PHYSICAL REVIEW D **87,** 076001 (2013)

Path integral approach to space-time probabilities: A theory without pitfalls but with strict rules

D. Sokolovski

Departmento de Química-Física, Universidad del País Vasco, UPV/EHU, 48940 Leioa, Spain and IKERBASQUE, Basque Foundation for Science, E-48011 Bilbao, Spain

(Received 13 December 2012; published 1 April 2013)

Following the renewed interest in the topic [Halliwell and Yearsley, Phys. Rev. D **86**, 024016 (2012)], we revisit the problem of assigning probabilities to classes of Feynman paths passing through specified space-time regions. We show that by assigning probabilities to interfering alternatives, one already makes the assumption that the interference has been destroyed through interacting with the environment or a meter. Including the effects of the meter allows one to construct a consistent theory, free of logical "pitfalls," such as those identified in Halliwell and Yearsley. Wherever a meter cannot be constructed, or cannot be set to effect the desired decoherence, formally constructed probabilities have no clear physical meaning and can violate the necessary sum rules. We illustrate the above approach by analysing the three examples considered in Halliwell and Yearsley.

DOI: 10.1103/PhysRevD.87.076001 PACS numbers: 03.65.−w, 02.50.−r, 03.75.Lm

I. INTRODUCTION

Recently, Halliwell and Yearsley [1] revisited the problem of assigning probabilities to amplitudes obtained by restricting Feynman paths to space-time regions. They emphasized that the method suffers from serious deficiencies: the "seemingly obvious notion of 'restricting paths' leads to the quantum Zeno effect and hence to unphysical results". This is the sense in which we would say that path integral constructions may suffer from pitfalls [1]. Their criticism appears to extend to the path integral analysis of the quantum traversal time [2–8]. In order to mitigate the Zeno effect, the authors of Ref. [1] followed Alonso *et al.* [9] in suggesting "temporal coarse graining," whereby the observed system is controlled only at discrete times, between which it is allowed to evolve freely. The problem is closely related to the more general question concerning the origin of quantum probabilities (see, for example, Refs. [10–15]).

The authors of Ref. [1] approach the problem from the point of view of a theory which attempts to assign probabilities without making a specific reference to measurements. One such theory is the decoherent histories approach (DHA) (Refs. [42]–[52] of Ref. [1]), designed to assign probabilities to possible scenarios available to a closed system where no external observation is possible.

Far from claiming to correct a technical error of Ref. [1] (for, indeed, there is no such error), we will attempt to analyze the space-time probability problem from the opposite prospective. We will advocate the view that probabilities assigned to certain classes for Feynman paths are only meaningful provided the interference between the classes can be destroyed by physical means. This restriction inevitably narrows the issue by bringing into the discussion such questions as the possibility of constructing a relevant "meter," the accuracy of the "measurement," and the back action exerted by the meter on the measured system. The argument can be inverted: if someone constructs—by restricting Feynman paths—probabilities corresponding to a possible measurement, he or she will also inherit the dynamical effects of the meter. Since no meter was considered in the first place, the unexpected properties of the probabilities can be perceived as failings of the path integral method. It is in this sense that we disagree with the conclusions of Ref. [1] with regards to the "pitfalls" of the path integral approach to quantum probabilities [16]. We also would like to restore (as much as possible) the good name of the quantum traversal time.

The main purpose of this paper is to show that the path integral approach is indeed a consistent theory with its own strict rules, yet free from "unphysical" features. We demonstrate that once the inevitable effects of the meter are properly taken into account, one has a theory free from logical contradictions.

The rest of the paper is organized as follows. In Sec. II we briefly describe the path decomposition of the propagator. In Sec. III we choose a variable, and construct its amplitude distribution by restricting Feynman paths. In Sec. IV we show that the chosen restriction automatically prescribes the type of meter required to destroy interference. In Sec. V we emphasize the distinction between the finite-time and continuous quantum measurements, and proceed to analyze the former type. In Sec. VI we formulate three questions which define a quantum measurement. Section VII describes the mixed state of the system after interference between the classes of Feynman paths has been destroyed. In Sec. VIII we prove a general result concerning the emergence of a kind of Zeno effect in high-accuracy "ideal" measurements. In Secs. IX, X, and XI we analyze some of the cases used in Ref. [1] to illustrate "unreasonable properties" of path integral amplitudes. Section XI contains our conclusions.

II. PATH DECOMPOSITION OF A TRANSITION AMPLITUDE

For a system governed by a Hamiltonian $\hat{H}$, consider the transition amplitude

$$K(F, I, t) = \langle \psi_F | \exp(-i\hat{H}t) | \psi_I \rangle, \quad (1)$$

between some initial and final states $|\psi_I\rangle$ and $|\psi_F\rangle$. In its Hilbert space, we choose a compete orthonormal basis $|x\rangle$, where x may take discrete or continuous values, so that (I is the unity)

$$\sum_x |x\rangle\langle x| = I. \quad (2)$$

Writing $\exp(-i\hat{H}t)$ as $\prod_{k=1}^{K} \exp[-i\hat{H}(tk/K)]$, inserting Eq. (2) between the exponentials and also before and after $|\psi_I\rangle$ and $|\psi_F\rangle$, and sending K to infinity yields the celebrated Feynman path integral [17] (or, as the case may be, path sum),

$$K(F, I, t) = \sum_{\text{paths}} A[\text{path}]. \quad (3)$$

Here a path is defined by the sequence of the x's labelling the states through which the system passes at the intermediate times $t' = 0, \epsilon, 2\epsilon, \ldots, t$, $\epsilon \equiv t/K$. In the continuous limit $K \to \infty$ we will denote it as $x(t')$. The functional $A[\text{path}] \equiv \lim_{K\to\infty} \langle \psi_F | x_K \rangle \langle x_K | \exp(-i\hat{H}\epsilon) | x_{K-1} \rangle \ldots \exp(-i\hat{H}\epsilon) | x_0 \rangle \langle x_0 | \psi_I \rangle$ is the amplitude the path contributes to $K(F, I, t)$. With the Feynman paths and their contributions $A[\text{path}]$ known, the task of evaluating the transition amplitudes reduces, at least formally, to the addition of complex numbers. The conceptual simplicity of Feynman quantum mechanics is matched by the difficulty of actually performing the path sum (3). It does, however, make a convenient starting point for a discussion of quantum measurements.

III. MEASURABLE PROPERTIES OF A QUANTUM SYSTEM

Next we may wish to enquire about the system's behavior in the time interval between its preparation in the state $|\psi_I\rangle$ and its subsequent detection in $|\psi_F\rangle$. We may not be interested in all the details, but rather just in the values of some quantity F. For the answer to exist, the value of F must be defined for each of the possible histories, i.e., it should be a functional $F[\text{path}]$ defined on the Feynman paths. It is reasonable to rearrange the paths according to the value of F, and define the probability amplitude to have this value equal to f as [$\delta(z)$ is the Dirac delta]

$$\Phi(F, I, t|f) = \sum_{\text{paths}} A[\text{path}] \delta(F[\text{path}] - f). \quad (4)$$

Equation (4) can be also written in an operator form,

$$\Phi(F, I, t|f) = \langle \psi_F | \hat{U}(t|f) | \psi_I \rangle, \quad (5)$$

where the restricted evolution operator propagates the initial state only along the paths which have the property $F[\text{path}] = f$. If there are several quantities $F_1, F_2, \ldots F_N$, the probability amplitude for them to have (jointly) the values $f_1, f_2 \ldots f_N$, $K(F, I, t|f_1, f_2, \ldots, f_N)$, can be constructed in a similar manner. With the part of the path summation (3) already performed, the full propagator is given by an ordinary quadrature,

$$K(F, I, t) = \int df_1 \int df_2 \cdots \int df_N \Phi(F, I, t|f_1, f_2, \ldots, f_N). \quad (6)$$

It is tempting to define the probability to have the property $F[\text{path}] = f$ as

$$P(F, I, t|f) \to |\Phi(F, I, t|f)|^2. \quad (7)$$

Next we will show that this temptation should (so far) be resisted.

IV. PATH RESTRICTION VS DYNAMICAL INTERACTION

The task of converting the amplitudes (4) into probabilities requires some additional care. Our rearrangement of the Feynman histories into classes was purely cosmetic. Like the paths themselves, the classes remain interfering alternatives. To assign the probabilities one must first destroy the interference. This must be done by bringing the system into contact with another quantum system or systems, and we must decide what type of an additional system (a meter) is suitable for the task.

Conveniently, the answer is already contained in our choice of the functional F. Consider the equation of motion satisfied by the restricted path integral (4). If this equation maps onto a Schrödinger equation describing the original system *plus* another system, we immediately obtain a recipe for constructing the meter with the desired properties. If, on the other hand, the equation of motion does not look like a Schrödinger equation, we must stop and admit that the interference cannot be destroyed by any means available to us. The question of whether further progress can be made in this latter case is beyond the scope of this paper.

The type of equation satisfied by the restricted propagator depends on the choice of the functional F, and must be established in each individual case. One general result was proven in Refs. [18]: let the functional be of the form

$$F[\text{path}] = \int_0^t \beta(t') a(x(t')) dt', \quad (8)$$

where $\beta(t)$ is a known function of time and $a(x)$ is some function of x. Then the probability amplitude for the system to arrive at a "location" $|x\rangle$ by taking only the paths with the properties $F[\text{path}] = f$, $\Phi(x, t|f)$, (we drop the implied dependence in the initial state $|\psi_I\rangle$ to simplify notations) satisfies the Schrödinger equation (SE)

$$\partial_t \Phi = [\hat{H} - i\partial_f \beta(t)\hat{A}]\Phi, \tag{9}$$

with the initial condition

$$\Phi(x, 0|f) = \langle x|\psi_I\rangle\delta(f). \tag{10}$$

Here the operator $\hat{A}$, diagonal in the chosen representation, is defined by the choice of $a(x)$ in Eq. (8),

$$\hat{A} \equiv \int dx|x\rangle a(x)\langle x|. \tag{11}$$

Equation (9) is, of course, an SE describing the system interacting with a von Neumann meter [19], with pointer position f, designed to measure the operator $\hat{A}$. Unlike in the original von Neumann approach, the pointer remains coupled to the system for a finite time, since the quantity F in Eq. (8) refers to the whole of the time interval $[0, t]$.

V. FINITE-TIME MEASUREMENTS VS THEIR CONTINUOUS COUNTERPARTS

Before proceeding we must emphasize an important distinction between two kinds of measurements. Suppose we have only one meter, and—in the spirit of Refs. [1,9]—activate it through a sequence of sharp strong pulses; that is, we choose

$$\beta(t') = \sum_{n=1}^{N} \delta(t' - t_n), \qquad t_n = nt/(N+1). \tag{12}$$

Assume, for simplicity, that the operator $\hat{A}$ in Eq. (11) is a projector onto a part of the Hilbert space Ω. Then the pointer would move one notch to the right each time the system is in Ω at $t = t_n$; otherwise, it remains where it is. At the time t we look at the pointer once, and, having found it shifted by m notches, conclude that the system was in Ω m times out of all N trials. We cannot, however, say exactly when this happened, as the outcome of the measurement is a single number yielding the value of a functional on otherwise unspecified virtual trajectories. Since the observation took t seconds to complete, we call it a *finite-time* measurement.

Suppose that instead we have N identical meters, only one of which is briefly activated at each $t = t_n$; that is, we choose

$$\beta_n(t') = \delta(t' - t_n). \tag{13}$$

As a result, at t we have N readings, all either 0 or 1, and the exact knowledge of the t_n at which the system was found in Ω. Accordingly, there is a set of probabilities $P(x_1, x_2, \ldots x_N)$, $x_n = \pm 1$. This is a prototype of a *continuous* measurement [20], whose outcome is a real trajectory followed by the observed system. Like the authors of Ref. [1] we are interested in finite-time measurements, which we will consider throughout the rest of the paper.

VI. THREE QUESTIONS TO DEFINE A QUANTUM MEASUREMENT

We see now that to describe a quantum measurement, one must provide clear answers to at least three questions:

(a) What is being measured?

(b) How is it being measured?

(c) To what accuracy is it being measured?

To answer the first question, we need to specify a variable that characterizes the system in the absence of a meter. In our case it is the functional in Eq. (8), whose physical meaning depends on the choice of the switching function $\beta(t')$. Thus, for $\beta(t') = 1/t = \text{const}$, it represents the time average of the quantity $\hat{A}$, and for $\beta(t') = \delta(t' - t_0)$, the instantaneous value of $\hat{A}$. Choosing $\beta(t') = \delta(t' - t_1) \pm \delta(t' - t_2)$ allows us to measure the sum or the difference of the values of A at t_1 and t_2, and so forth.

To answer the second question one must ask first if there is a suitable meter to be found. For a functional of the type (8) the answer is "yes". One should then prepare the system in the desired initial state, set the pointer to zero, turn on the interaction, and accurately measure the pointer's position at the time t.

The third question follows from the second. The amplitude Φ in Eq. (9) is not normalizable due to the presence of the $\delta(f)$ in Eq. (10), and cannot be used to construct physical probabilities. To obtain a physical amplitude $\Psi(x, 0|f)$, one should prepare the pointer in a physical state $G(f)$, $\int df|G(f)|^2 = 1$, and replace the initial condition (10) with

$$\Psi(x, 0|f) = \langle x|\psi_I\rangle G(f). \tag{14}$$

The result can be written in an equivalent form [18],

$$\Psi(x, t|f) = \int df' G(f - f')\Phi(x, t|f'), \tag{15}$$

which has a simple interpretation. The function $G(f - f')$ plays the role of a filter, selecting a limited range of the values of $F[x(t)]$'s which contribute to the pointer's advancement to position f. For an accurate measurement one should choose $G(f)$ sharply peaked around zero, with a small yet finite width Δf. Now

$$P(x, t|f) = |\Psi(x, t|f)|^2, \tag{16}$$

yields the probability of finding the system in $|x\rangle$ and the pointer at a location f. It is also the probability that the *observed* system arrives at x and F has the value in the interval $[f - \Delta f, f + \Delta f]$. Accordingly,

$$P(t|f) = \int dx P(x, t|f), \tag{17}$$

yields the probability of finding the pointer at x without asking about the state of the system. It is also the probability that, for the observed system, F has a value within $[f - \Delta f, f + \Delta f]$ regardless of where it ends up once the measurement is finished. The correct normalization of $P(x, t|f)$,

$$\int dx df P(x, t|f) = 1, \tag{18}$$

is guaranteed, since the evolution according to the SE (9) is unitary. We note that the limitation on the accuracy of a measurement is of a purely quantum nature: the values inside the interval $[f - \Delta f, f + \Delta f]$ cannot be distinguished since interference between them has not been destroyed.

VII. STATE OF THE SYSTEM AFTER A MEASUREMENT

We note that by restricting the evolution to paths with the property $F[\text{path}] = f$, we effectively "chop" the state of the system into sub-states

$$|\Phi(t|f)\rangle = \int dx|x\rangle\Phi(x, t|f), \tag{19}$$

which add up to the Schrödinger state of a freely evolving system,

$$\int df|\Phi(x, t|f)\rangle = \exp(-i\hat{H}t)|\psi_I\rangle. \tag{20}$$

This is the particular property of the interaction in Eq. (9). It can be interpreted as the quantum analog of the condition that a classical meter should monitor the measured system without affecting its evolution (for details see Ref. [21]). A meter of finite accuracy Δf uses linear combinations of the fine-grained sub-states (19)

$$|\Psi(t|f)\rangle = \int df' G(f - f')|\Phi(t|f)\rangle, \tag{21}$$

and then destroys the coherence between these "coarse-grained" states. The final mixed state of the system $\hat{\rho}$ after measurement, therefore, is

$$\hat{\rho}(t) = \int df|\Psi(t|f)\rangle\langle\Psi(t|f)|, \tag{22}$$

with $\mathrm{Tr}\hat{\rho}(t) = 1$, as follows from Eq. (18).

VIII. THE LIKELIHOOD OF A "ZENO EFFECT"

The convolution formula (15) has the advantage that if the main properties of the fine-grained amplitude $\Phi(x, t|f)$ are known, we can qualitatively estimate the result of smearing it with the function $G(f)$. For example, suppose that a transition is classically allowed. Then, in the semiclassical limit, a rapidly oscillating $\Phi(x, t|f)$ has a narrow stationary region around the classical value of $F[x(t)]$, f_{cl}. Then a not-too-accurate meter will always return the value f_{cl}, since if $G(f - f')$ is centred at any $f \neq f_{\mathrm{cl}}$ the integral (15) will vanish, destroyed by the oscillations [22].

It is possible to make another general statement about the properties of $\Phi(x, t|f)$. For a system confined to a finite interval (volume) $a \leq x \leq b$, the amplitude distribution $\Phi(x, t|f)$ cannot be a smooth function for all x. Rather, to ensure the conservation of probability in the high-accuracy limit $\Delta f \to 0$, it must have singularities, typically, of the Dirac delta type.

The proof is based on the conservation of probability. Suppose we have some reference function $G(f)$ and want to improve the accuracy by making it narrower. This can be achieved by a simple scaling,

$$G(f) \to G_\alpha(f) = \alpha^{1/2}G(\alpha f), \tag{23}$$

where we recall that $G(f)$ is also the initial state of the pointer, and as such must be normalized to unity, $\int df|G_\alpha(f)|^2 = 1$. Suppose now that $\Phi(x, t|f)$ is smooth. Then, as $\alpha \to \infty$, the width of $G_\alpha \sim \Delta f/\alpha \to 0$, and we should be able to take Φ outside of the integral (15),

$$\Psi(x, t|f) = \int df' G_\alpha(f - f')\Phi(x, t|f') \approx \Phi(x, t|f)\int G_\alpha(f')df' = \alpha^{-1/2}C\Phi(x, t|f), \tag{24}$$

where $C = \int G(f')df'$. In the high-accuracy (ideal measurement) limit $\alpha \to 0$, the rhs of Eq. (24) vanishes, and with it vanishes the probability to find the pointer at any location f, $P(t|f) = \int_a^b |\Psi(x, t|f)|^2 dx$, which is of course wrong. The way around this difficulty is to assume that in addition to its smooth part $\tilde{\Phi}(x, t|f)$, $\Phi(x, t|f)$ has a number of δ singularities,

$$\Phi(x, t|f) = \tilde{\Phi}(x, t|f) + \sum c_k(x)\delta(f - f_k). \tag{25}$$

Now, as the accuracy improves, the probability $P(x, t|f)$ becomes

$$\lim_{\alpha\to\infty} P(x, t|f) = \sum_k |c_k(x)|^2\delta(f - f_k) + \alpha^{-1}|C|^2|\tilde{\Phi}(x, t|f)|^2. \tag{26}$$

Suppose now that one is conducting an experiment on a large number of identical systems, all post-selected in the state $|x\rangle$, and receives a signal proportional to the number of cases in which the pointer is found in f. As the accuracy improves, the signal is dominated by strong peaks at $f = f_k$. In addition, there is a smaller signal revealing more and more details of the structure of $|\tilde{\Phi}(x, t|f)|^2$, and eventually fading altogether. The positions of the peaks f_k and the coefficients c_k must be determined for each particular case.

For example, we have previously shown [23] that for a system in a finite-dimensional Hilbert space, an attempt to determine precisely the value of the time average of an operator $\hat{A}$ inevitably leads to a Zeno effect, trapping the system in the eigenstates (eigen sub-spaces of $\hat{A}$). Another example of this behavior was given in Ref. [24], which analyzed a measurement of a qubit's residence time with a slightly more sophisticated variant of the von Neumann meter.

For a system in an infinite volume, e.g., for $-\infty < x < \infty$, there exists another possibility. While $|\Psi(x, t|f)|^2$ must vanish as $\hat{A} \to \infty$, the range of x's involved in the integral (17) may increase proportionally, so that the probability is conserved without $\Phi(x, t|f)$ acquiring singular terms. Physically this would mean that the meter scatters the observed system into a wide range of its final positions.

For the following, it suffices to note that—as follows from Eq. (26)—an accurate determination of the value of any quantity of the type (8) may suppress some of the transitions and lead to sharply defined values $F[\text{path}] = f_k$ in those transitions which survive. With some hesitation we follow the authors of Ref. [1] in also calling it a "Zeno effect" in the case where the measured system has a continuous spectrum. In a conventional Zeno effect [25–27], frequent observations trap the system in one of the eigenstates of the measured quantity. Here, the action of the meter restricts the system to a particular type of evolution without freezing it altogether. With this result we are ready to analyze some of the cases discussed in Ref. [1].

IX. THE PROBABILITY OF NOT ENTERING THE RIGHT HALF-SPACE

We start with the task of defining the probability that, in one dimension, a free particle of mass M does not enter the region $\Omega \equiv 0 \le x < \infty$ (see Sec. IV of Ref. [1]). Equivalently, one can ask: what is the probability that the time that the particle spends in Ω is zero, $\tau = 0$? The functional yielding the duration a Feynman paths spends in Ω is well known [2],

$$t_\Omega(t, [x(\cdot)]) = \int_0^t \theta_\Omega(x(t'))dt', \tag{27}$$

where $\theta_\Omega(z) = 1$ for z inside Ω, and zero otherwise. The operator in Eq. (15) is just the projector onto the right half-space, $\hat{A} = \hat{P}_\Omega = \int_0^\infty dx|x\rangle\langle x|$. The coupling $-i\partial_\tau \theta_\Omega(x)$ allows one to identify the meter as a continuous version of the Larmor clock [28–30], a large magnetic moment which precesses in a magnetic field for as long as the particle remains inside Ω. The solution to Eq. (9) is given by the Fourier integral [6]

$$\begin{aligned}\Phi(x, t|\tau) &= (2\pi)^{-1}\int_{-\infty}^{\infty} dV \exp(iV\tau)\psi_V(x, t)\\ \psi_V(x, t) &\equiv \langle x|\exp[-i\hat{H}_V t)|\psi_I\rangle,\end{aligned} \tag{28}$$

where [we use $\theta(x)$ for $\theta_{[0,\infty)}$]

$$\hat{H}_V = p^2/2m + V\theta(x). \tag{29}$$

In other words, to find the traversal time amplitude distribution one needs to know the results of evolving the initial state for all potential steps added in the right half-space—even though we are discussing the properties of a free particle. The transmission (T) and reflection (R) amplitudes for such a step at an energy E are easily found to be ($k = \sqrt{2ME}$)

$$T(k, V) = 2/[1 + (1 - V/E)^{1/2}], \qquad R = T - 1. \tag{30}$$

Now if the initial state is a wave packet initially in the left half-space and moving from left to right,

$$\begin{aligned}\langle x|\psi_I\rangle &= \int dk A(k)\exp[ikx - iE(k)t],\\ \langle x|\psi_I\rangle &\equiv 0 \quad \text{for } x \ge 0,\end{aligned} \tag{31}$$

it evolves into

$$\begin{aligned}\psi_V(x, t) = &\int dk T(k, V)A(k)\exp[ikx - iE(k)t]\\ &- \int dk A(k)\exp[-ikx - iE(k)t]\\ &+ \int dk T(k, V)A(k)\exp[-ikx - iE(k)t].\end{aligned} \tag{32}$$

Here the first term is the transmitted part, the second term describes the reflection from an infinite step, $V = \infty$, and the third reflected term accounts for the fact that the step is not, after all, infinite. We will consider t to be so large that the transmitted and reflected wave packets are well separated and do not overlap. With the help of Eq. (28) the amplitude distribution for the duration τ spent by a free particle in the right half-space becomes [cf. Eq. (29)]

$$\Phi(x, t|\tau) = \begin{cases} \tilde{\Phi}(x, t|\tau) & \text{for } x > 0\\ -\delta(\tau)\psi_0(-x, t) + \tilde{\Phi}(-x, t|\tau) & \text{for } x < 0. \end{cases} \tag{33}$$

Here $\psi_0(x, t) \equiv \langle x|\exp[-i\hat{H}t)|\psi_I\rangle$ is just the freely propagating wave packet, and $\tilde{\Phi}(x, t|\tau)$ is a smooth function involving the Fourier transform of $T(k, V)$ at all relevant energies. The last two terms clearly correspond to the particle being reflected; for example, we have

$$-\psi_0(-x, t) \equiv \psi_\infty(x, t), \tag{34}$$

where $\psi_\infty(x, t)$ is the wave packet reflected by an infinite potential wall, $V = \infty$. Again, this may seem strange since everything said so far has referred to a free particle.

This is, therefore, our central proposition: the assignment of probabilities to interfering classes of quantum histories should not be considered outside the context of measurements of the property of interest. We acknowledge that the statement is at odds with the decoherent histories approach, which would inevitably see it as too narrow. Earlier attempts to apply the DHA to the traversal time problem were made in Ref. [31]. Our criticism of the approach of Ref. [31] can be found in Ref. [32]. There, we argued that applying the DHA to most primitive closed systems, e.g., those consisting of a single particle, adds little to our understanding. We will briefly return to this issue in Sec. VII.

So far we have only contemplated a classification of Feynman paths according to the duration spent in Ω, yet the result already contains a reference to the reflection that would be caused by a Larmor clock, should we decide to employ one. Moreover, the fine-grained amplitude $\Phi(x, t|\tau)$ contains information about all measurement scenarios, which are, in turn, specified by the choice of the filter G in Eq. (21). Suppose, for example, that we decide to not make a measurement at all by making $G(f - f')$ in Eq. (10) so broad that it can be replaced by a constant. It is useful to note that, since $(2\pi)^{-1} \int d\tau \int dV \exp(iV\tau)T(k, V) = T(k, 0) = 1$, $\tilde{\Phi}(x, t|\tau)$ add up to the freely propagating pulse,

$$\int_0^t \tilde{\Phi}(x, t|\tau)d\tau = \psi_0(x, t). \tag{35}$$

Thus by not taking a measurement we make the last two terms in Eq. (33) cancel, leaving us, as it should, with only the free wave packet travelling to the right of the origin $x = 0$.

Alternatively, we may want to know the probability of spending *precisely* a duration τ in the right half-space. Again, we cannot avoid employing a highly accurate meter. The result of our attempt is known from Sec. VII: we can neglect all the terms except those that are singular in τ, thus obtaining

$$P(x, t|\tau) = |\psi_\infty(x, t)|^2 \delta(\tau), \tag{36}$$

which corresponds to the wave packet reflected as if by an infinite wall at the meter with only zero readings. In light of what was said above, this is hardly surprising. Classically, one can arrange a meter which would not perturb the measured system. One can also set the meter in such a way that it would act on the system, and then correctly measure the variable for the system perturbed by the measurement itself [21]. Quantally, the first option is not available since it is necessary to destroy interference. Our ideal measurement is, in this sense, correct: in order to measure τ to a great accuracy, the meter would need to exert a force which would not let the particle enter the region, and then confirm the zero result. In a similar way, we can analyze the "softer" measurements involving different shapes and widths of the function G. For example, a detailed analysis of Aharonov's "weak measurements" [33–35] can be found in Ref. [36].

Equally important are the restrictions that the above principle puts on what one can use the fine-grained amplitudes for. For example, simply summing $\tilde{\Phi}(x, t|\tau)$ over a certain range of τ, squaring the modulus of the result, and declaring it the probability of having τ within a range is dangerous. The authors of Ref. [1] have tried separating the amplitudes into just two classes—those for which τ is zero and those for which it is not—so that the two amplitudes are given by $A(x, t|\tau = 0) = -\psi_0(x, t)$ and $A(x, t|\tau \neq 0) = \int d\tau[\tilde{\Phi}(x, t|\tau) + \tilde{\Phi}(-x, t|\tau)] = \psi_0(x, t) + \psi_0(-x, t)$. Now the probability to have entered the right half-space appears to have the value

$$P(t|\tau \neq 0) \equiv \int dx |A(x, t|\tau)|^2 = \int dx |\psi_\infty(x, t|\tau)|^2 + \int dx |\psi_0(x, t|\tau)|^2 = 2, \tag{37}$$

which, as the authors of Ref. [1] pointed out, must be wrong. To find a reason for this discrepancy we revisit the measurement as defined by Eqs. (9) and (21). Apparently, no initial state of the meter $G(f)$ effects the coarse-graining of the fine-grained amplitudes into these two classes. We therefore no longer have Eq. (18), which is itself a consequence of the unitarity of the system-meter evolution. Moreover, the nonconservation of the number of particles in Eq. (37) suggests that the probabilities $P(t|\tau \neq 0)$ and $P(t|\tau = 0)$ are not measurable by any other scheme, short of injecting more particles into the system.

X. THE PROBABILITY OF NOT BEING ABSORBED IN THE RIGHT HALF-SPACE

Another case, discussed in Sec. III Ref. [1], is the absorption of a particle by an optical potential confined to the right half-space,

$$U(x) = iU\theta(x). \tag{38}$$

As before, we consider a wave packet incident on the absorbing potential from the left, and a time t so large that a free pulse would be fully contained to the right of the origin $x = 0$. Following Ref. [1], we wish to approximate the probability of not entering the right half-space with the probability of not being absorbed by $U(x)$. The problem is easily solved by the technique of the previous section. The amplitude of not being absorbed after travelling along a Feynman path $x(t')$ is obviously $\exp\{iS_0[x(t')] - Ut_\Omega[x(t')]\}$, with S_0 denoting the free-particle action. The amplitude of arriving in x by travelling along all paths satisfying $t_\Omega[x(t')] = \tau$ is, therefore, $\Psi(x, t|\tau)\exp(-U\tau)$, and the amplitude of arriving there at all is

$$\psi_U(x, t) = \int_0^t \exp(-U\tau')\Phi(x, t|\tau')d\tau', \tag{39}$$

where $\Phi(x, t|\tau')$ is given by Eq. (33). This is very similar to the amplitude of obtaining a zero reading if one measures the duration spent in the right half-space by a meter whose initial state $G(\tau)$ is $\theta(\tau)\exp(-U\tau)$. We therefore already know what will happen if one increases the absorption U

in order to eliminate the particles that have entered the right half-space, and then managed to survive until t. Since $\int d\tau'\theta(\tau')\exp(-U\tau') = 1/U$, which vanishes as $U \to \infty$, the contributions from the smooth part of $\Phi(x, t|\tau')$ will vanish, leaving us again with the particle fully reflected from the origin,

$$\psi_U(x, t) \approx \psi_\infty(x, t), \quad \text{as } U \to \infty. \tag{40}$$

This result is equivalent to two complimentary statements: (a) restricting an evolution to the Feynman paths that do not enter a spacial region leads to a perfect reflection from the region's boundary, and (b) such a restriction can be achieved by introducing a large absorbing potential, which is the physical cause of the reflection.

XI. THE TIME OF CROSSING INTO THE RIGHT HALF-SPACE FOR THE FIRST TIME

Another case discussed in Sec. I of Ref. [1] involves controlling the first time a system enters a given region of space Ω or, more generally, a certain sub-space of its Hilbert space. For a particle of mass M in one dimension, we construct a functional whose value gives the time a Feynman path enters Ω for the first time,

$$\Theta_\Omega[x(\cdot)] = \lim_{\gamma\to\infty}\int_0^t dt' \exp\{-\gamma t_\Omega(t', [x(\cdot)])\}, \tag{41}$$

where $t_\Omega([x(t'), t)]$ is the traversal time functional defined in Eq. (27) and γ is a positive constant. The functional adds up dt's for as long as $\exp\{-\gamma t_\Omega[x(t)]$ is not zero, i.e., for as long as the path has made no incursion into Ω, and marks the moment the border of Ω is crossed for the first time. If the path originates from inside Ω at $t = 0$, the value of $\Theta_\Omega[x(t)]$ is set to zero. If a path has not yet visited Ω by the time t, the value is set to t. In this way, every Feynman path is labeled by its first crossing time, and we can rearrange the paths into the classes, as was done previously. For the amplitude of crossing into $\Omega = [0, \infty)$ for the first time at a time τ, $0 \le \tau \le t$, we have

$$\Phi(x, t|\tau) = \int dx'\langle x'|\psi_I\rangle \times \int_{x(0)=x'}^{x(t)=x} Dx \exp\{iS_0[x(\cdot)\}\delta(\Theta_\Omega[x(\cdot)] - \tau), \tag{42}$$

where the path integration is over the paths starting at $t = 0$ in x' and ending in x at the time t. The first crossing-time expansion has been derived by many authors [37–40], and in the Appendix we offer yet another one based on the direct evaluation of the restricted path integral in Eq. (42). For a wave packet (31) approaching the origin from the left, from Eq. (A5) we have

$$\Phi(x, t|\tau) = \psi_\infty(x, t)\delta(\tau - t) + (i/2M)\theta(\tau)\theta(t - \tau) \times K(x, 0, t - \tau)\partial_x\psi_\infty(0, \tau), \tag{43}$$

where, as before, $\psi_\infty(x, t)$ is the wave packet scattered by an infinite wall at $x = 0$. We see that initially [0_- stands for $\lim_{\epsilon\to 0}(0 - \epsilon)$]

$$\Phi(x, 0_-|\tau) = \psi_I(x)\delta(\tau), \tag{44}$$

and, using Eq. (A4), find the equation of motion,

$$i\partial_t\Phi(x, t|\tau) = \hat{H}\Phi(x, t|\tau) - i\partial_\tau[\delta(\tau - t)\psi_\infty(x, t)\theta(t)]. \tag{45}$$

Integrating Eq. (45) over τ shows that

$$\int d\tau\Phi(x, t|\tau) = \psi(x, t), \tag{46}$$

which, together with Eq. (43), gives the standard first-crossing-time expansion, used for example in Ref. [39]. This completes the first task outlined in Sec. VI; that is, defining the quantity of interest.

We do, however, fail in the second task; that is, specifying a meter for the first crossing time. Indeed, Eq. (45) does not look like a SE describing the interaction between two quantum systems. It is nonhomogenous, with the source term fully determined by the evolution in the left half-space with an infinite wall at the origin. We cannot even guarantee that it conserves the probability. Indeed, with the help of Eq. (10), by constricting a square integrable solution to represent the particle and a potential meter,

$$\Psi(x, 0_-|\tau) = \psi_I(x)G(\tau), \qquad \int|\Psi(x, 0_-|\tau)|^2 d\tau dx = 1, \tag{47}$$

and using Eq. (45) to evaluate the rate of change of $P(t) = \int|\Psi(x, t|\tau)|^2 d\tau dx$, we find

$$\frac{dP(t)}{dt} = 2\mathrm{Re}\left[\int_0^\infty d\tau\partial_\tau G(\tau - t)\int dx\psi_\infty(x, t)\Psi^*(x, t|\tau)\right]. \tag{48}$$

It is unlikely that the rhs of Eq. (48) vanishes identically, and we abandon our attempts to find probabilities for the first crossing time (41), just as we promised we would in the second passage of Sec. IV.

XII. CONCLUSIONS AND DISCUSSION

In summary, assigning probabilities to interfering alternatives implies the destruction of coherence between the alternatives. The destruction of interference [e.g., the conversion of a pure state (20) into a statistical mixture (22)] is a physical process, and must be executed by a physical agent, which we call a meter. This puts serious restrictions on the probabilities that we may construct.

Firstly, such a meter must exist. The precise type of the interaction required to destroy the interference is prescribed by the property of Feynman paths that we wish to control. It may happen that it does not correspond to a coupling, which is physically acceptable.

Secondly, even if the meter exists, it must be capable of causing the desired separation of Feynman paths into classes. This requires finding an acceptable initial state for the meter, which may not always be possible.

Thirdly, a measurement must be classified according to its accuracy, which is determined by the width and shape of the initial meter's state. The range of possible measurements stretches for highly inaccurate "weak" measurements to highly accurate "ideal" ones.

A simple analysis of Sec. VIII shows that an ideal measurement of the type discussed in Ref. [1] may lead to a kind of Zeno effect. This is neither an "unphysical" result nor a "pitfall" of a theory, but rather a general quantum mechanical rule. To even contemplate the accurate value of a variable F, one must assume that the interference has been destroyed to the required degree, i.e., one must also consider the effects of an external meter. The meter would then perturb the system and yield a sharply defined value of F, which correctly describes this perturbed motion.

This "Zeno effect" cannot be avoided completely. It can, however, be mitigated, e.g., by requesting less information about F, and reducing the accuracy of the measurement Δf. The authors of Refs. [1,9] chose to consider an ideal measurement of a quantity obtained by replacing the integral (8) with a discrete sum (12). With this, the system is allowed to evolve freely between t_n and t_{n-1}, and is not reduced to the Zeno-like evolution if ϵ is kept sufficiently large. In general, one is lead to consider finite-time measurements of different quantities to different accuracies, varying Δf and ϵ, to achieve a desired ratio between the information obtained and the perturbation incurred.

We illustrate the above with a brief review of the examples considered in Sec. IX, X, and XI

(a) The probability of a free particle not entering the right half-space equals the probability of spending a zero net duration there. This duration is represented by the traversal time functional, and the relevant meter exists as a continuous version of the Larmor clock [28–30]. An accurate clock prevents the particle from entering the region and, under the circumstances, the probability of not entering is unity.

(b) The probability amplitude of entering the right half-space cannot be defined as the net probability of spending any duration other than zero there. Even though the meter (Larmor clock) exists, it cannot be set up to separate the Feynman paths into these two classes. One may also be confident that this cannot be done by any other means, since probabilities defined in this way do not add up to unity.

(c) The probability of not being absorbed in the right half-space is similar to the probability of not entering it, and tends to unity as the magnitude of the absorbing potential increases. This is a different way of saying that an infinite absorbing potential must reflect all incoming particles.

(d) We have found no meaningful probabilities associated with the first-crossing-time amplitudes (43), as no physical meter of the type discussed here can be realized. To identify the time of the first crossing, one needs the record of the particle's past. For this reason, even classically, we cannot construct a single pointer that would stop once the particle first crosses into the region of interest.

Finally, we note that in all the above cases we have failed to arrive at the classical limit. This may cause some concern [1]. Where a meter for our finite-time measurement exists, the remedy is simple [21]. We do not improve accuracy indefinitely, but stop while Δf exceeds the width of the stationary region of the fine-grained distribution in Eq. (9). This region occurs around the classical value of F, $f_{\rm cl}$, and is very narrow if the system is nearly classical. With the accuracy chosen to be high yet finite, only this region contributes to the integral (15). Thus the pointer always points at $f_{\rm cl}$, and we can replace the SE by the classical equations of motion.

Where no meter exists, the situation appears to be more difficult. Classically, one can always define the first crossing time, seemingly with no reference to meters or measurements. This is not quite so, as one always has at their disposal a classical trajectory $\bar{x}(t)$ from which all other quantities, including the first crossing time, can be derived. From the quantum mechanical point of view, $\bar{x}(t)$ comprises the results of measuring the particle's positions at all times. This suggests that the first crossing time should be determined in a continuous quantum measurement, yielding a sequence of the particle's positions $x(t)$, and then evaluating the functional (41) on this 'real" (rather than virtual) trajectory. A more detailed analysis will be given in our future work [41].

Finally, throughout this paper we have advocated a measurement-based approach incompatible, at first glance, with the decoherent histories approach, which makes no explicit mention of measurements. We believe that the two approaches can be reconciled by defining their respective domains of applicability. One possibility would be to apply the DHA not to a system on its own, but rather to the composite system plus meter in those cases where the meter can be constructed.

ACKNOWLEDGMENTS

We acknowledge support from the Basque Government (Grant No. IT-472-10), and the Ministry of Science and Innovation of Spain (Grant No. FIS2009-12773-C02-01). We are also grateful to J. N. L. Connor for bringing the issue to our attention.

APPENDIX

Consider the path integral in Eq. (42), keeping γ finite for the moment. Using the identity $\delta(z) = (2\pi)^{-1/2} \int \exp(i\lambda z)d\lambda$, we can rewrite Eq. (42) as

$$K(x, x', t|\tau) = (2\pi)^{-1/2} \int d\lambda \exp(i\lambda\tau) \int_{x(0)=x'}^{x(t)=x} Dx \times \exp\left\{ i\int_0^t dt' \left[L - \lambda \exp\left(-\gamma \int_0^{t'} \theta_\Omega(x(t''))dt'' \right) \right] \right\}, \tag{A1}$$

where $L(x, \dot{x})$ is the particle's Lagrangian. Expanding the second exponential we find the amplitude for a path $x(t')$ to be

$$A'[x(t')] = \exp\left(i\int_0^t L dt' \right) \left\{ 1 + \sum_{n=1}^{\infty} (-i\lambda)^n \times \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \times \exp\left[-\gamma \sum_{m=1}^{n} \int_0^{t_m} \theta_\Omega(x(t'))dt' \right] \right\}. \tag{A2}$$

The first term in the curly brackets corresponds to free motion. The integrand of the nth term corresponds to the particle moving in a time-dependent optical potential. For $0 \le t' < t_{n-1}$ we have $n\gamma\theta_{\omega(x)}$, for $t_{n-1} \le t' < t_{n-2}$ the potential is reduced to $(n-1)\gamma\theta_{\omega(x)}$, and so on. For $t \le t' < t_1$ the absorbing potential is turned off. As we send $\gamma \to \infty$, the distinction between, say, the terms containing $n\gamma$ and $(n-1)\gamma$ disappears, and the particle moves in an infinite absorbing potential until t_n, after which it is switched off. The integrals $\int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n$ can now be evaluated to yield $t_n^{n-1}/(n-1)!$, and by summing over n we have

$$A'[x(t')] = \exp\left(i\int_0^t L dt' \right) - i\lambda \int_0^t dt_1 \exp(-i\lambda t_1) \times \exp\left(i\int_{t_1}^t L dt' + \int_0^{t_1} L_\infty dt' \right), \tag{A3}$$

where $L_\infty = \lim_{\gamma\to\infty} L - \gamma\theta_\Omega(x)$ is the Lagrangian with an infinite absorbing potential introduced in the right half-space. Inserting Eq. (A3) into Eq. (A1) yields the net amplitude on all paths connecting x' and x and first crossing the origin $x = 0$ at the time τ,

$$K(x, x', t|\tau) = K(x, x', t)\delta(\tau) - \partial_\tau \int dx'' K(x, x'', t-\tau) \times K_\infty(x'', x', \tau)dx'', \tag{A4}$$

where K_∞ and K are the propagators with and without an infinite absorbing potential in the right half-space, i.e., $K(x,x',t-\tau) \equiv \theta(t-\tau)\langle x|\exp(-i\hat{H}t)|x'\rangle$ and $K_\infty(x, x', \tau) \equiv \theta(\tau)\langle x|\exp(-i\hat{H}_\infty\tau)|x'\rangle$ where $\hat{H}_\infty \equiv \lim_{\gamma\to\infty} \hat{H} - \gamma\theta(x)$. Note that the infinite absorbing potential is equivalent to an infinite potential wall introduced at $x = 0$, so that

$$K_\infty(x, x', t) \equiv 0 \quad \text{for } x, x' \ge 0.$$

Differentiating the product in Eq. (A4) and noting that $\hat{H}\theta(x) - \theta(x)\hat{H}_\infty = -(1/2M)[\partial_x^2, \theta(x)] = (1/2M)\delta(x)\partial_x$, we rewrite Eq. (A4) as

$$K(x, x', t|\tau) = K(x, x', t)\delta(\tau)\theta(x) + K_\infty(x, x', t)\delta(\tau - t) + (i/2M)K(x, 0, t-\tau)\partial_{x''}K_\infty(x''=0, x', \tau), \tag{A5}$$

which is the standard form of the first-crossing-time amplitude for a particle traveling between x' and x [37–40].

[1] J. J. Halliwell and J. M. Yearsley, Phys. Rev. D **86**, 024016 (2012).
[2] D. Sokolovski and L. M. Baskin, Phys. Rev. A **36**, 4604 (1987).
[3] D. Sokolovski and J. N. L. Connor, Solid State Commun. **89**, 475 (1994).
[4] D. Sokolovski, S. Brouard, and J. N. L. Connor, Phys. Rev. A **50**, 1240 (1994).
[5] D. Sokolovski, Phys. Rev. A **52**, R5 (1995).
[6] D. Sokolovski, Phys. Rev. Lett. **79**, 4946 (1997).
[7] D. Sokolovski, Proc. R. Soc. A **460**, 1505 (2004).
[8] D. Sokolovski, in *Time in Quantum Mechanics*, edited by J. G. Muga, R. Sala Mayato, and I. L. Egusquiza (Springer, Berlin, 2008), p. 195.
[9] D. Alonso, R. Sala Mayato, and J. G. Muga, Phys. Rev. A **67**, 032105 (2003).
[10] R. B. Griffiths, J. Stat. Phys. **36**, 219 (1984); Am. J. Phys. **55**, 11 (1987); Phys. Rev. Lett. **70**, 2201 (1993).
[11] M. Gell-Mann and J. B. Hartle, Phys. Rev. A **76**, 022104 (2007).
[12] S. Goldstein and D. N. Page, Phys. Rev. Lett. **74**, 3715 (1995).
[13] J. J. Halliwell, Phys. Rev. D **60**, 105031 (1999).
[14] L. Diosi, Phys. Rev. Lett. **92**, 170401 (2004).
[15] J. B. Hartle, Phys. Rev. A **78**, 012108 (2008).
[16] Since the appearance of this work in the arXiv, the authors of Ref. [1] have acknowledged and partially addressed our criticisms. See J. J. Halliwell and J. M. Yearsley, arXiv:1301.4373.

[17] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).
[18] D. Sokolovski and R. S. Mayato, Phys. Rev. A **71**, 042101 (2005); **73**, 052115 (2006); **76**, 039903(E) (2007).
[19] J. v. Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, 1955), p. 183.
[20] M. B. Mensky, *Quantum Measurements and Path Integrals* (IOP, Bristol, 1993); *Quantum Measurements and Decoherence: Models and Phenomenology* (Kluwer, Dordrecht, 2000).
[21] D. Sokolovski, Phys. Rev. A **66**, 032101 (2002).
[22] D. Sokolovski, Phys. Rev. A **59**, 1003 (1999).
[23] D. Sokolovski, Phys. Rev. A **84**, 062117 (2011).
[24] D. Sokolovski, Phys. Rev. Lett. **102**, 230405 (2009).
[25] D. Home and M. A. B. Whitaker, Ann. Phys. (N.Y.) **258**, 237 (1997).
[26] P. Facchi and S. Pascazio, Phys. Rev. Lett. **89**, 080401 (2002).
[27] P. Facchi, H. Nakazato, and S. Pascazio, Phys. Rev. Lett. **86**, 2699 (2001).
[28] A. I. Baz', Yad. Fiz. **4**, 252 (1966) [Sov. J. Nucl. Phys. **4**, 182 (1967)]; Yad. Fiz. **4**, 182 (1966) [Sov. J. Nucl. Phys. **5**, 161 (1967)]; V. F. Rybachenko, Yad. Fiz. **5**, 895 (1967) [Sov. J. Nucl. Phys. **5**, 635 (1967)].
[29] M. Biittiker, Phys. Rev. B **27**, 6178 (1983).
[30] D. Sokolovski and J. N. L. Connor, Phys. Rev. A **47**, 4677 (1993).
[31] N. Yamada, Phys. Rev. Lett. **83**, 3350 (1999).
[32] D. Sokolovski, Phys. Rev. A **66**, 032107 (2002).
[33] Y. Aharonov, D. Albert, and L. Vaidman, Phys. Rev. Lett. **60**, 1351 (1988).
[34] Y. Aharonov, J. Anandan, S. Popescu, and L. Vaidman, Phys. Rev. Lett. **64**, 2965 (1990).
[35] Y. Aharonov and L. Vaidman, in *Time in Quantum Mechanics*, edited by G. Muga, R. Sala Mayato, and I. Egusquiza (Springer, Berlin/Heidelberg, 2008), 2nd ed., Vol. 1.
[36] D. Sokolovski, Phys. Rev. A **76**, 042125 (2007).
[37] T. E. Feuchtwang, Phys. Rev. B **10**, 4121 (1974).
[38] A. Auerbach and S. Kivelson. Nucl. Phys. **B257**, 799 (1985).
[39] J. J. Halliwell, Phys. Lett. A **207**, 237 (1995).
[40] J. J. Halliwell and J. M. Yearsley, Phys. Rev. A **79**, 0621101 (2009).
[41] D. Sokolovski (unpublished).