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Stress tensors from trace anomalies in conformal field theories

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Using trace anomalies, we determine the vacuum stress tensors of arbitrary even-dimensional conformal field theories in Weyl flat backgrounds. We demonstrate a simple relation between the Casimir energy on $\mathbb{R}\times S^{d-1}$ and the type A anomaly coefficient. This relation generalizes earlier results in two and four dimensions. These field-theory results for the Casimir energy are shown to be consistent with holographic predictions in two, four, and six dimensions.

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I. INTRODUCTION

A conformal field theory (CFT) embedded in a curved spacetime background can be characterized by the trace anomaly coefficients of the stress tensor. Here we only consider even-dimensional CFTs because there is no trace anomaly in odd dimensions. The anomaly coefficients (or central charges) a_d and c_{dj} show up in the trace as follows:

$$\langle T^\mu_\mu\rangle = \frac{1}{(4\pi)^{d/2}}\left(\sum_j c_{dj} I_j^{(d)} - (-)^{\frac{d}{2}} a_d E_d\right). \tag{1}$$

Here E_d is the Euler density in d dimensions and $I_j^{(d)}$ are independent Weyl invariants of weight $-d$. The subscript "j" is used to index the Weyl invariants. Our convention for the Euler density is that

$$E_d = \frac{1}{2^{d/2}}\delta^{\nu_1\cdots\nu_d}_{\mu_1\cdots\mu_d} R^{\mu_1\mu_2}{}_{\nu_1\nu_2}\cdots R^{\mu_{d-1}\mu_d}{}_{\nu_{d-1}\nu_d}. \tag{2}$$

We will not need the explicit form of the $I_j^{(d)}$ in what follows, although we will discuss their form in $d\le 6$.

Note that we are working in a renormalization scheme where the trace anomaly is free of the so-called type D anomalies which are total derivatives that can be changed by adding local covariant but not Weyl-invariant counterterms to the effective action. For example, in four spacetime dimensions, a $\Box R$ in the trace can be eliminated by adding an R^2 term to the effective action.

The constraints of conformal symmetry mean that these central charges a_d and c_{dj} determine the behavior of other correlation functions as well. In this paper, for a conformally flat background, we show how to compute $\langle T^{\mu\nu}\rangle$ in terms of a_d and curvatures. In addition to their role in determining correlation functions, the central charges have attracted renewed interest as a way of ordering field theories under renormalization group flow. In two dimensions, the classic c-theorem [1] states that the central charge decreases through the renormalization group flow from the ultraviolet to the infrared. In four dimensions, the corresponding trace anomaly is defined by two types of central charge c_{41} and a_4. The conjecture that the Euler central charge a_4 is the analog of $c = 6a_2$ in two dimensions [2] was proven recently using dilaton fields to probe the trace anomaly [3]. The possibility of a six-dimensional a-theorem was explored in Ref. [4].

The properties of central charges in the six-dimensional case are of particular interest; the (2,0) theory, which describes the low-energy behavior of M5-branes in M-theory, is a six-dimensional CFT. From the AdS/CFT correspondence, it has been known for over a decade that quantities such as the thermal free energy [5] and the central charges [6] have an N^3 scaling for a large number N of M5-branes. However, a direct field-theory computation has proven difficult. Any results calculated from the field-theory side of the six-dimensional CFT without referring to AdS/CFT should be interesting. Such results also provide a nontrivial check of the holographic principle.

In this paper we study the general relation between the stress tensor and the trace anomaly of a CFT in a conformally flat background. Our main result, Eq. (21), is an expression for the vacuum stress tensor of an even-dimensional CFT in a conformally flat background in terms of a_d and curvatures.[1] We pay special attention to the general relation between the Casimir energy (ground-state energy) and a_d. Let ϵ_d be the Casimir energy on $\mathbb{R}\times S^{d-1}$. The well-known two-dimensional CFT result is [7]

$$\epsilon_2 = -\frac{c}{12\ell} = -\frac{a_2}{2\ell}, \tag{3}$$

where ℓ is the radius of S^1. This result is universal for an arbitrary two-dimensional CFT, independent of supersymmetry or other requirements. For general $\mathbb{R}\times S^{d-1}$, we find

$$\epsilon_d = \frac{1\cdot 3\cdots(d-1)}{(-2)^{d/2}}\frac{a_d}{\ell}. \tag{4}$$

[1]By vacuum, we have in mind a state with no spontaneous symmetry breaking, where the expectation values of the matter fields vanish.

II. STRESS TENSOR AND CONFORMAL ANOMALY

We would like to determine the contribution of the anomaly to the stress tensor of a field theory in a conformally flat background. The general strategy we use was originally developed in Ref. [8]. (See also Refs. [9–12] for related discussions.) The conformal (Weyl) transformation is parametrized by $\sigma(x)$ in the standard form,

$$\bar{g}_{\mu\nu}(x) = e^{2\sigma(x)} g_{\mu\nu}(x). \tag{5}$$

We denote the partition function as $Z[g_{\mu\nu}]$. The effective potential is given by

$$\Gamma[\bar{g}_{\mu\nu}, g_{\mu\nu}] = \ln Z[\bar{g}_{\mu\nu}] - \ln Z[g_{\mu\nu}]. \tag{6}$$

The expectation value of the stress tensor $\langle T^{\mu\nu}\rangle$ is defined by the variation of the effective potential with respect to the metric. Here we consider a conformally flat background, $\bar{g}_{\mu\nu}(x) = e^{2\sigma(x)}\eta_{\mu\nu}$, and we normalize the stress tensor in the flat spacetime to be zero. The (renormalized) stress tensor is given by

$$\langle T^{\mu\nu}(x)\rangle = \frac{2}{\sqrt{-\bar{g}}} \frac{\delta\Gamma[\bar{g}_{\alpha\beta}]}{\delta\bar{g}_{\mu\nu}(x)}, \tag{7}$$

which implies

$$\sqrt{-\bar{g}}\langle T^{\lambda}_{\lambda}(x)\rangle = 2\bar{g}_{\mu\nu}(x)\frac{\delta\Gamma[\bar{g}_{\alpha\beta}]}{\delta\bar{g}_{\mu\nu}(x)} = \frac{\delta\Gamma[\bar{g}_{\alpha\beta}]}{\delta\sigma(x')}. \tag{8}$$

We rewrite Eq. (7) as

$$\frac{\delta\left(\sqrt{-\bar{g}}\langle T^{\mu}_{\nu}(x)\rangle\right)}{\delta\sigma(x')} = 2\bar{g}_{\lambda\rho}(x')\frac{\delta}{\delta\bar{g}_{\lambda\rho}(x')} 2\bar{g}_{\nu\gamma}(x)\frac{\delta\Gamma[\bar{g}_{\alpha\beta}]}{\delta\bar{g}_{\mu\gamma}(x)}. \tag{9}$$

Then we use the commutative property

$$\left[\bar{g}_{\lambda\rho}(x')\frac{\delta}{\delta\bar{g}_{\lambda\rho}(x')}, \bar{g}_{\nu\gamma}(x)\frac{\delta}{\delta\bar{g}_{\mu\gamma}(x)}\right] = 0 \tag{10}$$

to obtain the following differential scale equation:

$$\frac{\delta\sqrt{-\bar{g}}\langle T^{\mu\nu}(x)\rangle}{\delta\sigma(x')} = 2\frac{\delta\sqrt{-\bar{g}}\langle T^{\lambda}_{\lambda}(x')\rangle}{\delta\bar{g}_{\mu\nu}(x)}. \tag{11}$$

This equation determines the general relation between the stress tensor (and hence the Casimir energy) and the trace anomaly.

Next we would like to rewrite the trace anomaly $\langle T^{\mu}_{\mu}\rangle$ in terms of a Weyl exact form, $\langle T^{\mu}_{\mu}\rangle = \frac{\delta}{\delta\sigma}$(something), so that we can factor out the sigma variation in Eq. (11) to simplify the calculation. The integration constant is fixed to zero by taking $\langle T^{\mu\nu}\rangle = 0$ in flat space. We use dimensional regularization and work in $n = d + \epsilon$ dimensions. While we do not alter E_d in moving away from d dimensions, we will alter the form of the $I^{(d)}_j$. Let $\lim_{n\to d}\mathcal{I}^{(d)}_j = I^{(d)}_j$, where the $\mathcal{I}^{(d)}_j$ continue to satisfy the defining relation $\delta_\sigma \mathcal{I}^{(d)}_j = -d\,\mathcal{I}^{(d)}_j$. We assume that in general the $\mathcal{I}^{(d)}_j$'s exist such that

$$\frac{\delta}{(n-d)\delta\sigma(x)}\int d^n x'\sqrt{-\bar{g}}E_d(x') = \sqrt{-\bar{g}}E_d, \tag{12}$$

$$\frac{\delta}{(n-d)\delta\sigma(x)}\int d^n x'\sqrt{-\bar{g}}\mathcal{I}^{(d)}_j(x') = \sqrt{-\bar{g}}\mathcal{I}^{(d)}_j. \tag{13}$$

We now make a brief detour to discuss the existence of $\mathcal{I}^{(d)}_j$ in $d = 2$, 4, and 6 [13], and also a general proof of the variation (12). In two dimensions, there are no Weyl invariants $I^{(2)}_j$ and we can ignore Eq. (13). In four dimensions, we have the single Weyl invariant $I^{(4)}_1 = C^{(n=4)}_{\mu\nu\lambda\rho}C^{(n=4)\mu\nu\lambda\rho}$, where $C^{(4)\mu\nu\lambda\rho}$ is the four-dimensional Weyl tensor. If we define the n-dimensional Weyl tensor

$$C^{(n)\mu\nu}{}_{\lambda\sigma} \equiv R^{\mu\nu}{}_{\lambda\sigma} - \frac{1}{n-2}\left[2(\delta^{\mu}_{[\lambda}R^{\nu}_{\sigma]} + \delta^{\nu}_{[\sigma}R^{\mu}_{\lambda]}) + \frac{R\delta^{\mu\nu}_{\lambda\sigma}}{(n-1)}\right], \tag{14}$$

then we find $\mathcal{I}^{(4)}_1 = C^{(n)}_{\mu\nu\lambda\rho}C^{(n)\mu\nu\lambda\rho}$ defined in terms of the n-dimensional Weyl tensor satisfies the eigenvector relation (13). At this point, our treatment differs somewhat from Ref. [8], where the authors instead vary $I^{(4)}_1$ with respect to σ. While Ref. [8] allows for an additional total derivative $\Box R$ term in the trace anomaly, in this paper we choose a renormalization scheme where the trace anomaly takes the minimal form (1). It turns out that this scheme is the one used to match holographic predictions, as we will discuss shortly. A $\Box R$ can be produced by varying $(n-4)R^2$ with respect to σ. Such an R^2 term appears in the difference between $\mathcal{I}^{(4)}_1$ and $I^{(4)}_1$ in Ref. [8].

In six dimensions, there are three Weyl invariants,

$$I^{(6)}_1 = C^{(6)}_{\mu\nu\lambda\sigma}C^{(6)\nu\rho\eta\lambda}C^{(6)\mu\sigma}_{\rho}{}_{\eta}, \tag{15}$$

$$I^{(6)}_2 = C^{(6)\lambda\sigma}_{\mu\nu}C^{(6)\rho\eta}_{\lambda\sigma}C^{(6)\mu\nu}_{\rho\eta}, \tag{16}$$

$$I^{(6)}_3 = C^{(6)}_{\mu\nu\lambda\sigma}\left(\Box\delta^{\mu}_{\rho} + 4R^{\mu}_{\rho} - \frac{6}{5}R\delta^{\mu}_{\rho}\right)C^{(6)\rho\nu\lambda\sigma} + D_\mu J^\mu. \tag{17}$$

To produce the $\mathcal{I}^{(6)}_j$ when $j = 1$, 2, we replace the six-dimensional Weyl tensor with its n-dimensional cousin as in the four-dimensional case. The variation (13) is then straightforward to show. For $j = 3$, Ref. [14] demonstrated the corresponding Weyl transformation for a linear combination of the three $I^{(6)}_j$, there denoted H. The full expression for $\mathcal{I}^{(6)}_3$ and the n-dimensional version of J^μ is not important; we refer the reader to Refs. [14,15] for details. For $d > 6$, we assume the Weyl invariants can be engineered in a similar fashion; see Ref. [16] for the $d = 8$ case.

To vary E_d, we write the corresponding integrated Euler density as

$$\int d^n x \sqrt{-\bar{g}} E_d = \int \frac{(\bigwedge_{j=1}^n dx^{\mu_j})}{2^{d/2}(n-d)!} R^{a_1 a_2}{}_{\mu_1\mu_2} \cdots R^{a_{d-1}a_d}{}_{\mu_{d-1}\mu_d} \times e^{a_{d+1}}_{\mu_{d+1}} \cdots e^{a_n}_{\mu_n} \epsilon_{a_1\cdots a_n}. \tag{18}$$

Recall that the variation of a Riemann curvature tensor with respect to the metric is a covariant derivative acting on the connection. After integration by parts, these covariant derivatives act on either the vielbeins e^a_μ or the other Riemann tensors and hence vanish by metricity or a Bianchi identity. Thus, in varying the integrated Euler density, we need only vary the vielbeins. We use the functional relation $2\delta/\delta g^\nu_\mu = e^a_{(\nu}\delta/\delta e^a_{\mu)}$. One finds

$$\frac{\delta}{\delta \bar{g}^\nu_\mu(x)} \int d^n x' \sqrt{-\bar{g}} E_d = \frac{\sqrt{-\bar{g}}}{2^{\frac{d}{2}+1}} R^{\nu_1\nu_2}{}_{\mu_1\mu_2} \cdots R^{\nu_{d-1}\nu_d}{}_{\mu_{d-1}\mu_d} \delta^{\mu_1\cdots\mu_d\mu}_{\nu_1\cdots\nu_d\nu}. \tag{19}$$

From this expression, the desired relation (12) follows after contracting with δ^ν_μ.

Given the variations (12) and (13), we can factor out the sigma variation in Eq. (11) to obtain[2]

$$\langle T^{\mu\nu}\rangle = \langle X^{\mu\nu}\rangle \equiv \lim_{n\to d} \frac{1}{(n-d)} \frac{2}{\sqrt{-\bar{g}}(4\pi)^{d/2}} \frac{\delta}{\delta \bar{g}_{\mu\nu}(x)} \times \int d^n x' \sqrt{-\bar{g}} \Big(\sum_j c_{dj} I^{(n)}_j - (-)^{\frac{d}{2}} a_d E_d \Big). \tag{20}$$

Comparing this with Eq. (7), we see that the effective action must contain terms proportional to $\langle T^\mu_\mu\rangle$. Indeed, these are precisely the counterterms that must be added to regularize divergences coming from placing the CFT in a curved spacetime [17]. We next perform the metric variation for a conformally flat background. The metric variation of the Weyl tensors $I^{(d)}_j$ vanishes for conformally flat backgrounds because the $I^{(d)}_j$ are all at least quadratic in the n-dimensional Weyl tensor. (Conformal flatness is used only after working out the metric variation.) Thus the stress tensor in a conformally flat background may be obtained by varying only the Euler density,

$$\langle T^\mu_\nu\rangle = -\frac{a_d}{(-8\pi)^{d/2}} \lim_{n\to d} \frac{1}{n-d} R^{\nu_1\nu_2}{}_{\mu_1\mu_2} \cdots R^{\nu_{d-1}\nu_d}{}_{\mu_{d-1}\mu_d} \delta^{\mu_1\cdots\mu_d\mu}_{\nu_1\cdots\nu_d\nu}. \tag{21}$$

Note that in a conformally flat background, by employing Eq. (14) the Riemann curvature can be expressed purely in terms of the Ricci tensor and Ricci scalar,

$$R^{\nu_1\nu_2}{}_{\mu_1\mu_2} = \frac{1}{n-2}\Big[2(\delta^{\nu_1}_{[\mu_1} R^{\nu_2}_{\mu_2]} + \delta^{\nu_2}_{[\mu_2} R^{\nu_1}_{\mu_1]}) - \frac{R\delta^{\nu_1\nu_2}_{\mu_1\mu_2}}{n-1}\Big].$$

Contracting a $\delta^{\nu_j}_{\mu_j}$ with the antisymmetrized Kronecker delta $\delta^{\mu_1\cdots\mu_d\mu}_{\nu_1\cdots\nu_d\nu}$ eliminates the factor of $(n-d)$ in Eq. (21).

In two and four dimensions, we can use Eq. (21) to recover the results of Ref. [8]. In two dimensions, the right-hand side of $\langle T^\mu_\nu\rangle$ is proportional to $R^\mu_\nu - \frac{1}{2}R\delta^\mu_\nu$, which vanishes in two dimensions. Thus we first must expand the Einstein tensor in terms of the Weyl factor σ, where $g_{\mu\nu} = e^{2\sigma}\eta_{\mu\nu}$ before taking the $n\to 2$ limit. The result is [8]

$$\langle T^{\mu\nu}\rangle = \frac{a_2}{2\pi}(\sigma^{\mu;\nu} + \sigma^\mu\sigma^\nu - g^{\mu\nu}(\sigma_{,\lambda}{}^{;\lambda} + \sigma_{,\lambda}\sigma^\lambda)). \tag{22}$$

In four dimensions, we obtain

$$\langle T^{\mu\nu}\rangle = \frac{-a_4}{(4\pi)^2}\Big[g^{\mu\nu}\Big(\frac{R^2}{2} - R^2_{\lambda\rho}\Big) + 2R^{\mu\lambda}R^\nu_\lambda - \frac{4}{3}RR^{\mu\nu}\Big]. \tag{23}$$

In six dimensions, we obtain (to our knowledge) a new result,

$$\langle T^{\mu\nu}\rangle = -\frac{a_6}{(4\pi)^3}\Big[\frac{3}{2}R^\mu_\lambda R^\nu_\sigma R^{\lambda\sigma} - \frac{3}{4}R^{\mu\nu}R^\lambda_\sigma R^\sigma_\lambda - \frac{1}{2}g^{\mu\nu}R^\sigma_\lambda R^\lambda_\rho R^\rho_\sigma - \frac{21}{20}R^{\mu\lambda}R^\nu_\lambda R + \frac{21}{40}g^{\mu\nu}R^\sigma_\lambda R^\lambda_\sigma R + \frac{39}{100}R^{\mu\nu}R^2 - \frac{1}{10}g^{\mu\nu}R^3\Big]. \tag{24}$$

As we work in Weyl flat backgrounds, there is no contribution from B-type anomalies. These $\langle T^{\mu\nu}\rangle$ are covariantly conserved, as they must be since they were derived from a variational principle.

III. CASIMIR ENERGY AND CENTRAL CHARGE

We would like to relate a_d to the Casimir energy,

$$\epsilon_d = \int_{S^{d-1}} \langle T^{00}\rangle \,\mathrm{vol}(S^{d-1}), \tag{25}$$

on $\mathbb{R}\times S^{d-1}$. In preparation, let us calculate E_d for the sphere S^d. For S^d with radius ℓ, the Riemann tensor is $R^{\nu_1\nu_2}{}_{\mu_1\mu_2} = \delta^{\nu_1\nu_2}_{\mu_1\mu_2}/\ell^2$. It follows from Eq. (2) that $E_d = \frac{d!}{\ell^d}$. We conclude that the trace of the vacuum stress tensor on S^d takes the form

$$\langle T^\mu_\mu\rangle = -\frac{a_d d!}{(-4\pi\ell^2)^{d/2}}. \tag{26}$$

Let us now calculate $\langle T^\mu_\nu\rangle$ for $S^1\times S^{d-1}$. The Riemann tensor on $S^1\times S^{d-1}$ is zero whenever it has a leg in the S^1 direction and looks like the corresponding Riemann tensor for S^{d-1} in the other directions. We can write $R^{i_1i_2}{}_{j_1j_2} = \delta^{i_1i_2}_{j_1j_2}/\ell^2$, where i and j index the S^{d-1}. The computation of

[2] While we specialize to conformally flat backgrounds, under a more general conformal transformation one has $\langle T^{\mu\nu}(\bar{g})\rangle - \langle X^{\mu\nu}(\bar{g})\rangle = e^{-(d+2)\sigma}(\langle T^{\mu\nu}(g)\rangle - \langle X^{\mu\nu}(g)\rangle)$.

$\langle T^0_0\rangle$ and $\langle T^i_j\rangle$ proceeds along similar lines to the computation of E_d,

$$\langle T^0_0\rangle = -\frac{a_d(d-1)!}{(-4\pi\ell^2)^{d/2}}, \qquad \langle T^i_j\rangle = \frac{a_d(d-2)!}{(-4\pi\ell^2)^{d/2}}\delta^i_j. \quad (27)$$

Note that $\langle T^\mu_\nu\rangle$ is traceless, consistent with a result of Ref. [11]. Using the definition (25), we compute the Casimir energy ϵ_d. We find that (for d even)

$$\epsilon_d = \frac{a_d(d-1)!}{(-4\pi\ell^2)^{d/2}}\mathrm{Vol}(S^{d-1}) = \frac{1\cdot 3\cdots(d-1)}{(-2)^{d/2}}\frac{a_d}{\ell}. \quad (28)$$

In two, four, and six dimensions, the ratios between the Casimir energy and a_d are $-\frac{1}{2\ell}$, $\frac{3}{4\ell}$, and $-\frac{15}{8\ell}$, respectively.

IV. HOLOGRAPHY AND DISCUSSION

In this section, we would like to use the AdS/CFT correspondence to check our relation between ϵ_d and a_d for $d = 2$, 4, and 6. For CFTs with a dual anti-de Sitter space description, the stress tensor can be calculated from a classical gravity computation [18–20]. The Euclidean gravity action is taken to be

$$\begin{aligned} S &= S_{\mathrm{bulk}} + S_{\mathrm{surf}} + S_{\mathrm{ct}},\\ S_{\mathrm{bulk}} &= -\frac{1}{2\kappa^2}\int_{\mathcal{M}} d^{d+1}x\sqrt{G}\left(\mathcal{R} + \frac{d(d-1)}{L^2}\right),\\ S_{\mathrm{surf}} &= -\frac{1}{\kappa^2}\int_{\partial\mathcal{M}} d^d x\sqrt{g}K,\\ S_{\mathrm{ct}} &= \frac{1}{2\kappa^2}\int_{\partial\mathcal{M}} d^d x\sqrt{g}\Big[\frac{2(d-1)}{L} + \frac{L}{d-2}R\\ &\quad + \frac{L^3}{(d-4)(d-2)^2}\left(R^{\mu\nu}R_{\mu\nu} - \frac{d}{4(d-1)}R^2\right) + \cdots\Big]. \end{aligned} \quad (29)$$

The Ricci tensor $R_{\mu\nu}$ is computed with respect to the boundary metric $g_{\mu\nu}$, while $\mathcal{R}$ is the Ricci scalar computed from the bulk metric G_{ab}. The object $K_{\mu\nu}$ is the extrinsic curvature of the boundary $\partial\mathcal{M}$. The counterterms S_{ct} render S finite, and we keep only as many as we need. The metrics with $S^{d-1}\times S^1$ conformal boundary,

$$ds^2 = L^2(\cosh^2 r dt^2 + dr^2 + \sinh^2 r d\Omega_{d-1}), \quad (30)$$

and S^d boundary,

$$ds^2 = L^2(dr^2 + \sinh^2 r d\Omega_d), \quad (31)$$

satisfy the bulk Einstein equations. Note that the S^{d-1} and S^d spheres have radius $\ell = \frac{L}{2}e^{r_0}$ at some large reference r_0, while we take the S^1 to have circumference β (hence the range of t is $0 < t < \beta/\ell$). We compute the stress tensor from the on-shell value of the gravity action using Eq. (7), making the identification $\Gamma = -S$ and using the boundary value of the metric in place of $\bar{g}_{\mu\nu}$. One has [19]

d	Γ_{S^d}	$\Gamma_{S^1\times S^{d-1}}$
2	$\frac{4\pi L}{\kappa^2}\log\ell$	$\frac{\pi\beta L}{\kappa^2\ell}$
4	$-\frac{4\pi^2L^3}{\kappa^2}\log\ell$	$-\frac{3\pi^2\beta L^3}{4\kappa^2\ell}$
6	$\frac{2\pi^3L^5}{\kappa^2}\log\ell$	$\frac{5\pi^3\beta L^5}{16\kappa^2\ell}$

We include only the leading log term of Γ_{S^d}. From Eq. (7), it follows that $\langle T^0_0\rangle\mathrm{Vol}(S^{d-1}) = \partial_\beta\Gamma_{S^1\times S^{d-1}}$ and $\langle T^\mu_\mu\rangle\mathrm{Vol}(S^d) = \partial_\ell\Gamma_{S^d}$. For a conformally flat manifold, we have from Eq. (1) that $\langle T^\mu_\mu\rangle = -a_d(-4\pi)^{-d/2}E_d$, which allows us to calculate a_d from $\langle T^\mu_\mu\rangle$ [6]. Defining the Casimir energy with respect to a time $\tilde{t} = \ell t$ whose range is the standard $0 < \tilde{t} < \beta$, we can deduce from Eq. (25) that $\epsilon_d = -\partial_\beta\Gamma_{S^1\times S^{d-1}}$ (see also Ref. [21]). We have a table,

	$\langle T^0_0\rangle$	ϵ_d		$\langle T^\mu_\mu\rangle$	E_d	a_d
$S^1\times S^1$	$\frac{L}{2\kappa^2\ell^2}$	$-\frac{\pi L}{\kappa^2\ell}$	S^2	$\frac{L}{\kappa^2\ell^2}$	$\frac{2}{\ell^2}$	$\frac{2\pi L}{\kappa^2}$
$S^1\times S^3$	$-\frac{3L^3}{8\kappa^2\ell^4}$	$\frac{3\pi^2L^3}{4\kappa^2\ell}$	S^4	$-\frac{3L^3}{2\kappa^2\ell^4}$	$\frac{24}{\ell^4}$	$\frac{\pi^2L^3}{\kappa^2}$
$S^1\times S^5$	$\frac{5L^5}{16\kappa^2\ell^6}$	$-\frac{5\pi^3L^5}{16\kappa^2\ell}$	S^6	$\frac{15L^5}{8\kappa^2\ell^6}$	$\frac{720}{\ell^6}$	$\frac{\pi^3L^5}{6\kappa^2}$

Comparing the ϵ_d and a_d columns, we can confirm the results from earlier in this paper, namely that[3]

$$\epsilon_2 = -\frac{a_2}{2\ell}, \qquad \epsilon_4 = \frac{3a_4}{4\ell}, \qquad \epsilon_6 = -\frac{15a_6}{8\ell}. \quad (32)$$

In the four-dimensional case, such a gravity model arises in type IIB string theory by placing a stack of N D3-branes at the tip of a six-dimensional Calabi-Yau cone. In this case, we can make the further identification [6,22] $a_4 = \frac{N^2}{4}\frac{\mathrm{Vol}(S^5)}{\mathrm{Vol}(SE_5)}$, where SE_5 is the five-dimensional base of the cone. These constructions are dual to four-dimensional quiver gauge theories with $\mathcal{N} = 1$ supersymmetry. In six dimensions, such a gravity model arises in M-theory by placing a stack of N M5-branes in flat space. In this case, we can make the further identification [6,15] (see also Ref. [23]) $a_6 = \frac{N^3}{9}$. The dual field theory is believed to be the non-Abelian (2,0) theory.

We would like to comment briefly on the Casimir energy calculated in the weak-coupling limit.[4] In typical regularization schemes, for example zeta-function regularization, the Casimir energy will not be related to the conformal anomaly via Eq. (4) because of the presence of total derivative terms (D type anomalies) in the trace of the stress tensor. For a conformally coupled scalar in four dimensions, Ref. [17] tells us that $a_4 = 1/360$. Our result (4) would then imply that $\epsilon_4 = 1/480L$, but naive zeta-function regularization instead

[3]These results indicate that any so-called type D anomalies present in the holographic renormalization scheme do not affect the relation between a_d and ϵ_d determined in a scheme where the type D anomalies are absent.

[4]We thank J. Minahan for discussions on this issue.

yields $\epsilon_4 = 1/240L$. The discrepancy can be resolved either by including a $\Box R$ term in the trace, and thus changing Eq. (4) [11], or by adding an R^2 counterterm to the effective action, thereby changing ϵ_4. Amusingly, in zeta-function regularization, the effect of the total derivative terms on ϵ_4 cancels for the full $\mathcal{N} = 4$ supersymmetric Yang-Mills multiplet, and the weak coupling results for ϵ_4 and a_4 are related via Eq. (4) [24,25]. In contrast, for the (2,0) multiplet in six dimensions, the total-derivative terms do not cancel [15]. The resulting discrepancy [26] in the relation between a_6 and ϵ_6 can presumably be cured either by adding counterterms to the effective action to eliminate the total derivatives or by improving Eq. (4) to include the effect of these derivatives. Generalizing our results to include the contribution of D type anomalies to the stress tensor would allow a more straightforward comparison of weak-coupling Casimir energies obtained via zeta-function regularization and the conformal anomaly a_d. We leave such a project for the future.

There are two other obvious calculations for future study. (i) Determine how $\langle T^{\mu\nu} \rangle$ transforms in nonconformally flat backgrounds. Such transformations would involve the type B anomalies. (ii) Check the full six-dimensional stress tensor (24) for any conformally flat background by the holographic method. A four-dimensional check of Eq. (23) was performed in Ref. [20].

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