PHYSICAL REVIEW D **87,** 084029 (2013)

Embedding hairy black holes in a magnetic universe

Marco Astorino*

Centro de Estudios Científicos (CECs), Valdivia 5110466, Chile

(Received 1 March 2013; published 15 April 2013)

Ernst's solution-generating technique is adapted to Einstein-Maxwell theory conformally (and minimally) coupled to a scalar field. This integrable system enjoys an SU(2,1) symmetry which enables one to move, by Kinnersley transformations, through the axisymmetric and stationary solution space, building an infinite tower of physically inequivalent solutions. As a specific application, metrics associated to scalar hairy black holes—such as the ones discovered by Bocharova, Bronnikov, Melnikov, and Bekenstein—are embedded in the external magnetic field of the Melvin universe.

DOI: 10.1103/PhysRevD.87.084029 PACS numbers: 04.20.Jb, 04.40.Nr, 04.70.Bw

I. INTRODUCTION

Exact solutions in the realm of general relativity are of immense interest and utility but—because of the nonlinear behavior of the theory—they are not easy to discover. For this scope some very powerful solution-generating techniques were built in the last decades, relying basically on the integrability properties of the system. The most famous branches in this field are the Ernst [1] and the Belinsky-Zakharov [2] approaches for stationary axisymmetrical spacetimes in the Einstein-Maxwell theory of gravity. These techniques are not only a useful tool for constructing nontrivial solutions—such as black holes in magnetic universes [3,4] or rotating multi-black hole solutions—but also were fundamental in the proof, by Helers and Ernst, of the Gerosh conjecture which states that, in principle, *all* electro-vacuum stationary, axisymmetric, spinning mass solutions could be generated by one particular solution (e.g., Minkowski spacetime) by means of an infinite sequence of transformations of a certain group [5,6]. All these approaches are strongly theory dependent and it is difficult to apply them even for small modifications of the theory's action. For instance, just the addition of the cosmological constant makes this method hard to generalize, as can be seen in Ref. [7] (in this case the problem is related to a reduction of symmetry of the moduli space, which makes the system not explicitly integrable anymore). Here we are interested in extending Ernst's technique for Einstein-Maxwell theory to the presence of a minimally and conformally coupled scalar field. In this case the integrability property is preserved so the Ernst approach can be directly extended, as can be seen in Secs. II and III. Besides the fact that the literature for gravitational systems coupled with a scalar field is wide from both a theoretical and a phenomenological point of view, actual astrophysical support for this kind of matter is not proven. Cosmologists use scalar fields in some models of inflation or employ them to describe dynamical models for dark matter or dark energy. Neither are fundamental scalar fields known in nature, apart from some recent footprint of the Higgs field found at CERN, which is anyway of a different kind than the ones considered here. Nevertheless, the theoretical interest for those conformally coupled scalar fields has arisen, at least since the 1970s when Bekenstein made use of it to find the first counterexample to the Wheeler's famous *"Black holes have not hairs"* conjecture. For an historical perspective see Ref. [8]. In fact, this matter is at least viable from a theoretical point of view in the sense that it does not violate most of the energy conditions, so if it is not endorsed at the moment by observation, it is at least plausible, possibly just at an effective level.

The black hole solution for general relativity conformally coupled with a scalar filed was first found by Bocharova, Bronnikov, and Melnikov in Ref. [9] and then independently studied by Bekenstein in Refs. [10,11] (henceforth we will call this metric BBMB). It is a static solution of Einstein-Maxwell theory, whose stationary rotating generalization is not known. The formalism developed in this paper could be of some utility in this direction or in other generalizations of the BBMB black hole as well, for instance embedding it in an external magnetic field, as was done by Ernst in Ref. [3] for the Schwarzschild and Reissner-Nordstrom black holes by means of a Harrison transformation. This point is addressed in Sec. III. Black holes embedded in an external magnetic source, such as that of the Melvin universe, are of some astrophysical interest because—especially at the center of galaxies—currents in the accretion disk around a black hole can likely generate such kinds of magnetic fields.

II. ERNST'S SOLUTION-GENERATING TECHNIQUE FOR EINSTEIN-MAXWELL THEORY WITH A MINIMALLY COUPLED SCALAR FIELD

A. Equations of motion

Consider the action for general relativity coupled to the Maxwell electromagnetic field and to a minimally coupled scalar field Ψ,

*marco.astorino@gmail.com

$$I[g_{\mu\nu}, A_\mu, \Psi] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\Big[\mathrm{R} - \frac{G}{\mu_0}F_{\mu\nu}F^{\mu\nu} - \kappa\nabla_\mu\Psi\nabla^\mu\Psi\Big]. \tag{2.1}$$

The gravitational, electromagnetic, and scalar field equations are obtained by extremizing with respect to the metric $g_{\mu\nu}$, the electromagnetic potential A_μ, and the scalar field Ψ, respectively,

$$\mathrm{R}_{\mu\nu} - \frac{\mathrm{R}}{2}g_{\mu\nu} = \frac{2G}{\mu_0}\Big(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\Big) + \kappa\Big(\partial_\mu\Psi\partial_\nu\Psi - \frac{1}{2}g_{\mu\nu}\partial_\sigma\Psi\partial^\sigma\Psi\Big), \tag{2.2}$$

$$\partial_\mu(\sqrt{-g}F^{\mu\nu}) = 0, \tag{2.3}$$

$$\Box\Psi = 0. \tag{2.4}$$

We are interested in stationary and axisymmetric spacetimes characterized by two commuting killing vectors ∂_t, ∂_φ that, for this minimal coupling, are given—in the most general way[1]—by the Lewis-Weyl-Papapetrou metric,

$$ds^2 = f(dt - \omega d\varphi)^2 - f^{-1}[r^2 d\varphi^2 + e^{2\gamma}(dr^2 + dz^2)], \tag{2.5}$$

where all the functions f, ω, γ depend on just the coordinates (r, z), and $\kappa = 8\pi G$. The most generic electromagnetic potential and scalar field compatible with this symmetry can be written as $A = A_t(r,z)dt + A_\varphi(r,z)d\varphi$ and $\Psi(r,z)$, respectively.

In terms of the form of the metric (2.5) the principal gravitational field equations ($GE^\mu{}_\nu$) are[2]

$$GE^\varphi{}_t\text{: } \vec\nabla\cdot\Big[r^{-2}f^2\vec\nabla\omega - 4\frac{G}{\mu_0}r^{-2}fA_t(\vec\nabla A_\varphi + \omega\vec\nabla A_t)\Big] = 0, \tag{2.6}$$

$$GE^t{}_\varphi - GE^\varphi{}_\varphi\text{: } f\nabla^2 f = (\vec\nabla f)^2 - r^{-2}f^4(\vec\nabla\omega)^2 + 2\frac{G}{\mu_0}f[(\vec\nabla A_t)^2 + r^{-2}f^2(\vec\nabla A_\varphi + \omega\vec\nabla A_t)^2], \tag{2.7}$$

while the Maxwell (ME^μ) and scalar (SE) field equations become:

$$ME^t\text{: } \vec\nabla\cdot[f^{-1}\vec\nabla A_t - r^{-2}f\omega(\vec\nabla A_\varphi + \omega\vec\nabla A_t)] = 0, \tag{2.8}$$

$$ME^\varphi\text{: } \vec\nabla\cdot[r^{-2}f(\vec\nabla A_\varphi + \omega\vec\nabla A_t)] = 0, \tag{2.9}$$

$$SE\text{: } \nabla^2\Psi = 0. \tag{2.10}$$

The differential vectorial operators appearing here are the standard flat ones in polar cylindrical coordinates. As can be seen, the scalar field remains decoupled from the gravitational (GE) and electromagnetic (ME) equations and the γ does not appear, so it can be obtained by quadrature after having detected the other functions. This set of equations, (2.6), (2.7), (2.8), (2.9), and (2.10), can be reduced to two complex equations and one real equation as follows:

$$(\mathrm{Re}\mathcal{E} + |\mathbf{\Phi}|^2)\nabla^2\mathcal{E} = (\vec\nabla\mathcal{E} + 2\mathbf{\Phi}^*\vec\nabla\mathbf{\Phi})\cdot\vec\nabla\mathcal{E}, \tag{2.11}$$

$$(\mathrm{Re}\mathcal{E} + |\mathbf{\Phi}|^2)\nabla^2\mathbf{\Phi} = (\vec\nabla\mathcal{E} + 2\mathbf{\Phi}^*\vec\nabla\mathbf{\Phi})\cdot\vec\nabla\mathbf{\Phi}, \tag{2.12}$$

$$\nabla^2\Psi = 0, \tag{2.13}$$

We take advantage of the system's integrability and introduce two complex $(\mathcal{E}, \Psi)$ fields, such that

$$\mathbf{\Phi} := A_t + i\tilde{A}_\varphi, \qquad \mathcal{E} := f - |\mathbf{\Phi}\mathbf{\Phi}^*| + ih, \tag{2.14}$$

where $\tilde{A}_\varphi$ and h are defined as

$$\vec\nabla\tilde{A}_\varphi := -fr^{-1}\vec{e}_\varphi\times(\vec\nabla A_\varphi + \omega\vec\nabla A_t), \tag{2.15}$$

$$\vec\nabla h := -f^2r^{-1}\vec{e}_\varphi\times\vec\nabla\omega - 2\,\mathrm{Im}(\mathbf{\Phi}^*\vec\nabla\mathbf{\Phi}). \tag{2.16}$$

Remarkably enough these equations of motion (2.11), (2.12), and (2.13) can be derived by an effective action principle,

$$S[\mathcal{E}, \mathbf{\Phi}, \Psi] = \int r dr dz d\varphi\Big[\frac{(\vec\nabla\mathcal{E} + 2\mathbf{\Phi}^*\vec\nabla\mathbf{\Phi})\cdot(\vec\nabla\mathcal{E}^* + 2\mathbf{\Phi}\vec\nabla\mathbf{\Phi}^*)}{(\mathcal{E} + \mathcal{E}^* + 2\mathbf{\Phi}\mathbf{\Phi}^*)^2} - \frac{2\vec\nabla\mathbf{\Phi}\cdot\vec\nabla\mathbf{\Phi}^*}{\mathcal{E} + \mathcal{E}^* + 2\mathbf{\Phi}\mathbf{\Phi}^*} - \frac{\kappa}{2}\vec\nabla\Psi\cdot\vec\nabla\Psi\Big]. \tag{2.17}$$

The homothetic symmetries of the action (2.17) are those that leave the equations of motion (2.11), (2.12), and (2.13) invariant. They form an $SU(2,1)\times U(1)$ group of nine real parameters, represented by these finite transformations:

(I) $\mathcal{E}\to\mathcal{E}' = \lambda\lambda^*\mathcal{E}, \quad \mathbf{\Phi}\to\mathbf{\Phi}' = \lambda\mathbf{\Phi}, \quad \Psi\to\Psi' = \Psi,$

(II) $\mathcal{E}\to\mathcal{E}' = \mathcal{E} + ib, \quad \mathbf{\Phi}\to\mathbf{\Phi}' = \mathbf{\Phi}, \quad \Psi\to\Psi' = \Psi,$

[1]As explained in Ref. [12], Sec. 3.4.

[2]To match the standard Ernst notation, G/μ_0 can be normalized to 1 without loss of generality. Any sign discrepancy with respect to Ref. [13] are due to several renowned typos of the latter, as admitted in Ref. [14].

(III) $\mathcal{E} \to \mathcal{E}' = \mathcal{E}/(1 + ic\mathcal{E})$, $\quad \mathbf{\Phi} \to \mathbf{\Phi}' = \mathbf{\Phi}/(1 + ic\mathcal{E})$, $\Psi \to \Psi' = \Psi$,

(IV) $\mathcal{E} \to \mathcal{E}' = \mathcal{E} - 2\beta^* \mathbf{\Phi} - \beta\beta^*$, $\quad \mathbf{\Phi} \to \mathbf{\Phi}' = \mathbf{\Phi} + \beta$, $\Psi \to \Psi' = \Psi$,

(V) $\mathcal{E} \to \mathcal{E}' = \frac{\mathcal{E}}{1 - 2\alpha^* \mathbf{\Phi} - \alpha\alpha^* \mathcal{E}}$, $\quad \mathbf{\Phi} \to \mathbf{\Phi}' = \frac{\mathbf{\Phi} + \alpha\mathcal{E}}{1 - 2\alpha^* \mathbf{\Phi} - \alpha\alpha^* \mathcal{E}}$, $\Psi \to \Psi' = \Psi$,

(VI) $\mathcal{E} \to \mathcal{E}' = \mathcal{E}$, $\quad \mathbf{\Phi} \to \mathbf{\Phi}' = \mathbf{\Phi}$, $\quad \Psi \to \Psi' = \Psi + d$,

where b, c, $d \in \mathbb{R}$ and α, λ, $\beta \in \mathbb{C}$. More generally, instead of the last term in the action (2.17), it is possible to have a sigma model for a collection of scalar fields Ψ_A: $\frac{\kappa}{2} G_{AB} \vec{\nabla}\Psi^A \cdot \vec{\nabla}\Psi^B$, as is done without the electromagnetic field in Refs. [12,15]. In this case the symmetry group is, at least, $SU(2,1) \times \mathcal{G}$, where $\mathcal{G}$ is the homothetic symmetry group of the scalar matter. The case we will consider here is just the simplest: $G_{AB} = 1$.

(I)–(V) are the standard SU(2,1) Kinnersley symmetries, while (VI) is just a trivial shift of U(1). Some of these transformations physically represent gauge transformations—that is, they can be reabsorbed by some diffeomorphism of the resulting metric—while some of them give inequivalent spacetimes that are in fact able to change the charges, the asymptotic behavior, the electromagnetic field content, etc. So the effective group of transformations is actually smaller than SU(2,1).

In principle, we suspect that any axisymmetric metric of the Einstein-Maxwell theory minimally coupled with a scalar field could be obtained, from a fixed seed, by means of the subsequent transformations $(I) - (VI)$. The case with a vanishing scalar field was proven by Hauser and Ernst in Ref. [6]. In practice it is not easy to find this sequence, and moreover not all transformations preserve the asymptotic behavior of the previous solution. In particular, in the next section, we will mostly be interested in the Harrison transformation (V), which is well known to enable one to embed one's favorite asymptotically flat spacetimes in a magnetic universe. Ernst was able to embed a Schwarzschild black hole and the whole Kerr-Newman family of black holes into the Melvin magnetic universe [3,4]. Note that, after being immersed in the external magnetic field, these black hole solutions are no longer of type D in the Petrov classification.

B. Magnetizing the Fisher, Janis, Robinson, and Winicour solution

Here we take advantage of the formalism of Sec. II A to embed the solution of Fisher and Janis, Robinson, Winicour (henceforth FJRW) [16,17] in a external magnetic field. That metric describes a static, asymptotically flat solution for Einstein gravity minimally coupled with a scalar field. Since it is plagued by some nonphysical features, it is not considered of physical interest, but our strategy is to use it as an intermediate step towards the more physical BBMB black hole family. The metric and the associated scalar field read

$$ds^2 = -\left(1 - \frac{2m}{R}\right)^A d\tau^2 + \frac{dR^2}{(1 - \frac{2m}{R})^A} + \left(1 - \frac{2m}{R}\right)^{1-A} R^2 (d\theta^2 + \sin^2\theta d\phi^2), \qquad (2.18)$$

$$\Psi = \sqrt{\frac{1 - A^2}{2\kappa}} \log\left(1 - \frac{2m}{R}\right), \qquad (2.19)$$

where the real parameter $A \in [0, 1]$. For $A \in [0, 1)$ the surface $R = 2m$ has the Ricci squared curvature invariant $\mathrm{R}_{\mu\nu}\mathrm{R}^{\mu\nu}$ unbounded, so it is a naked singularity, while for $A = 1$ it is evident that we have the Schwarzschild black hole. Another interesting value is $A = 1/2$ because—though it is a nonphysical solution in this minimal frame—it can be used, via the Bekenstein technique (which basically consists of a conformal rescaling), to obtain the BBMB black hole in the conformal frame. For this reason, henceforward in this section the parameter A will be fixed to $1/2$. The case with generic A for the magnetized FJRW can be extracted (as in the next section) by setting $e_0 = 0$ (or equivalently $b = 0$) in the metric (2.40).

Now we want to embed this solution, which will be considered as our seed metric, in the Melvin magnetic universe. In the absence of the scalar field the standard procedure consists of using the Harrison transformation (V), so we will do the same. For this purpose, it is more conventional (with respect to the standard literature [3,4]) to use another form of the Weyl-Lewis-Papapetrou metric (2.5) obtained by a double Wick rotation $(t, \varphi) \to (i\phi, i\tau)$,

$$ds^2 = -f(d\phi - \omega d\tau)^2 + f^{-1}[r^2 d\tau^2 - e^{2\gamma}(dr^2 + dz^2)]. \qquad (2.20)$$

Note that after the Wick rotation the electromagnetic complex potential (2.14) becomes $\mathbf{\Phi} = A_\phi + i\tilde{A}_\tau$. By comparing Eqs. (2.18) and (2.20) we get the complex seed potential associated with the killing vector ∂_ϕ,

$$\mathbf{\Phi}_0 = 0, \qquad \mathcal{E}_0 = f_0 = -\sqrt{R^4 - 2mR^3}\sin^2\theta. \qquad (2.21)$$

Then we apply the Harrison transformation to get

$$\mathbf{\Phi} = \frac{B}{2}\frac{\mathcal{E}_0}{\Lambda}, \qquad \mathcal{E} = \frac{\mathcal{E}_0}{\Lambda}, \qquad (2.22)$$

where we have defined[3]

$$\alpha = \frac{B}{2},$$
$$\Lambda = 1 - \alpha\alpha^*\mathcal{E}_0 = 1 + \frac{B^2}{4}\sqrt{R^4 - 2mR^3}\sin^2\theta.$$

[3]In Ernst's notation $\alpha = -\frac{B_0}{2}$ [3], which also implies a switch of the first minus sign in Eq. (2.39).

So the magnetized Janis-Robinson-Winicour spacetime becomes

$$ds^2 = \Lambda^2\left(-\sqrt{1-\frac{2m}{R}}d\tau^2 + \frac{dR^2}{\sqrt{1-\frac{2m}{R}}} + \sqrt{R^4 - 2mR^3}d\theta^2\right) + \frac{\sqrt{R^4-2mR^3}\sin^2\theta}{\Lambda^2}d\phi^2. \qquad (2.23)$$

The scalar field remains unchanged as in Eq. (2.19), while the magnetic field is given by

$$A_\phi = \mathbf{\Phi} = -\frac{B}{2}\frac{\sqrt{R^4-2mR^3}\sin^2\theta}{1+\frac{B^2}{4}\sqrt{R^4-2mR^3}\sin^2\theta}. \qquad (2.24)$$

This solution still contains nonphysical features—such as naked singularities—as with the nonmagnetic one, so it will only be considered a mathematical step towards a less pathological spacetime that will be analyzed in Sec. III A.

C. Magnetizing the Penney solution

In Ref. [18] Penney found, for the Einstein-Maxwell theory minimally coupled to a scalar field, a generalization of the FJRW metric in the presence of a non-null electric field. For our purposes it is best expressed as follows:

$$\Psi = \sqrt{\frac{1-A^2}{2\kappa}}\log\left(\frac{R-a}{R-b}\right), \qquad (2.25)$$

$$A_\tau = \frac{(b-a)(R-a)^A}{b(R-a)^A - a(R-b)^A}\sqrt{\frac{b}{a}\frac{\mu_0}{G}}, \qquad (2.26)$$

$$ds^2 = -e^{-\alpha}d\tau^2 + e^{\alpha}dR^2 + e^{\beta}(d\theta^2 + \sin^2\theta d\phi^2), \qquad (2.27)$$

where

$$e^{\alpha} = \frac{[b(R-a)^A - a(R-b)^A]^2}{(b-a)^2[(R-a)(R-b)]^A}, \qquad (2.28)$$

$$e^{\beta} = e^{\alpha}(R-a)(R-b). \qquad (2.29)$$

The real parameters a and b are related to the standard electric charge parameter e_0 and mass parameter m in this way as $2m = a + b$ and $ab = e_0^2 G/A^2\mu_0$ (again, the ratio G/μ_0 may be thought to be normalized to 1, without loss of generality, to match the standard Ernst notation). A is a constant parameter belonging to the real interval [0,1], as in Sec. II B. When $b = 0$ the FJRW solution (2.18) and (2.19) is retrieved. When $A \in [0, 1)$ the Penney solution displays naked singularities, but for $A = 1$ it is physically meaningful; in fact, it collapses into the Reissner-Nordstrom solution. When $A = 1/2$ it can be shifted into the conformal frame by a conformal transformation, giving the charged BBMB metric. For this reason, Eq. (2.27) represents a good seed to obtain a magnetized charged black hole in the conformal frame, as will be done in Sec. III B.

Comparing the Penney metric (2.27) to the Weyl-Lewis-Papapetrou metric (2.20), and according to the definitions (2.14), we can extract

$$r = \sqrt{(R-a)(R-b)}\sin\theta, \qquad (2.30)$$

$$f_0 = -e^{\beta}\sin^2\theta, \qquad (2.31)$$

$$\mathbf{\Phi}_0 = \tilde{A}_{\tau 0} = -iA\sqrt{ab}\cos\theta, \qquad (2.32)$$

$$\mathcal{E}_0 = -e^{\beta}\sin^2\theta - e_0^2\cos^2\theta. \qquad (2.33)$$

Now it is possible to apply the Harrison transformation (V) (with $\alpha = B/2$) to magnetize the solution (2.25), (2.26), and (2.27),

$$\mathcal{E} = \frac{\mathcal{E}_0}{1 - B\mathbf{\Phi}_0 - \frac{B^2}{4}\mathcal{E}_0} = \frac{-e^{\beta}\sin^2\theta - e_0^2\cos^2\theta}{1 + iBe_0\cos\theta + \frac{B^2}{4}(e^{\beta}\sin^2\theta + e_0^2\cos^2\theta)}, \qquad (2.34)$$

$$\mathbf{\Phi} = \frac{\mathbf{\Phi}_0 + \frac{B}{2}\mathcal{E}_0}{1 - B\mathbf{\Phi}_0 - \frac{B^2}{4}\mathcal{E}_0} = \frac{-ie_0\cos\theta - \frac{B}{2}(e^{\beta}\sin^2\theta + e_0^2\cos^2\theta)}{1 + iBe_0\cos\theta + \frac{B^2}{4}(e^{\beta}\sin^2\theta + e_0^2\cos^2\theta)}, \qquad (2.35)$$

This represents the Penney solution embedded in an external magnetic field, written in terms of the Ernst complex potentials. In case one wants to express it in terms of the more familiar metric, electromagnetic, and scalar field, it is sufficient to apply the definitions (2.14), (2.15), and (2.16),

$$A_\phi = \mathrm{Re}(\mathbf{\Phi}) = \frac{-\frac{B}{2}(e^{\beta}\sin^2\theta + e_0^2\cos\theta) - \frac{B^2}{8}(e^{\beta}\sin^2\theta + e_0^2\cos\theta)^2 - Be_0^2\cos^2\theta}{[1 + \frac{B^2}{4}(e^{\beta}\sin^2\theta + e_0^2\cos^2\theta)]^2 + B^2e_0^2\cos^2\theta}, \qquad (2.36)$$

$$\tilde{A}_\tau = \mathrm{Im}(\mathbf{\Phi}) = -e_0\cos\theta\frac{1 - \frac{B^2}{4}(e^{\beta}\sin^2\theta + e_0^2\cos\theta)}{[1 + \frac{B^2}{4}(e^{\beta}\sin^2\theta + e_0^2\cos^2\theta)]^2 + B^2e_0^2\cos^2\theta}, \qquad (2.37)$$

$$f = \mathrm{Re}(\mathcal{E}) + \mathbf{\Phi}\mathbf{\Phi}^* = \frac{-e^{\beta}\sin^2\theta}{[1 + \frac{B^2}{4}(e^{\beta}\sin^2\theta + e_0^2\cos^2\theta)]^2 + B^2 e_0^2\cos^2\theta}. \tag{2.38}$$

The last unknown metric function ω can be found, for this particular Harrison transformation (V), thanks to the relation

$$\nabla\omega = \Lambda\Lambda^*\vec{\nabla}\omega_0 - \frac{r}{f_0}(\Lambda^*\vec{\nabla}\Lambda - \Lambda\vec{\nabla}\Lambda^*), \tag{2.39}$$

where in this case $\omega_0 = 0$ because the seed (2.26), (2.27), (2.28), and (2.29) we have begun with is static, and where $\Lambda(R,\theta) = 1 + iBe_0\cos\theta + \frac{B^2}{4}(e^{\beta}\sin^2\theta + e_0^2\cos^2\theta)$. Thus, Eq. (2.39) gives

$$\partial_R\omega = -e_0\frac{B^3}{2}(1+\cos^2\theta) - e_0\frac{B}{2}(4 - B^2e_0^2\cos^2\theta)e^{-\beta},$$

$$\partial_\theta\omega = e_0\frac{B^3}{2}(R-a)(R-b)\frac{d\beta(R)}{dR}\sin\theta\cos\theta.$$

The latter equation can be integrated up to an arbitrary function $F(R)$, which can be found from the first,

$$\omega(R,\theta) = \frac{e_0B^3}{4}\sin^2\theta\left[2A\frac{b(R-a)^A(R-b) - a(R-b)^A(R-a)}{b(R-a)^A - a(R-b)^A} + (1-A)(2R-a-b)\right] + F(R),$$

$$F(R) = -B^3e_0R + (4Be_0 - B^3e_0^3)\frac{1}{2Aa}\frac{(a-b)(R-a)^A}{b(R-a)^A - a(R-b)^A} + F_0.$$

The magnetized Penney metric thus takes the final form

$$ds^2 = |\Lambda(R,\theta)|^2[-e^{-\alpha(R)}d\tau^2 + e^{\alpha(R)}dR^2 + e^{\beta(R)}d\theta^2] + \frac{e^{\beta(R)}\sin^2\theta}{|\Lambda(R,\theta)|^2}[d\phi - \omega(R,\theta)dt]^2. \tag{2.40}$$

The electric potential component A_τ follows from the double-Wick-rotated Eq. (2.15),

$$\nabla\tilde{A}_\tau := -fr^{-1}\vec{e}_\phi \times (\vec{\nabla}A_\tau + \omega\vec{\nabla}A_\phi), \tag{2.41}$$

which can be reduced to

$$\partial_R(A_\tau + \omega A_\phi) = \frac{|\Lambda|^2}{e^{\beta}\sin\theta}\partial_\theta\tilde{A}_\tau + A_\phi\partial_R\omega,$$

$$\partial_\theta(A_\tau + \omega A_\phi) = -\frac{|\Lambda|^2}{e^{\beta}\sin\theta}(R-a)(R-b)\partial_R\tilde{A}_\tau + A_\phi\partial_\theta\omega.$$

Finally, the electric potential becomes

$$A_\tau(R,\theta) = -\frac{3}{8}e_0B^2\sin^2\theta\left[2A\frac{b(R-a)^A(R-b) - a(R-b)^A(R-a)}{b(R-a)^A - a(R-b)^A} + (1-A)(2R-a-b)\right] + \frac{3}{2}B^2e_0R + \left(\frac{3}{4}e_0B^2Ab - \frac{e_0}{aA}\right)\frac{(a-b)(R-a)^A}{b(R-a)^A - a(R-b)^A} - \omega A_\phi + \text{const}. \tag{2.42}$$

In the next section we will combine these outcomes with the Bekenstein technique, in the presence of a conformal scalar field, to embed scalar hairy black holes of the BBMB type in an external magnetic field background.

III. EINSTEIN-MAXWELL THEORY WITH A CONFORMALLY COUPLED SCALAR FIELD

A. BBMB black hole in the Melvin magnetic universe

When the scalar filed is conformally coupled to the Einstein-Maxwell theory the action becomes[4]

$$\hat{I}[\hat{g}_{\mu\nu}, \hat{A}_\mu, \hat{\Psi}] = \frac{1}{16\pi G}\int d^4x\sqrt{-\hat{g}}\left[\hat{\mathrm{R}} - \hat{F}_{\mu\nu}\hat{F}^{\mu\nu} - \kappa\left(\nabla_\mu\hat{\Psi}\nabla^\mu\hat{\Psi} + \frac{\hat{\mathrm{R}}}{6}\hat{\Psi}^2\right)\right]. \tag{3.1}$$

We will denote all the quantities in this conformal frame with a hat: $\hat{g}_{\mu\nu}, \hat{A}_\mu, \hat{\Psi}, \ldots$.

It was discovered by Bekenstein in Ref. [10] that a solution $(g_{\mu\nu}, A_\mu, \Psi)$ of Einstein-Maxwell gravity minimally coupled to a scalar field can be mapped to a solution $(\hat{g}_{\mu\nu}, \hat{A}_\mu, \hat{\Psi})$ of the Einstein-Maxwell theory with a conformally coupled scalar field (3.1) by the following set of transformations:

$$\Psi \to \hat{\Psi} = \sqrt{\frac{6}{\kappa}}\tanh\left(\sqrt{\frac{\kappa}{6}}\Psi\right), \tag{3.2}$$

$$A_\mu \to \hat{A}_\mu = A_\mu, \tag{3.3}$$

$$g_{\mu\nu} \to \hat{g}_{\mu\nu} = \left(1 - \frac{\kappa}{6}\hat{\Psi}^2\right)^{-1}g_{\mu\nu}. \tag{3.4}$$

Actually, the original BBMB solution can be obtained by this technique from the $A = 1/2$ Fisher, Janis, Robinson, and Winicour one, Eq. (2.18). So here we play the same game, starting with the magnetized FJRW

[4]An extra conformally invariant potential term, such as $\alpha\hat{\Psi}^4$, might be included in the action (3.1), but we prefer to not consider it here because it would imply a potential term in the minimally coupled system (2.1), which spoils the integrability, and because is not necessary in the BBMB solutions that we will treat.

solution [Eqs. (2.23), (2.24), and (2.19)] and applying the transformations (3.2), (3.3), and (3.4) to pass to the conformal frame. After a coordinate transformation in the radial coordinate $R \longrightarrow \rho/(1-\frac{m}{2\rho})$, we obtain

$$\hat{\Psi} = \sqrt{\frac{6}{\kappa}}\left(1-\frac{2\rho}{m}\right)^{-1}, \tag{3.5}$$

$$A_\phi = -\frac{2B}{\Lambda}\rho^3\frac{\rho-m}{(2\rho-m)^2}\sin^2\theta, \tag{3.6}$$

$$\hat{ds}^2 = \Lambda^2\left[-\left(1-\frac{m}{2\rho}\right)^2 d\tau^2 + \frac{d\rho^2}{(1-\frac{m}{2\rho})^2} + \rho^2 d\theta^2\right] + \frac{\rho^2\sin^2\theta}{\Lambda^2}d\phi^2, \tag{3.7}$$

where

$$\Lambda(\rho,\theta) = 1 + B^2\rho^3\frac{\rho-m}{(2\rho-m)^2}\sin^2\theta. \tag{3.8}$$

This solution represents a BBMB black hole embedded in the Melvin magnetic universe. In fact, as can be easily seen from the limit of the mass parameter $m \longrightarrow 0$, the Melvin universe is exactly recovered,

$$A_\phi = -\frac{B^2}{2}\frac{\rho^2\sin^2\theta}{1+\frac{B^2}{4}\rho^2\sin^2\theta}, \qquad \Psi = 0, \tag{3.9}$$

$$ds^2 = \left(1+\frac{B^2}{4}\rho^2\sin^2\theta\right)^2[-d\tau^2 + d\rho^2 + \rho^2 d\theta^2] + \frac{\rho^2\sin^2\theta}{(1+\frac{B^2}{4}\rho^2\sin^2\theta)^2}d\phi^2. \tag{3.10}$$

The magnetic universe found by Melvin is a static, nonsingular, cylindrical symmetric spacetime in which there exists an axial magnetic field aligned with the z axis. It describes a universe containing a parallel bundle of electromagnetic flux held together by its own gravitational field. Actually, this magnetic universe also mimics the asymptotic behavior (for large ρ) of the metric (3.7).

The limit of a vanishing external magnetic field ($B \longrightarrow 0$) of the solution (3.5), (3.6), and (3.7) gives, as expected, the BBMB black hole,

$$\hat{\Psi} = \sqrt{\frac{6}{\kappa}}\left(1-\frac{2\rho}{m}\right)^{-1}, \tag{3.11}$$

$$\hat{ds}^2 = -\left(1-\frac{m}{2\rho}\right)^2 d\tau^2 + \frac{d\rho^2}{(1-\frac{m}{2\rho})^2} + \rho^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.12}$$

The global causal structure of of the magnetized black holes is generally very close to their nonmagnetized relatives, since any slice of constant ϕ gives a three-dimensional spacetime whose metric coincides with the nonmagnetized metric multiplied by a conformal factor that does not deform the casual structure. So the magnetized solutions share the same radial null geodesics, event horizons, and trapped surfaces with standard black holes (i.e., $B = 0$). In this particular case of BBMB black holes the analysis might be more subtle because the conformal factor $\Lambda(\rho,\theta)$ appears to be divergent on the surface $\rho = m/2$, in this set of coordinates. Anyway, the electric potential A_ϕ remains finite everywhere. The curvature invariants, such as $R^{\mu\nu}R_{\mu\nu}$ or $R^{\mu\nu\sigma\lambda}R_{\mu\nu\sigma\lambda}$, show divergences for $\rho = 0$—which is usual for BBMB black holes—but also on the poles ($\theta = 0, \pi$) of the surface $\rho = m/2$.[5] This surface constitutes the event horizon of the BBMB black hole, where it is also well known that the scalar field of that solution is divergent.[6] So it seems that embedding the BBMB black hole in an external magnetic field emphasizes its singular behavior. For these reasons one has to be very careful before considering the metric of the magnetized BBMB black hole (3.7) as a truly black hole spacetime, as it discloses naked singularities. However, the analysis of the spacetime's causal structure is beyond the scope of this work and will be done elsewhere.

The introduction of the cosmological term in the action usually helps to regularize these divergences because they can be hidden behind the horizon, as it occurs in the Martinez-Troncoso-Zanelli black hole [19] where the scalar field is regular on (and outside) the horizon. Moreover, the cosmological constant improves the astrophysical likelihood, but unfortunately a generating technique in the presence of a cosmological constant (nor even a Harrison transformation) is not available at the moment. As an alternative to the cosmological constant, it may be sufficient to consider the acceleration in order to regularize the solution. The mathematical reasoning behind this is due to the same asymptotic quadratic power scaling in the radial coordinate of the metric between the acceleration and the cosmological constant terms. In fact, as was observed in Ref. [20], the accelerating BBMB black hole has a more well-behaved scalar field on the horizon, because instead of being singular on the whole surface horizon, it is divergent on just one point—the pole ($\rho = m/2$, $\theta = \pi$). Moreover, the introduction of an external electromagnetic field into these accelerating BBMB black holes (as was recently found in Refs. [20,21]) will allow one to remove both of the conical singularities typical on the poles of this accelerating solution, as was first discovered by Ernst himself in Ref. [22] for the C metric. These

[5] At least reaching the poles along some particular directions.
[6] As widely discussed by de Witt and Bekenstein in Ref. [11], "the infinity in the scalar field is not physically pathological because it is not associated to an infinite potential barrier for test scalar charges, it does not cause the termination of any trajectory of these test particles at finite proper time and it is not connected with unbounded tidal accelerations between neighboring trajectories."

metrics, regularized by external magnetic fields, are of particular interest because they describe the creation of black hole pairs [23,24]. The solution-generating technique developed in this paper is able to generate such solutions in the presence of a conformally coupled scalar field [25].

We just want to comment on some features that are common with the nonmagnetized black hole. For instance, the surface gravity k—defined by $k^2 = -\frac{1}{2}\nabla^\mu \chi^\nu \nabla_\mu \chi_\nu$, where χ^μ is the Killing vector ∂_t—remains null (for $\rho = m/2$), as in the case of the BBMB metric. This is typical behavior for a double degenerate horizon, such as extremal black holes.

Also, the topology of constant radial slices remains the same as in the nonmagnetized case; in fact, consider the surface $\mathcal{S}$ described by the two-dimensional metric $\bar{g}_{\mu\nu}$ obtained by fixing $\rho = \bar{\rho} = \text{const}$ and $t = \text{const}$ in Eq. (3.7). Its Euler characteristic is

$$\chi(\mathcal{S}) = \frac{1}{4\pi}\int_{\mathcal{S}} \sqrt{\bar{g}}\,\bar{\mathrm{R}}\,d\theta\,d\phi = 2,$$

So, since $\chi(\mathcal{S}) = 2 - 2g$, the genus of the surface $\mathcal{S}$ is 0, which corresponds to a spherical topology, S^2. The area $\mathcal{A}$ of constant radial (and time) slices is remarkably unchanged by the presence of the external magnetic field,

$$\mathcal{A} = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sqrt{g_{\theta\theta}}\sqrt{g_{\phi\phi}} = 4\pi\bar{\rho}^2.$$

Of course the geometry of constant radial slices is not spherical anymore, but rather is stretched along the direction of the external magnetic field.

Furthermore note that, even though the BBMB metric precisely coincides with that of the extremal Reissner-Nordstrom, the resulting magnetized BBMB (3.7) differs from the magnetized Reissner-Nordstrom (which is not even static), since the generating technique is strongly theory dependent.

B. Charged BBMB black hole in the Melvin magnetic universe

The magnetized Penney solution (2.40) for the minimal scalar coupling, found in Sec. II C, can be uplifted as a solution of the Einstein-Maxwell with a conformally coupled scalar field by the set of transformations (3.2), (3.3), and (3.4),

$$\hat{\Psi} = \sqrt{\frac{6}{\kappa}}\left[\frac{\left(\frac{R-a}{R-b}\right)^{\sqrt{\frac{1-A^2}{3}}} - 1}{\left(\frac{R-a}{R-b}\right)^{\sqrt{\frac{1-A^2}{3}}} + 1}\right], \tag{3.13}$$

$$\hat{ds}^2 = \frac{1}{4}\left[\left(\frac{R-a}{R-b}\right)^{\sqrt{\frac{1-A^2}{3}}} + \left(\frac{R-a}{R-b}\right)^{-\sqrt{\frac{1-A^2}{3}}} + 2\right] ds^2_{(\mathrm{magn-Penney})}. \tag{3.14}$$

The electromagnetic potential A_μ remains unchanged—as in Eqs. (2.36) and (2.42)—because of the conformal invariance of the Maxwell coupling in four dimensions.

As summarized in the Table I (where some other notable spacetimes are also listed), this solution contains both the Ernst metrics family, such as the magnetized Reissner-Nordstrom black hole ($A = 1$) and the magnetized and charged BBMB metric for $A = 1/2$.

In order to analyze this point, A will henceforth be fixed to $1/2$. Moreover, we perform a change of the radial coordinate,

$$R \to \frac{4\rho^2 - ab}{4\rho - a - b}. \tag{3.15}$$

We prefer to express the $A = 1/2$ solution just in terms of the mass and charge parameters, m and e_0, instead of the less physical a and b. They are related (setting the coupling constant ratio $G/\mu_0 = 1$) as $2m = a + b$ and $e_0 = abA$. So Eqs. (3.13) and (3.14) take form

$$\hat{\Psi} = \sqrt{\frac{6}{\kappa}}\left(\frac{\sqrt{m^2 - 4e_0^2}}{2\rho - m}\right), \tag{3.16}$$

$$\hat{ds}^2 = |\Lambda|^2\left[-\left(1 - \frac{m}{2\rho}\right)^2 d\tau^2 + \frac{d\rho^2}{(1-\frac{m}{2\rho})^2} + \rho^2 d\theta^2\right] + \frac{\rho^2 \sin^2\theta}{|\Lambda|^2}(d\phi - \omega dt)^2, \tag{3.17}$$

where

$$e^\beta(\rho) = \frac{4\rho^2(\rho^2 - m\rho + e_0^2)}{(2\rho - m)^2}, \tag{3.18}$$

$$\Lambda(\rho,\theta) = 1 + iBe_0\cos\theta + \frac{B^2}{4}[e^\beta \sin^2\theta + e_0^2\cos^2\theta], \tag{3.19}$$

TABLE I. Some specializations of the metric (3.14) for some values of its parameters.

Spacetimes	A	B	e_0	m
Magnetized charged BBMB	1/2	√	√	√
Charged BBMB	1/2	0	√	√
BBMB black hole	1/2	0	0	√
Magnetized Reissner-Nordstrom	1	√	√	√
Reissner-Nordstrom	1	0	√	√
Magnetised Schwarzschild	1	√	0	√
Schwarzschild	1	0	0	√
Melvin magnetic universe	∀	√	0	0
Minkowski	∀	0	0	0

$$\omega(\rho,\theta) = \frac{B^3}{4} e_0 \sin^2\theta\left(2\rho - 2m + \frac{2e_0^2}{\rho} + \frac{m^2-4e_0^2}{2\rho - m}\right) - B^3 e_0 \frac{2\rho^2 - 2e_0^2}{2\rho - m} + \frac{1}{2\rho}(4Be_0 - B^3 e_0^3),$$
$$A_\phi(\rho,\theta) = -\frac{\frac{B}{2}(e^\beta \sin^2\theta + e_0^2\cos^2\theta)[1 + \frac{B^2}{4}(e^\beta\sin^2\theta + e_0^2\cos^2\theta)] + Be_0^2\cos^2\theta}{|\Lambda|^2},$$
$$A_\tau(\rho,\theta) = -\frac{3}{8}B^2\sin^2\theta\left(2\rho - 2m + \frac{2e_0^2}{\rho} + \frac{m^2-4e_0^2}{2\rho-m}\right) + \frac{3e_0B^2(\rho^2 - e_0^2)}{2\rho - m} + \frac{3e_0^3B^2}{4\rho} - \frac{e_0}{\rho}. \tag{3.20}$$

This solution describes a charged BBMB black hole embedded in an axial external magnetic field. The same considerations of the previous section regarding the appearance of curvature singularities on the poles of the surface $\rho = m/2$ have to be taken into account with caution. The immersion into a background magnetic field is, therefore, not so physically smooth as for more standard back holes—such as the Kerr-Newman family—although it is mathematically similar.

The fact that the seed black hole is charged and immersed into an external magnetic field leads to frame-dragging effects, due to the $\vec{E}\times\vec{B}$ circulating momentum flux in the stress-energy tensor, which serves as a source for a twist potential. Thus, although the seed metric is static, the Harrison-transformed one is stationary. The angular momentum is proportional to the intrinsic electric charge of the black hole e_0 and the external magnetic field B, so the rotation can be detained by switching off either the black hole electric charge e_0 or the external magnetic field B. In the first case, we will retrieve the static uncharged magnetized BBMB metric of Sec. III A, while in the latter case we retrieve the standard charged BBMB black hole. This is a property shared by the magnetised Reissner-Nordstrom too.

Another property in common with the magnetized Reissner-Nordstrom black hole is, after the process of magnetization, the appearance of a conical singularity on the polar axis (which can be interpreted as a string with positive energy density and negative tension associated with some additional and singular stress-energy tensor on the right-hand side of Einstein's equations). This can be seen by expanding the $g_{\theta\theta}$ and $g_{\phi\phi}$ components of the metric (3.17) in powers of θ for a small circle around $\theta = 0$ and $\theta = \pi$. Eventually, it is possible to avoid this extra feature and obtain a regular spacetime by simply rescaling the angular coordinate $\phi \rightsquigarrow \bar{\phi} = \phi/F$, where

$$F = 1 + \frac{3}{2}B^2e_0^2 + \frac{B^4e_0^4}{16}. \tag{3.21}$$

This value is exactly in agreement with the result of Ref. [26] for the Reissner-Nordstrom black hole. This feature is not present (i.e., $F = 1$) when the intrinsic electric charge of the black hole e_0 is null, as in the metric (3.7).

Note that, while the static magnetized BBMB metric (3.7) approaches the Melvin magnetic universe asymptotically, the stationary solution (3.17) does not reach the Melvin universe globally (i.e., for all θ) because the electric field on the symmetry axis is not null for $\rho \to \infty$, as occurs in the magnetic universe. This is a common fact for magnetized charged black holes, as pointed out in Ref. [26]. The generalization to the dyonic black hole (that is, using as a seed a metric with an intrinsic magnetic potential in addition to the electric one) is trivial because of the electromagnetic duality of the Maxwell field in four dimensions. In this section we proposed just a couple of examples, but we note that by applying the transformations (I)–(VI) one can generate an infinite tower of physically inequivalent solutions, which is exactly what happens for the Einstein-Maxwell theory.

IV. COMMENTS AND CONCLUSIONS

In this paper we have applied Ernst's solution-generating technique to Einstein gravity coupled with Maxwell electromagnetism and a minimally coupled scalar field. We have found that, for axisymmetric and stationary spacetimes, the SU(2,1) symmetry group behind the Kinnersley transformation is preserved and can be used to generate an infinite tower of solutions. A couple of examples are provided and worked out to show how the machinery works. In particular, the Fisher, Janis, Robinson, and Winicour metric and the Penney metrics are embedded in an external magnetic field thanks to a Harrison transformation.

These magnetized naked singularities, by means of a conformal transformation, were then mapped to uncharged and charged BBMB black holes embedded in an external Melvin magnetic universe for the Einstein-Maxwell theory of gravitation with a conformally coupled scalar field. The "intrinsic" charged metric is stationary rotating, while the uncharged metric remains static after the Harrison transformation. The external magnetic field seems to sharpen the singular behavior of the standard BBMB black holes, because singularities not covered by event horizons come out.

Therefore, Ernst's solution-generating technique can also be stretched in the presence of a conformally coupled scalar field, acting on the seed metric through a sequence of three steps: (i) a conformal transformation f that brings the seed metric to the minimally coupled (MC) system, (ii) any (let us say n) sequence of generalized Kinnersley $g_1 \circ g_2 \circ \cdots \circ g_n$ transformations can be performed on the MC system, and finally (iii) coming back to the conformally coupled (CC) system with a conformal transformation f^{-1}, as is represented in following figure:

$$\begin{array}{ccc} MC & \xleftarrow{f} & CC \\ {\scriptstyle g_1\circ g_2\circ\cdots\circ g_n}\downarrow & & \downarrow{\scriptstyle \hat{g}=f^{-1}\circ g_1\circ g_2\circ\cdots\circ g_n\circ f} \\ MC & \xrightarrow[f^{-1}]{} & CC \end{array}$$

We suspect that, similarly to what occurs in the case with a vanishing scalar field (Gerosh theorem), for the scalar field minimally (and conformally) coupled to Einstein-Maxwell gravity *all* spacetime solutions might be generated by the set of generalized Kinnersley transformations (I)–(VI). The biggest issue we are concerned about is the suitability of the conformally rescaled Lewis-Weyl-Papapetrou metric for describing the most general stationary, axisymmetric spacetimes for the Einsten-Maxwell theory with a conformally coupled scalar field.

It is worth noticing that in this paper we have only taken advantage of the duality between the minimally and conformally coupled scalar fields, but Ernst's solution-generating technique considered here can be applied to many other theories connected with the minimally coupled scalar matter, such as some classes of Brans-Dicke or F(R) gravities.

Furthermore, the same procedure can also be directly extended to more general matter, such as harmonic map coupling—which consists of a collection of scalar fields arranged in a nonlinear sigma-model fashion—and all conformally related theories. Generally, in that case the symmetry group is enlarged.

So the mechanism developed here is able to generate an infinite number of physically inequivalent axisymmetric stationary solutions for a wide range of gravitational theories related to the scalar coupling (and eventually to Maxwell electromagnetism).

For future perspectives, we would like to explore the possibility of exploiting the integrability and the symmetries of the system directly in the conformally coupled system[7] without passing through the minimally coupled one, and to try to also apply this formalism for a possible generalization of the BBMB metrics, including the Kerr family. Work in this direction, and in magnetizing the accelerating BBMB black hole, is in progress. Also, a better understanding of the causal structure and eventually the thermodynamic properties of magnetized BBMB spacetimes might be interesting.

For people interested in higher-dimensional gravity, the generalization of the present work to five dimensions is straightforward by following the lines of Refs. [27,28], where Ernst's formalism, without scalar fields, was extended to five dimensions.

ACKNOWLEDGMENTS

I would like to thank Hideki Maeda, Cristián Martínez, and Patricia Ritter for fruitful discussions. This work has been funded by the Fondecyt grant 3120236. The Centro de Estudios Científicos (CECs) is funded by the Chilean Government through the Centers of Excellence Base Financing Program of Conicyt.

[7] This procedure can be done in principle for any theory related to Einstein-Maxwell gravity minimally coupled with a scalar field, not just for a conformally coupled one.

[1] F. J. Ernst, Phys. Rev. **167**, 1175 (1968).
[2] V. Belinski and E. Verdaguer, *Gravitational solitons* (Cambridge University Press, Cambridge, 2001).
[3] F. J. Ernst, J. Math. Phys. (N.Y.) **17**, 54 (1976).
[4] F. J. Ernst and W. Wild, J. Math. Phys. (N.Y.) **17**, 182 (1976).
[5] F. J. Ernst, Lecture Notes, http://members.localnet.com/~atheneum/exact/preface.html (2004).
[6] M. Hauser and F. J. Ernst, in *Galaxies, Axisymmetric Systems and Relativity*, edited by M. A. H. MacCallum (Cambridge University Press, Cambridge, 1985), p. 115.
[7] M. Astorino, J. High Energy Phys. 06 (2012) 086.
[8] J. D. Bekenstein, in *Proceedings of the Second International A. D. Sakharov Conference on Physics: Moscow, Russia, 1996* (World Scientific, Singapore, 1997).
[9] N. Bocharova, K. Bronnikov, and V. Melnikov, Vestn. Mosk. Univ. Fiz. Astron. **6**, 706 (1970).
[10] J. D. Bekenstein, Ann. Phys. (N.Y.) **82**, 535 (1974).
[11] J. D. Bekenstein, Ann. Phys. (N.Y.) **91**, 75 (1975).
[12] M. Heusler, *Black hole uniqueness theorems* (Cambridge University Press, Cambridge, 1996), Vol. 6.
[13] F. J. Ernst, Phys. Rev. **168**, 1415 (1968).
[14] F. J. Ernst, J. Math. Phys. (N.Y.) **15**, 1409 (1974).
[15] M. Heusler, Classical Quantum Gravity **12**, 2021 (1995).
[16] I. Z. Fisher, Zh. Eksp. Teor. Fiz. **18**, 636 (1948).
[17] A. I. Janis, D. C. Robinson, and J. Winicour, Phys. Rev. **186**, 1729 (1969).
[18] R. Penney, Phys. Rev. **182**, 1383 (1969).
[19] C. Martinez, R. Troncoso, and J. Zanelli, Phys. Rev. D **67**, 024008 (2003).
[20] C. Charmousis, T. Kolyvaris, and E. Papantonopoulos, Classical Quantum Gravity **26**, 175012 (2009).
[21] A. Anabalon and H. Maeda, Phys. Rev. D **81**, 041501 (2010).
[22] F. J. Ernst, J. Math. Phys. (N.Y.) **17**, 515 (1976).
[23] S. W. Hawking, G. T. Horowitz, and S. F. Ross, Phys. Rev. D **51**, 4302 (1995).

[24] F. Dowker, J. P. Gauntlett, D. A. Kastor, and J. H. Traschen, Phys. Rev. D **49**, 2909 (1994).
[25] M. Astorino (work in progress).
[26] W. A. Hiscock, J. Math. Phys. (N.Y.) **22**, 1828 (1981).
[27] D. Ida and Y. Uchida, Phys. Rev. D **68**, 104014 (2003).
[28] T. Mishima and H. Iguchi, Phys. Rev. D **73**, 044030 (2006).