Modified-gravity wormholes without exotic matter

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A fundamental ingredient in wormhole physics is the flaring-out condition at the throat which, in classical general relativity, entails the violation of the null energy condition. In this work, we present the most general conditions in the context of modified gravity, in which the matter threading the wormhole throat satisfies all of the energy conditions, and it is the higher order curvature terms, which may be interpreted as a gravitational fluid, that support these nonstandard wormhole geometries. Thus, we explicitly show that wormhole geometries can be theoretically constructed without the presence of exotic matter but are sustained in the context of modified gravity.

I. INTRODUCTION

A fundamental property in classical general relativistic wormhole geometries is that these spacetimes are supported by *exotic matter*, which involves a stress-energy tensor $T_{\mu\nu}$ that violates the null energy condition (NEC), i.e., $T_{\mu\nu}k^\mu k^\nu < 0$, where k^μ is *any* null vector [1]. More specifically, the NEC imposes that $T_{\mu\nu}k^\mu k^\nu \geq 0$. Indeed, using the theory of embedded hypersurfaces to place restrictions on the Riemann tensor and stress-energy tensor at the throat of the wormhole, Hochberg and Visser demonstrated that the wormhole throat generically violates the NEC and provided several theorems that generalize the Morris-Thorne results on exotic matter [2,3]. In this context, wormhole geometries violate all the pointwise energy conditions and the averaged energy conditions [4]. The latter permit localized violations of the energy conditions, as long as on average they hold when integrated along timelike or null geodesics. However, the averaged energy conditions involve a line integral, and therefore do not provide useful information regarding the "total amount" of energy-condition violating matter. This fact prompted the proposal of a "volume integral quantifier," which provides information about the "total amount" of energy-condition violating matter in the spacetime [5]. The amount of energy-condition violations is then the extent that these integrals become negative.

Indeed, classical forms of matter are believed to obey the energy conditions, although they are violated by certain quantum fields such as the Casimir effect and Hawking evaporation [1]. Thus, due to its problematic nature, it is useful to minimize the usage of exotic matter,

and a wide variety of wormhole solutions have been analyzed in the literature to this effect, ranging from thin-shell wormholes [6], to rotating [7] and dynamic wormhole geometries [3,8], and in modified theories of gravity [9–12]. In the latter context, more specifically in $f(R)$ gravity, it was shown that in principle the matter threading the wormholes can be imposed to obey all the energy conditions, and it is the higher order curvature terms that are responsible for supporting the geometries [9]. In braneworlds, the local high-energy bulk effects and the nonlocal corrections from the Weyl curvature in the bulk may induce a NEC violating signature on the brane, while the stress-energy tensor confined on the brane, threading the wormhole, is imposed to satisfy the energy conditions [11]. In the curvature-matter coupled generalization of $f(R)$ gravity, exact solutions were found where the nonminimal coupling minimizes the violation of the NEC of matter at the throat [12].

It is the purpose of the present paper to generalize the above analysis presenting the most general conditions coming from various modified theories of gravity that are imposed in order to support wormhole geometries. We note that in this context, in principle, one may impose that the matter stress-energy tensor satisfies the NEC and the respective violations arise from the higher order curvature terms.

II. WORMHOLE GEOMETRIES AND THE ENERGY CONDITIONS

Consider the following wormhole line element in curvature coordinates [1]:

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

(1)

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The redshift function $\Phi(r)$ must be finite everywhere to avoid the presence of event horizons. In order to have a wormhole geometry, the shape function $b(r)$ must obey the flaring-out condition of the throat r_0, which is translated by $(b - b'r)/b^2 > 0$ [1]. At the throat, we have $b(r_0) = r_0$, and the condition $b'(r_0) < 1$ is imposed. Note that the flaring-out condition has a purely geometric nature. However, in classical general relativity, through the Einstein field equation, one can deduce that the matter threading the wormhole throat violates the NEC.

Generally, the NEC arises when one refers back to the Raychaudhuri equation given by

$$\frac{d\theta}{d\tau} = -\frac{1}{2}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}k^\mu k^\nu, \quad (2)$$

where $R_{\mu\nu}$ is the Ricci tensor, and θ, $\sigma^{\mu\nu}$, and $\omega_{\mu\nu}$ are, respectively, the expansion, shear, and rotation associated to the congruence defined by the null vector field k^μ. The Raychaudhuri equation is also a purely geometric statement, and as such it makes no reference to any gravitational field equations. Now, the shear is a "spatial" tensor with $\sigma^2 \equiv \sigma_{\mu\nu}\sigma^{\mu\nu} \geq 0$ and $\omega_{\mu\nu} \equiv 0$ for any hypersurface orthogonal congruences, so that the condition for attractive gravity reduces to $R_{\mu\nu}k^\mu k^\nu \geq 0$. The positivity of this latter quantity ensures that geodesic congruences focus within a finite value of the parameter labeling points on the geodesics. In general relativity, contracting both sides of the Einstein field equation $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa^2 T_{\mu\nu}$ with any null vector k^μ, one can write the above condition in terms of the stress-energy tensor given by $T_{\mu\nu}k^\mu k^\nu \geq 0$.

In modified theories of gravity, the gravitational field equations can be rewritten as an effective Einstein equation given by $G_{\mu\nu} = \kappa^2 T^{\text{eff}}_{\mu\nu}$, where $T^{\text{eff}}_{\mu\nu}$ is an effective stress-energy tensor containing the matter stress-energy tensor $T_{\mu\nu}$ and the curvature quantities arising from the specific modified theory of gravity considered. Now, the positivity condition $R_{\mu\nu}k^\mu k^\nu \geq 0$ in the Raychaudhuri equation provides the *generalized* NEC, $T^{\text{eff}}_{\mu\nu}k^\mu k^\nu \geq 0$, through the modified gravitational field equation.

By definition (see Ref. [2]) the wormhole throat has to defocus a null geodesic congruence. Therefore, the necessary condition to have a wormhole geometry is the violation of the generalized NEC, i.e., $T^{\text{eff}}_{\mu\nu}k^\mu k^\nu < 0$. In classical general relativity this simply reduces to the violation of the usual NEC, i.e., $T_{\mu\nu}k^\mu k^\nu < 0$. However, in modified theories of gravity, one may in principle impose that the matter stress-energy tensor satisfies the standard NEC, $T_{\mu\nu}k^\mu k^\nu \geq 0$, while the respective generalized NEC is necessarily violated, $T^{\text{eff}}_{\mu\nu}k^\mu k^\nu < 0$, in order to ensure the flaring-out condition.

Note that instead of a null geodesic congruence, we can consider a congruence of timelike geodesics. In this case the corresponding Raychaudhuri equation reads

$$\frac{d\hat{\theta}}{d\tau} = -\frac{1}{3}\hat{\theta}^2 - \hat{\sigma}_{\mu\nu}\sigma^{\mu\nu} + \hat{\omega}_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}u^\mu u^\nu, \quad (3)$$

where $\hat{\theta}$, $\hat{\sigma}^{\mu\nu}$, and $\hat{\omega}_{\mu\nu}$ are, respectively, the expansion, shear, and twist of the congruence defined by the timelike unit vector field u^μ normalized to unit length, $u_\mu u^\mu = -1$. The positivity condition $R_{\mu\nu}u^\mu u^\nu \geq 0$ focuses the timelike congruence and ensures an attractive nature of gravity. In classical general relativity, using Einstein's equation, we can write this condition as $T_{\mu\nu}u^\mu u^\nu \geq -\frac{1}{2}T$, where u^μ is any timelike vector. This assumption is known as the strong energy condition (SEC). Its violation is a necessary condition to have a wormhole geometry.

III. WORMHOLES IN GENERALIZED MODIFIED GRAVITY

Consider the generalized gravitational field equations for a large class of modified theories of gravity given by the following field equation

$$g_1(\Psi^i)(G_{\mu\nu} + H_{\mu\nu}) - g_2(\Psi^j)T_{\mu\nu} = \kappa^2 T_{\mu\nu}, \quad (4)$$

where $H_{\mu\nu}$ is an additional geometric term that includes the geometrical modifications inherent in the modified gravitational theory under consideration; $g_i(\Psi^j)$ $(i = 1, 2)$ are multiplicative factors that modify the geometrical sector of the field equations, and Ψ^j denote generically curvature invariants or gravitational fields such as scalar fields; the term $g_2(\Psi^i)$ covers the coupling of the curvature invariants or the scalar fields with the matter stress-energy tensor, $T_{\mu\nu}$.

It is useful to rewrite this field equation as an effective Einstein field equation, as mentioned above, with the effective stress-energy tensor $T^{\text{eff}}_{\mu\nu}$ given by

$$T^{\text{eff}}_{\mu\nu} \equiv \frac{1 + \bar{g}_2(\Psi^j)}{g_1(\Psi^i)}T_{\mu\nu} - \bar{H}_{\mu\nu}, \quad (5)$$

where $\bar{g}_2(\Psi^j) = g_2(\Psi^j)/\kappa^2$ and $\bar{H}_{\mu\nu} = H_{\mu\nu}/\kappa^2$ are defined for notational convenience.

In modified gravity, the violation of the generalized NEC $T^{\text{eff}}_{\mu\nu}k^\mu k^\nu < 0$ implies the following restriction:

$$\frac{1 + \bar{g}_2(\Psi^j)}{g_1(\Psi^i)}T_{\mu\nu}k^\mu k^\nu < \bar{H}_{\mu\nu}k^\mu k^\nu. \quad (6)$$

For general relativity, with $g_1(\Psi^j) = 1$, $g_2(\Psi^j) = 0$, and $H_{\mu\nu} = 0$, we recover the standard violation of the NEC for the matter threading the wormhole, i.e., $T_{\mu\nu}k^\mu k^\nu < 0$.

If the additional condition $[1 + \bar{g}_2(\Psi^j)]/g_1(\Psi^i) > 0$ is met, then one obtains a general bound for the normal matter threading the wormhole, in the context of modified theories of gravity given by

$$0 \leq T_{\mu\nu}k^\mu k^\nu < \frac{g_1(\Psi^i)}{1 + \bar{g}_2(\Psi^j)}\bar{H}_{\mu\nu}k^\mu k^\nu. \quad (7)$$

Analogously, in modified gravity, wormhole solutions also violate the generalized SEC, i.e., $T^{\text{eff}}_{\mu\nu}u^\mu u^\nu < -\frac{1}{2}T^{\text{eff}}$, which implies the following bound on the matter stress-energy tensor:

$$\frac{1 + \bar{g}_2(\Psi^j)}{g_1(\Psi^i)}\left[T_{\mu\nu}u^\mu u^\nu - \frac{1}{2}T\right] < \bar{H}_{\mu\nu}u^\mu u^\nu - \frac{1}{2}\bar{H}. \quad (8)$$

Now, one may demand that the latter condition is fulfilled even if the matter stress-energy tensor satisfies the usual SEC, $T_{\mu\nu}u^\mu u^\nu - \frac{1}{2}T \geq 0$, or the weak energy condition (WEC) $T_{\mu\nu}u^\mu u^\nu \geq 0$.

In order for normal matter to satisfy the WEC, to have a positive energy density, one also needs to impose the following relationship:

$$T_{\mu\nu}u^\mu u^\nu = \frac{g_1(\Psi^i)}{\kappa^2 + g_2(\Psi^j)}(G_{\mu\nu} + H_{\mu\nu})u^\mu u^\nu \geq 0. \quad (9)$$

Imposing $T_{\mu\nu}u^\mu u^\nu \geq 0$ entails a restriction on the geometry arising from the modified gravity under consideration. If the normal matter is given by a diagonal stress-energy tensor, i.e., $T^\mu{}_\nu = \text{diag}[-\rho(r), p_r(r), p_t(r), p_t(r)]$, one can physically interpret $T_{\mu\nu}u^\mu u^\nu$ as the energy density measured by any timelike observer with four-velocity u^μ. This definition is useful, as using local Lorentz transformations it is possible to show that $T_{\mu\nu}u^\mu u^\nu \geq 0$ implies that the energy density is positive in all local frames of reference. Thus, the standard WEC imposes that $\rho \geq 0$ and $\rho + p_i \geq 0$ (where $i = r, t$).

A. $f(R)$ gravity

A well-known modification of general relativity is $f(R)$ gravity with the following action:

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}f(R) + \mathcal{L}_m\right\}. \quad (10)$$

The stress-energy tensor of matter is defined as [13] $T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}}$, and we consider that the matter Lagrangian density $\mathcal{L}_m$ only depends on the metric tensor components $g_{\mu\nu}$, and not on its derivatives.

In this case, the field equations are given by Eq. (4), with the relationships $g_1(\Psi^i) = f_R(R)$, $g_2(\Psi^j) = 0$, and

$$H_{\mu\nu} = \frac{1}{f_R}\left[\frac{1}{2}(Rf_R - f)g_{\mu\nu} - \nabla_\mu\nabla_\nu f_R + g_{\mu\nu}\Box f_R\right], \quad (11)$$

where $f_R = df/dR$.

The generic condition for the violation of the generalized NEC in $f(R)$ gravity is given by

$$\frac{1}{f_R}T_{\mu\nu}k^\mu k^\nu < -\frac{1}{\kappa^2 f_R}k^\mu k^\nu \nabla_\mu\nabla_\nu f_R. \quad (12)$$

It is worth noting that, depending on a particular form of $f(R)$, the latter inequality could be fulfilled even if

$T_{\mu\nu}k^\mu k^\nu > 0$. In particular, if $f_R > 0$, then the bound $0 \leq \kappa^2 T_{\mu\nu}k^\mu k^\nu < -k^\mu k^\nu \nabla_\mu\nabla_\nu f_R$ is imposed.

As a specific example, consider R^2 gravity with $f(R) = R + \frac{1}{2}\alpha R^2$, so that the second term dominates for strong curvatures as is the case at the wormhole throat and its neighborhood. In this case, Eq. (12) reads

$$\frac{T_{\mu\nu}k^\mu k^\nu}{1 + \alpha R} < -\frac{\alpha k^\mu k^\nu R_{;\mu\nu}}{\kappa^2(1 + \alpha R)}. \quad (13)$$

In particular, in the wormhole metric (1) with $\Phi(r) \equiv 0$, so that the curvature scalar is given by $R = 2b'/r^2$, then inequality (13) evaluated at the throat r_0 takes the form

$$\frac{T_{\mu\nu}k^\mu k^\nu|_{r_0}}{r_0^2 + 2\alpha b_0'} < \frac{\alpha(1 - b_0')(2b_0' - r_0 b_0'')}{\kappa^2 r_0^4(r_0^2 + 2\alpha b_0')}. \quad (14)$$

It is obvious that for $\alpha = 0$ we have $T_{\mu\nu}k^\mu k^\nu|_{r_0} < 0$. However, generally we can choose α so that Eq. (14) is fulfilled even if $T_{\mu\nu}k^\mu k^\nu \geq 0$.

B. Curvature-matter coupling

Now let us consider the modified theory of gravity with an explicit curvature-matter coupling given by the following action [14]:

$$S = \int d^4x\sqrt{-g}\left\{\frac{1}{2\kappa^2}f(R) + [1 + \lambda h(R)]\mathcal{L}_m\right\}, \quad (15)$$

where $f(R)$ and $h(R)$ are arbitrary functions of the Ricci scalar R. The coupling constant λ characterizes the strength of the interaction between $h(R)$ and the matter Lagrangian.

Taking into account the modified Einstein equation, the effective stress-energy tensor is given by

$$T^{\text{eff}}_{\mu\nu} = \frac{1}{\mathcal{L}_c}\left\{(1 + \lambda h)T_{\mu\nu} + \frac{1}{2\kappa^2}(f - \mathcal{L}_c R)g_{\mu\nu} - \frac{1}{\kappa^2}(g_{\mu\nu}\nabla_\alpha\nabla^\alpha - \nabla_\mu\nabla_\nu)\mathcal{L}_c\right\}, \quad (16)$$

where $f_R = df/dR$ and $h_R = dh/dR$, and we have defined $\mathcal{L}_c \equiv f_R + 2\lambda\kappa^2 h_R\mathcal{L}_m$ for notational simplicity.

The general condition to have wormhole geometries $T^{\text{eff}}_{\mu\nu}k^\mu k^\nu < 0$ reduces to

$$\frac{1 + \lambda h}{\mathcal{L}_c}T_{\mu\nu}k^\mu k^\nu < -\frac{k^\mu k^\nu \nabla_\mu\nabla_\nu \mathcal{L}_c}{\kappa^2\mathcal{L}_c}. \quad (17)$$

As a specific example, consider the model with $f(R) = h(R) = R$ and $\mathcal{L}_m = -\rho$ [15], where ρ is the energy density of matter. In this case, Eq. (17) yields

$$\frac{1 + \lambda R}{1 - 2\kappa^2\lambda\rho}T_{\mu\nu}k^\mu k^\nu < \frac{2\lambda k^\mu k^\nu \rho_{;\mu\nu}}{1 - 2\kappa^2\lambda\rho}. \quad (18)$$

For the wormhole geometry (1) with $\Phi(r) \equiv 0$, and choosing the null vector $k^\mu = (1, \sqrt{1 - b/r}, 0, 0)$, so that

$k^\mu k^\nu \nabla_\mu \nabla_\nu \rho = [2(r^2 - rb)\rho'' + (b - rb')\rho']/(2r^2)$, the inequality (18) evaluated at the throat r_0, is as follows:

$$\frac{r_0^2 + 2\lambda b_0'}{1 - 2\kappa^2 \lambda \rho_0} T_{\mu\nu} k^\mu k^\nu|_{r_0} < \frac{\lambda r_0 \rho_0' (1 - b_0')}{1 - 2\kappa^2 \lambda \rho_0}. \quad (19)$$

Again, it is seen that $T_{\mu\nu} k^\mu k^\nu|_{r_0} < 0$ provided $\lambda = 0$. However, generally one can choose the specific parameters of the model so that the restriction (19) could be fulfilled even if $T_{\mu\nu} k^\mu k^\nu \gtreqqless 0$.

C. $F(R, \mathcal{L}_m)$ gravity

In this section, consider the following action for the generalized modified gravitational theory [16]:

$$S = \int d^4 x \sqrt{-g} F(R, \mathcal{L}_m), \quad (20)$$

where $F(R, \mathcal{L}_m)$ is an arbitrary function of the Ricci scalar R, and of matter Lagrangian density, $\mathcal{L}_m$. This theory generalizes the $f(R)$ gravity models and the curvature-matter couplings [14].

For $F(R, \mathcal{L}_m)$ gravity, we have the following relationships: $H_{\mu\nu} = \frac{1}{F_R} [\frac{1}{2} \bar{F} g_{\mu\nu} + \square_{\mu\nu} F_R]$, $g_1(\Psi^i) = F_R$, and $g_2(\Psi^j) = F_{\mathcal{L}_m}/2 - \kappa^2$, where we have defined $\bar{F} \equiv (F_R R + F_{\mathcal{L}_m} \mathcal{L}_m - F)$, $\square_{\mu\nu} \equiv g_{\mu\nu} \nabla_\alpha \nabla^\alpha - \nabla_\mu \nabla_\nu$, and denoted $F_R = \partial F/\partial R$ and $F_{\mathcal{L}_m} = \partial F/\partial \mathcal{L}_m$, respectively.

The gravitational field equation may be written as an effective Einstein field equation, with $T_{\mu\nu}^{\text{eff}}$ given by

$$T_{\mu\nu}^{\text{eff}} = \frac{1}{2F_R} [\bar{F} g_{\mu\nu} - 2\square_{\mu\nu} F_R + F_{\mathcal{L}_m} T_{\mu\nu}]. \quad (21)$$

Note that the divergence of the stress-energy tensor $T_{\mu\nu}$ is given by $\nabla^\mu T_{\mu\nu} = 2\nabla^\mu \ln(F_{\mathcal{L}_m}) \frac{\partial \mathcal{L}_m}{\partial g^{\mu\nu}}$, which translates an explicit exchange of energy and momentum between the matter and the higher order curvature terms. The covariant conservation of the stress-energy tensor $\nabla^\mu T_{\mu\nu} = 0$ provides an effective functional relation between the matter Lagrangian density and the function $F_{\mathcal{L}_m}(R, \mathcal{L}_m)$ given by $\nabla^\mu \ln(F_{\mathcal{L}_m}) \partial \mathcal{L}_m / \partial g^{\mu\nu} = 0$. Thus, in principle, once the matter Lagrangian density is known, by an appropriate choice of the function $F(R, \mathcal{L}_m)$ one can construct conservative models with arbitrary curvature-geometry couplings [17].

Now, contracting $T_{\mu\nu}^{\text{eff}}$ with *any* null vector k^μ, the essential condition to support wormhole geometries is $T_{\mu\nu}^{\text{eff}} k^\mu k^\nu < 0$, which is given by

$$\frac{F_{\mathcal{L}_m}}{F_R} T_{\mu\nu} k^\mu k^\nu < -\frac{2k^\mu k^\nu \nabla_\mu \nabla_\nu F_R}{F_R}. \quad (22)$$

For $f(R, \mathcal{L}_m) = R/2\kappa^2 + \mathcal{L}_m$, then (22) reduces to the standard violation of the NEC in general relativity [1].

Thus, the general condition for wormhole spacetimes is given by inequality (22), and in principle one may now construct specific solutions, either by considering a specific form for $F(R, \mathcal{L}_m)$, and by imposing an equation of state of the matter stress-energy tensor, and/or considering choices for the metric functions $\Phi(r)$ and $b(r)$. We leave this analysis for a future publication.

IV. DISCUSSIONS AND FINAL REMARKS

Despite the fact that the flaring-out condition, in classical general relativity, through the Einstein field equation necessarily entails the violation of the NEC, in modified theories of gravity it is the *generalized* NEC that is violated. The latter involves an effective stress-energy tensor that includes the matter stress-energy tensor and higher curvature terms, which may be interpreted as a gravitational fluid. Thus, in this work, we have explicitly shown that one may impose that the normal matter stress-energy tensor satisfies all the standard energy conditions and presented very general restrictions on the wormhole geometry arising from the modified theory of gravity under consideration. Indeed, this is translated through general inequalities showing that the higher curvature terms sustain the wormhole geometries.

In addition to this, we have considered specific cases of modified theories of gravity considered in the literature, namely, $f(R)$ gravity, the curvature-matter coupling, and the $F(R, \mathcal{L}_m)$ generalization. In the first two theories, we analyzed specific cases and showed explicitly that one may choose the parameters of the theory such that the matter threading the wormhole throat satisfies the energy conditions. Thus, this shows that one may theoretically construct wormhole geometries without the use of exotic matter, although it is the higher order curvature terms arising in the modified theories of gravity that sustain these exotic spacetimes.

It would also be interesting to analyze whether the various modified gravitational models required for a wormhole have any serious instabilities. For instance, in $R + \alpha R^2$ gravity, it was claimed in Ref. [18] that a Lorentzian wormhole would require $\alpha < 0$, i.e., the existence of the sign that corresponds to an unbounded from below potential or tachyonic kinetic term in the conformally related scalar field model (see also Ref. [19]). In this context, it would be interesting to consider whether a modified gravitational model could effectively violate the null energy condition without such a related instability. Although this analysis lies outside the scope of the present paper, work along these lines is currently under way.

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