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Definition of the covariant lattice Dirac operator

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In the continuum the definitions of the covariant Dirac operator and of the gauge covariant derivative operator are tightly intertwined. We point out that the naive discretization of the gauge covariant derivative operator is related to the existence of local unitary covariant ladder operators which allow the definition of a natural lattice gauge covariant derivative. The associated lattice Dirac operator has all the properties of the classical continuum Dirac operator, in particular anti-Hermiticity and chiral invariance in the massless limit, but is of course nonlocal in accordance to the Nielsen-Ninomiya theorem. We show that this lattice Dirac operator coincides in the limit of an infinite lattice volume with the naive gauge covariant generalization of the SLAC derivative, but contains nontrivial boundary terms for finite-size lattices. Its numerical complexity compares pretty well on finite lattices with smeared lattice Dirac operators.

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I. INTRODUCTION

The standard mathematical description of the dynamics of the strong interactions, a description called quantum chromodynamics (QCD), is obtained by writing the partition function of an Euclidean $SU(3)$ quantum gauge field theory interacting with $N_f \geq 2$ fermions in the fundamental representation of the gauge group. The partition function of QCD is postulated by analogy with the path integral formalism of quantum electrodynamics (QED) which has been shown to be successful with a very high accuracy. Using the rules of Grassmannian integration, the QCD partition function can be formally written as a functional integral over the non-Abelian gauge degrees of freedom only,

$$Z = \int \mathcal{D}A_\mu \Big(\prod_f \det \hat{\mathcal{D}}_f \Big) e^{-S_G}. \tag{1}$$

The measure of integration in Eq. (1) can be interpreted as a formal probability measure over the space of gauge configurations because the Euclidean Dirac operator $\hat{\mathcal{D}}_f$ of each fermion flavor is anti-Hermitian (with the right boundary conditions) and chirally invariant in the massless limit,

$$\begin{aligned} \hat{\mathcal{D}}_f = \gamma_\mu D_\mu + m_f, \qquad D_\mu = \partial_\mu + igA_\mu, \\ \{\gamma_5, \gamma_\mu D_\mu\} = 0, \qquad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \qquad \gamma_\mu^\dagger = \gamma_\mu. \end{aligned} \tag{2}$$

Hence the eigenvalues of each operator $\hat{\mathcal{D}}_f$ come in complex conjugate pairs, up to a possible discrete set of zero modes of the operator $\gamma_\mu D_\mu$, which guarantees reality and positiveness of their determinant for massive fermions, $\det \hat{\mathcal{D}}_f > 0$.

The Euclidean gauge action (summation over repeated indices is implied throughout),

$$\begin{aligned} S_G &= \frac{1}{4} F_{\mu\nu} F_{\mu\nu}, \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu], \end{aligned} \tag{3}$$

is invariant under the local gauge transformations $G(x) \in SU(3)$,

$$A_\mu(x) \to G(x) A_\mu(x) G^{-1}(x) + \frac{i}{g} (\partial_\mu G(x)) G^{-1}(x), \tag{4}$$

whereas the operator $\hat{\mathcal{D}}_f$ transforms covariantly.

Physical observables can then be related to expectation values with respect to the probability measure (1) of certain gauge-invariant matrix elements $\mathcal{O}(A)$ of operators built out of the Dirac operators and their inverses, provided that the formal measure $\mathcal{D}A_\mu$ in (1) be given a precise gauge-invariant meaning through a constructive procedure.

Section II recalls briefly the main properties of Wilson's lattice regularization which is the only constructive proposal known to date [1]. The discretization of space-time in a finite box allows for the nonperturbative calculation of physical observables by means of numerical simulations. The approach has been very successful in describing many features of hadronic physics, except for one thing. It proves difficult to reproduce the continuum physics with the physical light quark masses.

The reason is well understood [2] and resides in the formulation of lattice fermions. The Lorentz invariant regularization of quantum fluctuations in continuum QCD generate a chiral anomaly which cannot be duplicated on the lattice with a discretization of the Dirac operator which is local, chirally symmetric and contains the correct number of fermionic degrees of freedom in the continuum limit. This result is usually referred to as the no-go theorem. The standard avoidance is to hold to local fermions and break chiral symmetry explicitly.

In Sec. III we reconsider the naive discretization of the covariant derivative and identify a set of local unitary

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covariant ladder operators which enables the definition of another lattice gauge covariant derivative with the same algebraic properties as in the continuum. This lattice derivative is nonlocal in accordance to the no-go theorem and coincides in the free limit with the SLAC derivative [3]. The lattice fermion formulation based on the SLAC derivative has been extensively discussed three decades ago. A general consensus has emerged according to which formulations of nonlocal lattice fermions coupled to gauge fields lead to various inconsistencies in weak coupling perturbation theory [4–8], and cannot reproduce the continuum limit properly, in particular the axial anomaly [9]. In fact we are not aware of a single numerical study of the functional integral of a SLAC-type fermion coupled to a compact lattice gauge field.

Nonetheless we think it is fair to state that none of the objections to nonlocal fermions has the status of a no-go theorem. In all studies to date, the coupling of a SLAC fermion to a gauge field on the lattice has been written by mimicking the textbook derivation of a local gauge symmetry in the continuum. The result is correct only for infinite lattices. The boundary conditions on finite-size lattices are not taken into account by the standard technique. The unitary covariant ladder operators exhibited in Sec. III are the right tools to include boundary conditions. As an example, in Sec. IV we diagonalize these unitary operators on periodic lattices and express in Sec. V the matrix elements of the associated lattice Dirac operator in configuration space. We find nontrivial boundary contributions which vanish only in the limit of infinite physical volume.

We discuss in Sec. VI the localization properties of this nonlocal lattice Dirac operator, particularly with respect to the standard locality condition put forward to guarantee the universality of the continuum limit [10]. We prove that the approach to the continuum limit of the proposed operator is in fact far better than for conventional local operators when the action is upon smooth enough classical field configurations. The exact matrix elements obtained in Sec. V are crucial to derive this result. More generally, the constraints of the underlying locality of the exponentiated nonlocal lattice gauge covariant derivative probably cannot be neglected in analyzing the weak coupling perturbation theory and its renormalization, even in the infinite lattice volume limit. The intent of the present work is not to address this complicated issue, which deserves a separate work, but to stress its existence.

In the concluding remarks we point out that the numerical complexity of the associated finite-size lattice Dirac operator is similar to a five-dimensional local Dirac operator. In fact the algorithmic implementation is much simpler. Moreover this nonlocal Dirac operator can be naturally interpreted as smeared over the Wilson lines. This smearing has the virtue to be completely analytic. Numerical tests of the meaningfulness of the finite-size formulation of nonlocal fermions coupled to gauge fields can easily be performed on a single desktop computer in the quenched approximation up to four dimensions.

II. LATTICE REGULARIZATION

As is well known, Wilson's formulation consists of regularizing the Euclidean continuum gauge theory on a finite four-dimensional lattice $\mathcal{L}$ with hypercubic cells of spacing a, and of replacing the continuum gauge degrees of freedom, the gauge potential $A_\mu(x)$ which belongs to the $SU(3)$ Lie algebra, by variables $U_{x,\mu}$ associated to each link $(x, x + a\hat{\mu})$ of the lattice and which belong to the $SU(3)$ group manifold. Then Eq. (1) becomes

$$\mathcal{Z}_{\mathcal{L}} = \int \bigg(\prod_{(x,x+a\hat{\mu})\in\mathcal{L}} dU_{x,\mu} \bigg) \bigg(\prod_f \det \hat{\mathcal{D}}_f(U) \bigg) e^{-S_G(U)}, \tag{5}$$

where the integration measure is now a perfectly well-defined finite product of gauge-invariant Haar measures over the $SU(3)$ group manifold, and $S_G(U)$ and $\hat{\mathcal{D}}_f(U)$ are discretized versions of the continuum gauge action and Dirac operators. This measure can be evaluated numerically by stochastic importance sampling.

There is a large arbitrariness in the choice of lattice gauge action and lattice Dirac operators. The main constraint is that the lattice regularized model possess a second-order critical point which reproduces the asymptotic freedom of QCD in the continuum limit, with critical exponents predicted by perturbation theory. This requires in particular that the lattice operators reproduce the naive continuum definitions when the lattice spacing vanishes, $a \to 0$. Scaling theory then suggests that there exists a whole universality class of lattice actions which correspond to different regularizations of the same continuum theory and whose critical properties are related by renormalization group transformations.

The most direct *ab initio* approach is to consider lattice QCD actions with the same number of parameters as continuum QCD. The simplest such lattice gauge action, the Wilson action [1], has discretization errors of $\mathcal{O}(a^2)$,

$$S_w(U) = \beta \sum_{x,\mu<\nu} \bigg(\mathbf{1} - \frac{1}{6} \mathrm{Tr}(P_{\mu\nu}(x) + P^\dagger_{\mu\nu}(x)) \bigg), \quad \text{with}$$

$$P_{\mu\nu}(x) = U_{x,\mu} U_{x+a\hat{\mu},\nu} U^\dagger_{x+a\hat{\nu},\mu} U^\dagger_{x,\nu}, \qquad \beta = \frac{6}{g^2}. \tag{6}$$

Local gauge invariance is preserved on the lattice provided that the variables $U_{x,\mu}$ transform as

$$U_{x,\mu} \to G(x) U_{x,\mu} G^{-1}(x + a\hat{\mu}). \tag{7}$$

The simplest candidate for a lattice Dirac operator is expressed in terms of the naive discretization of the covariant derivative operator, namely,

$$\hat{\mathcal{D}}_l(U) = \gamma_\mu D_{l,\mu}(U) + m,$$
$$(D_{l,\mu}(U)\psi)_x = \frac{1}{a}(U_{x,\mu}\psi_{x+a\hat{\mu}} - \psi_x), \tag{8}$$

which has the correct covariant transformation law under (7) but does not have a spectrum with definite transformation properties under the conjugation operation. Hence the determinant $\det \hat{\mathcal{D}}_l(U)$ is complex in general and does not define a probability measure.

An obvious work around would be to introduce anti-Hermitian covariant difference operators,

$$(D_{s,\mu}(U)\psi)_x = \frac{1}{2a}(U_{x,\mu}\psi_{x+a\hat{\mu}} - U^\dagger_{x-a\hat{\mu},\mu}\psi_{x-a\hat{\mu}}), \tag{9}$$

which have the same conjugation properties as the continuum operators and produces a valid probability measure. But the operator $\hat{\mathcal{D}}_s = \gamma_\mu D_{s,\mu}$ is plagued by the famous fermion doubling problem, due to the use of a central difference operator, and does not describe a single fermion flavor even in the continuum limit.

Wilson proposed [11] to add to $\hat{\mathcal{D}}_s$ a piece proportional to the finite difference approximation to the Laplacian operator Δ, which lifts the mass degeneracy of the fermion doublers by terms of order $1/a$ at the expense of breaking chiral invariance explicitly. However the Wilson operator still possess the same pseudo-Hermiticity property as the continuum Dirac operator,

$$\hat{\mathcal{D}}_w(U) = \gamma_\mu D_{s,\mu}(U) - r\Delta_L \quad (0 < r \le 1),$$
$$\hat{\mathcal{D}}_w^\dagger = \gamma_5 \hat{\mathcal{D}}_w \gamma_5, \tag{10}$$

which guarantees the invariance of its spectrum under conjugacy and the interpretation of (5) as a probability measure. It was later realized [2] that it is not possible to devise a (ultra-)local lattice Dirac operator with the correct classical continuum limit and without fermion doublers while preserving exact chiral invariance on the lattice in the massless case.

Various alternative lattice Dirac operators have been put forward during the subsequent three decades, some of which with a presently viable ecosystem. The reader can find all references in a recent, and very nice, review Ref. [12] of the state-of-the-art of numerical simulations of lattice gauge theories.

III. A CHIRALLY INVARIANT LATTICE DIRAC OPERATOR

If one examines the definition (8) of the naive lattice covariant derivative operator, one realizes immediately that the four operators,

$$\mathcal{S}_\mu = \mathbf{1} + aD_{l,\mu}, \quad (\mathcal{S}_\mu)_{ix,jy} = (U_{x,x+a\hat{\mu}})_{ij}\delta_{y,x+a\hat{\mu}}, \tag{11}$$

are, for periodic gauge field configurations, unitary ladder operators which translate by one lattice unit in direction $\hat{\mu}$ each slice of the lattice field they act upon, while rotating locally their color degrees of freedom,

$$\mathcal{S}_\mu \mathcal{S}_\mu^\dagger = \mathcal{S}_\mu^\dagger \mathcal{S}_\mu = \mathbf{1}, \quad \forall\ \mu. \tag{12}$$

The set of operators $\mathcal{S}_\mu$ transforms covariantly under the local gauge transformations (7),

$$\mathcal{S}_\mu \to \mathcal{G}\mathcal{S}_\mu\mathcal{G}^{-1}, \qquad (\mathcal{G})_{ix,jy} = G(x)_{ij}\delta_{xy}, \quad \forall\ \mu, \tag{13}$$

and encodes all the space-time and color degrees of freedom of a gauge field configuration on a four-dimensional lattice. For instance, the Wilson action (6) can be written, up to a constant term, as

$$S_w(U) = -\frac{\beta}{6}\mathrm{Tr}(\mathcal{S}_\mu \mathcal{S}_\nu \mathcal{S}_\mu^\dagger \mathcal{S}_\nu^\dagger). \tag{14}$$

Expressing the unitary operators $\mathcal{S}_\mu(U)$ as exponentials of anti-Hermitian operators $D_{r,\mu}(U)$,

$$\mathcal{S}_\mu(U) = e^{aD_{r,\mu}(U)}, \qquad D_{r,\mu} + D^\dagger_{r,\mu} = 0, \quad \forall\ \mu, \tag{15}$$

singles out the operators $D_{r,\mu}(U)$ as the natural definition of the lattice covariant derivative. Indeed the symmetric covariant difference operators (9) are just the leading approximation in the series expansion of these exponentials with respect to the lattice spacing a,

$$aD_{s,\mu}(U) = \frac{1}{2}(e^{aD_{r,\mu}} - e^{-aD_{r,\mu}}). \tag{16}$$

Then we can define the lattice Dirac operator

$$\hat{\mathcal{D}}_r(U) = \gamma_\mu D_{r,\mu}(U) + m, \tag{17}$$

which is anti-Hermitian, chirally symmetric in the massless limit, and transforms covariantly under the local gauge transformations (7). The eigenvalues of the operator $\hat{\mathcal{D}}_r$ come in complex conjugate pairs, $m \pm i\lambda$, up to a possible set of zero modes for the imaginary part which ensures, like in the continuum, reality and positiveness of the determinant for massive fermions, $\det \hat{\mathcal{D}}_r > 0$.

The operator $\hat{\mathcal{D}}_r(U)$ is nonlocal since each lattice covariant derivative operator $D_{r,\mu}$ is a series expansion in the local covariant derivative operator $D_{l,\mu}$,

$$aD_{r,\mu} = \log(\mathbf{1} + aD_{l,\mu}) = \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}a^n}{n}D^n_{l,\mu}. \tag{18}$$

From the convergence properties of the series expansion (18), the prospect of a practical numerical implementation of the lattice Dirac operator $\hat{\mathcal{D}}_r$ might seem very slim.

On the other hand, in the free case, the eigenvectors of the operators $\mathcal{S}_\mu$ are just plane waves and their $3N^3$ degenerate spectrum is simply

$$\mathrm{Spec}\mathcal{S}_\mu(\mathbf{1}) = \left\{ e^{iap_\mu}, p_\mu = \frac{2\pi n}{aN_\mu}, -\frac{N_\mu}{2} \le n < \frac{N_\mu}{2}, n \in \mathbb{Z} \right\}. \tag{19}$$

In this limit the lattice Dirac operator $\hat{\mathcal{D}}_r$ has a discrete Fourier representation which has the same form as in the continuum,

$$\dot{\hat{\mathcal{D}}}_r(\mathbf{1}) = i\gamma_\mu p_\mu + m, \tag{20}$$

which shows that the operator $\hat{\mathcal{D}}_r(\mathbf{1})$ is not afflicted with the fermion doubling problem. In compliance to the Nielsen-Ninomya theorem, the price to pay is the nonlocality of the operator $\hat{\mathcal{D}}_r(U)$. The operator $D_{r,\mu}(\mathbf{1})$ coincides with the SLAC derivative introduced long ago [3] which reads, in the limit of an infinite lattice volume,

$$D_{\infty,\mu}(x-y) = \int_{-\pi/a}^{\pi/a} \frac{d^4 p}{(2\pi)^4} ip_\mu e^{ip\cdot(x-y)}. \tag{21}$$

The usual recipe to couple a SLAC-type fermion to a $SU(3)$ gauge field consists in restoring gauge invariance by inserting, if $x-y$ has a nonvanishing component only in direction $\hat{\mu}$, the Wilson $SU(3)$ ordered straight line integral $W_\mu(U,x,y)$ between x and y, which yields the covariant lattice Dirac operator

$$(\hat{\mathcal{D}}_S(U))_{\alpha ix,\beta jy} = m\delta_{\alpha\beta}\delta_{ij}\delta_{xy} + \sum_\mu (\gamma_\mu)_{\alpha\beta} D_{\infty,\mu}(x-y) \times (W_\mu(U,x,y))_{ij}, \tag{22}$$

$$W_\mu(U,x,y) = \delta_{\mathbf{x}_\mu^\perp,\mathbf{y}_\mu^\perp} \prod_{k=0}^{(y_\mu - x_\mu - 1)/a} U_{x+ka\hat{\mu},x+(k+1)a\hat{\mu}}, \quad \text{if } y_\mu > x_\mu, \tag{23}$$

where $x = (\mathbf{x}_\mu^\perp, x_\mu)$, $y = (\mathbf{y}_\mu^\perp, y_\mu)$, and $\mathbf{x}_\mu^\perp, \mathbf{y}_\mu^\perp$ label the sites in the three-dimensional slices orthogonal to the μth direction. If $y_\mu < x_\mu$, we have of course $W_\mu(U,x,y) = W_\mu(U,y,x)^\dagger$.

The matrix elements (22) are certainly correct in the limit of an infinite lattice. However the boundary conditions on finite-size lattices are not taken into account by the conventional prescription. The operators $\mathcal{S}_\mu$ have such a simple structure that it makes possible their explicit diagonalization for an arbitrary background periodic lattice gauge field configuration and rather general twisted boundary conditions for the matter fields. We shall find nontrivial boundary terms in the matrix elements of the operator $\hat{\mathcal{D}}_r$ on finite-size lattices.

To the best of our knowledge, the underlying local unitary structure of the lattice gauge covariant generalization of the SLAC derivative does not seem to have been appreciated since its introduction.

IV. EXPLICIT DIAGONALIZATION OF THE UNITARY COVARIANT LADDER OPERATORS

For definiteness we shall assume the lattice to be hypercubic, $N_1 = N_2 = N_3 = N_4 \equiv N$, and we shall impose periodic boundary conditions on both the lattice gauge field configuration and the lattice matter fields which can be scalars, fermions, ... The operators $\mathcal{S}_\mu$ do not commute in general and have different eigenspectra $\{\lambda_\mu, \psi_{\lambda_\mu}\}$. Iterating the eigenvalue equation for $\mathcal{S}_\mu$,

$$\mathcal{S}_\mu \psi_{\lambda_\mu} = \lambda_\mu \psi_{\lambda_\mu}, \tag{24}$$

yields

$$(\mathcal{S}_\mu^n \psi_{\lambda_\mu})_{ix} = \left(\prod_{k=0}^{n-1} U_{x+ka\hat{\mu},x+(k+1)a\hat{\mu}} \right)_{ij} (\psi_{\lambda_\mu})_{j,x+na\hat{\mu}}, \\ = \lambda_\mu^n (\psi_{\lambda_\mu})_{ix}. \tag{25}$$

Imposing the periodic boundary condition,

$$\psi_{x+Na\hat{\mu}} = \psi_x, \quad \forall\ x, \tag{26}$$

implies that the eigenvectors ψ_{λ_μ} satisfy the equations

$$(W_\mu(U,x,x+Na\hat{\mu}))_{ij}(\psi_{\lambda_\mu})_{jx} = \lambda_\mu^N (\psi_{\lambda_\mu})_{ix}, \quad \forall\ x, \tag{27}$$

where $W_\mu(U,x,x+Na\hat{\mu})$, defined in (23), is the Wilson line from x in direction $\hat{\mu}$ which wraps the lattice. We shall use the shorthand $W_{\mu,x}(U)$ for such Wilson lines and the argument U will be implicit most of the time.

Hence each nonzero space-time component of the eigenvector ψ_{λ_μ} is a color triplet which is an eigenvector of some Wilson line $W_{\mu,x}(U)$. The Wilson lines are covariant objects under the local gauge transformations (7) and their eigenvalues are gauge invariant and do not depend on the choice of base points x which differ only by the x_μ component along their direction. Indeed a change of base point along a Wilson line is nothing but a similarity transformation.

Therefore the eigenspectra of the Wilson lines $W_{\mu,x}(U)$ can be labeled by the points $\vec{x}$ of the lattice slice $x_\mu = 0$, $\vec{x} \equiv (\mathbf{x}_\mu^\perp, 0)$,

$$\{e^{i\delta^c_{\mu,\vec{x}}}, \eta^c_{\mu,\vec{x}}\}, \qquad -\pi < \delta^c_{\mu,\vec{x}} \le \pi, \qquad c = 1,2,3, \tag{28}$$

where $\eta^c_{\mu,\vec{x}}$ are the color triplet eigenvectors of $W_{\mu,\vec{x}}(U)$. Their calculation requires only $4N^4$ $SU(3)$ matrix multiplications and $4N^3$ $SU(3)$ matrix diagonalizations.

Barring accidental degeneracies, the eigenvalues λ_μ and eigenvectors ψ_{λ_μ} fall into families labeled by the $4 \times 3 \times N^3$ eigenvalues of the Wilson lines and defined by the equations

$$\lambda_\mu^N = e^{i\delta^c_{\mu,\vec{x}}}. \tag{29}$$

The general solution for λ_μ reads

$$\lambda^c_{\mu,\vec{x},p_\mu} = e^{i(ap_\mu + \delta^c_{\mu,\vec{x}}/N)},$$
$$p_\mu = \frac{2\pi n}{Na} - \frac{\pi}{a}, \qquad 0 \leq n < N. \tag{30}$$

The N nonvanishing components of the corresponding eigenvector $\psi^c_{\mu,\vec{x},p_\mu}$ are, with $n = 0, \ldots, N-1$,

$$(\psi^c_{\mu,\vec{x},p_\mu})_{jy} = (\lambda^c_{\mu,\vec{x},p_\mu})^n (\eta^c_{\mu,y})_j \delta_{y,\vec{x}+na\hat{\mu}}, \quad \text{with}$$
$$\eta^c_{\mu,\vec{x}+y_\mu\hat{\mu}} = W^\dagger_\mu(\vec{x}, \vec{x} + y_\mu\hat{\mu})\eta^c_{\mu,\vec{x}}. \tag{31}$$

These components can be computed sequentially and the calculation of each of the $12N^4$ eigenvectors requires only N $SU(3)$ matrix-vector multiplications. So the total computational complexity of the complete diagonalization of every operator $\mathcal{S}_\mu$ is of order $\mathcal{O}(N^5)$. A complete diagonalization has to be performed only once for each lattice gauge field configuration and its storage requirement is in practice proportional to the lattice volume since it is more efficient to recompute the N components of each eigenvector when needed.

The eigenvectors of $\mathcal{S}_\mu$ are also eigenvectors of $D_{r,\mu}$ and the action of $D_{r,\mu}$ on an eigenvector $\psi^c_{\mu,\vec{x},p_\mu}$ has a remarkably simple continuumlike expression,

$$aD_{r,\mu}\psi^c_{\mu,\vec{x},p_\mu} = i(ap_\mu + \bar{\alpha}^c_{\mu,\vec{x}})\psi^c_{\mu,\vec{x},p_\mu}, \qquad \bar{\alpha}^c_{\mu,\vec{x}} = \frac{\delta^c_{\mu,\vec{x}}}{N}. \tag{32}$$

There is an inherent ambiguity in the logarithmic definition of $D_{r,\mu}$. We have defined rather arbitrarily the phases $\delta^c_{\mu,\vec{x}}$ as the principal argument of the eigenvalues of Wilson lines. Other prescriptions are possible for unitary gauge groups.

V. MATRIX ELEMENTS OF THE OPERATOR $\hat{\mathcal{D}}_r(U)$

The kernel operation which enters most algorithms involving fermions, such as the calculation of the fermion propagator, is the action of the lattice Dirac operator on an arbitrary lattice fermion field. The action of $\hat{\mathcal{D}}_r$ on a fermion Ψ can be written spin-component-wise as

$$(\hat{\mathcal{D}}_r(U)\Psi)_\alpha = (\gamma_\mu)_{\alpha\beta}D_{r,\mu}(U)\Psi_\beta + m\Psi_\alpha. \tag{33}$$

To perform this calculation we just need to expand each spin component of the fermion field over the complete eigensystem of every operator $D_{r,\mu}$,

$$\Psi_\beta = \sum_{c,\vec{x},p_\mu} C^c_{\beta,\mu,\vec{x},p_\mu}\psi^c_{\mu,\vec{x},p_\mu}, \quad \forall\ \mu. \tag{34}$$

We get $4 \times 4 \times 3 \times N^4$ equations, with $x = (\vec{x}, x_\mu)$ and $a = 1$ throughout this section,

$$(\Psi_\beta)_{jx} = \sum_{c,p_\mu} C^c_{\beta,\mu,\vec{x},p_\mu}(\lambda^c_{\mu,\vec{x},p_\mu})^{x_\mu}(\eta^c_{\mu,x})_j. \tag{35}$$

We can always choose all color triplet eigensystems $\{\eta^c_{\mu,\vec{x}}\}$ to be orthonormal. Then we observe that all eigensystems $\{\eta^c_{\mu,x}\}$ along the same Wilson line are simultaneously orthonormal,

$$\sum_j (\eta^{\star a}_{\mu,x})_j(\eta^b_{\mu,x})_j = \delta^{ab}, \quad \forall\ x = (\vec{x}, x_\mu), \tag{36}$$

since they are related by unitary transformations which preserve the scalar product. Thus we can transform each Eq. (35) into a simple one-dimensional Fourier series,

$$(\Psi'^c_\beta)_x = e^{-ix_\mu\bar{\alpha}^c_{\mu,\vec{x}}}\sum_j(\eta^{\star c}_{\mu,x})_j(\Psi_\beta)_{jx},$$
$$= \sum_{p_\mu} C^c_{\beta,\mu,\vec{x},p_\mu}e^{ip_\mu x_\mu}, \quad \forall\ x = (\vec{x}, x_\mu). \tag{37}$$

Therefore the coefficients $C_{\beta,c,\mu,\vec{x},p_\mu}$ are the one-dimensional inverse discrete Fourier transforms,

$$C^c_{\beta,\mu,\vec{x},p_\mu} = \frac{1}{N}\sum_{x_\mu}(\Psi'^c_\beta)_x e^{-ip_\mu x_\mu}. \tag{38}$$

It follows that the total computational complexity of the action of the operator $\hat{\mathcal{D}}_r(U)$ on a fermion field is of order $\mathcal{O}(N^5)$ which is only a factor N more expensive than the action of a local operator like the Wilson operator $\hat{\mathcal{D}}_w(U)$.

Plugging (38) into (34) yields, with $x = \vec{x} + x_\mu\hat{\mu}$ and $\vec{x} \equiv (\mathbf{x}^\perp_\mu, 0)$ as before,

$$(D_{r,\mu}\Psi_\beta)_{jy}$$
$$= \frac{1}{N}\sum_{c,\vec{x},x_\mu,p_\mu} i(p_\mu + \bar{\alpha}^c_{\mu,\vec{x}})e^{-i(p_\mu+\bar{\alpha}^c_{\mu,\vec{x}})x_\mu}$$
$$\times\left(\sum_k(\eta^{\star c}_{\mu,x})_k(\Psi_\beta)_{kx}\right)(\psi^c_{\mu,\vec{x},p_\mu})_{jy}. \tag{39}$$

Inserting (31) gives, with $y = \vec{y} + y_\mu\hat{\mu}$,

$$(D_{r,\mu}\Psi_\beta)_{jy}$$
$$= \frac{i}{N}\sum_{c,\vec{x},x_\mu,p_\mu}\delta_{\vec{y},\vec{x}}(p_\mu + \bar{\alpha}^c_{\mu,\vec{x}})e^{i(p_\mu+\bar{\alpha}^c_{\mu,\vec{x}})(y_\mu - x_\mu)}$$
$$\times\left(\sum_k(\eta^{\star c}_{\mu,x})_k(\Psi_\beta)_{kx}\right)(\eta^c_{\mu,y})_j. \tag{40}$$

The summation over p_μ brings in the finite-size SLAC derivative,

$$D_{N,\mu}(x_\mu) = \frac{1}{N}\sum_{p_\mu} ip_\mu e^{ip_\mu x_\mu}, \tag{41}$$

and the summation over $\vec{x}$ produces

$$(D_{r,\mu}\Psi_\beta)_{jy} = \delta_{\vec{x},\vec{y}}\sum_{x_\mu} D_{N,\mu}(y_\mu - x_\mu)\sum_k \times\left(\sum_c e^{i\bar{\alpha}^c_{\mu,\vec{y}}(y_\mu - x_\mu)}(\eta^{\star c}_{\mu,x})_k(\eta^c_{\mu,y})_j\right)(\Psi_\beta)_{kx} + i\sum_k\left(\sum_c \bar{\alpha}^c_{\mu,\vec{y}}(\eta^{\star c}_{\mu,y})_k(\eta^c_{\mu,y})_j\right)(\Psi_\beta)_{ky}. \tag{42}$$

The last step is to insert (31) and use the identity,

$$(W^\dagger_\mu(\vec{y},\vec{y}+y_\mu\hat{\mu}))_{jl}\left(\sum_c f(e^{i\delta^c_{\mu,\vec{y}}})(\eta^c_{\mu,\vec{y}})_l(\eta^{\star c}_{\mu,\vec{y}})_m\right) \times (W_\mu(\vec{y},\vec{y}+y_\mu\hat{\mu}))_{mk} = (f(W_{\mu,y}))_{jk}. \tag{43}$$

Collecting everything together, the matrix elements of the operator $\hat{\mathcal{D}}_r(U)$ read finally,

$$(\hat{\mathcal{D}}_r(U))_{\alpha jy,\beta ix} = \delta_{xy}\left(m\delta_{\alpha\beta}\delta_{ij} + \frac{1}{Na}\sum_\mu(\gamma_\mu)_{\alpha\beta}(\log(W_{\mu,y}(U)))_{ji}\right) + \sum_\mu(\gamma_\mu)_{\alpha\beta}D_{N,\mu}(y_\mu - x_\mu)(W_\mu(U,y,x) \times (W_{\mu,x}(U))^{(y_\mu - x_\mu)/Na})_{ji}, \tag{44}$$

where we reintroduced the lattice spacing a to make apparent the physical dimensions. The covariance of the operator $\hat{\mathcal{D}}_r(U)$ under the local gauge transformations (7) is clearly satisfied,

$$\hat{\mathcal{D}}_r(U) \to \mathcal{G}\hat{\mathcal{D}}_r(U)\mathcal{G}^{-1}. \tag{45}$$

We find two additional boundary contributions with respect to the infinite volume expression (22). The first one is a diagonal term in configuration space which vanishes proportionally to the inverse physical lattice size. The second one is an insertion in the open Wilson line $W_\mu(U,y,x)$ of the closed Wilson line $W_{\mu,x}(U)$ raised to a power the variation of which is also proportional to the distance $y - x$ in physical units. The insertion point can be covariantly transported anywhere along the open Wilson line. Anti-Hermiticity of the massless operator $\hat{\mathcal{D}}_r(U)$ then follows from the change of sign of the diagonal logarithmic term under conjugation and from the odd parity of the SLAC derivative $D_{N,\mu}(x_\mu)$ under sign inversion of x_μ.

The lattice action for a single flavor of color triplet, four-component Dirac fermions Ψ_x,

$$S_r(U) = \sum_{x,y}\bar{\Psi}_y(\hat{\mathcal{D}}_r(U))_{y,x}\Psi_x, \tag{46}$$

is invariant under the transformations of the hypercubic group $H(4)$ which is the discrete subgroup of $O(4)$ which survives after the lattice discretization of Euclidean space-time with hypercubic cells.

The group $H(4)$, which has the semidirect product structure $Z_2^4 \rtimes S_4$, has the same set of improper discrete transformations, in particular parity $\mathcal{P}$ and time reversal $\mathcal{T}$, as the Euclidean orthogonal group $O(4)$. There is no distinction between the temporal and spatial directions of an Euclidean lattice, and time reversal is just like inversion about any of the other directions. We have $\mathcal{T} = \mathcal{P}_1\mathcal{P}_2\mathcal{P}_3$, with the reflection transformations $\mathcal{P}_\mu$ defined by

$$\Psi_x \xrightarrow{\mathcal{P}_\mu} \gamma_\mu\Psi_{x_{P_\mu}}, \qquad \bar{\Psi}_x \xrightarrow{\mathcal{P}_\mu} \bar{\Psi}_{x_{P_\mu}}\gamma_\mu, \qquad U_{\mu,x} \xrightarrow{\mathcal{P}_\mu} U_{\mu,x_{P_\mu}}, \qquad U_{\nu,x} \xrightarrow{\mathcal{P}_\mu} U^\dagger_{\nu,x_{P_\mu}-\hat{\nu}}\ (\nu \neq \mu), \tag{47}$$

where x_{P_μ} is the site with all components of x reversed except along direction μ.

Similarly, charge conjugation is defined by the usual transformation on the lattice fields,

$$\Psi_x \xrightarrow{\mathcal{C}} C^{-1}\bar{\Psi}^\top_x, \qquad \bar{\Psi}_x \xrightarrow{\mathcal{C}} -\Psi^\top_x C, \qquad U_{\mu,x} \xrightarrow{\mathcal{C}} U^\star_{\mu,x}, \tag{48}$$

where the charge conjugation operator C obeys the relations $C\gamma_\mu C^{-1} = -\gamma^\top_\mu$. An explicit realization in the Euclidean representation (2) for gamma matrices, where γ_5 is diagonal, is $C = i\gamma_2\gamma_0$.

The proof of invariance of the lattice action (46) under the transformations (47) and (48) follows in a standard way from the periodicity of the lattice fields, anti-Hermiticity of the massless operator $\hat{\mathcal{D}}_r(U)$ and anticommutation relations of the fermion field.

VI. CONVERGENCE TO THE CLASSICAL CONTINUUM LIMIT

It has always been taken for granted that lattice Dirac operators $\hat{\mathcal{D}}_L(U)$ had to be local, up to exponential tails, in order to reproduce the correct continuum limit when the number of lattice points is sent to infinity at fixed physical lattice size. This intuitive locality condition, defined in Ref. [10], has become the criterion of the standard wisdom to discriminate admissible lattice Dirac operators in numerical simulations of lattice gauge theories. This condition reads

$$\|a\hat{\mathcal{D}}_L(U)_{y,x}\| \underset{d(x,y)\gg a}{\leq} c_L\exp(-k_L d(x,y)/a), \qquad d(x,y) = \sum_\mu |x_\mu - y_\mu|, \tag{49}$$

where $\|\cdot\|$ is a matrix norm in spin and color space, $d(x,y)$ is the "taxi driver distance" between two sites on the lattice and c_L, k_L are dimensionless constants. In other words, the matrix elements of $\hat{\mathcal{D}}_L$ should be exponentially decaying at large distances with a rate proportional to the cutoff $1/a$. The condition (49) on the operator norm of $\hat{\mathcal{D}}_L$ can be properly qualified as a *strong* locality condition. However it must be remembered that the localization of a lattice Dirac operator is measured in practice through its action upon lattice fields. For example, the numerical test performed in Ref. [10] consists in computing the action of the overlap operator upon a point source, i.e., a periodic lattice field which vanishes everywhere except for a finite discontinuity at some point.

In the case at hand, the matrix elements (44) of the operator $\hat{\mathcal{D}}_r(U)$ vanish whenever the sites x and y do not belong to a same Wilson line and

$$\|a\hat{\mathcal{D}}_r(U)_{y,x}\| \underset{x\neq y}{=} \begin{cases} 0 & \text{if} \sum_\mu \delta_{\mathbf{x}_\mu^\perp,\mathbf{y}_\mu^\perp}=0 \\ |aD_{N,\mu}(y_\mu - x_\mu)| & \text{for some } \mu \text{ otherwise.} \end{cases} \tag{50}$$

We drop the notation of the index μ until further notice. The finite-size SLAC derivative $D_N(x)$ is defined in (41) as the discrete Fourier expansion of the sawtooth function in momentum space, which reads for $x_n = na$ and $n = 1, 2, \ldots, N-1$,

$$aD_N(x_n) = \begin{cases} \frac{\pi}{N}\frac{(-1)^n}{\sin\frac{\pi n}{N}} & \text{for } N \text{ odd} \\ -i\frac{\pi}{N} + \frac{\pi}{N}\frac{(-1)^n}{\tan\frac{\pi n}{N}} & \text{for } N \text{ even.} \end{cases} \tag{51}$$

For N even there is a zero mode contribution which is purely imaginary and causes certain technical difficulties for numerical simulations via Monte Carlo techniques. A way to remove this zero mode would be to use antiperiodic boundary conditions for matter fields and N even. Anyhow it is already manifest that the operators $D_N(x)$ and $\hat{\mathcal{D}}_r(U)$ do not satisfy the strong locality condition (49) for finite odd N since $|\csc(x)| > 1/|x| > \exp(-|x|)$. It will be convenient to suppose N odd in the following and continue to work with periodic boundary conditions.

On the other hand, numerical simulations of the two-dimensional Wess-Zumino model at weak and intermediate couplings find that SLAC fermions display a better approach to the continuum limit than Wilson fermions [13]. These results hint at the existence of a more general convergence criterion to the continuum limit than the strong locality condition (49). Thus it is instructive to understand more precisely the convergence of the continuum extrapolation first in the free theory.

By construction, the operator $D_N(y-x)$ reproduces the exact derivative at the lattice sites $x_n = na$ when acting on complex trigonometric polynomials $T_N(x)$ of order less or equal to $N/2$,

$$T_N(x) = \sum_{k=-N/2}^{(N-1)/2} c_k \exp\left(\frac{2i\pi kx}{Na}\right) \Rightarrow \sum_{m=0}^{N-1} D_N(x_n - x_m)T_N(x_m) = T_N'(x_n). \tag{52}$$

By the Weierstrass theorem any continuous function, defined on a finite interval, can be approximated uniformly on that interval by an infinite sequence of trigonometric polynomials of ever increasing orders. In particular, let $\psi(x)$ be a differentiable periodic function on the interval $[0, L]$ with Fourier coefficients $\hat{\psi}_n$, and let us denote by $\psi_N(x)$ the partial sum of order $N/2$ of its Fourier expansion and by $\epsilon_N(x)$ the remainder. Choosing a lattice subdivision of $[0, L]$ with equidistant points such that $L = Na$, and since ψ_N is a trigonometric polynomial, one has at the lattice sites $x_k = ka$, $k = 1, \ldots, N-1$,

$$|\nabla\psi(x_k) - (D_N\psi)(x_k)| \le |\nabla\psi(x_k) - \nabla\psi_N(x_k)| + |(D_N\psi)(x_k) - (D_N\psi_N)(x_k)|. \tag{53}$$

There are well-known mathematical theorems [14] on the rapidity of convergence of Fourier series of functions according to their class of differentiability. In particular if $\psi \in C^p([0, L])$ and if, moreover, its pth derivative satisfies the Lipshitz condition,

$$|\psi^{(p)}(x_2) - \psi^{(p)}(x_1)| \le \lambda|x_2 - x_1|, \tag{54}$$

for all values of x_1 and x_2, λ being a constant, then

$$|\psi^{(m)}(x) - \psi_N^{(m)}(x)| \le \lambda K_m \frac{\log N}{N^{p-m+1}}, \qquad 0 \le m \le p, \qquad 0 \le x \le L, \tag{55}$$

where K_m is a constant which depends only on m. In other words, all the bounds depend upon the function ψ only through λ. It follows from (51) that

$$|\nabla\psi(x_k) - (D_N\psi)(x_k)| \le \lambda\frac{\log N}{N^p}(K_1 + K_0 \max_{x_n}|D_N(x_n)|) \sim a^p \log a, \tag{56}$$

for all values of x_k. Therefore, the action of the finite-size SLAC derivative upon smooth enough matter fields approaches the continuum limit uniformly and faster than local discretizations of the gradient operator. This approach is even exponentially fast for infinitely differentiable fields. The property that the discretization errors of the operator $D_N(x)$ are much smaller should not be a surprise. Indeed the momentum-space representation of this operator coincides by design with the continuum expression for the discretized momenta within the first Brillouin zone. Therefore the Lorentz symmetry violation at finite lattice spacing is minimal. For instance, when the lattice is hypercubic, the momentum-space propagator $\dot{\hat{\mathcal{D}}}_N^{-1}(p)$ is not spoiled by the characteristic orbit pattern present in the free Wilson propagator $\dot{\hat{\mathcal{D}}}_w^{-1}(p)$, which is due to the occurrence in the sine terms of invariants of the hypercubic group $H(4)$ which are different from the Lorentz invariant p^2.

We are now in position to consider the action of the operator $\hat{\mathcal{D}}_r(U)$ on a periodic massless fermion field Ψ in the presence of an arbitrary periodic background gauge field. The leading approximation of $\hat{\mathcal{D}}_r(U)$ at small lattice spacing is the local operator $\hat{\mathcal{D}}_s(U)$ and it would appear that the discretization errors are of order a^2 in the naive classical continuum limit. However truncating the series expansion (18) destroys the unitarity structure and removes all the constraints it comprises, such as anti-Hermiticity, which can lead to serious pitfalls even if they are enforced by hand.

So let us start from the exact matrix elements (44) and introduce a family of fermion fields $\tilde{\Psi}^{(\mu,y)}$ labeled by the lattice points y and the direction μ, which we make explicit again. The components of these fermion fields $\tilde{\Psi}^{(\mu,y)}$ are defined in terms of those of the original field Ψ by the formulas

$$\tilde{\Psi}^{(\mu,y)}_{\beta x} = W_\mu(U, y, x)(W_{\mu,x}(U))^{(y_\mu - x_\mu)/Na}\Psi_{\beta x}, \tag{57}$$

where we drop the notation of the color indices. The components of $\tilde{\Psi}^{(\mu,y)}$ are nonzero only at the lattice points along the Wilson line which goes through y in direction $\hat{\mu}$. Then the action of $\hat{\mathcal{D}}_r(U)$ on Ψ reads

$$\begin{aligned}(\hat{\mathcal{D}}_r(U)\Psi)_{\alpha y} &= \sum_\mu (\gamma_\mu)_{\alpha\beta}\Big(\frac{1}{Na}\log(W_{\mu,y}(U))\Psi_{\beta y} \\ &\quad + \sum_x D_{N,\mu}(y_\mu - x_\mu)\tilde{\Psi}^{(\mu,y)}_{\beta x}\Big).\end{aligned} \tag{58}$$

If the fermion field Ψ and lattice gauge field U are samplings at the lattice sites and links of a continuum periodic fermion field $\psi(x)$ and continuum gauge field $A_\mu(x)$ which are sufficiently smooth, then we can apply the convergence theorems satisfied by the SLAC derivatives $D_{N,\mu}$,

$$\begin{aligned}&\lim_{N\to\infty}(\hat{\mathcal{D}}_r(U)\Psi)_{\alpha y} \\ &\quad = \sum_\mu (\gamma_\mu)_{\alpha\beta}\Big(\frac{1}{L}\log(W_{\mu,y}(A))\psi_\beta(y) + \lim_{x\to y}\nabla_\mu \tilde{\psi}^{(\mu,y)}_\beta(x)\Big),\end{aligned} \tag{59}$$

with

$$\begin{aligned}W_{\mu,y}(A) &= P\exp\Big(\int_{y_\mu}^{y_\mu+L} ds A_\mu(s)\Big), \\ W_\mu(A, y, x) &= \delta_{\mathbf{x}_\mu^\perp,\mathbf{y}_\mu^\perp} P\exp\Big(\int_{x_\mu}^{y_\mu} ds A_\mu(s)\Big), \\ \tilde{\psi}^{(\mu,y)}_\beta(x) &= W_\mu(A, y, x)(W_{\mu,x}(A))^{(y_\mu - x_\mu)/L}\psi_\beta(x).\end{aligned} \tag{60}$$

We have

$$\begin{aligned}\nabla_\mu \tilde{\psi}^{(\mu,y)}_\beta(x) &= W_\mu(A,y,x)(W_{\mu,x}(A))^{(y_\mu-x_\mu)/L}\nabla_\mu\psi_\beta(x) + A_\mu(x)W_\mu(A,y,x)(W_{\mu,x}(A))^{(y_\mu-x_\mu)/L}\psi_\beta(x) \\ &\quad + W_\mu(A,y,x)\Big(-\frac{1}{L}\log(W_{\mu,x}(A)) + \frac{(y_\mu - x_\mu)}{L}(A_\mu(x) - W_{\mu,x}(A)A_\mu(x)W^{-1}_{\mu,x}(A))\Big)(W_{\mu,x}(A))^{(y_\mu-x_\mu)/L}\psi_\beta(x).\end{aligned} \tag{61}$$

It follows that the two finite-size contributions in the operator $\hat{\mathcal{D}}_r$ cancel out precisely in the right-hand side of (59) when $x = y$, and we get the correct continuum limit,

$$\begin{aligned}\lim_{N\to\infty}(\hat{\mathcal{D}}_r(U)\Psi)_{\alpha y} &= \sum_\mu (\gamma_\mu)_{\alpha\beta}(\nabla_\mu + A_\mu(y))\psi_\beta(y) \\ &\equiv (\hat{\mathcal{D}}(A)\psi)_\alpha(y).\end{aligned} \tag{62}$$

This result is of course expected, but what is novel and nontrivial is the dependence of the rate of convergence upon the smoothness of the classical field configurations. Indeed let $\mathcal{L}$ be a lattice subdivision of a continuum hypercubic box of size L such that $L = Na$, then we get from Eqs. (58) and (59),

$$\begin{aligned}&\|(\hat{\mathcal{D}}(A)\psi)_\alpha(y) - (\hat{\mathcal{D}}_r(U)\Psi)_{\alpha y}\| \\ &\quad \le \sum_\mu \Big\|(\gamma_\mu)_{\alpha\beta}(\frac{1}{L}(\log(W_{\mu,y}(A))\psi_\beta(y) \\ &\qquad - \log(W_{\mu,y}(U))\Psi_{\beta y}) + \nabla_\mu\tilde{\psi}^{(\mu,y)}_\beta(y) \\ &\qquad - (D_{N,\mu}\tilde{\Psi}^{(\mu,y)}_\beta)_y)\Big\|,\end{aligned} \tag{63}$$

for any lattice point y. The logarithmic terms cancel out exactly by definition of the gauge field and fermion field discretizations,

$$U_{x,x+a\hat{\mu}} \equiv P\exp\Big(\int_x^{x+a\hat{\mu}} ds A_\mu(s)\Big), \quad \Psi_{\beta x} \equiv \psi_\beta(x). \tag{64}$$

It follows that we can use Eq. (56) if all fermion fields $\tilde{\Psi}^{(\mu,y)}$ satisfy the Lipshitz condition (54), where the Lipshitz constant now depends on y and μ,

$$\|(\hat{\mathcal{D}}(A)\psi)_\alpha(y) - (\hat{\mathcal{D}}_r(U)\Psi)_{\alpha y}\| \underset{a\to 0}{\sim} a^p \log a. \tag{65}$$

Note however that we still have uniform convergence over the lattice with $\lambda \equiv \max \lambda^{(\mu,y)}$.

Equation (65) shows that, for smooth enough classical gauge field configurations, the convergence of the action of the operator $\hat{\mathcal{D}}_r$ towards the continuum limit is much better, hence the discretization errors much smaller, than for conventional local lattice Dirac operators.

One could still object that the convergence theorems (65) are not useful for the quantum continuum limit. The lattice quantum observables are built out of matrix elements of the lattice fermion propagator. The calculation of these matrix elements are usually performed by an iterative algorithm which requires the evaluation of the action of the lattice Dirac operator on a discrete unit point source. It is clear that, for the operator $\hat{\mathcal{D}}_r$, the discontinuity of the point source will produce an oscillatory behavior in the raw lattice data, the well-known Gibbs phenomenon, which would seem to prevent a good analytical control of the continuum limit.

From a numerical standpoint however, Eq. (65) allows us to develop smearing filters of the raw lattice data. The better a semiclassical approximation is, the better the filters are expected to be. If a semi-classical approximation to the quantum continuum limit were exact, then it should be possible to devise a filter which produces the exact continuum result at finite lattice spacing. For example in the free case, where the semiclassical approximation to the quantum functional integral is exact, it is possible to extract the continuum fermion mass at finite lattice spacing by a mere cosh fit to the raw two-point correlator $C(t)$ of SLAC fermions in position space [13].

From the mathematical standpoint, a discrete unit point source becomes a Dirac distribution in the continuum limit. It is well known that a periodic Dirac distribution on the interval $[0, L]$ can be defined as the limit of a sequence of Dirichlet kernels $K_n(x)$,

$$aK_n(x) = \frac{1}{2n+1} \frac{\sin\left((2n+1)\frac{\pi x}{L}\right)}{\sin\left(\frac{\pi x}{L}\right)}, \tag{66}$$

which are infinitely differentiable trigonometric polynomials peaked at the point source, here located at the origin, and which satisfy the convolution property

$$\lim_{n\to\infty} \int_0^L dt K_n(x-t) f(t) = f(x), \tag{67}$$

for all periodic smooth functions f on the interval $[0, L]$. The Dirichlet kernel $K_n(x)$ coincides with the unit discrete point source when sampled at the sites of a lattice subdivision $[0, L]$ with $N = 2n + 1$ points. Moreover, the values at all the lattice sites of the derivative of $K_n(x)$ coincide by definition with the matrix elements (51) of the operator $D_N(x)$. But the sequence of derivatives $K_n'(x)$ converges, in the sense of the theory of distributions, towards the derivative of the Dirac distribution

$$\lim_{n\to\infty} \int_0^L dt K_n'(x-t) f(t) = f'(x). \tag{68}$$

Hence the action of the operator $D_N(x)$ upon a discrete point source does converge in the continuum limit in the sense of the theory of distributions.

We can thus apply Eq. (62) choosing $\psi(y) = K_{n_0}(y)$ as a continuum source with n_0 held fixed in the limiting process $N \to \infty$. The source is infinitely differentiable, but the measure of integration in the quantum functional integral (1) is over arbitrary continuous gauge fields. Therefore the corresponding fields $\tilde{\psi}^{(\mu,y)}_\beta(x)$ defined in (60) are only continuously differentiable and, letting $L = (2n_0 + 1)a$, we have for large enough n_0,

$$\|(\hat{\mathcal{D}}(A)\psi_{n_0})_\alpha(y) - (\hat{\mathcal{D}}_r(U)\Psi_{n_0})_{\alpha y}\| \sim a \log a, \tag{69}$$

provided that the derivatives $\nabla_\mu \tilde{\psi}^{(\mu,y)}_\beta(x)$ satisfy also the Lipshitz condition. The limit $n_0 \to \infty$ is defined only in the sense of the theory of distributions because of the dependence of the source on n_0. In fact, to get the bound (69), it is sufficient to assume that the modulus of continuity of the derivatives $\nabla_\mu \tilde{\psi}^{(\mu,y)}_\beta(x)$ satisfy the following condition [14]:

$$\omega(a) = \max_{|x_{2\mu}-x_{1\mu}|\leq a} \|\nabla_\mu \tilde{\psi}^{(\mu,y)}_\beta(x_2) - \nabla_\mu \tilde{\psi}^{(\mu,y)}_\beta(x_1)\| \sim \frac{a}{L}, \tag{70}$$

which implies the same condition on the continuum gauge field along every Wilson line,

$$\max_{|x_{2\mu}-x_{1\mu}|\leq a} \|A_\mu(x_2) - A_\mu(x_1)\| \sim \frac{a}{L}. \tag{71}$$

This condition is usually satisfied by the continuum gauge fields which interpolate lattice gauge configurations produced via a local Monte Carlo process such as the Metropolis algorithm or the heat bath algorithm.

To summarize, we have achieved more than would have conveyed just a numerical check of exponential locality for the action of the operator $\hat{\mathcal{D}}_r(U)$ upon a discrete point source. Assuredly, there is no exponential convergence to the continuum limit far from the point source, but we do have proven convergence everywhere, even at the point source in a precise sense, and for the lattice discretization of rather general continuous gauge fields, for which we have exposed the rate of convergence.

VII. OUTLOOK

As recalled in the introductory section, the lattice fermion formulation based on the SLAC derivative has been rather controversial. There have been one-loop calculations with the SLAC operator (22) in weak coupling perturbation theory of lattice QED in four dimensions, which have shown the occurrence of singularities in the fermion triangle graph [4] and in the vacuum polarization [5], that lead to nonlocal, non-Lorentz covariant expressions. These singularities are generated by the discontinuities of the SLAC derivative at the edges of the Brillouin zone, $p_\mu = \pm\pi/a$. It has been claimed [6] that these divergences could not be renormalized while keeping in the continuum limit $a \to 0$ (at $L = Na \to \infty$), both chiral invariance without extra states and Lorentz invariance. However it has also been suggested [7] that QED could be recovered in the continuum limit by a proper, nonperturbative, treatment of the infrared singularities and by imposing a finite number of nonlocal renormalization conditions. But the application of such empirical prescriptions to the lattice Schwinger model, namely two-dimensional quantum electrodynamics with massless fermions, which is a completely solvable model in the continuum [15], has still generated a spectrum doubling, a vanishing anomaly, a vanishing vacuum expectation value for $\langle \bar{\psi} \psi \rangle$, and a noncovariant axial-vector current [8].

Despite all these early negative results, and the fact that the SLAC derivative does not satisfy the strong locality condition (49), there has been a recent claim that the discretized Wess-Zumino model in two dimensions with the SLAC derivative has a renormalizable continuum limit [16], with a check of renormalizability to first-order in perturbation theory. We have shown in the previous section how essential the properties of the lattice operator $\hat{\mathcal{D}}_r$ are, in order to get the correct classical continuum limit with an improved convergence. Clearly such properties must be critical as well for a rigorous treatment of the quantum continuum limit, either nonperturbatively or via a perturbative expansion. As already emphasized, the underlying unitarity structure of the nonlocal lattice covariant derivative has not been taken into account in existing studies and the issue of renormalizability in such a formulation, which depends crucially upon the handling of singularities in the infinite volume limit, has to be settled accordingly.

Meanwhile, we advocate a pragmatic approach, *à la* Wilson. We have presented a covariant and chirally invariant lattice Dirac operator on finite-size lattices which has certainly no spectrum doubling at the classical level. Moreover, the expressions (44) of the matrix elements of the operator $\hat{\mathcal{D}}_r$ are quite convenient for an actual computer implementation. If there is a vanishing axial anomaly, one could always introduce an explicit chiral symmetry breaking term. Our derivation reveals that the nonlocal operator $\hat{\mathcal{D}}_r$ is a smeared operator with a controllable analytic averaging over the links of Wilson lines (whereas the original smearing proposal [17], as well as its many variants, are empirical thickenings of the links). Smeared operators are smoother and it is widely known [12] that their inversion has better convergence properties than local operators. So the accelerated convergence near the chiral limit, whose qualitative nature can be studied in the quenched approximation, may even turn out to compensate for the additional computational complexity of $\hat{\mathcal{D}}_r$ with respect to local Dirac operators such as $\hat{\mathcal{D}}_w$.

[1] K. G. Wilson, Phys. Rev. D **10**, 2445 (1974).
[2] H. B. Nielsen and M. Ninomiya, Nucl. Phys. **B185**, 20 (1981).
[3] S. D. Drell, M. Weinstein, and S. Yankielowicz, Phys. Rev. D **14**, 1627 (1976).
[4] L. Karsten and J. Smit, Nucl. Phys. **B144**, 536 (1978).
[5] L. Karsten and J. Smit, Phys. Lett. **85B**, 100 (1979).
[6] L. Karsten and J. Smit, Nucl. Phys. **B183**, 103 (1981).
[7] J. M. Rabin, Phys. Rev. D **24**, 3218 (1981).
[8] G. T. Bodwin and E. V. Kovacs, Phys. Rev. D **35**, 3198 (1987).
[9] M. Ninomiya and C. Tan, Phys. Rev. Lett. **53**, 1611 (1984).
[10] P. Hernández, K. Jansen, and M. Lüscher, Nucl. Phys. **B552**, 363 (1999).
[11] K. G. Wilson, in *Proceedings of the International School of Subnuclear Physics*, Erice (Report No. CLNS-321, 1975).
[12] Z. Fodor and C. Hoelbling, Rev. Mod. Phys. **84**, 449 (2012).
[13] T. Kästner, G. Bergner, S. Uhlmann, A. Wipf, and C. Wozar, Phys. Rev. D **78**, 095001 (2008).
[14] D. Jackson, *The Theory of Approximation*, AMS Colloquium Publication Vol. XI (American Mathematical Society, Providence, 1930).
[15] J. Schwinger, Phys. Rev. **128**, 2425 (1962).
[16] G. Bergner, T. Kästner, S. Uhlmann, and A. Wipf, Ann. Phys. (Amsterdam) **323**, 946 (2008).
[17] M. Albanese *et al.* (APE Collaboration), Phys. Lett. B **192**, 163 (1987).