Time-dependent *q*-deformed coherent states for generalized uncertainty relations

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We investigate properties of generalized time-dependent q-deformed coherent states for a noncommutative harmonic oscillator. The states are shown to satisfy a generalized version of Heisenberg's uncertainty relations. For the initial value in time the states are demonstrated to be squeezed, i.e., the inequalities are saturated, whereas when time evolves the uncertainty product oscillates away from this value albeit still respecting the relations. For the canonical variables on a noncommutative space we verify explicitly that Ehrenfest's theorem holds at all times. We conjecture that the model exhibits revival times to infinite order. Explicit sample computations for the fractional revival times and superrevival times are presented.

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I. INTRODUCTION

The algebras satisfied by the canonical variables resulting from q-deformed oscillator algebras have been shown to be related to noncommutative spacetime structures leading to minimal lengths and minimal momenta as a result of a generalized version of Heisenberg's uncertainty relations [1–4]. An important question to address in this context is whether explicit states satisfying these relations actually exist and how they can be constructed. Recently two of the present authors [5] have investigated this problem for a nontrivial limit of the q-deformed oscillator algebra. Using generalized coherent states, so-called Klauder-coherent states [6–9], it was shown in Ref. [5] for a noncommutative harmonic oscillator to first order perturbation theory in the deformation parameter that these states not only satisfy the generalized uncertainty relations, but even saturate them at all times. The main purpose of this paper is to extend this type of analysis to the case for generic deformation parameter q.

II. GENERALIZED TIME-DEPENDENT q-DEFORMED COHERENT STATES

Following Refs. [10–14], up to minor differences in the conventions, we consider a one dimensional q-deformed oscillator algebra for the creation and annihilation operators $A^\dagger$ and A in the form

$$AA^\dagger - q^2 A^\dagger A = 1, \quad \text{for } q \leq 1. \tag{2.1}$$

Defining a q-deformed version of the Fock space involving q-deformed integers $[n]_q$ as

$$|n\rangle_q := \frac{(A^\dagger)^n}{\sqrt{[n]_q!}}|0\rangle, \qquad [n]_q := \frac{1-q^{2n}}{1-q^2},$$
$$[n]_q! := \prod_{k=1}^{n}[k]_q, \qquad A|0\rangle = 0, \qquad \langle 0|0\rangle = 1, \tag{2.2}$$

it follows immediately that the operators $A^\dagger$ and A act indeed as raising and lowering operators, respectively,

$$A^\dagger|n\rangle_q = \sqrt{[n+1]_q}|n+1\rangle_q \quad \text{and}$$
$$A|n\rangle_q = \sqrt{[n]_q}|n-1\rangle_q. \tag{2.3}$$

Furthermore, one deduces from (2.1) and (2.2) that the states $|n\rangle_q$ form an orthonormal basis, i.e., ${}_q\langle n|m\rangle_q = \delta_{n,m}$. As was first argued in Ref. [10], the q-deformed Hilbert space $\mathcal{H}_q$ is then spanned by the vectors $|\psi\rangle := \sum_{n=0}^{\infty} c_n|n\rangle_q$ with $c_n \in \mathbb{C}$, such that $\langle\psi|\psi\rangle = \sum_{n=0}^{\infty}|c_n|^2 < \infty$.

Using these states we can construct the Klauder-coherent states introduced in Refs. [6–9]. In general, these states are defined for a Hermitian Hamiltonian H with discrete bounded below and nondegenerate eigenspectrum and orthonormal eigenstates $|\phi_n\rangle$ as a two parameter set

$$|J,\gamma\rangle = \frac{1}{\mathcal{N}(J)}\sum_{n=0}^{\infty}\frac{J^{n/2}\exp(-i\gamma e_n)}{\sqrt{\rho_n}}|\phi_n\rangle,$$
$$J \in \mathbb{R}_0^+, \qquad \gamma \in \mathbb{R}. \tag{2.4}$$

The probability distribution and normalization constant,

$$\rho_n := \prod_{k=0}^{n} e_k \quad \text{and} \quad \mathcal{N}^2(J) := \sum_{k=0}^{\infty}\frac{J^k}{\rho_k}, \tag{2.5}$$

are expressed in terms of the scaled energy eigenvalues e_n resulting from $H|\phi_n\rangle = \hbar\omega e_n|\phi_n\rangle$. The key properties of these states are their continuity in the two variables (J,γ), the fact that they provide a resolution of the identity and

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that they are temporarily stable satisfying the action angle identity $\langle J,\gamma|H|J,\gamma\rangle = \hbar\omega J$. The time evolution is governed by a shift in the parameter γ, i.e., $\exp(-iHt/\hbar)|J,\gamma\rangle = |J,\gamma+t\omega\rangle$.

As a concrete system let us now consider the noncommutative harmonic oscillator Hamiltonian $H = \hbar\omega(A^\dagger A + 1)$, where the operators $A^\dagger$ and A obey (2.1). With the rescaled eigenvalues $e_n = [n]_q$ and eigenstates $|\phi_n\rangle = |n\rangle_q$ for this Hamiltonian, we obtain the probability distribution $\rho_n = [n]_q!$. We use standard conventions $[0]_q! = 1$. Furthermore, the normalization condition $\langle J,\gamma|J,\gamma\rangle = 1$ yields the q-deformed exponential $E_q(J)$ as the normalization constant

$$E_q(J) := \sum_{n=0}^{\infty} \frac{J^n}{[n]_q!} = \mathcal{N}^2(J). \tag{2.6}$$

Thus our normalized coherent state,

$$|J,\gamma\rangle_q := \frac{1}{\sqrt{E_q(J)}} \sum_{n=0}^{\infty} \frac{J^{n/2}\exp(-i\gamma e_n)}{\sqrt{[n]_q!}} |n\rangle_q, \tag{2.7}$$

coincides with the coherent state $|z\rangle$, as defined already in Ref. [10], for the specific choice $|z^2,0\rangle_q$ and $z\in\mathbb{R}$, that is for $t=0$. Let us now investigate some properties of these states and in particular investigate to which kind of expectation values they lead for observables and compare with the results for the nontrivial $q\to 1$ limit studied in Ref. [5]. In the latter case these states were found to be squeezed states up to first order in perturbation theory in τ when parametrizing the deformation parameter as $q = e^{2\kappa_6^2\tau}$, where κ_6 is explained in Ref. [4]. Most importantly we wish to investigate whether these states respect the generalized uncertainty relations.

III. GENERALIZED HEISENBERG'S UNCERTAINTY RELATIONS

To verify the uncertainty relations projected onto these states we commence by recalling [1,4,15] that the analogues of the canonical variables expressed in terms of the q-deformed oscillator algebra generators

$$X = \alpha(A^\dagger + A) \quad \text{and} \quad P = i\beta(A^\dagger - A), \tag{3.1}$$

with $\alpha = 1/2\sqrt{1+q^2}\sqrt{\hbar/(m\omega)}$ and $\beta = 1/2\sqrt{1+q^2}\sqrt{\hbar m\omega}$, satisfy the deformed canonical commutation relations

$$[X,P] = i\hbar + i\frac{q^2-1}{q^2+1}\left(m\omega X^2 + \frac{1}{m\omega}P^2\right). \tag{3.2}$$

The interesting feature about this version of a noncommutative spacetime is that it leads to a minimal length as well as a minimal momentum. Let us first analyze the generalized version of Heisenberg's uncertainty relation for a simultaneous measurement of the two observables X and P projected onto the normalized coherent states $|J,\gamma\rangle_q$ as defined in Eq. (2.7)

$$\Delta X\Delta P|_{|J,\gamma\rangle_q} \geq \frac{1}{2}|({}_q\langle J,\gamma|[X,P]|J,\gamma\rangle_q)_\eta|. \tag{3.3}$$

The uncertainty for X is computed as $\Delta X^2 = ({}_q\langle J,\gamma|X^2|J,\gamma\rangle_q)_\eta - ({}_q\langle J,\gamma|X|J,\gamma\rangle_q)_\eta^2$ and analogously for P with $X\to P$. The η indicates that we might have to change to a nontrivial metric when X and/or P are non-Hermitian following the prescriptions provided in the recent literature on non-Hermitian systems [16–20] or more specifically for this particular setting in Ref. [5].

Notice that when we assume that the conjugation of A and $A^\dagger$ yield $A^\dagger$ and A, respectively, the operators X and P can be seen as Hermitian. In that case the metric η is taken to be the standard one, possibly with some change to ensure proper self-adjointness and the convergence of the inner products. Indeed, in Refs. [12,21] such a representation on a unit circle acting on Rogers-Szëgo polynomials [22] was derived,

$$A = \frac{i}{\sqrt{1-q^2}}(e^{-i\check{x}} - e^{-i\check{x}/2}e^{2\tau\check{p}}) \quad \text{and} \\ A^\dagger = \frac{-i}{\sqrt{1-q^2}}(e^{i\check{x}} - e^{2\tau\check{p}}e^{i\check{x}/2}). \tag{3.4}$$

Here we used the dimensionless quantities $\check{x} = x\sqrt{m\omega/\hbar}$ and $\check{p} = p/\sqrt{m\omega\hbar}$ with x, p being the standard canonical coordinates satisfying $[x,p] = i\hbar$ and parametrize the deformation parameter $q = e^\tau$. Evidently $A^\dagger$ is the conjugate of A for $q<1$ and consequently with (3.1) follows that also the canonical variables satisfying (3.2) are Hermitian in this representation, i.e., $X^\dagger = X$, $P^\dagger = P$. We notice further that for the representation (3.4) the $\mathcal{PT}$ symmetry of the standard canonical variables $\mathcal{PT}$: $x\to -x$, $p\to p$, $i\to -i$ is inherited by canonical variables on the noncommutative space $\mathcal{PT}$: $X\to -X$, $P\to P$, $i\to -i$.

There exist also alternative representations [23]

$$A = \frac{1}{1-q^2}D_q \quad \text{and} \quad A^\dagger = (1-x) - x(1-q^2)D_q, \tag{3.5}$$

in terms of Jackson derivatives $D_q f(x) := [f(x) - f(q^2x)]/[x(1-q^2)]$ introduced in Ref. [24]. The operators in (3.5) commute to (2.1) when acting on eigenvectors constructed from normalized Rogers-Szëgo polynomials. It is less obvious to see whether this representation can be made Hermitian. For our purposes it is important that at least one such representation exists, and we may compute expectation values on the q-deformed Fock space with the standard metric.

To verify the inequality (3.3) for the states (2.7) we compute first the expectation values for the creation and annihilation operators

$$
{}_q\langle J,\gamma|A|J,\gamma\rangle_q = J^{1/2}\frac{F_q(J,-\gamma)}{E_q(J)} \quad \text{and} \quad {}_q\langle J,\gamma|A^\dagger|J,\gamma\rangle_q = J^{1/2}\frac{F_q(J,\gamma)}{E_q(J)}, \tag{3.6}
$$

where we introduced the function

$$
F_q(J,\gamma) := \sum_{n=0}^{\infty}\frac{J^n e^{i\gamma q^{2n}}}{[n]_q!} = \sum_{n=0}^{\infty}\frac{i^n}{n!}E_q(q^{2n}J)\gamma^n. \tag{3.7}
$$

Notice that this function reduces to the q-deformed exponential $F_q(J,0) = E_q(J)$ and also the duality in the derivatives with respect to the two parameters. The standard derivative with respect to γ corresponds to a q deformation in the parameter J,

$$
-i\frac{\partial}{\partial\gamma}F_q(J,\gamma) = F_q(q^2J,\gamma), \tag{3.8}
$$

and in turn the Jackson derivative acting on J is identical to a deformation in the second parameter

$$
D_qF_q(J,\gamma) = \frac{F_q(J,\gamma) - F_q(q^2J,\gamma)}{J(1-q^2)} = F_q(J,q^2\gamma). \tag{3.9}
$$

These identities are easily derived from the defining relations for F_q and will be made use of below. Using the representations for X and P in terms of the creation and annihilation operators (3.1), it follows directly with the help of (3.6) that

$$
{}_q\langle J,\gamma|X|J,\gamma\rangle_q = \frac{\alpha J^{1/2}}{E_q(J)}[F_q(J,\gamma) + F_q(J,-\gamma)], \tag{3.10}
$$

$$
{}_q\langle J,\gamma|P|J,\gamma\rangle_q = \frac{i\beta J^{1/2}}{E_q(J)}[F_q(J,\gamma) - F_q(J,-\gamma)]. \tag{3.11}
$$

To compute the expectation values for X^2 and P^2, we use once again (3.1) to express them in terms of the $A^\dagger$ and A. Thus we evaluate

$$
{}_q\langle J,\gamma|A^\dagger A^\dagger|J,\gamma\rangle_q = J\frac{F_q(J,\gamma(1+q^2))}{E_q(J)}, \tag{3.12}
$$

$$
{}_q\langle J,\gamma|AA|J,\gamma\rangle_q = J\frac{F_q(J,-\gamma(1+q^2))}{E_q(J)}, \tag{3.13}
$$

$$
{}_q\langle J,\gamma|A^\dagger A|J,\gamma\rangle_q = J, \tag{3.14}
$$

$$
{}_q\langle J,\gamma|AA^\dagger|J,\gamma\rangle_q = 1 + q^2J, \tag{3.15}
$$

and with $X^2 = \alpha^2(A^\dagger A^\dagger + A^\dagger A + AA^\dagger + AA)$ and $P^2 = -\beta^2(A^\dagger A^\dagger - A^\dagger A - AA^\dagger + AA)$ we assemble this to

$$
{}_q\langle J,\gamma|X^2|J,\gamma\rangle_q = \alpha^2\left[J\frac{F_q(J,\gamma(1+q^2)) + F_q(J,-\gamma(1+q^2))}{E_q(J)} + 1 + J + q^2J\right], \tag{3.16}
$$

$$
{}_q\langle J,\gamma|P^2|J,\gamma\rangle_q = -\beta^2\left[J\frac{F_q(J,\gamma(1+q^2)) + F_q(J,-\gamma(1+q^2))}{E_q(J)} - 1 - J - q^2J\right]. \tag{3.17}
$$

From these expressions we find that the right hand side of the generalized Heisenberg's inequality (3.3) is always a constant value independent of γ, i.e., time,

$$
\frac{1}{2}\left|{}_q\langle J,\gamma|\hbar + \frac{q^2-1}{q^2+1}\left(m\omega X^2 + \frac{1}{m\omega}P^2\right)|J,\gamma\rangle_q\right| = \frac{\hbar}{4}(1+q^2)|1+(q^2-1)J|. \tag{3.18}
$$

The square of the left hand side of (3.3) can be written as

$$
\Delta X^2\Delta P^2|_{|J,0\rangle_q} = \alpha^2\beta^2[1+(1+q^2)J + G_q - G_c^2(\gamma)] \times[1+(1+q^2)J - G_q - G_s^2(\gamma)], \tag{3.19}
$$

where we introduced the functions

$$
G_c(\gamma) := \frac{2\sqrt{J}}{E_q(J)}\sum_{n=0}^{\infty}\frac{J^n}{[n]_q!}\cos(\gamma q^{2n}), \quad G_s(\gamma) := \frac{2i\sqrt{J}}{E_q(J)}\sum_{n=0}^{\infty}\frac{J^n}{[n]_q!}\sin(\gamma q^{2n}), \tag{3.20}
$$

and $G_q := \sqrt{J}G_c(\gamma + \gamma q^2)$. Noting that $\lim_{\gamma\to 0}G_q = 2J$, $\lim_{\gamma\to 0}G_c(\gamma) = 2\sqrt{J}$ and $\lim_{\gamma\to 0}G_s(\gamma) = 0$, it is easy to see that for $\gamma = 0$ the expression (3.19) becomes the square of (3.18), such that the minimal uncertainty product for the observables X and P is saturated. From the expressions in (3.20) we deduce that the range for these functions is $-2J \leq G_q \leq 2J$, $0 \leq G_c^2(\gamma) \leq 4J$ and $-4J \leq G_s^2(\gamma) \leq 0$. Recognizing next that the inequality holds when each of the brackets in (3.19) is greater than $1 + (q^2-1)J$, this requires that $2J \geq G_c^2(\gamma) - G_q$ and at the same time $2J \geq G_s^2(\gamma) + G_q$. This means $4J \geq G_c^2(\gamma) + G_s^2(\gamma)$, which by the previous estimates is indeed the case. Overall this implies that for $\gamma \neq 0$ the uncertainty relation (3.3) is always respected.

Next we verify Ehrenfest's theorem. For the time evolution of the operator X we compute directly

$$i\hbar \frac{d}{dt}{}_q\langle J, \omega t|X|J, \omega t\rangle_q = -\frac{\omega\hbar\alpha J^{1/2}}{E_q(J)}[F_q(q^2 J, \omega t) - F_q(q^2 J, -\omega t)], \quad (3.21)$$

and compare it to

$${}_q\langle J, \omega t|[X, H]|J, \omega t\rangle_q = -\frac{\omega\hbar\alpha J^{1/2}}{E_q(J)} \sum_{s=\pm\omega t} \frac{s}{\omega t} F_q(J, s) + \frac{s}{\omega t} J(q^2 - 1) F_q(J, q^2 s), \quad (3.22)$$

with $H = A^\dagger A$, which is easily computed from the expectation values

$${}_q\langle J, \gamma|A^\dagger A^\dagger A|J, \gamma\rangle_q = J^{3/2}\frac{F_q(J, q^2\gamma)}{E_q(J)}, \quad (3.23)$$

$${}_q\langle J, \gamma|A^\dagger A A^\dagger|J, \gamma\rangle_q = J^{1/2}\frac{F_q(J, \gamma)}{E_q(J)} + q^2 J^{3/2}\frac{F_q(J, q^2\gamma)}{E_q(J)}, \quad (3.24)$$

$${}_q\langle J, \gamma|A^\dagger A A|J, \gamma\rangle_q = J^{3/2}\frac{F_q(J, -q^2\gamma)}{E_q(J)}, \quad (3.25)$$

$${}_q\langle J, \gamma|A A^\dagger A|J, \gamma\rangle_q = J^{1/2}\frac{F_q(J, -\gamma)}{E_q(J)} + q^2 J^{3/2}\frac{F_q(J, -q^2\gamma)}{E_q(J)}. \quad (3.26)$$

The equality of (3.21) and (3.22) follows from the identities (3.8) and (3.9). Similarly we verified the validity of Ehrenfest's theorem also for the operator P.

IV. REVIVAL TIMES

As previously argued [5,9,25], revival time structures are very interesting and important quantities of time-dependent states as in principle they are measurable quantities; see for instance Ref. [26]. The structure is directly linked to the dependence of the energy eigenvalues E_n on the quantum number n, i.e., the existence of the kth derivative $d^k E_{\bar{n}}/d\bar{n}^k$ with respect to some average value $\bar{n}$ at which the wave packet $\psi = \sum c_n \phi_n$ is well localized. For the case at hand these derivatives exist to all orders, such that we expect infinitely many revival times to exist.

At the smallest scale one obtains the classical period $T_{\rm cl} = 2\pi\hbar/|E'_{\bar{n}}|$, thereafter at larger scale the fractional revivals for the revival time $T_{\rm rev} = 4\pi\hbar/|E''_{\bar{n}}|$, then the superrevival time $T_{\rm suprev} = 12\pi\hbar/|E'''_{\bar{n}}|$, etc. For the case at hand the peak of the wave packet is computed to $\bar{n} := \langle n\rangle = J d \ln \mathcal{N}^2(J)/dJ$. Noting that $d^k E_n/dn^k = \hbar\omega 2^k q^{2n} \ln^k q/(q^2 - 1)$ we obtain the times

$$T_{\rm cl} = \frac{\pi}{\omega}\left|\frac{q^2 - 1}{q^{2\bar{n}}\ln q}\right|, \quad T_{\rm rev} = \frac{\pi}{\omega}\left|\frac{q^2 - 1}{q^{2\bar{n}}\ln^2 q}\right|, \quad \text{and} \quad T_{\rm suprev} = \frac{3\pi}{2\omega}\left|\frac{q^2 - 1}{q^{2\bar{n}}\ln^3 q}\right|. \quad (4.1)$$

In Fig. 1 we present the autocorrelation function $A(t) := |\langle J, 0, \phi|J, t\omega, \phi\rangle|^2$ as a function of time at different scales. In panel 1(a) the revival after the classical period is clearly visible. The parameters have been chosen in a way that $T_{\rm rev}/T_{\rm cl} \approx 200$, such that at the revival time scale the revivals due to the classical periods have died out and only the revivals due to $T_{\rm rev}$ are exhibited as clearly visible in the computation presented in panel 1(b). With $T_{\rm suprev}/T_{\rm rev} \approx 300$ this type of behavior is repeated at the superrevival time scale as seen in panel 1(c). Because of the aforementioned dependence of the energy eigenvalues on n, we conjecture here that this behavior is repeated order by order. However, the verification of this feature poses a more and more challenging numerical problem, which we leave for future investigations.

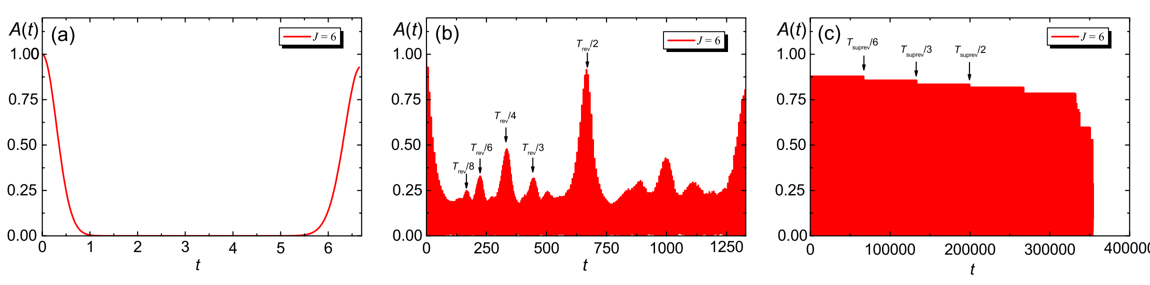

FIG. 1 (color online). Autocorrelation function as a function of time at different scales for $\hbar = 1$, $\omega = 1$, $q = e^{-0.005}$, $J = 6$ and $\bar{n} = 6.1875$. (a) Classical period at $T_{\rm cl} = 6.65$, (b) fractional revival times for $T_{\rm rev} = 1330.19$ and (c) fractional superrevival times for $T_{\rm suprev} = 3999056$.

V. CONCLUSIONS

By extending the analysis of Ref. [5], from a perturbative treatment to the generic case for $q < 1$, we have computed time-dependent q-deformed coherent states for a harmonic oscillator on a noncommutative space. We demonstrated that all key requirements for coherent states are satisfied. A direct comparison with the results obtained in Ref. [5] is not possible as the analysis in there relates to a nontrivial limit $q \to 1$, which is not directly obtainable from the setting presented here; see Refs. [1,4]. However, qualitatively we found a somewhat different behavior with regard to the key question addressed in this manuscript. Whereas the perturbative treatment in Ref. [5] indicated a saturation for the generalized version of Heisenberg's uncertainty relation at all times, the generic q-deformed states exhibit this feature only for $t = 0$, but do respect the inequality thereafter. We have also presented explicit computations for the verification of Ehrenfest's theorem for the coordinate and momentum operator at all times. By computing the autocorrelation functions we have shown that besides a fractional revival time structure this system also exhibits a superrevival structure at a much larger time scale.

Clearly there are various open problems left for future investigations, such as the study of different types of models on the type of noncommutative spaces investigated here. Especially an extension to higher dimensional models would be very interesting. It would also be interesting to study representations for which the operators X and P are non-Hermitian, as for instance in (3.5), in analogy to the analysis presented in Ref. [5]. More computational power should also allow us to confirm our conjecture about the existence of revival time structure at much larger time scales, such as supersuperrevival time structures.

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