

Hedgehog ansatz and its generalization for self-gravitating SkyrmonsFabrizio Canfora^{1,2,*} and Hideki Maeda^{1,†}¹*Centro de Estudios Científicos (CECs), Casilla 1469, Valdivia, Chile*²*Universidad Andres Bello, Avenida Republica 440, Santiago, Chile*

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The hedgehog ansatz for spherically symmetric spacetimes in self-gravitating nonlinear sigma models and Skyrme models is revisited and its generalization for nonspherically symmetric spacetimes is proposed. The key idea behind our construction is that, even if the matter fields depend on the Killing coordinates in a nontrivial way, the corresponding energy-momentum tensor can still be compatible with spacetime symmetries. Our generalized hedgehog ansatz reduces the Skyrme equations to coupled differential equations for two scalar fields together with several constraint equations between them. Some particular field configurations satisfying those constraints are presented in several physically important spacetimes, including stationary and axisymmetric spacetimes. Incidentally, new exact solutions are obtained under the standard hedgehog ansatz, one of which represents a global monopole inside a black hole with the Skyrme effect.

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I. INTRODUCTION

Nonlinear sigma models are among the most important nonlinear field theories due to their many applications, ranging from quantum field theory to statistical mechanics. (See Ref. [1] for a detailed review.) Examples are quantum magnetism, the quantum hall effect, mesons, and string theory. It has also been successfully applied as an effective field theory to super fluid ³He. A sigma model in D -dimensional spacetime $(\mathcal{M}^D, g_{\mu\nu})$ is defined by a set of n real scalar fields $Y^i (i = 1, \dots, n)$ which take on values in a flat manifold, called the target manifold. It is called a nonlinear sigma model if the target manifold is nonflat, the Lagrangian density of which is given by

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} G_{ij} (\nabla_\mu Y^i) (\nabla_\nu Y^j),$$

where $G_{ij}(Y)$ is the metric on the target manifold.

Actually, nonlinear sigma models do not admit any static soliton solutions in $3 + 1$ dimensions, which is shown by a scaling argument. (See Ref. [1] for instance). For this reason, Skyrme introduced his famous term, which allows the existence of static solutions with finite energy called *Skyrmions* [2]. Remarkably, excitations around Skyrme solitons may represent Fermionic degrees of freedom suitable to describe nucleons. The Skyrme model is therefore one of the most important nonlinear field theories in nuclear and high-energy physics.

However, it is difficult to obtain exact solutions in nonlinear sigma models or Skyrme models, due to their highly nonlinear characters. Therefore one often adopts a certain ansatz to make the field equations more tractable. Under such ansätze, the results can be interpreted more clearly

and the simplified equations are also useful for numerical studies. Among others, the best known one for Skyrme models is the hedgehog ansatz for spherically symmetric systems, which reduces the field equations to a single scalar equation.

Because of its great advantage, the hedgehog ansatz has been also adopted in self-gravitating Skyrme models. The Einstein-Skyrme system has attracted considerable attention since Droz, Heusler, and Straumann numerically found spherically symmetric black hole solutions with a nontrivial Skyrme field, namely a Skyrme hair [3]. (Before them, Luckock and Moss numerically constructed such hairy configurations in the Schwarzschild background spacetime [4]). This was the first counterexample to the black hole no-hair conjecture, and it is stable against spherical linear perturbations [5]. Regular particle-like configurations [6] and dynamical properties of the system have also been investigated numerically [7].

In this decade, not only spherically symmetric configurations [8] but also more realistic black holes or regular configurations with axisymmetry have been studied in the Einstein-Skyrme system [9]. In those studies, one mostly relies on numerical analyses because of the complexity of the system. (See Ref. [10] for a review.) Under these circumstances, it would be helpful for both analytic and numerical investigations to provide a new useful ansatz which also makes the field equations much simpler and tractable. In the present paper, we generalize the hedgehog ansatz in an applicable way not only to spherically symmetric spacetimes but also to other symmetric spacetimes.

In the following section, we review the Einstein-Skyrme system in the presence of a cosmological constant. In Sec. III, we revisit the standard hedgehog ansatz in spherically symmetric spacetimes and obtain a new exact black hole solution. In Sec. IV, we present the generalized hedgehog ansatz and derive the basic equations. We also present

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some particular configurations which are compatible with a variety of symmetric spacetimes. Concluding remarks and future prospects are summarized in Sec. V. Our basic notation follows Ref. [11]. The conventions for curvature tensors are $[\nabla_\rho, \nabla_\sigma]V^\mu = \mathcal{R}^\mu{}_{\nu\rho\sigma}V^\nu$ and $\mathcal{R}_{\mu\nu} = \mathcal{R}^\rho{}_{\mu\rho\nu}$. The signature of the Minkowski spacetime is $(-, +, +, +)$ and Greek indices run over all spacetime indices. We adopt the units such that $c = \hbar = 1$.

II. THE EINSTEIN-SKYRME SYSTEM

In the present paper, we study the Einstein-Skyrme system with a cosmological constant Λ in four dimensions. A Skyrme field is described by a nonlinear sigma model with additional terms and can be conveniently written in terms of an SU(2) group-valued scalar field U . The dynamical sector in the total action of this system is written as [10]

$$S = S_G + S_{\text{Skyrme}}, \quad (2.1)$$

where the gravitational action S_G and the Skyrme action S_{Skyrme} are given by

$$S_G = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (\mathcal{R} - 2\Lambda), \quad (2.2)$$

$$S_{\text{Skyrme}} = \int d^4x \sqrt{-g} \text{Tr} \left(\frac{F_\pi^2}{16} R^\mu R_\mu + \frac{1}{32e^2} F_{\mu\nu} F^{\mu\nu} \right). \quad (2.3)$$

Here R_μ and $F_{\mu\nu}$ are defined by

$$R_\mu := U^{-1} \nabla_\mu U, \quad (2.4)$$

$$F_{\mu\nu} := [R_\mu, R_\nu], \quad (2.5)$$

while G is the Newton constant and the parameters F_π and e are fixed by comparison with experimental data. The first and the second terms in S_{Skyrme} , respectively, represent a nonlinear sigma model and the Skyrme term. Skyrme fields satisfy the dominant energy condition and the strong energy condition [12].

The Skyrme Lagrangian describes the low-energy nonlinear interactions of pions or baryons. The deep observation of Skyrme [2] was that if one adds a suitable term (the Skyrme quartic term) to the Lagrangian of a nonlinear sigma model the resulting action describes not only the low-energy interactions of pions but also of baryons. This observation is remarkable in that it was the first example of a purely bosonic Lagrangian that is able to describe both bosons and fermions.

For convenience, defining $K := F_\pi^2/4$ and $\lambda := 4/(e^2 F_\pi^2)$, we write the Skyrme action as

$$S_{\text{Skyrme}} = \frac{K}{2} \int d^4x \sqrt{-g} \text{Tr} \left(\frac{1}{2} R^\mu R_\mu + \frac{\lambda}{16} F_{\mu\nu} F^{\mu\nu} \right). \quad (2.6)$$

The resulting Einstein equations are

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad (2.7)$$

where $G_{\mu\nu}$ is the Einstein tensor and

$$T_{\mu\nu} = -\frac{K}{2} \text{Tr} \left[\left(R_\mu R_\nu - \frac{1}{2} g_{\mu\nu} R^\alpha R_\alpha \right) + \frac{\lambda}{4} \left(g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) \right]. \quad (2.8)$$

The Skyrme equations are written as

$$\nabla^\mu R_\mu + \frac{\lambda}{4} \nabla^\mu [R^\nu, F_{\mu\nu}] = 0. \quad (2.9)$$

Here R_μ is expressed as

$$R_\mu = R_\mu^i t_i \quad (2.10)$$

in the basis of the SU(2) generators t^i (where the Latin index $i = 1, 2, 3$ corresponds to the group index, which is raised and lowered with the flat metric δ_{ij}), which satisfy

$$t^i t^j = -\delta^{ij} \mathbf{1} - \varepsilon^{ijk} t^k, \quad (2.11)$$

where $\mathbf{1}$ is the identity 2×2 matrix and ε_{ijk} and ε^{ijk} are the totally antisymmetric Levi-Civita symbols with $\varepsilon_{123} = \varepsilon^{123} = 1$. t^i are related to the Pauli matrices as $t_i = -i\sigma_i$. Using the identity

$$\varepsilon^{ijk} \varepsilon^{mnk} = \delta_i^m \delta_j^n - \delta_i^n \delta_j^m, \quad (2.12)$$

we obtain the commutation relation of R_μ ,

$$[R_\mu, R_\nu]^i = -2\varepsilon_{ijk} R_\mu^j R_\nu^k. \quad (2.13)$$

Hereafter we will use the following standard parametrization of the SU(2)-valued scalar $U(x^\mu)$:

$$U(x^\mu) = Y^0 \mathbf{1} + Y^i t_i, \quad U^{-1}(x^\mu) = Y^0 \mathbf{1} - Y^i t_i, \quad (2.14)$$

where $Y^0 = Y^0(x^\mu)$ and $Y^i = Y^i(x^\mu)$ satisfy

$$(Y^0)^2 + Y^i Y_i = 1. \quad (2.15)$$

From the definition (2.4), R_μ^k is written as

$$R_\mu^k = \varepsilon^{ijk} Y_i \nabla_\mu Y_j + Y^0 \nabla_\mu Y^k - Y^k \nabla_\mu Y^0. \quad (2.16)$$

Using the quadratic combination

$$\mathcal{S}_{\mu\nu} := \delta_{ij} R_\mu^i R_\nu^j = G_{ij}(Y) \nabla_\mu Y^i \nabla_\nu Y^j, \quad (2.17)$$

where

$$G_{ij} := \delta_{ij} + \frac{Y_i Y_j}{1 - Y^k Y_k}, \quad (2.18)$$

we obtain

$$\text{Tr}(R_\mu R_\nu) = -2\mathcal{S}_{\mu\nu}, \quad (2.19)$$

$$\text{Tr}(F_{\mu\alpha} F_\nu{}^\alpha) = 8\mathcal{S}_{\mu\alpha} \mathcal{S}_\nu{}^\alpha - 8\mathcal{S}_{\mu\nu} \mathcal{S}. \quad (2.20)$$

Using these results, we can write the Skyrme action (2.6) only with Y^i as

$$S_{\text{Skyrme}} = -K \int d^4x \sqrt{-g} \left[\frac{1}{2} G_{ij} (\nabla_\mu Y^i) (\nabla^\mu Y^j) + \frac{\lambda}{4} \{ (G_{ij} (\nabla_\mu Y^i) (\nabla^\mu Y^j))^2 - G_{ij} (\nabla_\mu Y^i) (\nabla_\nu Y^j) G_{kl} (\nabla^\mu Y^k) (\nabla^\nu Y^l) \} \right], \quad (2.21)$$

while the energy-momentum tensor (2.8) is expressed as

$$T_{\mu\nu} = K \left[\mathcal{S}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{S} + \lambda \left\{ \mathcal{S} \mathcal{S}_{\mu\nu} - \mathcal{S}_{\mu\alpha} \mathcal{S}_\nu^\alpha - \frac{1}{4} g_{\mu\nu} (\mathcal{S}^2 - \mathcal{S}_{\alpha\beta} \mathcal{S}^{\alpha\beta}) \right\} \right]. \quad (2.22)$$

It is seen that the contribution of the Skyrme term to the energy-momentum tensor is traceless (in four dimensions) and shares some characteristics of a Yang-Mills field.

Here G_{ij} is the metric corresponding to the group (target) manifold, which is S^3 in the present case. It is worth noting here that if one considers a configuration with vanishing Y^0 , then G_{ij} becomes δ_{ij} .

III. HEDGEHOG ANSATZ FOR SPHERICALLY SYMMETRIC SPACETIMES

A. Tensorial formulation of the basic equations

In this section we will derive the field equations under the standard hedgehog ansatz for spherically symmetric spacetimes. The most general metric with spherical symmetry may be written as

$$ds^2 = g_{AB}(y) dy^A dy^B + r(y)^2 \gamma_{ab}(z) dz^a dz^b, \quad (3.1)$$

where g_{AB} ($A, B = 0, 1$) and y^A are the metric and coordinates on a two-dimensional Lorentzian manifold M^2 , respectively, while γ_{ab} ($a, b = 2, 3$) and z^a are the metric and coordinates on a two-dimensional unit sphere S^2 , respectively. We are going to derive the basic equations under the hedgehog ansatz in a covariant form on (M^2, g_{AB}) .

In terms of the group element U , the usual hedgehog ansatz reads

$$U = \mathbf{1} \cos \alpha + \hat{n}^i t_i \sin \alpha, \quad U^{-1} = \mathbf{1} \cos \alpha - \hat{n}^i t_i \sin \alpha, \quad (3.2)$$

where $\hat{n}^i = \hat{n}^i(z)$ ($i = 1, 2, 3$) are given by

$$\hat{n}^1 = \sin \theta \cos \phi, \quad \hat{n}^2 = \sin \theta \sin \phi, \quad \hat{n}^3 = \cos \theta, \quad (3.3)$$

and $\alpha = \alpha(y)$. Here we have adopted the coordinates on (S^2, γ_{ab}) such that

$$\gamma_{ab} dz^a dz^b = d\theta^2 + \sin^2 \theta d\phi^2. \quad (3.4)$$

In terms of the variables Y^0 and Y^i , this ansatz corresponds to

$$Y^0 = \cos \alpha, \quad Y^i = \hat{n}^i \sin \alpha. \quad (3.5)$$

\hat{n}^i are normalized as $\delta_{ij} \hat{n}^i \hat{n}^j = 1$ so as to satisfy Eq. (2.15). It is also possible to define the normalized internal vectors \hat{n}^i by

$$\bar{D}^2 \hat{n}^i = -2 \hat{n}^i, \quad (3.6)$$

where \bar{D}_a is the covariant derivative on S^2 and $\bar{D}^2 := \bar{D}_a \bar{D}^a$. Namely, \hat{n}^i are the eigenvectors of the Laplacian operator on S^2 with the eigenvalue -2 . They satisfy $\delta_{ij} (\bar{D}_a \hat{n}^i) (\bar{D}_b \hat{n}^j) = \gamma_{ab}$, which will be used in the following calculations.

Let us derive the expression of the energy-momentum tensor (2.22) in a tensorial way on M^2 . Using Eqs. (2.16) and (3.5), we obtain the following expression of R_μ^k :

$$R_\mu^k dx^\mu = (\hat{n}^k D_A \alpha) dy^A + \left(\sin^2 \alpha \delta^{sk} \varepsilon_{ijs} \hat{n}^i \bar{D}_a \hat{n}^j + \frac{1}{2} \sin(2\alpha) \bar{D}_a \hat{n}^k \right) dz^a, \quad (3.7)$$

where D_A is the covariant derivative on M^2 . Using Eqs. (2.17), (3.5), and (3.7), we obtain

$$\mathcal{S}_{\mu\nu} dx^\mu dx^\nu = (D_A \alpha) (D_B \alpha) dy^A dy^B + \sin^2 \alpha \gamma_{ab} dz^a dz^b, \quad (3.8)$$

and finally derive the energy-momentum tensor (2.22) as

$$T_{\mu\nu} dx^\mu dx^\nu = K \left[(1 + 2\lambda r^{-2} \sin^2 \alpha) \left((D_A \alpha) (D_B \alpha) - \frac{1}{2} g_{AB} (D\alpha)^2 \right) - g_{AB} r^{-2} \sin^2 \alpha \left(1 + \frac{\lambda}{2} r^{-2} \sin^2 \alpha \right) \right] dy^A dy^B - \frac{1}{2} K \left((D\alpha)^2 - \lambda r^{-4} \sin^4 \alpha \right) r^2 \gamma_{ab} dz^a dz^b, \quad (3.9)$$

where $(D\alpha)^2 := g^{AB} (D_A \alpha) (D_B \alpha)$. The Einstein equations are written down with the following expression of the Einstein tensor with the Λ -term:

$$(G_{\mu\nu} + \Lambda g_{\mu\nu}) dx^\mu dx^\nu = \left[-2 \frac{D_A D_B r}{r} + g_{AB} \left(2 \frac{D^2 r}{r} - \frac{1 - (Dr)^2}{r^2} + \Lambda \right) \right] dy^A dy^B + \frac{1}{2} \left(2 \frac{D^2 r}{r} - {}^{(2)}\mathcal{R} + 2\Lambda \right) r^2 \gamma_{ab} dz^a dz^b, \quad (3.10)$$

where ${}^{(2)}\mathcal{R}$ is the Ricci scalar on M^2 and $D^2 := D_A D^A$.

Next we derive the expression of the field Eq. (2.9). Using the formula $\nabla^\mu u_\mu = D^A u_A + r^{-2} \bar{D}^a u_a + 2r^{-1} (D^A r) u_A$, we obtain the divergence of R_μ^k as

$$\nabla^\mu R_\mu^k = \hat{n}^k [D^2 \alpha - r^{-2} \sin(2\alpha) + 2r^{-1} (D^A r) (D_A \alpha)], \quad (3.11)$$

where we have used Eq. (3.6). Hence, the field Eq. (2.9) without the Skyrme term reduce to the following single scalar equation on M^2 :

$$D^2\alpha + 2r^{-1}(D^A r)(D_A\alpha) - r^{-2}\sin(2\alpha) = 0. \quad (3.12)$$

This is a very nontrivial characteristic of the hedgehog ansatz, which reduces a system of coupled nonlinear partial differential equations (2.9) to a single Eq. (3.12). Actually, this still holds even with the Skyrme term, as shown below.

It is straightforward to show that

$$[R^\nu, F_{\mu\nu}] = 4(SR_\mu^k - S^\nu{}_\mu R_\nu^k)t_k. \quad (3.13)$$

Using the two expressions

$$\begin{aligned} SR_\mu^k dx^\mu &= ((D\alpha)^2 + 2r^{-2}\sin^2\alpha)(D_A\alpha)\hat{n}^k dy^A \\ &+ ((D\alpha)^2 + 2r^{-2}\sin^2\alpha) \\ &\times \left(\sin^2\alpha \delta^{sk} \varepsilon_{ijs} \hat{n}^i \bar{D}_a \hat{n}^j + \frac{1}{2} \sin(2\alpha) \bar{D}_a \hat{n}^k \right) dz^a, \end{aligned} \quad (3.14)$$

$$\begin{aligned} S^\nu{}_\mu R_\nu^k dx^\mu &= (D\alpha)^2 (D_A\alpha)\hat{n}^k dy^A + r^{-2}\sin^2\alpha \\ &\times \left(\sin^2\alpha \delta^{sk} \varepsilon_{ijs} \hat{n}^i \bar{D}_a \hat{n}^j + \frac{1}{2} \sin(2\alpha) \bar{D}_a \hat{n}^k \right) dz^a, \end{aligned} \quad (3.15)$$

we obtain

$$\begin{aligned} \nabla^\mu [R^\nu, F_{\mu\nu}]^k &= 4r^{-2} [2(D^2\alpha)\sin^2\alpha + ((D\alpha)^2 - r^{-2}\sin^2\alpha)\sin(2\alpha)] \hat{n}^k, \end{aligned} \quad (3.16)$$

and finally the Skyrme Eq. (2.9) reduce to the following single scalar equation on M^2 :

$$\begin{aligned} 0 &= (1 + 2\lambda r^{-2}\sin^2\alpha)D^2\alpha + 2r^{-1}(D^A r)(D_A\alpha) \\ &- r^{-2}\sin(2\alpha) \left[1 - \lambda((D\alpha)^2 - r^{-2}\sin^2\alpha) \right]. \end{aligned} \quad (3.17)$$

Equations (3.9), (3.10), and (3.17) give a complete set of the basic equations in this system.

B. Exact monopole black hole

The simplest nontrivial solution of the master Eq. (3.17) is $\alpha = \pi/2 + N\pi$, where N is an integer. The energy-momentum tensor (3.9) then becomes

$$\begin{aligned} T_{\mu\nu} dx^\mu dx^\nu &= -K g_{AB} r^{-2} \left(1 + \frac{1}{2} \lambda r^{-2} \right) dy^A dy^B \\ &+ \frac{1}{2} K \lambda r^{-2} \gamma_{ab} dz^a dz^b. \end{aligned} \quad (3.18)$$

It is shown that the most general solution with $\alpha = \pi/2 + N\pi$ and $(Dr)^2 \neq 0$ is given by

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (3.19)$$

$$f(r) := 1 - 8\pi GK - \frac{2GM}{r} + \frac{4\pi GK\lambda}{r^2} - \frac{1}{3}\Lambda r^2. \quad (3.20)$$

This solution with $\lambda = 0$ (without the Skyrme term) was obtained in Ref. [13] and represents a global monopole inside a black hole. In the present solution, there is the Skyrme contribution in the metric function which, at first glance, is similar to the Maxwell term in the Reissner-Nordström solution. However, unlike the Maxwell case, the coefficient of the $1/r^2$ term is *not an integration constant* since it is fixed by the couplings of the theory. (This is similar to the case of the meron black hole [14]). To the best of the authors' knowledge, the above solution has not been mentioned in any literature. The metric (3.19) with $M = \Lambda = \lambda = 0$ is the same as the Barriola-Vilenkin monopole spacetime [15].

It is worth emphasizing that there are also nonspherical exact solutions with $\alpha = \pi/2 + N\pi$ such as the following Taub-NUT-type solution;

$$\begin{aligned} ds^2 &= -F(r)(dt - 2n \cos\theta d\phi)^2 + F(r)^{-1}dr^2 \\ &+ (r^2 + n^2)(d\theta^2 + \sin^2\theta d\phi^2), \end{aligned} \quad (3.21)$$

$$\begin{aligned} F(r) &:= \frac{r}{r^2 + n^2} \left((1 - 8\pi GK - 2\Lambda n^2)r - 2M \right. \\ &\left. + \frac{4\pi GK\lambda + \Lambda n^4 - n^2(1 - 8\pi GK)}{r} - \frac{1}{3}\Lambda r^3 \right) \end{aligned} \quad (3.22)$$

and the (Euclidean) Eguchi-Hanson-type solution;

$$\begin{aligned} ds^2 &= g(r) \frac{r^2}{4} (dt + \cos\theta d\phi)^2 + g(r)^{-1} dr^2 \\ &+ \frac{r^2}{4} (d\theta^2 + \sin^2\theta d\phi^2), \end{aligned} \quad (3.23)$$

$$g(r) := 1 - 8\pi GK - \frac{32\pi GK\lambda}{r^2} - \frac{a}{r^4} - \frac{1}{6}\Lambda r^2, \quad (3.24)$$

where n and a are constants.

We discuss the properties of the spacetime (3.19) with $\Lambda = 0$ for simplicity. Although this solution can represent a black hole, the spacetime is not asymptotically flat but asymptotic to the global monopole spacetime for $K \neq 0$. The location of the Killing horizon is given by $f(r_h) = 0$, which is solved to give

$$r_h = \frac{GM}{1 - 8\pi GK} \left(1 \pm \sqrt{1 - \frac{4\pi K\lambda(1 - 8\pi GK)}{GM^2}} \right). \quad (3.25)$$

The relation between M and r_h is

$$M = \frac{1}{2G} \left((1 - 8\pi GK)r_h + \frac{4\pi GK\lambda}{r_h} \right). \quad (3.26)$$

In addition to $K > 0$ and $\lambda \geq 0$, we also assume $0 < 8\pi GK < 1$ in order to have an outer Killing horizon defined by $df/dr|_{r=r_h} > 0$, which coincides with the black hole event horizon. The location of the outer Killing horizon is given by Eq. (3.25) with the upper sign, and it satisfies

$$\frac{1}{r_h^2} < \frac{1 - 8\pi GK}{4\pi GK\lambda}. \quad (3.27)$$

The temperature of the black hole is given by

$$T = \frac{1}{4\pi} \frac{df}{dr} \Big|_{r=r_h} = \frac{1}{4\pi} \left(\frac{1 - 8\pi GK}{r_h} - \frac{4\pi GK\lambda}{r_h^3} \right), \quad (3.28)$$

while the Wald entropy is

$$S = \frac{1}{4G} A_h = \frac{\pi}{G} r_h^2. \quad (3.29)$$

There is a subtle problem about the global mass of this monopole black hole. Since there is no free parameter except for M , the first law must have the form of $\delta E = T\delta S$ for some global mass E . The parameter M coincides with the ADM mass and satisfies $\delta M = T\delta S$ if and only if $K = 0$. On the other hand, Nucamendi and Sudarsky showed that if the spacetime approaches the metric

$$ds^2 = -g(r)dt^2 + g(r)^{-1}dr^2 + (1 - \alpha)r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (3.30)$$

$$g(r) \simeq 1 - \frac{2G\tilde{M}}{r}, \quad (3.31)$$

\tilde{M} is identified as the global mass in the monopole spacetime [16]. For our monopole black hole spacetime, the Nucamendi-Sudarsky mass is $\tilde{M} = M/(1 - 8\pi GK)^{3/2}$ and—as can be seen directly—it does not satisfy the first law. Instead, by integrating $\delta E = T\delta S$, we obtain the following expression of E :

$$E = \frac{1}{2G} \left((1 - 8\pi GK)r_h + \frac{4\pi GK\lambda}{r_h} \right) + E_0, \quad (3.32)$$

where E_0 is a constant and $M = E - E_0$ is satisfied.

Once the first law is fulfilled, it is possible to discuss the thermodynamical properties of the present black hole with the above energy. The heat capacity C and the free energy F read

$$C = \frac{dE}{dr_h} \Big/ \frac{dT}{dr_h} = \frac{2\pi}{G} r_h^2 \left((1 - 8\pi GK) - \frac{4\pi GK\lambda}{r_h^2} \right) \times \left(-(1 - 8\pi GK) + \frac{12\pi GK\lambda}{r_h^2} \right)^{-1}, \quad (3.33)$$

$$F = E - TS = \frac{1}{4G} \left((1 - 8\pi GK)r_h + \frac{12\pi GK\lambda}{r_h} \right) + E_0. \quad (3.34)$$

Although it is difficult to discuss the global thermodynamical stability due to the fact that we have no *a priori* argument to fix the integration constant E_0 in Eq. (3.32), the local thermodynamical stability can be analyzed. It is seen that $C < 0$ is satisfied for

$$r_h^2 > \frac{12\pi GK\lambda}{1 - 8\pi GK}, \quad (3.35)$$

while $C > 0$ holds for

$$\frac{4\pi GK\lambda}{1 - 8\pi GK} < r_h^2 < \frac{12\pi GK\lambda}{1 - 8\pi GK}. \quad (3.36)$$

This result shows the local thermodynamical stability of a small monopole black hole with the Skyrme term. Without the Skyrme term, we have $C < 0$ and the black hole is always thermodynamically unstable.

IV. GENERALIZED HEDGEHOG ANSATZ

A. The ansatz

In this section, we propose a generalization of the hedgehog ansatz for self-gravitating Skyrme fields and derive the basic equations in a covariant form. We start from the following configuration:

$$Y^0 = \cos \alpha, \quad Y^i = \hat{n}^i \sin \alpha, \quad (4.1)$$

which is the same as the hedgehog ansatz, and then R_μ^k is given by

$$R_\mu^k = \sin^2 \alpha \varepsilon^{ijk} \hat{n}^i (\nabla_\mu \hat{n}^j) + \frac{1}{2} \sin(2\alpha) (\nabla_\mu \hat{n}^k) + \hat{n}^k (\nabla_\mu \alpha). \quad (4.2)$$

We now assume the following form of \hat{n}^i :

$$\hat{n}^1 = \cos \Theta, \quad \hat{n}^2 = \sin \Theta, \quad \hat{n}^3 = 0, \quad (4.3)$$

which satisfy $\delta_{ij} \hat{n}^i \hat{n}^j = 1$ and hence Eq. (2.15). Here α and Θ are scalar functions. Using the above expressions, we obtain $\mathcal{S}_{\mu\nu}$ defined by Eq. (2.17) as

$$\mathcal{S}_{\mu\nu} = (\nabla_\mu \alpha)(\nabla_\nu \alpha) + \sin^2 \alpha (\nabla_\mu \Theta)(\nabla_\nu \Theta), \quad (4.4)$$

and hence

$$S = \sin^2 \alpha (\nabla \Theta)^2 + (\nabla \alpha)^2. \quad (4.5)$$

The energy-momentum tensor for the nonlinear sigma model is then given by

$$T_{\mu\nu} = K \left[(\nabla_\mu \alpha)(\nabla_\nu \alpha) + \sin^2 \alpha (\nabla_\mu \Theta)(\nabla_\nu \Theta) - \frac{1}{2} g_{\mu\nu} ((\nabla \alpha)^2 + \sin^2 \alpha (\nabla \Theta)^2) \right]. \quad (4.6)$$

Next let us see the field equations for the nonlinear sigma model. It is shown that they reduce to a single scalar equation under the following assumptions:

$$\nabla^2 \hat{n}^i = L \hat{n}^i, \quad (4.7)$$

$$(\nabla_\mu \Theta)(\nabla^\mu \alpha) = 0, \quad (4.8)$$

where L is a scalar function. This ansatz for the nonlinear sigma model was first introduced on flat backgrounds in Ref. [17] with a particular choice of Θ . From Eq. (4.3), the condition (4.7) gives $\nabla^2 \Theta = 0$ and $L = -(\nabla \Theta)^2$ and then we obtain

$$\nabla^\mu R_\mu^k = \left((\nabla^2 \alpha) - \frac{1}{2} (\nabla \Theta)^2 \sin(2\alpha) \right) \hat{n}^k. \quad (4.9)$$

In summary, the field equations for the nonlinear sigma model (2.9) (with $\lambda = 0$) have been decomposed into the following equations for Θ and α :

$$\nabla^2 \Theta = 0, \quad (4.10)$$

$$(\nabla^2 \alpha) - \frac{1}{2} (\nabla \Theta)^2 \sin(2\alpha) = 0, \quad (4.11)$$

with a constraint, Eq. (4.8). The corresponding Einstein equations are sourced by the energy-momentum tensor (4.6). We call the set of conditions (4.1), (4.3), and (4.8) the *generalized hedgehog ansatz for nonlinear sigma models*. Unlike the standard hedgehog ansatz, it also works in systems without spherical symmetry, as shown in the following subsections.

At first glance, the simplest nontrivial solution $\alpha = \pi/2 + N\pi$ of the field Eq. (4.11) is very similar to the Einstein-Klein-Gordon system since the energy-momentum tensor (4.6) becomes

$$T_{\mu\nu} = K \left[(\nabla_\mu \Theta)(\nabla_\nu \Theta) - \frac{1}{2} g_{\mu\nu} (\nabla \Theta)^2 \right] \quad (4.12)$$

and Θ is governed by Eq. (4.10). However, as shown in Sec. IV C, the present system allows a larger class of solutions than the Einstein-Klein-Gordon system.

Let us add the Skyrme term to the system under the generalized hedgehog ansatz. Using the expressions

$$\begin{aligned} \mathcal{S}_{\mu\alpha} \mathcal{S}_\nu^\alpha &= \sin^4 \alpha (\nabla \Theta)^2 (\nabla_\mu \Theta)(\nabla_\nu \Theta) \\ &+ (\nabla \alpha)^2 (\nabla_\mu \alpha)(\nabla_\nu \alpha), \end{aligned} \quad (4.13)$$

$$\mathcal{S}^2 - \mathcal{S}_{\alpha\beta} \mathcal{S}^{\alpha\beta} = 2 \sin^2 \alpha (\nabla \Theta)^2 (\nabla \alpha)^2, \quad (4.14)$$

we obtain the energy-momentum tensor as

$$\begin{aligned} T_{\mu\nu} &= K \left[(\nabla_\mu \alpha)(\nabla_\nu \alpha) + \sin^2 \alpha (\nabla_\mu \Theta)(\nabla_\nu \Theta) + \lambda \sin^2 \alpha \right. \\ &\quad \times ((\nabla \Theta)^2 (\nabla_\mu \alpha)(\nabla_\nu \alpha) + (\nabla \alpha)^2 (\nabla_\mu \Theta)(\nabla_\nu \Theta)) \\ &\quad - \frac{1}{2} g_{\mu\nu} ((\nabla \alpha)^2 + \sin^2 \alpha (\nabla \Theta)^2 \\ &\quad \left. + \lambda \sin^2 \alpha (\nabla \Theta)^2 (\nabla \alpha)^2 \right]. \end{aligned} \quad (4.15)$$

Now we derive the Skyrme equations. We will show that they reduce to a single scalar equation under the assumptions (4.8) and (4.10) and the following additional conditions:

$$(\nabla^\mu \nabla^\nu \Theta)(\nabla_\mu \Theta)(\nabla_\nu \Theta) = 0, \quad (4.16)$$

$$(\nabla^\mu \nabla^\nu \alpha)(\nabla_\mu \alpha)(\nabla_\nu \Theta) = 0. \quad (4.17)$$

From Eqs. (4.1), (4.3), and (4.16), we obtain

$$(\nabla_\mu \Theta)(\nabla^\nu \Theta)(\nabla^\mu \nabla_\nu \hat{n}^k) = -(\nabla \Theta)^4 \hat{n}^k. \quad (4.18)$$

It is a trivial computation to derive the following expressions:

$$\begin{aligned} \mathcal{S} R_\mu^k &= (\sin^2 \alpha (\nabla \Theta)^2 + (\nabla \alpha)^2) \left(\sin^2 \alpha \varepsilon^{ijk} \hat{n}^i (\nabla_\mu \hat{n}^j) \right. \\ &\quad \left. + \frac{1}{2} \sin(2\alpha) (\nabla_\mu \hat{n}^k) + \hat{n}^k (\nabla_\mu \alpha) \right), \end{aligned} \quad (4.19)$$

$$\begin{aligned} \mathcal{S}^\nu_\mu R_\nu^k dx^\mu &= \sin^2 \alpha (\nabla_\mu \Theta)(\nabla^\nu \Theta) \left(\sin^2 \alpha \varepsilon^{ijk} \hat{n}^i (\nabla_\nu \hat{n}^j) \right. \\ &\quad \left. + \frac{1}{2} \sin(2\alpha) (\nabla_\nu \hat{n}^k) \right) + (\nabla_\mu \alpha)(\nabla \alpha)^2 \hat{n}^k, \end{aligned} \quad (4.20)$$

from which it follows

$$\begin{aligned} \nabla^\mu (\mathcal{S} R_\mu^k) &= \left[(\nabla_\mu \alpha) \nabla^\mu (\sin^2 \alpha (\nabla \Theta)^2 + (\nabla \alpha)^2) \right. \\ &\quad \left. + (\sin^2 \alpha (\nabla \Theta)^2 + (\nabla \alpha)^2) \right. \\ &\quad \left. \times \left((\nabla^2 \alpha) - \frac{1}{2} \sin(2\alpha) (\nabla \Theta)^2 \right) \right] \hat{n}^k, \end{aligned} \quad (4.21)$$

$$\begin{aligned} \nabla^\mu (\mathcal{S}^\nu_\mu R_\nu^k) &= \left[\nabla^\mu ((\nabla_\mu \alpha)(\nabla \alpha)^2) \right. \\ &\quad \left. - \frac{1}{2} \sin(2\alpha) \sin^2 \alpha (\nabla \Theta)^4 \right] \hat{n}^k. \end{aligned} \quad (4.22)$$

Finally, the Skyrme field Eq. (2.9) reduce to the following scalar equation:

$$\begin{aligned}
 0 &= (\nabla^2 \alpha) - \frac{1}{2} \sin(2\alpha) (\nabla \Theta)^2 \\
 &+ \lambda \left[(\nabla_\mu \alpha) \nabla^\mu (\sin^2 \alpha (\nabla \Theta)^2) + \sin^2 \alpha (\nabla \Theta)^2 \right. \\
 &\left. \times (\nabla^2 \alpha) - \frac{1}{2} \sin(2\alpha) (\nabla \alpha)^2 (\nabla \Theta)^2 \right]. \quad (4.23)
 \end{aligned}$$

In summary, the set of conditions in Eqs. (4.1), (4.3), (4.8), (4.16), and (4.17), define the *generalized hedgehog ansatz for Skyrme models*, under which the Skyrme equations are decomposed into Eqs. (4.10) and (4.23). Again, $\alpha = \pi/2 + N\pi$ is a solution of Eq. (4.23). In this case, the Skyrme term does not appear directly in the geometry, as seen in Eq. (4.15). This system is also not equivalent to the Einstein-Klein-Gordon system because of the constraints (4.16) and (4.17). (See also the discussion in Sec. IV C.)

In the following subsections, we will present several spacetimes with suitable isometries which are compatible with the generalized hedgehog ansatz and, in particular, with the constraints (4.8), (4.16), and (4.17), for α and Θ .

B. Spherically, plane, hyperbolically, and cylindrically symmetric spacetimes

The metric in the most general spacetime with spherical ($k = 1$), plane ($k = 0$), or hyperbolic ($k = -1$) symmetry is given by

$$ds^2 = g_{AB}(y) dy^A dy^B + r(y)^2 \gamma_{ab}(z) dz^a dz^b. \quad (4.24)$$

We assume $\alpha = \alpha(y)$ and $\Theta = \Theta(y, z)$ in Eqs. (4.1), (4.3), (4.16), and (4.17). The canonical coordinates on the submanifold (K^2, γ_{ab}) are

$$\gamma_{ab}(z) dz^a dz^b = d\theta^2 + h(\theta)^2 d\phi^2, \quad (4.25)$$

where $h(\theta) = \sin \theta$, 1 , and $\sinh \theta$ for $k = 1, 0, -1$, respectively. The most general energy-momentum tensor compatible with this symmetry is given by

$$T_{\mu\nu} dx^\mu dx^\nu = T_{AB}(y) dy^A dy^B + P(y) \gamma_{ab} dz^a dz^b, \quad (4.26)$$

where P is a scalar on M^2 . The compatibility of the energy-momentum tensor (4.15) with the above form requires $\Theta = \Theta(y)$ or $\Theta = \Theta(z)$.

In the case of $\Theta = \Theta(z)$, the conditions (4.8) and (4.16) are fulfilled, while Eq. (4.10) becomes

$$\bar{D}^2 \Theta = 0, \quad (4.27)$$

where $\bar{D}^2 := \bar{D}_a \bar{D}^a$. It is still not clear if there exist solutions of the above equation which give the energy-momentum tensor in the form of Eq. (4.26) and fulfill the condition (4.17).

In the case of $\Theta = \Theta(y)$, Eqs. (4.10) and (4.8) become

$$D^2 \Theta + 2r(D^A r)(D_A \Theta) = 0, \quad (D^A \alpha)(D_A \Theta) = 0, \quad (4.28)$$

where $D^2 := D_A D^A$. There are two interesting solutions of the above equations which give the energy-momentum tensor in the form of (4.26). One is the static spacetime

$$ds^2 = -g_{tt}(\rho) dt^2 + g_{\rho\rho}(\rho) d\rho^2 + r(\rho)^2 \gamma_{ab} dz^a dz^b, \quad (4.29)$$

$$\alpha = \alpha(\rho), \quad \Theta = \varpi t, \quad (4.30)$$

and the other is the cosmological spacetime

$$ds^2 = -g_{tt}(t) dt^2 + g_{\rho\rho}(t) d\rho^2 + r(t)^2 \gamma_{ab} dz^a dz^b, \quad (4.31)$$

$$\alpha = \alpha(t), \quad \Theta = \varpi \rho, \quad (4.32)$$

where ϖ is a constant. In both cases, the conditions (4.16) and (4.17) for Skyrme fields are fulfilled.

Our ansatz works also in nonrotating cylindrically symmetric spacetimes. We consider the most general nonrotating cylindrically symmetric space-time,

$$ds^2 = g_{AB}(y) dy^A dy^B + r(y)^2 d\theta^2 + s(y)^2 d\phi^2, \quad (4.33)$$

and assume $\alpha = \alpha(y)$ and $\Theta = \Theta(y, \theta, \phi)$ in Eqs. (4.1) and (4.3). The most general energy-momentum tensor compatible with this symmetry is given by

$$T_{\mu\nu} dx^\mu dx^\nu = T_{AB}(y) dy^A dy^B + P_1(y) d\theta^2 + P_2(y) d\phi^2, \quad (4.34)$$

where P_1 and P_2 are scalars on M^2 . The compatibility of the energy-momentum tensor (4.15) with the above form requires $\Theta = \Theta(y)$, $\Theta = \Theta(\theta)$, or $\Theta = \Theta(\phi)$. Actually, the configuration $\Theta = m\phi$ or $\Theta = m\theta$ is compatible with the generalized hedgehog ansatz, namely, it satisfies the conditions (4.8), (4.16), and (4.17), and gives the energy-momentum tensor in the form of Eq. (4.34). In the case of $\Theta = \Theta(y)$, the following configurations are compatible with or without the Skyrme term:

$$ds^2 = -g_{tt}(\rho) dt^2 + g_{\rho\rho}(\rho) d\rho^2 + r(\rho)^2 d\theta^2 + s(\rho)^2 d\phi^2, \quad (4.35)$$

$$\alpha = \alpha(\rho), \quad \Theta = \varpi t, \quad (4.36)$$

and

$$ds^2 = -g_{tt}(t) dt^2 + g_{\rho\rho}(t) d\rho^2 + r(t)^2 d\theta^2 + s(t)^2 d\phi^2, \quad (4.37)$$

$$\alpha = \alpha(t), \quad \Theta = \varpi \rho. \quad (4.38)$$

C. Axisymmetric spacetimes and nontrivial realization of symmetries

It is shown that the conditions (4.8), (4.16), and (4.17) are satisfied for the following configuration:

$$ds^2 = h_{ab}(v) dv^a dv^b + g_{AB}(v) dv^A dv^B, \quad (4.39)$$

$$\alpha = \alpha(v), \quad \Theta = \varpi t + m\phi, \quad (4.40)$$

where ϖ and m are constants and $v^A = r, z$. $w^a = t, \phi$ are Killing coordinates and h_{ab} is the induced metric on the Killing leaves, which are the $\{r = \text{const}, z = \text{const}\}$ surfaces. An interesting example of the above metric is the well-known Weyl-Papapetrou metric for stationary and axisymmetric spacetimes,

$$ds^2 = -Ae^{\Omega/2}(dt + \omega d\phi)^2 + \frac{e^{2\nu}}{\sqrt{A}}(dr^2 + dz^2) + Ae^{-\Omega/2}d\phi^2, \quad (4.41)$$

where the metric functions depend only on r and z (see Ref. [18]). We will follow the notation in Ref. [19].

Here let us focus on the nonlinear sigma model. The nonzero components of its energy-momentum tensor are given by

$$T^z_z + T^r_r = -KA^{-1}e^{\Omega/2}[-\varpi^2e^{-\Omega} + (m - \varpi\omega)^2]\sin^2\alpha, \quad (4.42)$$

$$T^z_z - T^r_r = K\sqrt{A}e^{-2\nu}[(\partial_z\alpha)^2 - (\partial_r\alpha)^2], \quad (4.43)$$

$$T^\phi_t = K\varpi A^{-1}e^{\Omega/2}(m - \varpi\omega)\sin^2\alpha, \quad (4.44)$$

$$T^\phi_\phi = -KmA^{-1}e^{\Omega/2}[\omega(m - \varpi\omega) + \varpi e^{-\Omega}]\sin^2\alpha, \quad (4.45)$$

$$T^t_t - T^\phi_\phi = -KA^{-1}e^{\Omega/2}[\varpi^2e^{-\Omega} + (m + \varpi\omega)(m - \varpi\omega)]\sin^2\alpha, \quad (4.46)$$

$$T^t_t + T^\phi_\phi = -K\sqrt{A}e^{-2\nu}[(\partial_r\alpha)^2 + (\partial_z\alpha)^2]. \quad (4.47)$$

The relevant combinations of the tensor $\mathcal{G}^\mu_\nu := G^\mu_\nu + \Lambda g^\mu_\nu$ are

$$\mathcal{G}^z_z + \mathcal{G}^r_r = \frac{e^{-2\nu}}{\sqrt{A}}(\Delta A + 2\Lambda\sqrt{A}e^{2\nu}), \quad (4.48)$$

$$\begin{aligned} \mathcal{G}^z_z - \mathcal{G}^r_r = \frac{e^{-2\nu}}{\sqrt{A}} \left[\partial_r^2 A - \partial_z^2 A + \frac{1}{8}A((\partial_r\Omega)^2 - (\partial_z\Omega)^2) \right. \\ \left. - \frac{1}{2}Ae^\Omega((\partial_r\omega)^2 - (\partial_z\omega)^2) - 2((\partial_r A)(\partial_r\nu) - (\partial_z A)(\partial_z\nu)) \right], \end{aligned} \quad (4.49)$$

$$\mathcal{G}^\phi_t = \frac{e^{-2\nu}}{2\sqrt{A}}\vec{\nabla} \cdot (Ae^\Omega\vec{\nabla}\omega), \quad (4.50)$$

$$\begin{aligned} \mathcal{G}^t_\phi = -\frac{e^{-2\nu}}{2\sqrt{A}}[A\omega\Delta\Omega + \omega(\vec{\nabla}A) \cdot (\vec{\nabla}\Omega) + 2A\omega e^\Omega(\vec{\nabla}\omega)^2 \\ + (1 + \omega^2e^\Omega)\{\alpha\Delta\omega + \vec{\nabla}\omega \cdot (\vec{\nabla}A + A\vec{\nabla}\Omega)\}], \end{aligned} \quad (4.51)$$

$$\begin{aligned} \mathcal{G}^t_t - \mathcal{G}^\phi_\phi = -\frac{e^{-2\nu}}{\sqrt{A}} \left[\frac{1}{2}\vec{\nabla} \cdot (A\vec{\nabla}\Omega) + \omega\vec{\nabla} \cdot (Ae^\Omega\vec{\nabla}\omega) \right. \\ \left. + Ae^\Omega(\vec{\nabla}\omega)^2 \right], \end{aligned} \quad (4.52)$$

$$\begin{aligned} \mathcal{G}^t_t + \mathcal{G}^\phi_\phi = \frac{e^{-2\nu}}{\sqrt{A}} \left[\frac{1}{2}\Delta A + 2A\Delta\nu + \frac{1}{8}A(\vec{\nabla}\Omega)^2 \right. \\ \left. - \frac{1}{2}Ae^\Omega(\vec{\nabla}\omega)^2 + 2\Lambda\sqrt{A}e^{2\nu} \right], \end{aligned} \quad (4.53)$$

where $\Delta = \partial_r^2 + \partial_z^2$ and $\vec{\nabla} = (\partial_z, \partial_r)$. Equations (4.42)–(4.53) provide a complete set of the Einstein equations. It is seen that, in the static case ($\omega = 0$), we have $\mathcal{G}^t_\phi = \mathcal{G}^\phi_t = 0$ and hence the Einstein equations require $\varpi m = 0$.

The master Eq. (4.11) for $\alpha = \alpha(r, z)$ is written as

$$\begin{aligned} \frac{e^{-2\nu}}{\sqrt{A}}(A\Delta\alpha + \vec{\nabla}A \cdot \vec{\nabla}\alpha) \\ + \frac{e^{\Omega/2}}{2A}(\varpi^2e^{-\Omega} - (\varpi\omega - m)^2)\sin 2\alpha = 0. \end{aligned} \quad (4.54)$$

$\alpha = \pi/2 + N\pi$ is again a special solution and gives the following energy-momentum tensor:

$$T^z_z + T^r_r = -KA^{-1}e^{\Omega/2}(-\varpi^2e^{-\Omega} + (m - \varpi\omega)^2), \quad (4.55)$$

$$T^z_z - T^r_r = T^t_t + T^\phi_\phi = 0, \quad (4.56)$$

$$T^\phi_t = K\varpi A^{-1}e^{\Omega/2}(m - \varpi\omega), \quad (4.57)$$

$$T^\phi_\phi = -KmA^{-1}e^{\Omega/2}(\omega(m - \varpi\omega) + \varpi e^{-\Omega}), \quad (4.58)$$

$$T^t_t - T^\phi_\phi = -KA^{-1}e^{\Omega/2}(\varpi^2e^{-\Omega} + (m + \varpi\omega)(m - \varpi\omega)). \quad (4.59)$$

At first glance, the above form of the energy-momentum tensor can also be realized by a massless Klein-Gordon field. A linear configuration

$$\psi(t, \phi) = p_1 t + p_2 \phi \quad (4.60)$$

certainly solves the Klein-Gordon equation $\square\psi = 0$ in the axisymmetric spacetime (4.41), where p_1 and p_2 are constants. This configuration gives the following energy-momentum tensor:

$$T^z_z + T^r_r = -A^{-1}e^{\Omega/2}(-p_1^2e^{-\Omega} + (p_2 - p_1\omega)^2), \quad (4.61)$$

$$T^z_z - T^r_r = T^t_t + T^\phi_\phi = 0, \quad (4.62)$$

$$T^\phi{}_\tau = p_1 A^{-1} e^{\Omega/2} (p_2 - p_1 \omega), \quad (4.63)$$

$$T^t{}_\phi = -p_2 A^{-1} e^{\Omega/2} (\omega(p_2 - p_1 \omega) + p_1 e^{-\Omega}), \quad (4.64)$$

$$T^t{}_\tau - T^\phi{}_\phi = -A^{-1} e^{\Omega/2} (p_1^2 e^{-\Omega} + (p_2 + p_1 \omega)(p_2 - p_1 \omega)), \quad (4.65)$$

which are indeed the same as Eqs. (4.55)–(4.59) with $p_1 = \sqrt{K}\varpi$ and $p_2 = \sqrt{K}m$. However, there is a crucial difference between the present system and the Klein-Gordon system: unlike the generalized hedgehog ansatz constructed here, the configuration (4.60) of the Klein-Gordon field *is not physical*. Indeed, the configuration with $p_2 \neq 0$ is not compatible with the axisymmetric spacetime because the periodic boundary condition $\psi(t, \phi) = \psi(t, \phi + 2\pi)$ is not satisfied. Even in the case with $p_2 = 0$, if one assumes that the scalar field is observable, the configuration $\psi = p_1 t$ is not quite realistic due to the obvious unboundedness of ψ for $t \rightarrow \pm\infty$. In contrast, one obtains the same energy-momentum tensor in which the fields Y^i are completely smooth and bounded in the case of the nonlinear sigma model and automatically satisfy the boundary conditions, as can be seen in Eqs. (4.1) and (4.3).

Thus, the configuration (4.40) discloses a new sector of research of stationary and axisymmetric spacetimes. Such spacetimes have been deeply analyzed until now and the solution-generating techniques have been established for the self-gravitating nonlinear sigma models [20,21]. By adopting the powerful techniques introduced in Ref. [18], however, one assumes that the nonlinear sigma model does *not* depend on the Killing coordinates. (See also the recent paper [22] on exact solutions with this assumption.) Indeed, in such a case, the corresponding energy-momentum tensor is trivially compatible with the spacetime symmetry. In the configuration (4.40), in contrast, the nonlinear sigma model (both with and without the Skyrme term) depends on the Killing coordinates in a nontrivial way such that the energy-momentum tensor is still compatible with the spacetime symmetry.

V. SUMMARY AND PERSPECTIVES

In the present paper, we have reinvestigated the hedgehog ansatz for spherically symmetric spacetimes and considered its generalization for nonspherically symmetric

spacetimes for self-gravitating nonlinear sigma models and Skyrme models. Our main results are broadly classified into two types.

In Sec. III, we derived the basic equations under the hedgehog ansatz for future investigations in a fully covariant form on the two-dimensional orbit spacetime under the spherical isometries. We then obtained an exact solution representing a global monopole inside a black hole and briefly discussed its thermodynamical properties. The Skyrme term in the metric function resembles the Maxwell term but its coefficient is fixed by the coupling constants.

In Sec. IV, we proposed the generalized hedgehog ansatz. Under this new ansatz, the field equations reduce to coupled partial differential equations for two scalar fields α and Θ with several constraint equations between them. We have presented some particular configurations compatible with the generalized hedgehog ansatz in physically interesting spacetimes, including stationary and axisymmetric spacetimes. In those configurations, the Skyrme fields depend on the Killing coordinates but the corresponding energy-momentum tensor does not depend on the Killing coordinates. As a result, they allow one to implement the spacetime symmetries in a nontrivial way.

For this reason, the field configurations constructed here are quite different from the usual ones and it is still unknown at present what kind of solutions they allow. For this purpose, to extend the solution-generating techniques to this new sector is an important subject. Also, the generalized hedgehog ansatz is useful to construct black hole or regular solutions numerically. Those studies will shed new light on the nature of Skyrmons.

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