PHYSICAL REVIEW D **87,** 074014 (2013)

Symmetries of the three-heavy-quark system and the color-singlet static energy at next-to-next-to-leading logarithmic order

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(Received 15 January 2013; published 12 April 2013)

We study the symmetries of the three-heavy-quark system under exchange of the quark fields within the effective field theory framework of potential nonrelativistic QCD. The symmetries constrain the form of the matching coefficients in the effective theory. We then focus on the color-singlet sector and determine the so far unknown leading ultrasoft contribution to the static potential, which is of order $\alpha_s^4 \ln \mu$, and consequently to the static energy, which is of order $\alpha_s^4 \ln \alpha_s$. Finally, in the case of an equilateral geometry, we solve the renormalization group equations and resum the leading ultrasoft logarithms for the static potential of three quarks in a color singlet, octet and decuplet representation.

DOI: 10.1103/PhysRevD.87.074014 PACS numbers: 12.38.−t, 12.38.Bx, 14.20.−c, 14.40.Pq

I. INTRODUCTION

Bound states of a heavy quark Q and an antiquark $\bar{Q}$ have been the subject of extensive theoretical studies since the early days of quantum chromodynamics (QCD). Relatively less attention has been paid to bound states of three heavy quarks (QQQ), also referred to as triple heavy baryons, as a consequence of their still missing experimental evidence. Nevertheless there is an ongoing theoretical activity devoted to their study mostly driven by lattice computations [1–13], but also by phenomenological analyses (for a review, see Ref. [14]) and more recently by effective field theory methods [15–17]. The theoretical interest is mainly triggered by the geometry of these systems, which allows to address questions that are inaccessible with two-body systems. Examples are the minimal energy configuration of three quarks in the presence of a confining potential or the origin of a three-body interaction. In this paper we will further explore the geometrical properties of the three-heavy-quark system.

Systems of heavy quarks are conveniently studied within an effective field theory (EFT) framework, a treatment motivated by the observation that these systems are nonrelativistic and, therefore, characterized by, at least, three separated and hierarchically ordered energy scales: a hard scale of the order of the heavy-quark mass, m, a soft scale of the order of the typical relative momenta of the heavy quarks, which are much smaller than m, and an ultrasoft (US) scale of the order of the typical binding energy, which is much smaller than the relative momenta.[1] We further assume that these scales are much larger than the typical hadronic scale $\Lambda_{\rm QCD}$, in this way justifying a perturbative treatment for all of them. By integrating out modes associated with the different energy scales one goes through a sequence of EFTs [18]: nonrelativistic QCD (NRQCD), obtained from integrating out hard modes [19,20] and potential nonrelativistic QCD (pNRQCD), derived from integrating out gluons with soft momenta from NRQCD [21,22]. Potential NRQCD provides a formulation of the nonrelativistic system in terms of potentials and US interactions; for this reason it has proven a convenient framework for calculating US corrections. Although originally designed for the study of $Q\bar{Q}$ bound states, i.e., quarkonia, pNRQCD has been subsequently applied also to baryons with two and three heavy quarks [15,16].

In this paper we study the symmetry properties of three-heavy-quark systems under exchange of the heavy-quark fields and their implications for the form of the pNRQCD Lagrangian. We also calculate the US corrections of order $\alpha_s^4 \ln \alpha_s$ to the singlet static energy and of order $\alpha_s^4 \ln \mu$ to the singlet static potential of a triple heavy baryon. Whereas this has been achieved for the case of $Q\bar{Q}$ systems more than ten years ago [23], the result for QQQ systems will be new.

The paper is organized as follows. Section II is devoted to set up pNRQCD for systems made of three static quarks. The explicit construction and color structure of the heavy-quark composite fields, pNRQCD is conventionally formulated in, is outlined in detail. In Sec. III, we discuss the symmetry under exchange of the heavy-quark fields and analyze its implications for the various matching coefficients, i.e., the potentials, of pNRQCD. In Sec. IV, we determine the correction of order $\alpha_s^4 \ln \alpha_s$ to the singlet static energy. Restricting ourselves to an equilateral configuration of the heavy quarks, we finally solve in Sec. V the renormalization group equations for the singlet, octet and decuplet static potentials at leading logarithmic accuracy. We conclude in Sec. VI.

[1] In a three-body system, we may in general expect to have more than one typical relative momentum and more than one US energy scale. To keep our discussion simple, we assume all relative momenta to be of the same order and so for all US energy scales. In the dynamical case, this is realized when the masses of the heavy quarks are of the same order. In the static limit, which will be our main concern in the following, this condition is realized by locating the three quarks at distances of the same order. We emphasize that this condition may be (also largely) violated in different geometrical configurations.

II. pNRQCD FOR QQQ

In this section, we shortly review the basic steps that lead to pNRQCD for systems made of three static quarks. Finite mass corrections may be systematically added to the static Lagrangian in the form of irrelevant operators, some of which have been considered in Ref. [15]. The nonrelativistic nature of the system ensures that, apart from the kinetic energy, which is of the same order as the static potential, $1/m$ corrections are small.

A. NRQCD

Our starting point is NRQCD in the static limit. In the quark sector the Lagrangian is identical with the heavy-quark effective theory Lagrangian [24] and reads

$$\mathcal{L}_{\mathrm{NRQCD}} = Q^\dagger iD^0 Q + \sum_l \bar{q}^l i\not{D} q^l - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}. \quad (1)$$

The heavy-quark fields Q ($Q^\dagger$), which annihilate (create) a heavy quark, are described by Pauli spinors, whereas q^l are the Dirac spinors that describe light (massless) quarks of flavor l. The quantity $iD^0 = i\partial^0 - gA^0$ denotes the time component of the covariant derivative, where g is the strong gauge coupling, $\alpha_s \equiv g^2/(4\pi)$, and A^0 is the time component of the gauge field. The Lagrangian (1) is insensitive to the flavor assignment of the heavy-quark fields, a property known as heavy-quark symmetry. We have omitted the heavy-antiquark sector, as it is irrelevant to our scope.

B. pNRQCD

For the purpose of studying heavy-quark bound states, it is convenient to employ an EFT where the heavy-quark potentials are explicit rather than encoded in dynamical gluons, as it is the case in NRQCD. Such an EFT is pNRQCD, which is obtained from NRQCD by integrating out gluons whose momenta are soft. The degrees of freedom of pNRQCD are heavy-quark fields, light quarks and US gluons. As it is unnecessary to resolve the individual heavy quarks, pNRQCD is often formulated in terms of heavy-quark composite fields. The matching coefficients of pNRQCD multiplying operators bilinear in the composite fields may then be interpreted as the heavy-quark potentials in the corresponding color configurations.

The derivation of pNRQCD involves identifying the heavy-quark composite fields in NRQCD, matching them to pNRQCD, and explicitly ensuring that the resulting pNRQCD field content is ultrasoft. We start with the construction of the heavy-quark composite fields. This is the point where the specific heavy-quark state that the EFT is meant to describe has to be specified. In our case, this is a QQQ state.

C. Geometry of a three-quark state

To characterize the geometry of a QQQ state, we call $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ the positions of the quarks and define the vectors $\mathbf{r}_i (i = 1, 2, 3)$ as follows (cf. Fig. 1):

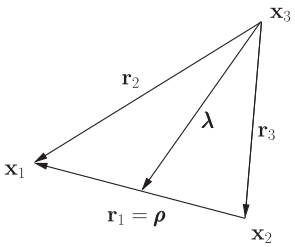


FIG. 1. Triangle formed by three heavy quarks located at the positions $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$. The vector $\boldsymbol{\lambda}$ points from the heavy quark at $\mathbf{x}_3$ to the center of mass of the two heavy quarks at $\mathbf{x}_1$ and $\mathbf{x}_2$.

$$\mathbf{r}_1 = \mathbf{x}_1 - \mathbf{x}_2, \qquad \mathbf{r}_2 = \mathbf{x}_1 - \mathbf{x}_3, \qquad \mathbf{r}_3 = \mathbf{x}_2 - \mathbf{x}_3. \quad (2)$$

Note that the three vectors are not independent, for $\mathbf{r}_1 + \mathbf{r}_3 = \mathbf{r}_2$. Moreover, for three quarks of equal mass or static, it is useful to define the vectors

$$\boldsymbol{\rho} = \mathbf{r}_1, \qquad \boldsymbol{\lambda} = \frac{\mathbf{r}_2 + \mathbf{r}_3}{2}. \quad (3)$$

D. Heavy-quark composite fields

Quarks transform under the fundamental representation, 3, of the (color) gauge group $\mathrm{SU}(3)_c$. Hence, a generic three (heavy) quark field made of fields located at the same point, $Q_i Q_j Q_k$ ($i, j, k = 1, 2, 3$ denote color indices), transforms as a representation of $3 \otimes 3 \otimes 3$. The direct product can be decomposed into a sum of irreducible representations of $\mathrm{SU}(3)_c$, namely,

$$3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10. \quad (4)$$

In general, however, the three quarks are located at different spatial positions $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$. Under an $\mathrm{SU}(3)_c$ gauge transformation, each heavy-quark field $Q_i(\mathbf{x}, t)$ transforms as $Q_i(\mathbf{x}, t) \to U_{ii'}(\mathbf{x}, t) Q_{i'}(\mathbf{x}, t)$, where $U(\mathbf{x}, t) = \exp[i\theta^a(\mathbf{x}, t) T^a]$, and $T^a = \lambda^a/2$ ($a = 1, \ldots, 8$) denote the eight generators of $\mathrm{SU}(3)_c$ in the fundamental representation; λ^a are the Gell-Mann matrices. The decomposition (4) requires the fields to be linked to a common point $\mathbf{R}$. For a multiquark system a natural choice is the system's center of mass. A way to link the quark fields to another point is through an equal-time straight Wilson string,

$$\phi(\mathbf{y}, \mathbf{x}, t) = \mathcal{P} \exp\left\{ ig \int_0^1 \mathrm{d}s (\mathbf{y} - \mathbf{x}) \cdot \mathbf{A}(\mathbf{x} + (\mathbf{y} - \mathbf{x})s, t) \right\}, \quad (5)$$

where $\mathbf{A} = \mathbf{A}^a T^a$ is the color gauge field, and $\mathcal{P}$ denotes path ordering of the color matrices. Because of its transformation property under $\mathrm{SU}(3)_c$ gauge transformations, $\phi(\mathbf{y}, \mathbf{x}, t) \to U(\mathbf{y}, t) \phi(\mathbf{y}, \mathbf{x}, t) U^\dagger(\mathbf{x}, t)$, the Wilson string acts as a gauge transporter, and $\phi(\mathbf{R}, \mathbf{x}, t) Q(\mathbf{x}, t) \to U(\mathbf{R}, t) \phi(\mathbf{R}, \mathbf{x}, t) Q(\mathbf{x}, t)$ indeed transforms like a quark field located at $\mathbf{R}$. Hence, the following three-quark field,

$$\mathcal{M}_{ijk}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) = \phi_{ii'}(\mathbf{R}, \mathbf{x}_1, t) Q_{i'}(\mathbf{x}_1, t) \phi_{jj'}(\mathbf{R}, \mathbf{x}_2, t) Q_{j'}(\mathbf{x}_2, t) \phi_{kk'}(\mathbf{R}, \mathbf{x}_3, t) Q_{k'}(\mathbf{x}_3, t), \tag{6}$$

transforms as a $3 \otimes 3 \otimes 3$ representation of the $\mathrm{SU}(3)_c$ gauge group, and, following Eq. (4), can be decomposed into a singlet, two octets and a decuplet field with respect to gauge transformations in $\mathbf{R}$.

Since the quark fields do not commute, the order of the quark fields in Eq. (6) matters. This observation will play a crucial role in Sec. III. For simplicity, we have omitted an explicit reference to $\mathbf{R}$ in the argument of $\mathcal{M}$, which includes the time coordinate t and the list of position coordinates $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ of the heavy-quark fields in the order (from left to right) of their appearance on the right-hand side of Eq. (6). The same convention is used for the color indices (i, j, k).

The composite field $\mathcal{M}_{ijk}$ may be decomposed into a singlet, S, two octets, O^A and O^S, and a decuplet, Δ, according to

$$\begin{aligned}\mathcal{M}_{ijk}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) &= S(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) \underline{\mathbf{S}}_{ijk} + \sum_{a=1}^{8} O^{Aa}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) \underline{\mathbf{O}}^{Aa}_{ijk} \\ &\quad + \sum_{a=1}^{8} O^{Sa}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) \underline{\mathbf{O}}^{Sa}_{ijk} + \sum_{\delta=1}^{10} \Delta^{\delta}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) \underline{\mathbf{\Delta}}^{\delta}_{ijk},\end{aligned} \tag{7}$$

where $\underline{\mathbf{S}}_{ijk}$, $\underline{\mathbf{O}}^{Aa}_{ijk}$, $\underline{\mathbf{O}}^{Sa}_{ijk}$ and $\underline{\mathbf{\Delta}}^{\delta}_{ijk}$ are orthogonal and normalized color tensors that satisfy the relations

$$\begin{aligned}&\underline{\mathbf{S}}_{ijk}\underline{\mathbf{S}}_{ijk} = 1, \qquad \underline{\mathbf{O}}^{Aa*}_{ijk}\underline{\mathbf{O}}^{Ab}_{ijk} = \delta^{ab}, \\ &\underline{\mathbf{O}}^{Sa*}_{ijk}\underline{\mathbf{O}}^{Sb}_{ijk} = \delta^{ab}, \qquad \underline{\mathbf{\Delta}}^{\delta}_{ijk}\underline{\mathbf{\Delta}}^{\delta'}_{ijk} = \delta^{\delta\delta'}, \\ &\underline{\mathbf{S}}_{ijk}\underline{\mathbf{O}}^{Aa}_{ijk} = \underline{\mathbf{S}}_{ijk}\underline{\mathbf{O}}^{Sa}_{ijk} = \underline{\mathbf{S}}_{ijk}\underline{\mathbf{\Delta}}^{\delta}_{ijk} = \underline{\mathbf{O}}^{Aa*}_{ijk}\underline{\mathbf{O}}^{Sb}_{ijk} \\ &\qquad = \underline{\mathbf{O}}^{Aa*}_{ijk}\underline{\mathbf{\Delta}}^{\delta}_{ijk} = \underline{\mathbf{O}}^{Sa*}_{ijk}\underline{\mathbf{\Delta}}^{\delta}_{ijk} = 0,\end{aligned} \tag{8}$$

with $a, b \in \{1, \ldots, 8\}$, and $\delta, \delta' \in \{1, \ldots, 10\}$ [15]. If the octet tensors $\underline{\mathbf{O}}^{Aa}_{ijk}$ and $\underline{\mathbf{O}}^{Sa}_{ijk}$ have the above properties, the following linear combinations also do:

$$\underline{\mathbf{O}}'^{Aa}_{ijk} = \mathrm{e}^{i\varphi_A}(\underline{\mathbf{O}}^{Aa}_{ijk} \cos\omega - \underline{\mathbf{O}}^{Sa}_{ijk} \sin\omega), \tag{9}$$

$$\underline{\mathbf{O}}'^{Sa}_{ijk} = \mathrm{e}^{i\varphi_S}(\underline{\mathbf{O}}^{Aa}_{ijk} \sin\omega + \underline{\mathbf{O}}^{Sa}_{ijk} \cos\omega), \tag{10}$$

where ω is an arbitrary angle and φ_A, φ_S denote generic phases. The octet tensors $\underline{\mathbf{O}}'^{Aa}_{ijk}$ and $\underline{\mathbf{O}}'^{Sa}_{ijk}$ hence form an alternative basis for the $8 \oplus 8$ sector. Requiring

$$O^{Aa}\underline{\mathbf{O}}^{Aa}_{ijk} + O^{Sa}\underline{\mathbf{O}}^{Sa}_{ijk} = O'^{Aa}\underline{\mathbf{O}}'^{Aa}_{ijk} + O'^{Sa}\underline{\mathbf{O}}'^{Sa}_{ijk}, \tag{11}$$

the associated octet fields are related to the original ones through the dual relations

$$O'^{Aa}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) = \mathrm{e}^{-i\varphi_A}[O^{Aa}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t)\cos\omega - O^{Sa}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t)\sin\omega], \tag{12}$$

$$O'^{Sa}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) = \mathrm{e}^{-i\varphi_S}[O^{Aa}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t)\sin\omega + O^{Sa}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t)\cos\omega]. \tag{13}$$

To work out the pNRQCD Lagrangian explicitly, we choose a specific (matrix) representation of the color tensors, namely that given in Ref. [15], Appendix B2. In order to keep this paper self-contained, we reproduce it here. Sticking to this particular choice, the color-octet tensors are given by

$$\underline{\mathbf{O}}^{Aa}_{ijk} = \frac{1}{2}\epsilon_{ijl}\lambda^a_{kl}, \tag{14}$$

and

$$\underline{\mathbf{O}}^{Sa}_{ijk} = \frac{1}{2\sqrt{3}}(\epsilon_{jkl}\lambda^a_{il} + \epsilon_{ikl}\lambda^a_{jl}). \tag{15}$$

The choice in Eqs. (13) and (14) is such that $\underline{\mathbf{O}}^{Aa}_{ijk}$ and $\underline{\mathbf{O}}^{Sa}_{ijk}$ are antisymmetric and symmetric in the first two color indices, respectively. Consequently, O^A and O^S will be referred to as the antisymmetric and symmetric octets. Moreover, the color-singlet tensor $\underline{\mathbf{S}}_{ijk}$ is chosen to be totally antisymmetric,

$$\underline{\mathbf{S}}_{ijk} = \frac{1}{\sqrt{6}}\epsilon_{ijk}, \tag{16}$$

whereas the color-decuplet tensor $\underline{\mathbf{\Delta}}^{\delta}_{ijk}$ is totally symmetric (an alternative decuplet is in Ref. [16]),

$$\begin{aligned}&\underline{\mathbf{\Delta}}^{1}_{111} = \underline{\mathbf{\Delta}}^{4}_{222} = \underline{\mathbf{\Delta}}^{10}_{333} = 1, \qquad \underline{\mathbf{\Delta}}^{6}_{\{123\}} = \frac{1}{\sqrt{6}}, \\ &\underline{\mathbf{\Delta}}^{2}_{\{112\}} = \underline{\mathbf{\Delta}}^{3}_{\{122\}} = \underline{\mathbf{\Delta}}^{5}_{\{113\}} = \underline{\mathbf{\Delta}}^{7}_{\{223\}} \\ &\qquad = \underline{\mathbf{\Delta}}^{8}_{\{133\}} = \underline{\mathbf{\Delta}}^{9}_{\{233\}} = \frac{1}{\sqrt{3}}.\end{aligned} \tag{17}$$

The symbol $\{ijk\}$ denotes all permutations of the indices ijk; all components not listed explicitly in Eq. (17) are zero. Note that $\underline{\mathbf{S}}_{ijk}$ and $\underline{\mathbf{\Delta}}^{\delta}_{ijk}$ are real-valued quantities.

From Eq. (6), it follows that the three-quark field

$$\Phi_{ijk}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) \equiv Q_i(\mathbf{x}_1, t) Q_j(\mathbf{x}_2, t) Q_k(\mathbf{x}_3, t) \tag{18}$$

can be written as

$$\Phi_{ijk}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) = \phi_{ii'}(\mathbf{x}_1, \mathbf{R}, t)\phi_{jj'}(\mathbf{x}_2, \mathbf{R}, t)\phi_{kk'}(\mathbf{x}_3, \mathbf{R}, t)\mathcal{M}_{i'j'k'}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t), \tag{19}$$

where we have used that $\phi^{-1}(\mathbf{y}, \mathbf{x}, t) = \phi^\dagger(\mathbf{y}, \mathbf{x}, t) = \phi(\mathbf{x}, \mathbf{y}, t)$. Finally, plugging Eq. (7) into Eq. (19) we may express the three-quark field Φ_{ijk} in terms of the composite singlet, octet and decuplet fields. The next step will consist in matching these composite fields to the corresponding ones in pNRQCD.

E. Matching and multipole expansion

We denote with $|\Omega\rangle$ a generic Fock state containing no heavy quarks, but an arbitrary number of US gluons and light quarks: $Q_i(\mathbf{x}, t)|\Omega\rangle = 0$. Therewith we define the three-heavy-quark Fock state

$$|QQQ\rangle = \frac{1}{\mathcal{N}} \int \mathrm{d}^3x_1 \int \mathrm{d}^3x_2 \int \mathrm{d}^3x_3 \Phi_{ijk}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) Q_k^\dagger(\mathbf{x}_3, t) Q_j^\dagger(\mathbf{x}_2, t) Q_i^\dagger(\mathbf{x}_1, t)|\Omega\rangle, \tag{20}$$

where $\mathcal{N}$ is a normalization factor and the composite field is now interpreted as independent of the heavy-quark fields. One can match NRQCD to pNRQCD by equating the expectation value of the NRQCD Hamiltonian in the state $|QQQ\rangle$ with the pNRQCD Hamiltonian (see Refs. [18,21] for the matching in the $Q\bar{Q}$ case). Thus, the heavy-quark fields in pNRQCD are cast into singlet, S, octet, O^{Aa} and O^{Sa}, and decuplet, Δ^δ, fields. The gluons in pNRQCD are explicitly rendered US by multipole expanding the gluon fields in the relative coordinates $\mathbf{r}_i (i = 1, 2, 3)$ with respect to the center of mass coordinate $\mathbf{R}$. The reason is that the center of mass coordinate (the "location" of the three-heavy-quark system) scales like the inverse of the recoiling total momentum of the three quarks, which is of the order of the US energy scale, while the relative coordinates of the three quarks (describing the "extension" of the triple heavy baryon) scale like the inverse of the typical relative momenta of the heavy quarks, which are of the order of the soft scale. As a result, ultrasoft gluons in pNRQCD are invariant under US gauge transformations, i.e., gauge transformations localized in $\mathbf{R}$. A Legendre transform of the pNRQCD Hamiltonian finally provides us with the pNRQCD Lagrangian.

In the same way as NRQCD can be understood as an expansion of QCD in terms of the inverse of the heavy-quark masses, pNRQCD can be understood as an expansion of the gluon fields of NRQCD, projected onto the specific (two or three) heavy-quark Fock space, in powers of the relative coordinates of the heavy quarks. Quantum corrections of the order of the soft scale are encoded in the matching coefficients of pNRQCD in the same way as quantum corrections of the order of the heavy-quark masses are encoded in the matching coefficients of NRQCD. The matching coefficients of pNRQCD are typically nonanalytic functions of the relative coordinates.

F. The pNRQCD Lagrangian

The pNRQCD Lagrangian is organized as an expansion in $1/m$ and in the relative coordinates $\mathbf{r}_i$. Up to zeroth order in the $1/m$ expansion (static limit) and first order in the multipole expansion, the pNRQCD Lagrangian for QQQ systems reads

$$\mathcal{L}_{\mathrm{pNRQCD}} = \mathcal{L}^{(0,0)}_{\mathrm{pNRQCD}} + \mathcal{L}^{(0,1)}_{\mathrm{pNRQCD}}. \tag{21}$$

An explicit derivation of this Lagrangian can be found in Ref. [15]; here we recall its expression. The term $\mathcal{L}^{(0,0)}_{\mathrm{pNRQCD}}$ describes at zeroth order in the multipole expansion the propagation of light quarks and US gluons as well as the temporal evolution of the static quarks, which are cast into singlet, $S \equiv S(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t)$, octet, $O^A \equiv O^A(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t)$ and $O^S \equiv O^S(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t)$, and decuplet, $\Delta \equiv \Delta(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t)$, fields (cf. Sec. II D),

$$\begin{aligned}\mathcal{L}^{(0,0)}_{\mathrm{pNRQCD}} = \int \mathrm{d}^3\rho \mathrm{d}^3\lambda \{ & S^\dagger [i\partial_0 - V^s] S + \Delta^\dagger [iD_0 - V^d] \Delta + O^{A\dagger} [iD_0 - V_A^o] O^A \\ & + O^{S\dagger} [iD_0 - V_S^o] O^S + O^{A\dagger} [-V_{AS}^o] O^S + O^{S\dagger} [-V_{AS}^o] O^A \} + \sum_l \bar{q}^l i \not{D} q^l - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}.\end{aligned} \tag{22}$$

The matching coefficients V^s, V_A^o, V_S^o and V^d correspond to singlet, (antisymmetric and symmetric) octet and decuplet potentials. The coefficient V_{AS}^o is an octet mixing potential. The term $\mathcal{L}^{(0,1)}_{\mathrm{pNRQCD}}$ accounts for the interactions between static quarks and US gluons at first order in the multipole expansion,

$$\begin{aligned}\mathcal{L}^{(0,1)}_{\mathrm{pNRQCD}} = \int \mathrm{d}^3\rho \mathrm{d}^3\lambda \Big\{ & V^{(0,1)}_{S\boldsymbol{\rho}\cdot\mathbf{E}O^S} \sum_{a=1}^{8} \frac{1}{2\sqrt{2}} [S^\dagger \boldsymbol{\rho} \cdot g\mathbf{E}^a O^{Sa} + O^{Sa\dagger} \boldsymbol{\rho} \cdot g\mathbf{E}^a S] \\ & - V^{(0,1)}_{S\boldsymbol{\lambda}\cdot\mathbf{E}O^A} \sum_{a=1}^{8} \frac{1}{\sqrt{6}} [S^\dagger \boldsymbol{\lambda} \cdot g\mathbf{E}^a O^{Aa} + O^{Aa\dagger} \boldsymbol{\lambda} \cdot g\mathbf{E}^a S] - V^{(0,1)}_{O^A\boldsymbol{\lambda}\cdot\mathbf{E}O^A} \sum_{a,b,c=1}^{8} \left(i \frac{f^{abc}}{6} + \frac{d^{abc}}{2} \right) O^{Aa\dagger} \boldsymbol{\lambda} \cdot g\mathbf{E}^b O^{Ac}\end{aligned}$$

$$+ V^{(0,1)}_{O^S \boldsymbol{\lambda}\cdot\mathbf{E} O^S} \sum_{a,b,c=1}^{8} \left(i\frac{f^{abc}}{6} + \frac{d^{abc}}{2} \right) O^{Sa\dagger} \boldsymbol{\lambda} \cdot g\mathbf{E}^b O^{Sc}$$

$$- V^{(0,1)}_{O^A \boldsymbol{\rho}\cdot\mathbf{E} O^S} \sum_{a,b,c=1}^{8} \left(\frac{if^{abc} + 3d^{abc}}{4\sqrt{3}} \right) [O^{Aa\dagger} \boldsymbol{\rho} \cdot g\mathbf{E}^b O^{Sc} + O^{Sa\dagger} \boldsymbol{\rho} \cdot g\mathbf{E}^b O^{Ac}]$$

$$+ V^{(0,1)}_{O^A \boldsymbol{\rho}\cdot\mathbf{E}\Delta} \sum_{a,b=1}^{8} \sum_{\delta=1}^{10} [(\epsilon_{ijk} T^a_{ii'} T^b_{jj'} \underline{\Delta}^{\delta}_{i'j'k}) O^{Aa\dagger} \boldsymbol{\rho} \cdot g\mathbf{E}^b \Delta^{\delta} - (\underline{\Delta}^{\delta}_{ijk} T^b_{ii'} T^a_{jj'} \epsilon_{i'j'k}) \Delta^{\delta\dagger} \boldsymbol{\rho} \cdot g\mathbf{E}^b O^{Aa}]$$

$$+ V^{(0,1)}_{O^S \boldsymbol{\lambda}\cdot\mathbf{E}\Delta} \sum_{a,b=1}^{8} \sum_{\delta=1}^{10} \frac{2}{\sqrt{3}} [(\epsilon_{ijk} T^a_{ii'} T^b_{jj'} \underline{\Delta}^{\delta}_{i'j'k}) O^{Sa\dagger} \boldsymbol{\lambda} \cdot g\mathbf{E}^b \Delta^{\delta} - (\underline{\Delta}^{\delta}_{ijk} T^b_{ii'} T^a_{jj'} \epsilon_{i'j'k}) \Delta^{\delta\dagger} \boldsymbol{\lambda} \cdot g\mathbf{E}^b O^{Sa}] \Big\}, \quad (23)$$

where $\mathbf{E} = \mathbf{E}^a T^a$ denotes the chromoelectric field evaluated at $\mathbf{R}$ and the coefficients $V^{(0,1)}_{\cdots}$ are matching coefficients associated to chromoelectric dipole interactions between QQQ fields in different color representations. The covariant derivatives, whose time components act on the octet and decuplet fields in Eq. (22), are understood to be in the octet and decuplet representations, respectively. They are given explicitly in Appendix A.

A mixing term, $-V^o_{AS}(O^{A\dagger} O^S + O^{S\dagger} O^A)$, has been included in $\mathcal{L}^{(0,0)}_{\text{pNRQCD}}$. Such a term was not considered in Ref. [15], but was first recognized in Ref. [16]. The mixing potential will play a crucial role in the study of the symmetry of pNRQCD under exchange of the heavy-quark fields (see Sec. III) and in the calculation of the US corrections to the singlet static energy (see Sec. IV).

For completeness, we list here the leading-order (LO) expressions for the various matching coefficients appearing in Eqs. (22) and (23). At order α_s the potentials in Eq. (22) are given by (cf. Refs. [15,16])

$$V^s(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = -\frac{2}{3}\alpha_s \left(\frac{1}{|\mathbf{r}_1|} + \frac{1}{|\mathbf{r}_2|} + \frac{1}{|\mathbf{r}_3|} \right), \quad (24)$$

$$V^d(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \frac{1}{3}\alpha_s \left(\frac{1}{|\mathbf{r}_1|} + \frac{1}{|\mathbf{r}_2|} + \frac{1}{|\mathbf{r}_3|} \right), \quad (25)$$

$$V^o_A(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \alpha_s \left(-\frac{2}{3}\frac{1}{|\mathbf{r}_1|} + \frac{1}{12}\frac{1}{|\mathbf{r}_2|} + \frac{1}{12}\frac{1}{|\mathbf{r}_3|} \right), \quad (26)$$

$$V^o_S(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \alpha_s \left(\frac{1}{3}\frac{1}{|\mathbf{r}_1|} - \frac{5}{12}\frac{1}{|\mathbf{r}_2|} - \frac{5}{12}\frac{1}{|\mathbf{r}_3|} \right), \quad (27)$$

$$V^o_{AS}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = -\frac{\sqrt{3}}{4}\alpha_s \left(\frac{1}{|\mathbf{r}_2|} - \frac{1}{|\mathbf{r}_3|} \right), \quad (28)$$

whereas all matching coefficients in Eq. (23) are equal to one at LO. The expressions for V^s up to next-to-next-to-leading order (NNLO), and for V^d, V^o_A, V^o_S and V^o_{AS} up to next-to-leading order (NLO) can be found in Ref. [16] (the expression for V^s up to NNLO is also in Appendix B).

III. SYMMETRY UNDER EXCHANGE OF THE HEAVY-QUARK FIELDS

As outlined in detail in Sec. II D, the heavy-quark fields in the pNRQCD Lagrangian are written in terms of composite fields, which are proportional to $Q_i(\mathbf{x}_1, t) Q_j(\mathbf{x}_2, t) \times Q_k(\mathbf{x}_3, t)$. However, as there is no preferred ordering, and the heavy-quark fields anticommute, different orderings of the heavy quarks lead to different composite fields. The orderings are however arbitrary and the pNRQCD Lagrangian should be invariant under different orderings of the heavy-quark fields. We call this invariance symmetry under exchange of the heavy-quark fields or, in short, exchange symmetry. A special case of exchange symmetry is the symmetry under permutation of the heavy-quark fields. A different ordering of the heavy-quark fields can be realized either (a) by relabeling the heavy-quark coordinates in the pNRQCD Lagrangian or (b) by anticommuting the heavy-quark fields in the composite fields. Since the two procedures lead to the same Lagrangian, this constrains the form of the heavy-quark potentials. In fact, the invariance of the Lagrangian under (a) is trivially realized due to the additional integrations over the quark locations x_1, x_2 and x_3, and only (b) results in nontrivial transformations.

(a) We may relabel the coordinates $\mathbf{x}_i$ and the relative vectors $\mathbf{r}_i$ in the pNRQCD Lagrangian according to one of the following two possibilities (other relabelings follow from these):

$$\mathbf{x}_1 \leftrightarrow \mathbf{x}_2, \ \mathbf{x}_3: \quad \begin{cases} \mathbf{r}_1 \to -\mathbf{r}_1 \\ \mathbf{r}_2 \to \mathbf{r}_3 \\ \mathbf{r}_3 \to \mathbf{r}_2, \end{cases} \quad (29)$$

$$\mathbf{x}_1 \leftrightarrow \mathbf{x}_3, \ \mathbf{x}_2: \quad \begin{cases} \mathbf{r}_1 \to -\mathbf{r}_3 \\ \mathbf{r}_2 \to -\mathbf{r}_2 \\ \mathbf{r}_3 \to -\mathbf{r}_1. \end{cases} \quad (30)$$

The relabelings affect the pNRQCD potentials and the ordering of the quark fields in the composite fields of pNRQCD.

(b) Because the heavy-quark fields $Q_i(\mathbf{x})$ of NRQCD satisfy equal-time anticommutation relations, $\{Q_i(\mathbf{x}, t), Q_j(\mathbf{y}, t)\} = 0$, from Eq. (18) it follows that

$$\Phi_{ijk}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) = -\Phi_{jik}(\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3, t), \tag{31}$$

$$\Phi_{ijk}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) = -\Phi_{kji}(\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1, t). \tag{32}$$

These identities hold also for $\mathcal{M}_{ijk}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t)$, which is related to $\Phi_{ijk}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t)$ through Eq. (19):

$$\mathcal{M}_{ijk}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) = -\mathcal{M}_{jik}(\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3, t), \tag{33}$$

$$\mathcal{M}_{ijk}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) = -\mathcal{M}_{kji}(\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1, t). \tag{34}$$

In turn, the identities for $\mathcal{M}_{ijk}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t)$ enable us to derive corresponding identities for the singlet, octet and decuplet fields just by multiplying Eqs. (33) and (34) with $\underline{\mathbf{S}}_{ijk}$, $\underline{\mathbf{\Delta}}^{\delta}_{ijk}$, $\underline{\mathbf{O}}^{Aa*}_{ijk}$, or $\underline{\mathbf{O}}^{Sa*}_{ijk}$, respectively, and summing over i, j, k:

$$\begin{cases} S(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) = S(\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3, t) \\ \Delta^{\delta}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) = -\Delta^{\delta}(\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3, t) \\ O^{Aa}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) = O^{Aa}(\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3, t) \\ O^{Sa}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) = -O^{Sa}(\mathbf{x}_2, \mathbf{x}_1, \mathbf{x}_3, t), \end{cases} \tag{35}$$

and

$$\begin{cases} S(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) = S(\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1, t) \\ \Delta^{\delta}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) = -\Delta^{\delta}(\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1, t) \\ O^{Aa}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) = -\frac{1}{2} O^{Aa}(\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1, t) + \frac{\sqrt{3}}{2} O^{Sa}(\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1, t) \\ O^{Sa}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, t) = \frac{\sqrt{3}}{2} O^{Aa}(\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1, t) + \frac{1}{2} O^{Sa}(\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1, t). \end{cases} \tag{36}$$

At variance with the relabeling (a), anticommuting the heavy-quarks in the composite fields only indirectly affects the pNRQCD potentials. Note that the octet transformations in (35) and (36) may be interpreted as a special case of the transformations (12) and (13) for $\varphi_S = 0$, $\varphi_A = \pi$ and $\omega = \pi/3$.

By relabeling (a) or by anticommuting the heavy-quark fields (b) we get two versions of the pNRQCD Lagrangian that must be the same. This requires the pNRQCD potentials to change in a well defined manner under the transformations (29) and (30). In particular, if we restrict ourselves to the potentials in Eq. (22), the singlet and decuplet potentials remain invariant, whereas the octet potentials transform as

$$\begin{cases} V_A^o(-\mathbf{r}_1, \mathbf{r}_3, \mathbf{r}_2) = V_A^o(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \\ V_S^o(-\mathbf{r}_1, \mathbf{r}_3, \mathbf{r}_2) = V_S^o(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \\ V_{AS}^o(-\mathbf{r}_1, \mathbf{r}_3, \mathbf{r}_2) = -V_{AS}^o(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3), \end{cases} \tag{37}$$

and

$$\begin{cases} V_A^o(-\mathbf{r}_3, -\mathbf{r}_2, -\mathbf{r}_1) = \frac{1}{4} V_A^o(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) + \frac{3}{4} V_S^o(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) - \frac{\sqrt{3}}{2} V_{AS}^o(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \\ V_S^o(-\mathbf{r}_3, -\mathbf{r}_2, -\mathbf{r}_1) = \frac{3}{4} V_A^o(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) + \frac{1}{4} V_S^o(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) + \frac{\sqrt{3}}{2} V_{AS}^o(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \\ V_{AS}^o(-\mathbf{r}_3, -\mathbf{r}_2, -\mathbf{r}_1) = \frac{\sqrt{3}}{4} [V_S^o(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) - V_A^o(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)] + \frac{1}{2} V_{AS}^o(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3), \end{cases} \tag{38}$$

for transformations of type (29) and (30), respectively. We emphasize that the above transformations are general and do not rely on any specific geometry of the three quarks. They also do not rely on any perturbative expansion. Furthermore, they are valid also beyond the static limit for any order in $1/m$.[2] As a simple application of the above formulas, let us consider for instance the LO expression of $V_{AS}^o(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ given in Eq. (28). Under (38) it transforms into

$$V_{AS}^o(-\mathbf{r}_3, -\mathbf{r}_2, -\mathbf{r}_1) = -\frac{\sqrt{3}}{4} \alpha_s \left(\frac{1}{|\mathbf{r}_2|} - \frac{1}{|\mathbf{r}_1|} \right), \tag{39}$$

which is the result expected from relabeling the coordinates according to Eq. (30). Let us emphasize again that the inclusion of the octet mixing potential V_{AS}^o in Eq. (22) is essential for reproducing the correct transformation properties of the octet potentials.

Finally, it is interesting to apply relations (37) and (38) to the most simple case of an equilateral geometry. In such a geometry we have a single length scale $r = |\mathbf{r}_1| = |\mathbf{r}_2| = |\mathbf{r}_3|$ and a single angle $\hat{\mathbf{r}}_1 \cdot \hat{\mathbf{r}}_2 = -\hat{\mathbf{r}}_1 \cdot \hat{\mathbf{r}}_3 = \hat{\mathbf{r}}_2 \cdot \hat{\mathbf{r}}_3 = \cos(\pi/3)$. Whenever the potentials are invariant under the transformations (29) and (30), which is surely the case for two-body interactions but may not hold at higher orders, from Eq. (37) it follows that $V_{AS}^o = 0$ and from Eq. (38) that

$$V_A^o(r) = V_S^o(r) \equiv V^o(r). \tag{40}$$

[2]Note however that a generalization to finite heavy-quark masses, m_1, m_2 and m_3, would also require some adjustment in Eqs. (29) and (30), as—besides the heavy-quark locations—also the masses would have to be exchanged, e.g., in Eq. (29), $m_1 \leftrightarrow m_2$, etc.

IV. THE QQQ SINGLET STATIC ENERGY AT $\mathcal{O}(\alpha_s^4 \ln \alpha_s)$

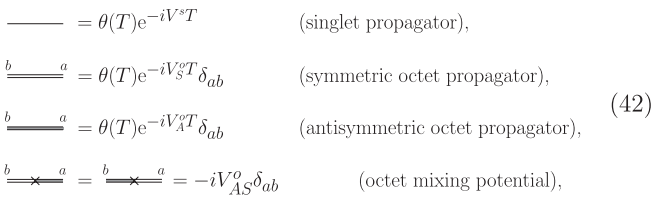
The potentials of pNRQCD depend in general on a factorization scale μ separating soft from US contributions,[3] whereas the singlet static energy E^s is an observable and therewith independent of μ. As in the $Q\bar{Q}$ case [23], the QQQ singlet static potential V^s is expected to become μ dependent at next-to-next-to-next-to leading order (NNNLO), i.e., at order α_s^4 [15]. The difference between the singlet static energy and the singlet static potential is encoded in an ultrasoft contribution denoted $\delta^s_{\rm US}$, which starts contributing at order α_s^4. It depends on μ in such a way that E^s, given by

$$E^s(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = V^s(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3; \mu) + \delta^s_{\rm US}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3; \mu), \tag{41}$$

is μ independent. The cancelation of the μ dependence of V^s against $\delta^s_{\rm US}$ at NNNLO leaves in E^s a remnant, which is a contribution of order $\alpha_s^4 \ln \alpha_s$. This is the leading perturbative contribution to E^s that is nonanalytic in α_s. The most convenient way to calculate the $\alpha_s^4 \ln \mu$ term in V^s, and the $\alpha_s^4 \ln \alpha_s$ term in E^s, is by looking at the leading divergence of $\delta^s_{\rm US}$. This requires the one-loop calculation of the color-singlet self energy as opposed to the three-loop calculation necessary to extract the term $\alpha_s^4 \ln \mu$ directly from V^s. We will perform this calculation in the following section.

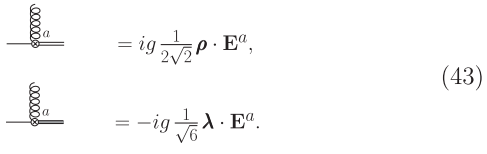
A. Determination of $\delta^s_{\rm US}$

We aim at calculating $\delta^s_{\rm US}$ up to order α_s^4. For this purpose we need the singlet and octet propagators, and the octet mixing potential at leading order [cf. Eq. (22)],



(42)

as well as the singlet-to-octet interaction vertices at order $\mathbf{r}_i$ in the multipole expansion [cf. Eq. (23), note that the singlet couples differently to the symmetric and antisymmetric octets],

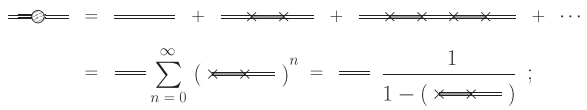


(43)

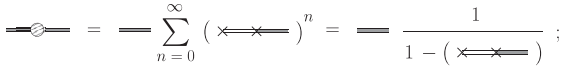
The parameter T in Eq. (42) is the propagation time. The wavy lines in Eq. (43) represent ultrasoft gluons; note that we have written the vertices with US gluons treating the gluons as external fields.

The most noteworthy difference with respect to the calculation of $\delta^s_{\rm US}$ in the $Q\bar{Q}$ case is that here the singlet couples to two distinct octet fields and that the octet fields mix. For this reason the calculation in the baryonic case exhibits some novel features with respect to the analogous mesonic case. Since the mixing of the octet fields is an effect of the same order as the energies of the octets, it must be accounted for to all orders when computing the physical octet-to-octet propagators. The resummation of the octet mixing potential gives rise to three different types of resummed octet propagators:

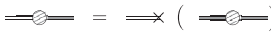
(1) a resummed octet propagator, G^o_S, that describes the propagation from a symmetric initial state to a symmetric final state:



(2) a resummed octet propagator, G^o_A, that describes the propagation from an antisymmetric initial state to an antisymmetric final state:



(3) a resummed octet propagator, G^o_{AS}, that describes the propagation from a symmetric initial state to an antisymmetric final state or vice versa:



The explicit expressions for the resummed octet propagators are most conveniently computed in momentum space and read

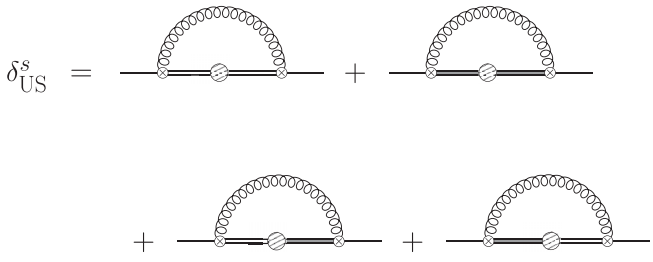


FIG. 2. Leading-order contributions to $\delta^s_{\rm US}$. As there is no direct coupling between decuplet and singlet fields at first order in the multipole expansion, we do not have contributions involving decuplet degrees of freedom.

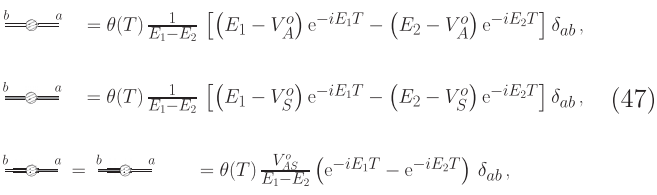
[3]This dependence, which will be displayed explicitly in the following, has been dropped in Eqs. (37) and (38).

$$-i[G_S^o(E)]_{ab} = \frac{i\delta_{ab}(E - V_A^o)}{(E - V_S^o + i\epsilon)(E - V_A^o + i\epsilon) - (V_{AS}^o)^2}, \tag{44}$$

$$-i[G_A^o(E)]_{ab} = \frac{i\delta_{ab}(E - V_S^o)}{(E - V_S^o + i\epsilon)(E - V_A^o + i\epsilon) - (V_{AS}^o)^2}, \tag{45}$$

$$-i[G_{AS}^o(E)]_{ab} = \frac{i\delta_{ab}V_{AS}^o}{(E - V_S^o + i\epsilon)(E - V_A^o + i\epsilon) - (V_{AS}^o)^2}, \tag{46}$$

with $\epsilon \to 0^+$. After performing a Fourier transform from energy E to time T, we obtain



(47)

where

$$E_{1,2} = \frac{V_A^o + V_S^o}{2} \pm \sqrt{\left(\frac{V_A^o - V_S^o}{2}\right)^2 + (V_{AS}^o)^2} - i\epsilon. \tag{48}$$

The US contribution $\delta_{\rm US}^s$ is given at LO by the color-singlet self-energy diagrams shown in Fig. 2. Because the singlet couples to two distinct octet fields and they mix, we have four such diagrams [cf. Eq. (47)]. They give

$$\begin{aligned}\delta_{\rm US}^s = &-ig^2\left(\frac{1}{2\sqrt{2}}\right)^2 \int_0^\infty dt \frac{1}{E_1 - E_2}[(E_1 - V_A^o)e^{-it(E_1 - V^s)} - (E_2 - V_A^o)e^{-it(E_2 - V^s)}]\langle \boldsymbol{\rho}\cdot\mathbf{E}^a(t)\boldsymbol{\rho}\cdot\mathbf{E}^a(0)\rangle \\ &- ig^2\left(\frac{1}{\sqrt{6}}\right)^2 \int_0^\infty dt \frac{1}{E_1 - E_2}[(E_1 - V_S^o)e^{-it(E_1 - V^s)} - (E_2 - V_S^o)e^{-it(E_2 - V^s)}]\langle \boldsymbol{\lambda}\cdot\mathbf{E}^a(t)\boldsymbol{\lambda}\cdot\mathbf{E}^a(0)\rangle \\ &+ 2ig^2 \frac{1}{2\sqrt{2}}\frac{1}{\sqrt{6}} \int_0^\infty dt \frac{V_{AS}^o}{E_1 - E_2}[e^{-it(E_1 - V^s)} - e^{-it(E_2 - V^s)}]\langle \boldsymbol{\rho}\cdot\mathbf{E}^a(t)\boldsymbol{\lambda}\cdot\mathbf{E}^a(0)\rangle,\end{aligned} \tag{49}$$

where $\langle\cdots\rangle$ stands for a vacuum expectation value. In writing the various contributions in Eq. (49), we have kept the same order as in Fig. 2: the first two terms correspond to the two diagrams shown in the first line of Fig. 2, and the last contribution is the sum of the two diagrams in the second line of Fig. 2, which are equal.

The vacuum expectation value of two chromoelectric fields reads in dimensional regularization ($d = 4 - 2\varepsilon$ is the number of dimensions)

$$\langle \mathbf{a}\cdot\mathbf{E}^a(t)\mathbf{b}\cdot\mathbf{E}^a(0)\rangle = \mathbf{a}\cdot\mathbf{b}\frac{4(d-2)}{(d-1)}\mu^{4-d}\int \frac{d^{d-1}q}{(2\pi)^{d-1}}|\mathbf{q}|e^{-i|\mathbf{q}|t} + \mathcal{O}(\alpha_s), \tag{50}$$

where $\mathbf{a}$ and $\mathbf{b}$ are two generic vectors and $t > 0$. Performing the integrals in (49) we obtain

$$\begin{aligned}\delta_{\rm US}^s = \frac{4}{3}\frac{\alpha_s}{\pi}\frac{1}{E_1 - E_2}\Bigg[&\left(\frac{|\boldsymbol{\rho}|^2}{4}(E_1 - V_A^o) + \frac{|\boldsymbol{\lambda}|^2}{3}(E_1 - V_S^o) - \frac{\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}{\sqrt{3}}V_{AS}^o\right)(E_1 - V^s)^3\left(\frac{1}{\varepsilon} - \gamma_E - \ln\frac{(E_1 - V^s)^2}{\pi\mu^2} + \frac{5}{3}\right) \\ &- \left(\frac{|\boldsymbol{\rho}|^2}{4}(E_2 - V_A^o) + \frac{|\boldsymbol{\lambda}|^2}{3}(E_2 - V_S^o) - \frac{\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}{\sqrt{3}}V_{AS}^o\right)(E_2 - V^s)^3\left(\frac{1}{\varepsilon} - \gamma_E - \ln\frac{(E_2 - V^s)^2}{\pi\mu^2} + \frac{5}{3}\right)\Bigg],\end{aligned} \tag{51}$$

where γ_E is the Euler-Mascheroni constant. Equation (51) comprises the entire US contribution up to order α_s^4. The explicit expressions may be obtained by replacing E_1 and E_2 with the right-hand side of Eq. (48), and V^s, V_A^o, V_S^o and V_{AS}^o by the LO expressions given in Eqs. (24) and (26)–(28), respectively. Equation (51) corrects the expression derived in Ref. [15] where the mixing of the octet fields was not taken into account. Hence, the result of Ref. [15] is retained from Eq. (51) by setting $V_{AS}^o = 0$.

B. Invariance of $\delta_{\rm US}^s$ under exchange symmetry

The US correction, $\delta_{\rm US}^s$, calculated in the previous section is expected to be invariant under the exchange symmetry discussed in Sec. III. To verify this we observe that according to Eqs. (37) and (38) the combinations $(V_A^o + V_S^o)$ and $[(V_A^o - V_S^o)^2/4 + (V_{AS}^o)^2]$ are each invariant. This implies that both E_1 and E_2 are invariant according to the definition (48). Also the singlet static potential, V^s, is invariant at LO [see Eq. (24)]. If we rewrite explicitly the expression $|\boldsymbol{\rho}|^2/4 + |\boldsymbol{\lambda}|^2/3$ in terms of the positions of the heavy quarks with the help of Eqs. (2) and (3),

$$\frac{|\boldsymbol{\rho}|^2}{4} + \frac{|\boldsymbol{\lambda}|^2}{3} = \frac{1}{3}(\mathbf{x}_1^2 + \mathbf{x}_2^2 + \mathbf{x}_3^2 - \mathbf{x}_1\cdot\mathbf{x}_2 - \mathbf{x}_1\cdot\mathbf{x}_3 - \mathbf{x}_2\cdot\mathbf{x}_3), \tag{52}$$

it is evident that this expression is invariant under the transformations (29) and (30). Finally, we have to show that the expression

$$V_A^o \frac{|\boldsymbol{\rho}|^2}{4} + V_S^o \frac{|\boldsymbol{\lambda}|^2}{3} + V_{AS}^o \frac{\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}{\sqrt{3}} \tag{53}$$

is also invariant. This is a straightforward, although not manifest, consequence of the transformations (29), (30), (37), and (38), which completes the proof that $\delta^s_{\rm US}$ is invariant under the exchange symmetry. The invariance of $\delta^s_{\rm US}$ is directly inherited by the contribution to V^s at order $\alpha_s^4 \ln \mu$ and the singlet static energy E^s at order $\alpha_s^4 \ln \alpha_s$.

C. The QQQ singlet static potential and energy

According to Eq. (41), the divergence and the $\alpha_s^4 \ln \mu$ term in $\delta^s_{\rm US}$ must cancel against a divergence and a term $\alpha_s^4 \ln \mu$ in the singlet static potential V^s. Therefore the $\alpha_s^4 \ln \mu$ part of the potential may be read off from Eq. (51). In a minimal subtraction scheme, the singlet static potential up to order $\alpha_s^4 \ln \mu$ is then given by

$$\begin{aligned} V^s(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3;\mu) = V^s_{\rm NNLO}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) - \frac{\alpha_s^4}{3\pi}\ln\mu\Bigg[&\left(\mathbf{r}_1^2 + \frac{(\mathbf{r}_2+\mathbf{r}_3)^2}{3}\right)\left(\frac{1}{|\mathbf{r}_1|^2}+\frac{1}{|\mathbf{r}_2|^2}+\frac{1}{|\mathbf{r}_3|^2}-\frac{1}{4}\frac{|\mathbf{r}_1|+|\mathbf{r}_2|+|\mathbf{r}_3|}{|\mathbf{r}_1||\mathbf{r}_2||\mathbf{r}_3|}\right)\left(\frac{1}{|\mathbf{r}_1|}+\frac{1}{|\mathbf{r}_2|}+\frac{1}{|\mathbf{r}_3|}\right)\\ &+\left(\mathbf{r}_1^2 - \frac{(\mathbf{r}_2+\mathbf{r}_3)^2}{3}\right)\left(\frac{1}{|\mathbf{r}_1|^2}+\frac{1}{|\mathbf{r}_2|^2}+\frac{1}{|\mathbf{r}_3|^2}+\frac{5}{4}\frac{|\mathbf{r}_1|+|\mathbf{r}_2|+|\mathbf{r}_3|}{|\mathbf{r}_1||\mathbf{r}_2||\mathbf{r}_3|}\right)\left(\frac{1}{|\mathbf{r}_1|}-\frac{1}{2|\mathbf{r}_2|}-\frac{1}{2|\mathbf{r}_3|}\right)\\ &+\mathbf{r}_1\cdot(\mathbf{r}_2+\mathbf{r}_3)\left(\frac{1}{|\mathbf{r}_1|^2}+\frac{1}{|\mathbf{r}_2|^2}+\frac{1}{|\mathbf{r}_3|^2}+\frac{5}{4}\frac{|\mathbf{r}_1|+|\mathbf{r}_2|+|\mathbf{r}_3|}{|\mathbf{r}_1||\mathbf{r}_2||\mathbf{r}_3|}\right)\left(\frac{1}{|\mathbf{r}_2|}-\frac{1}{|\mathbf{r}_3|}\right)\Bigg]. \end{aligned} \tag{54}$$

The singlet static potential up to order α_s^3, which we have denoted by $V^s_{\rm NNLO}$, has been calculated in Ref. [16] and is reproduced in Appendix B. At order α_s^3, $V^s_{\rm NNLO}$ contains the leading three-body potential; also the new term proportional to $\alpha_s^4 \ln \mu$ that we have added here is a genuine three-body potential.

Summing up the singlet static potential (54) with the US contribution (51) we obtain the singlet static energy up to order $\alpha_s^4 \ln \alpha_s$, which reads

$$\begin{aligned} E^s(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) = V^s_{\rm NNLO}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) - \frac{\alpha_s^4}{3\pi}\ln\alpha_s\Bigg[&\left(\mathbf{r}_1^2 + \frac{(\mathbf{r}_2+\mathbf{r}_3)^2}{3}\right)\left(\frac{1}{|\mathbf{r}_1|^2}+\frac{1}{|\mathbf{r}_2|^2}+\frac{1}{|\mathbf{r}_3|^2}-\frac{1}{4}\frac{|\mathbf{r}_1|+|\mathbf{r}_2|+|\mathbf{r}_3|}{|\mathbf{r}_1||\mathbf{r}_2||\mathbf{r}_3|}\right)\left(\frac{1}{|\mathbf{r}_1|}+\frac{1}{|\mathbf{r}_2|}+\frac{1}{|\mathbf{r}_3|}\right)\\ &+\left(\mathbf{r}_1^2 - \frac{(\mathbf{r}_2+\mathbf{r}_3)^2}{3}\right)\left(\frac{1}{|\mathbf{r}_1|^2}+\frac{1}{|\mathbf{r}_2|^2}+\frac{1}{|\mathbf{r}_3|^2}+\frac{5}{4}\frac{|\mathbf{r}_1|+|\mathbf{r}_2|+|\mathbf{r}_3|}{|\mathbf{r}_1||\mathbf{r}_2||\mathbf{r}_3|}\right)\left(\frac{1}{|\mathbf{r}_1|}-\frac{1}{2|\mathbf{r}_2|}-\frac{1}{2|\mathbf{r}_3|}\right)\\ &+\mathbf{r}_1\cdot(\mathbf{r}_2+\mathbf{r}_3)\left(\frac{1}{|\mathbf{r}_1|^2}+\frac{1}{|\mathbf{r}_2|^2}+\frac{1}{|\mathbf{r}_3|^2}+\frac{5}{4}\frac{|\mathbf{r}_1|+|\mathbf{r}_2|+|\mathbf{r}_3|}{|\mathbf{r}_1||\mathbf{r}_2||\mathbf{r}_3|}\right)\left(\frac{1}{|\mathbf{r}_2|}-\frac{1}{|\mathbf{r}_3|}\right)\Bigg]. \end{aligned} \tag{55}$$

The logarithm of α_s signals that an ultraviolet divergence from the US scale has canceled against an infrared divergence from the soft scale.

Finally, it may be useful to express Eqs. (54) and (55) in a way that makes manifest the invariance under exchange symmetry proven in Sec. IV B. First, we recall that $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ are not independent (cf. Sec. II C) and write

$$E^s(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) = E^s(\mathbf{r}_2-\mathbf{r}_3,\mathbf{r}_2,\mathbf{r}_3) \equiv E^s(\mathbf{r}_2,\mathbf{r}_3), \tag{56}$$

then we observe that

$$E^s(\mathbf{r}_2,\mathbf{r}_3) = E^s(\mathbf{r}_3,\mathbf{r}_2). \tag{57}$$

Hence an expression of the singlet static energy, which is manifestly invariant under exchange symmetry, is

$$E^s(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) = \frac{E^s(\mathbf{r}_2,\mathbf{r}_3) + E^s(\mathbf{r}_1,-\mathbf{r}_3) + E^s(-\mathbf{r}_2,-\mathbf{r}_1)}{3}. \tag{58}$$

Similarly one can obtain a manifestly invariant expression of the singlet static potential.

V. RENORMALIZATION GROUP IMPROVEMENT OF THE SINGLET STATIC POTENTIAL IN AN EQUILATERAL GEOMETRY

The US logarithms that start appearing in the static potential at NNNLO may be resummed to all orders by solving the corresponding renormalization group equations. These are a set of equations that describe the scale dependence of the static potentials in the different color representations. They follow from requiring that the static energies of the QQQ system and its gluonic excitations are independent of the renormalization scheme. The potentials in the different color representations mix under renormalization. This may be easily understood by looking at the renormalization group equation for the singlet potential that can be derived from $\mu {\rm d}V^s/{\rm d}\mu = -\mu {\rm d}\delta^s_{\rm US}/{\rm d}\mu$ and Eq. (51),

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}V^s = -\frac{8}{3}\frac{\alpha_\mathrm{s}}{\pi}\left\{\left[\frac{V_S^o - V_A^o}{2}\left(\frac{|\boldsymbol{\rho}|^2}{4} - \frac{|\boldsymbol{\lambda}|^2}{3}\right) - V_{AS}^o\frac{\boldsymbol{\rho}\cdot\boldsymbol{\lambda}}{\sqrt{3}}\right]\left[3\left(\frac{V_S^o + V_A^o}{2} - V^s\right)^2 + \frac{(V_S^o - V_A^o)^2}{4} + (V_{AS}^o)^2\right]\right.$$
$$\left. + \left(\frac{V_S^o + V_A^o}{2} - V^s\right)\left(\frac{|\boldsymbol{\rho}|^2}{4} + \frac{|\boldsymbol{\lambda}|^2}{3}\right)\left[\left(\frac{V_S^o + V_A^o}{2} - V^s\right)^2 + 3\frac{(V_S^o - V_A^o)^2}{4} + 3(V_{AS}^o)^2\right]\right\}. \quad (59)$$

It shows the explicit dependence of the running of V^s on the octet potentials and octet mixing potential.

In the $Q\bar{Q}$ case the renormalization group equations have been solved for the singlet static potential at next-to-next-to-leading logarithmic (NNLL) accuracy in Ref. [25] and at next-to-next-to-next-to-leading logarithmic (NNNLL) accuracy in Ref. [26].[4] In the QQQ case similar results can be obtained by solving Eq. (59) with the corresponding renormalization group equations for the octet and decuplet potentials. There is however a difference between the $Q\bar{Q}$ and the QQQ case that is worth highlighting. While in a $Q\bar{Q}$ system there is just one length, the distance between the heavy quark and antiquark, the generic three-body system is characterized by more than one length. For a general three-body geometry, therefore, logarithmic corrections in the US scale could be numerically as important as finite logarithms involving ratios among the different lengths of the system. The calculation of these finite logarithms requires the calculation of the QQQ static Wilson loop. However, these logarithms are unimportant if the distances between the heavy quarks are similar. In the following, we will therefore restrict ourselves to the simplest case of three static quarks located at the corners of an equilateral triangle. In this situation, the three-body system is characterized, like the two-body one, by just one fundamental length, which can be identified with the length of each side of the triangle: $|\mathbf{r}_1| = |\mathbf{r}_2| = |\mathbf{r}_3| = r$.

In the equilateral limit at least up to NLO, the different octet fields do not mix, moreover, as has been shown in Eq. (40), the two octet potentials V_S^o and V_A^o are equal. The US contribution for the singlet static energy follows by specializing the general formula (51) to the equilateral limit. The US contributions for the octet and decuplet static energies can be derived along the same lines (cf. also the calculation of the US corrections for the $Q\bar{Q}$ octet potential in Ref. [22]). In particular, in the equilateral limit one has to consider only the diagrams shown in Fig. 3, since octet-to-octet diagrams with an intermediate octet propagator in the loop are scaleless for $V_S^o = V_A^o = V^o$, and thus vanish in dimensional regularization. Moreover, the US leading-order contribution for the symmetric octet is equal to the one for the antisymmetric octet; we call it, δ_{US}^o. The divergent parts of the diagrams shown in Fig. 3 give rise to the following renormalization group equations valid for the singlet, octet and decuplet static potentials of three quarks located at the corners of an equilateral triangle of side length r:

$$\begin{cases} \mu\frac{\mathrm{d}}{\mathrm{d}\mu}V^s = -\frac{4}{3\pi}\alpha_\mathrm{s}r^2(V^o - V^s)^3 + \mathcal{O}(\alpha_\mathrm{s}^5) \\ \mu\frac{\mathrm{d}}{\mathrm{d}\mu}V^o = \frac{1}{12\pi}\alpha_\mathrm{s}r^2[(V^o - V^s)^3 + 5(V^o - V^d)^3] + \mathcal{O}(\alpha_\mathrm{s}^5) \\ \mu\frac{\mathrm{d}}{\mathrm{d}\mu}V^d = -\frac{2}{3\pi}\alpha_\mathrm{s}r^2(V^o - V^d)^3 + \mathcal{O}(\alpha_\mathrm{s}^5) \\ \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\alpha_\mathrm{s} = \alpha_\mathrm{s}\beta(\alpha_\mathrm{s}). \end{cases} \quad (60)$$

The first equation is just the equilateral limit of Eq. (59). The last equation describes the running of the strong coupling constant, where $\beta(\alpha_\mathrm{s}) = -\alpha_\mathrm{s}\beta_0/(2\pi) + \mathcal{O}(\alpha_\mathrm{s}^2)$ is the beta function; the first coefficient of the beta function is $\beta_0 = 11 - 2/3n_l$ with n_l the number of light-quark flavors. By observing that

$$V^o - V^s = -(V^o - V^d) + \mathcal{O}(\alpha_\mathrm{s}^3), \quad (61)$$

as follows straightforwardly from the results of Ref. [16], the system of equations (60) can be split into two sets of decoupled equations:

$$\begin{cases} \mu\frac{\mathrm{d}}{\mathrm{d}\mu}V^s = -\frac{4}{3\pi}\alpha_\mathrm{s}r^2(V^o - V^s)^3 + \mathcal{O}(\alpha_\mathrm{s}^5) \\ \mu\frac{\mathrm{d}}{\mathrm{d}\mu}V^o = -\frac{1}{3\pi}\alpha_\mathrm{s}r^2(V^o - V^s)^3 + \mathcal{O}(\alpha_\mathrm{s}^5) \\ \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\alpha_\mathrm{s} = \alpha_\mathrm{s}\beta(\alpha_\mathrm{s}), \end{cases} \quad (62)$$

and

$$\begin{cases} \mu\frac{\mathrm{d}}{\mathrm{d}\mu}V^d = -\frac{2}{3\pi}\alpha_\mathrm{s}r^2(V^o - V^d)^3 + \mathcal{O}(\alpha_\mathrm{s}^5) \\ \mu\frac{\mathrm{d}}{\mathrm{d}\mu}V^o = \frac{1}{3\pi}\alpha_\mathrm{s}r^2(V^o - V^d)^3 + \mathcal{O}(\alpha_\mathrm{s}^5) \\ \mu\frac{\mathrm{d}}{\mathrm{d}\mu}\alpha_\mathrm{s} = \alpha_\mathrm{s}\beta(\alpha_\mathrm{s}). \end{cases} \quad (63)$$

The two sets of equations can be solved as in Ref. [25] leading to[5]

$$V^s(r;\mu) = V^s_{\mathrm{NNLO}}(r) - 9\frac{\alpha_\mathrm{s}^3(1/r)}{\beta_0 r}\ln\frac{\alpha_\mathrm{s}(1/r)}{\alpha_\mathrm{s}(\mu)}, \quad (64)$$

$$V^o(r;\mu) = V^o_{\mathrm{NNLO}}(r) - \frac{9}{4}\frac{\alpha_\mathrm{s}^3(1/r)}{\beta_0 r}\ln\frac{\alpha_\mathrm{s}(1/r)}{\alpha_\mathrm{s}(\mu)}, \quad (65)$$

$$V^d(r;\mu) = V^d_{\mathrm{NNLO}}(r) + \frac{9}{2}\frac{\alpha_\mathrm{s}^3(1/r)}{\beta_0 r}\ln\frac{\alpha_\mathrm{s}(1/r)}{\alpha_\mathrm{s}(\mu)}. \quad (66)$$

[4] An NNLL accuracy amounts at resumming $\alpha_\mathrm{s}^3(\alpha_\mathrm{s}\ln\mu)^n$ terms and an NNNLL accuracy amounts at resumming $\alpha_\mathrm{s}^4(\alpha_\mathrm{s}\ln\mu)^n$ terms, with $n \in \mathbb{N}_0$.

[5] All coupling constants in $V^s_{\mathrm{NNLO}}(r)$, $V^o_{\mathrm{NNLO}}(r)$ and $V^d_{\mathrm{NNLO}}(r)$ are evaluated at the scale $1/r$.

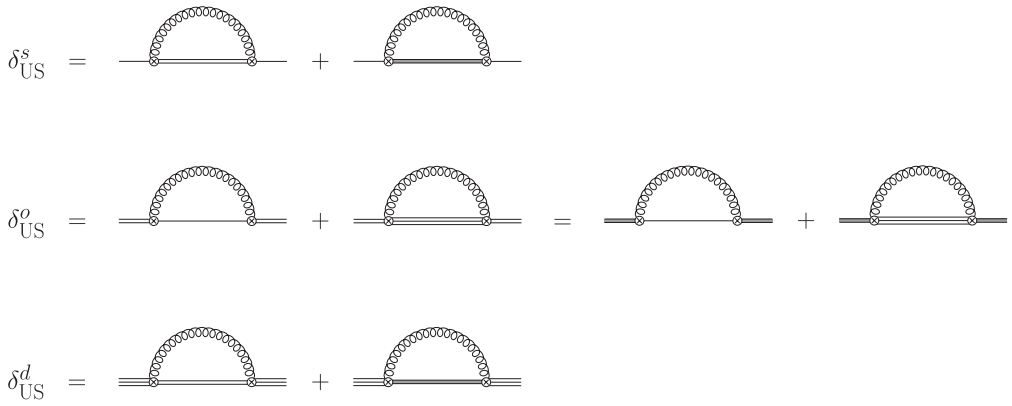


FIG. 3. Leading-order ultrasoft contributions to the singlet, $\delta^s_{\rm US}$, octet, $\delta^o_{\rm US}$, and decuplet, $\delta^d_{\rm US}$, energies in an equilateral geometry. The triple lines represent the decuplet propagator, $\theta(T)e^{-iV^dT}\delta_{\delta\delta'}$; the decuplet can couple to a symmetric octet, with vertex $ig\frac{2}{\sqrt{3}}(\epsilon_{ijk}T^a_{ii'}T^b_{jj'}\underline{\Delta}^{\delta}_{i'j'k})\boldsymbol{\lambda}\cdot\mathbf{E}^b$, or to an antisymmetric octet, with vertex $ig(\epsilon_{ijk}T^a_{ii'}T^b_{jj'}\underline{\Delta}^{\delta}_{i'j'k})\boldsymbol{\rho}\cdot\mathbf{E}^b$; the other propagators and vertices have been introduced in Eqs. (42) and (43).

The singlet static potential is known at NNLO, hence Eq. (64) provides the complete expression of the singlet static potential at NNLL accuracy in an equilateral geometry. This is the most accurate perturbative determination of this quantity. Instead neither the octet nor the decuplet potentials are known beyond NLO (see Ref. [16]).

VI. CONCLUSIONS

In the paper, we have reconsidered the construction of pNRQCD for systems made of three heavy quarks with equal masses. We have, in particular, rederived the pNRQCD Lagrangian in the static limit and put special attention to the symmetry under exchange of the heavy-quark fields. Although the symmetry is an obvious property of these systems, its consequences for the pNRQCD Lagrangian and in particular for its octet sector have been explored here for the first time. Three static quarks may be cast either in a color-singlet, two distinct color-octets or a color-decuplet configuration. Whereas the color singlet is completely antisymmetric and the color decuplet is completely symmetric in the color-indices, the color-octet transformations depend on the color indices that are exchanged. The fact that color-octet fields are specially sensitive to the ordering of the quarks reflects in the fact that they mix, in general, under exchange of the heavy-quark fields and dynamically through a one-gluon exchange. As a consequence, also the octet potentials and the mixing potential transform nontrivially under exchange symmetry; we have listed their transformation properties in Eqs. (37) and (38).

Thereafter, we have computed the leading ultrasoft contribution to the QQQ singlet static energy, $\delta^s_{\rm US}$. Its expression can be found in (51). Because of the two different octet fields and their mixing, the calculation of $\delta^s_{\rm US}$ requires the evaluation of four diagrams and the resummation of the octet mixing potential for all of them. The calculation is therefore more involved than the analogous one of the US contribution in the $Q\bar{Q}$ case. The expression for $\delta^s_{\rm US}$ in the QQQ case offers also a nontrivial test for the invariance under exchange symmetry; this has been performed in Sec. IV B. A consequence of the calculation of $\delta^s_{\rm US}$ at leading order is that we can determine the singlet static potential at order $\alpha_{\rm s}^4\ln\mu$, see Eq. (54), and the singlet static energy at order $\alpha_{\rm s}^4\ln\alpha_{\rm s}$, see Eq. (55). These results represent the new computational outcome of this work and so far the most accurate determinations of the QQQ singlet static potential and energy in perturbative QCD. The new contribution computed for the potential is valid for any configuration in space that the three quarks may take and it is a three-body interaction. Together with the three-body interaction at two-loop order computed in Ref. [16] it may provide new insight on the emergence of a long-range three-body interaction governed by just one fundamental length that is observed in lattice studies (see, e.g., Refs. [2,3,10]).

In the last part of the paper, we have focused on the special situation where the three quarks are located at the corners of an equilateral triangle of side length r. In this limit, where the two octet potentials become degenerate, we have solved the renormalization group equations for the color singlet, octet and decuplet potentials at NNLL accuracy. The corresponding expressions can be found in Eqs. (64)–(66). Hence, for an equilateral geometry, the QQQ singlet static potential is now known up to order $\alpha_{\rm s}^3(\alpha_{\rm s}\ln\mu r)^n$ for all $n\in\mathbb{N}_0$.

ACKNOWLEDGMENTS

Work supported in part by DFG and NSFC (CRC 110), and by the DFG cluster of excellence "Origin and structure of the universe" (www.universe-cluster.de). F. K. gratefully acknowledges financial support from the FAZIT foundation and inspiring discussions with E. Thoma.

APPENDIX A: COVARIANT DERIVATIVE OPERATORS

In this Appendix, we list the explicit matrix representations for the covariant derivative operators in the octet and decuplet representations of $\mathrm{SU}(3)_c$ that appear in Eq. (22). The $\mathrm{SU}(3)_c$ covariant derivative is of the general form

$$D_\mu = \partial_\mu + igA^a_\mu T^a_r, \tag{A1}$$

where $a = 1, \ldots, 8$ and T^a_r refers to the $\mathrm{SU}(3)_c$ generators in the representation r. The generators in the octet ($r = 8$) and in the decuplet ($r = 10$) representation are [15]

$$\begin{aligned} (T^a_8)_{bc} &= -if^{abc}, \qquad b, c = 1, \ldots, 8, \\ (T^a_{10})_{\delta\delta'} &= \frac{3}{2}\underline{\Delta}^{\delta}_{ijk}\lambda^a_{ii'}\underline{\Delta}^{\delta'}_{i'jk}, \qquad \delta, \delta' = 1, \ldots, 10, \end{aligned} \tag{A2}$$

where f^{abc} are the structure constants of $\mathrm{SU}(3)_c$. An explicit representation of the decuplet tensor $\underline{\Delta}^{\delta}_{ijk}$ is in (17).

APPENDIX B: THE SINGLET STATIC POTENTIAL UP TO ORDER α_s^3

We reproduce here for completeness the expression of the singlet static potential up to order α_s^3 computed in [16]:

$$\begin{aligned} &V^s_{\mathrm{NNLO}}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \\ &= -\frac{2}{3}\sum_{i=1}^{3}\frac{\alpha_s(1/|\mathbf{r}_i|)}{|\mathbf{r}_i|}\left[1 + \tilde{a}_1\frac{\alpha_s(1/|\mathbf{r}_i|)}{4\pi}\right] \\ &\quad - \alpha_s\left(\frac{\alpha_s}{4\pi}\right)^2\left[\frac{2}{3}\tilde{a}_{2,s}\left(\frac{1}{|\mathbf{r}_1|} + \frac{1}{|\mathbf{r}_2|} + \frac{1}{|\mathbf{r}_3|}\right)\right. \\ &\quad \left. + v_{\mathcal{H}}(\mathbf{r}_2, \mathbf{r}_3) + v_{\mathcal{H}}(\mathbf{r}_1, -\mathbf{r}_3) + v_{\mathcal{H}}(-\mathbf{r}_2, -\mathbf{r}_1)\right]. \end{aligned} \tag{B1}$$

The one-loop and two-loop coefficients $\tilde{a}_1$ and $\tilde{a}_{2,s}$ depend on the number of light (massless) quark flavors, n_l, and are given by

$$\tilde{a}_1 = \frac{31}{3} + 22\gamma_E - \left(\frac{10}{3} + 4\gamma_E\right)\frac{n_l}{3}, \tag{B2}$$

$$\begin{aligned} \tilde{a}_{2,s} &= \frac{4343}{18} + \frac{3\pi^4}{4} + \frac{121\pi^2}{3} + 66\zeta(3) - 484\gamma_E^2 + 204\gamma_E \\ &\quad - \left(\frac{1229}{9} + \frac{44\pi^2}{3} + 52\zeta(3) - 176\gamma_E^2 + 76\gamma_E\right)\frac{n_l}{3} \\ &\quad + \left(\frac{100}{9} + \frac{4\pi^2}{3} - 16\gamma_E^2\right)\left(\frac{n_l}{3}\right)^2 + 4\gamma_E\left(11 - 2\frac{n_l}{3}\right)\tilde{a}_1. \end{aligned} \tag{B3}$$

At two loop, a genuine three-body potential shows up. It is encoded in the function $v_{\mathcal{H}}$ defined as

$$\begin{aligned} &v_{\mathcal{H}}(\mathbf{r}_2, \mathbf{r}_3) \\ &= 16\pi\int_0^1 dx\int_0^1 dy\left\{\frac{\hat{\mathbf{r}}_2\cdot\hat{\mathbf{r}}_3}{|\mathbf{R}|}\left[\left(1 - \frac{M^2}{|\mathbf{R}|^2}\right)\arctan\frac{|\mathbf{R}|}{M} + \frac{M}{|\mathbf{R}|}\right]\right. \\ &\quad \left. + \frac{(\hat{\mathbf{r}}_2\cdot\hat{\mathbf{R}})(\hat{\mathbf{r}}_3\cdot\hat{\mathbf{R}})}{|\mathbf{R}|}\left[\left(1 + 3\frac{M^2}{|\mathbf{R}|^2}\right)\arctan\frac{|\mathbf{R}|}{M} - 3\frac{M}{|\mathbf{R}|}\right]\right\}, \end{aligned} \tag{B4}$$

with $\mathbf{R}(\mathbf{r}_2, \mathbf{r}_3) \equiv x\mathbf{r}_2 - y\mathbf{r}_3$ and $M(\mathbf{r}_2, \mathbf{r}_3) \equiv |\mathbf{r}_2|\sqrt{x(1-x)} + |\mathbf{r}_3|\sqrt{y(1-y)}$. Note that the three-body potential in (B1) is manifestly invariant under the transformations (29) and (30).

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