Pi_-^{\mu\nu}(k)$. Using $\mathrm{tr}(\gamma_5\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta)=-4i\epsilon^{\mu\alpha\nu\beta}$, we have

$$\begin{aligned}\Pi_-^{\mu\nu}(k) = &-\frac{1}{2}\int\frac{d^3q}{(2\pi)^3}\frac{1}{4|\mathbf{q}|^2}\\ &\times\bigg[\mathbf{k}\cdot\mathbf{v}\delta(\mu-|\mathbf{q}|)\frac{4i\epsilon^{\mu\nu\alpha\beta}k_\alpha v_\beta|\mathbf{q}|}{k\cdot v+i\epsilon}\\ &-\theta(\mu-|\mathbf{q}|)2i\epsilon^{\mu\nu\alpha\beta}(k_\alpha\bar{v}_\beta+\bar{k}_\alpha v_\beta)\bigg].\end{aligned}\tag{82}$$

The second term is vanishing while the first term remains nonzero only when $(\alpha,\beta)=(0,k)$ and $(k,0)$. Collecting both contributions, we obtain the expression

$$\Pi_-^{ij}(k)=\frac{\mu}{4\pi^2}\bigg(i\epsilon^{ijk}k^k+i\epsilon^{ijk}\omega\int\frac{d\mathbf{v}}{4\pi}\frac{\omega v^k-k^k}{k\cdot v+i\epsilon}\bigg),\tag{83}$$

where i, j, k denote the spatial indices [$\Pi_-^{\mu\nu}(k)$ is vanishing otherwise]. Performing the angular integration, we finally arrive at

$$\Pi_-^{ij}(k)=\frac{\mu}{4\pi^2}i\epsilon^{ijk}k^k\bigg(1-\frac{\omega^2}{|\mathbf{k}|^2}\bigg)[1-\omega L(k)],\tag{84}$$

where

$$L(k)=\frac{1}{2|\mathbf{k}|}\ln\frac{\omega+|\mathbf{k}|}{\omega-|\mathbf{k}|}.\tag{85}$$

Equation (84) reduces to a simple form in the static or long wavelength limit:

$$\Pi_-^{ij}(k)=\begin{cases}\frac{\mu}{4\pi^2}i\epsilon^{ijk}k^k & (\omega\ll|\mathbf{k}|)\\ \frac{\mu}{12\pi^2}i\epsilon^{ijk}k^k & (\omega\gg|\mathbf{k}|).\end{cases}\tag{86}$$

That the latter $\lim_{|\mathbf{k}|/\omega\to 0}\Pi_-^{ij}(k)$ is smaller than the former $\lim_{\omega/|\mathbf{k}|\to 0}\Pi_-^{ij}(k)$ by a factor of 3 is consistent with the result of the "chiral magnetic conductivity" in Ref. [41].

B. Kinetic theory with Berry curvature

We now compute the same retarded correlation function from the kinetic theory (15) constructed in Sec. II B through the linear response theory

$$j^\mu(x)=\int d^4y\,\Pi^{\mu\nu}(x-y)A_\nu(y),\tag{87}$$

or in the momentum space

$$j^\mu(k)=\Pi^{\mu\nu}(k)A_\nu(k).\tag{88}$$

Here we are interested in the current induced by a linear-order deviation of the gauge field A_μ. For definiteness, we set up the following power counting scheme: $A_\mu=O(\epsilon)$ and $\partial_x=O(\delta)$, where ϵ and δ are small and independent expansion parameters. Under this counting scheme, we compute the deviation of the distribution function $\delta n_\mathbf{p}$ and the current j^μ to $O(\epsilon\delta)$.

Remembering that $\partial n_\mathbf{p}/\partial\mathbf{x}$ or $\partial\epsilon_\mathbf{p}/\partial\mathbf{x}$ can be nonvanishing at least when $\partial n_\mathbf{p}/\partial\mathbf{x}=O(\epsilon)$ or $\partial\epsilon_\mathbf{p}/\partial\mathbf{x}=O(\epsilon)$ in

Eq. (15), it is sufficient to consider the following kinetic equation of order $O(\epsilon\delta^2)$ [or $n_{\mathbf{p}}$ of order $O(\epsilon\delta)$],

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}\right)n_{\mathbf{p}}+\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}-\frac{\partial \epsilon_{\mathbf{p}}}{\partial \mathbf{x}}\right)\cdot\frac{\partial n_{\mathbf{p}}}{\partial \mathbf{p}}=0, \quad (89)$$

in which the $\mathbf{v}\times\mathbf{B}$ term does not contribute since $(\mathbf{v}\times\mathbf{B})\cdot\mathbf{v}=0$. The distribution function $n_{\mathbf{p}}$ is decomposed as

$$n_{\mathbf{p}}=n_{\mathbf{p}}^{(0)}+n_{\mathbf{p}}^{(\epsilon)}+n_{\mathbf{p}}^{(\epsilon\delta)}+\cdots, \quad (90)$$

where

$$n_{\mathbf{p}}^{(0)}=\theta(\mu-\epsilon_{\mathbf{p}})\simeq\theta(\mu-|\mathbf{p}|)+\frac{\mathbf{B}\cdot\mathbf{v}}{2\mu}\delta(\mu-|\mathbf{p}|), \quad (91)$$

which follows from the dispersion relation (43). Note that the second term in Eq. (91) is also $O(\epsilon\delta)$, but this is separated from $n_{\mathbf{p}}^{(\epsilon\delta)}$ in our definition. In the calculations below, we have to add both contributions at the same order of $O(\epsilon\delta)$ [see Eq. (93) below].

The kinetic equation can be written down at each order as

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}\right)n_{\mathbf{p}}^{(\epsilon)}=\mathbf{E}\cdot\mathbf{v}\delta(\mu-|\mathbf{p}|), \quad (92)$$

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}\right)\left(n_{\mathbf{p}}^{(\epsilon\delta)}+\frac{\mathbf{B}\cdot\mathbf{v}}{2\mu}\delta(\mu-|\mathbf{p}|)\right)-\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}\left(\frac{\mathbf{B}\cdot\mathbf{v}}{2\mu}\right)\delta(\mu-|\mathbf{p}|)=0. \quad (93)$$

The second equation is further simplified to

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}}\right)n_{\mathbf{p}}^{(\epsilon\delta)}=-\frac{\partial}{\partial t}\left(\frac{\mathbf{B}\cdot\mathbf{v}}{2\mu}\right)\delta(\mu-|\mathbf{p}|). \quad (94)$$

Using the method of characteristics, we can solve these equations; the operator in the left-hand sides of equations $v\cdot\partial_x$ is the time derivative along the characteristic $\mathbf{v}=d\mathbf{x}/dt$. The solutions are given by

$$n_{\mathbf{p}}^{(\epsilon)}=\delta(\mu-|\mathbf{p}|)\int_0^\infty d\tau e^{-\eta\tau}\mathbf{v}\cdot\mathbf{E}(x-v\tau), \quad (95)$$

$$n_{\mathbf{p}}^{(\epsilon\delta)}=-\delta(\mu-|\mathbf{p}|)\int_0^\infty d\tau e^{-\eta\tau}\frac{1}{2\mu}\mathbf{v}\cdot\dot{\mathbf{B}}(x-v\tau), \quad (96)$$

where η is a small positive parameter which ensures $\mathbf{E}(t\to-\infty,\mathbf{x})\to 0$ and $\mathbf{B}(t\to-\infty,\mathbf{x})\to 0$.

Now let us compute the current defined in Eq. (18). The current can be written down at the orders of $O(\epsilon)$ and $O(\epsilon\delta)$, respectively, as

$$j^{\mu(\epsilon)}(x)=\int\frac{d^3p}{(2\pi)^3}v^\mu n_{\mathbf{p}}^{(\epsilon)}, \quad (97)$$

$$j^{i(\epsilon\delta)}(x)=\int\frac{d^3p}{(2\pi)^3}\left[v^i n_{\mathbf{p}}^{(\epsilon\delta)}+\frac{B^i}{2\mu}\delta(\mu-|\mathbf{p}|)-\epsilon^{ijk}\frac{v^j}{2\mu}\frac{\partial n_{\mathbf{p}}^{(\epsilon)}}{\partial x^k}\right], \quad (98)$$

where $v^\mu=(1,\mathbf{v})$. The zeroth component of the four current of order $O(\epsilon\delta)$, i.e., the number density $n^{(\epsilon\delta)}(x)$, is found to vanish after the angular integration.

First consider the four current $j^{\mu(\epsilon)}$. Substituting Eq. (95) into Eq. (97), the current reads

$$j^{\mu(\epsilon)}(x)=\int\frac{d^3p}{(2\pi)^3}v^\mu\delta(\mu-|\mathbf{p}|)\int_0^\infty d\tau e^{-\eta\tau}\mathbf{v}\cdot\mathbf{E}(x-v\tau). \quad (99)$$

Using the useful formula

$$\int d^4x e^{ik\cdot x}\int_0^\infty d\tau e^{-\eta\tau}f(x-v\tau)=\frac{if(k)}{v\cdot k+i\eta}, \quad (100)$$

and the linear response theory (88), we obtain the retarded parity-even polarization tensor [42]

$$\Pi_+^{\mu\nu}(k)=-\frac{\mu^2}{2\pi^2}\left[\delta^{\mu 0}\delta^{\nu 0}-\omega\int\frac{d\mathbf{v}}{4\pi}\frac{v^\mu v^\nu}{v\cdot k+i\epsilon}\right], \quad (101)$$

which agrees with Eq. (81) derived from the perturbation theory.

We then turn to the subleading three current $j^{i(\epsilon\delta)}$. Substituting Eqs. (95) and (96) into (98), and using the formula (100), the current reads

$$j^{i(\epsilon\delta)}(k)=\int\frac{d^3p}{(2\pi)^3}\delta(\mu-|\mathbf{p}|)\left[-\frac{i\epsilon^{klm}v^iv^k\omega k^lA^m}{2\mu(v\cdot k+i\epsilon)}+\frac{i\epsilon^{ijk}k^jA^k}{2\mu}+\frac{i\epsilon^{ikl}v^kk^l[\omega(\mathbf{v}\cdot\mathbf{A})-(\mathbf{v}\cdot\mathbf{k})A_0]}{2\mu(v\cdot k+i\epsilon)}\right], \quad (102)$$

among which the A_0 term in the brackets vanishes after the angular integration. Using the linear response theory (88), we obtain the parity-odd polarization tensor

$$\Pi_-^{ij}(k)=\frac{\mu}{4\pi^2}\left[i\epsilon^{ijk}k^k+i\omega\int\frac{d\mathbf{v}}{4\pi}\frac{(\epsilon^{jkl}v^i-\epsilon^{ikl}v^j)v^kk^l}{v\cdot k+i\epsilon}\right]. \quad (103)$$

After the angular integration, this reduces to the form

$$\Pi_-^{ij}(k)=\frac{\mu}{4\pi^2}i\epsilon^{ijk}k^k\left(1-\frac{\omega^2}{|\mathbf{k}|^2}\right)[1-\omega L(k)], \quad (104)$$

where $L(k)$ is defined in Eq. (85). This is equivalent to Eq. (84) derived from the perturbation theory, which confirms that the physics in the next-to-leading order hard dense loop approximation can be described by the kinetic theory with Berry curvature corrections. Note that contributions of the inhomogeneous term in the current (21) and the magnetic moment in Eq. (43) are necessary for the matching of the correlation functions.

V. CONCLUSION

In this paper, we have shown a way to bridge between quantum field theories and the kinetic theory with Berry curvature corrections that exhibits triangle anomalies and the chiral magnetic effect. The field theoretic procedure to derive such a kinetic theory developed in this paper can, in principle, be generalized to higher order in gauge fields and/or derivatives. We have also computed the parity-odd correlation function using this kinetic theory, which was found to agree with the perturbative result beyond the leading-order hard dense loop approximation.

It should be remarked that our derivation of the kinetic theory from underlying quantum field theories is limited to low temperature region $T \ll \mu$ where the Fermi surface is well defined and the high density effective theory is applicable; a generalization to higher temperature regime would be desirable. We also remark that our formulation based on the Berry curvature is not manifestly Lorentz covariant by construction. It might be possible to formulate the kinetic theory in a Lorentz covariant way similarly to the usual Vlasov equation. Without referring to the Berry curvatures, such a solution of the kinetic equation in the hydrodynamic regime was obtained in Ref. [11], which also reproduces the chiral vortical effect [20,43–46].

The inclusion of collision terms in the kinetic theory (15) is straightforward. This gives the modified Boltzmann equation taking into account the anomalous effects. We hope that our work motivates numerical applications of the new kinetic equation (15) with or without collisions in various systems such as hot and dense quark matter, neutrino gas, the early Universe at large lepton chemical potential, and doped Weyl semimetals.

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