

Worksheet propagator on the lightcone worksheet lattice

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We develop new more powerful techniques, based on an almost closed form for the lattice worksheet propagator, for analyzing planar open string worldsheets defined on a lightcone lattice. We show that results obtained in earlier work are easily reproduced with far more precision. In particular, consistency checks that required numerical analysis in the earlier work can now be confirmed exactly.

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I. INTRODUCTION

The lightcone worksheet [1–3] lattice was proposed long ago [4] as a method to digitize the summation of planar open string multiloop diagrams. Because the open string spectrum includes a massless spin 1 particle, this sum of diagrams should have the infrared behavior of large N [5] gauge theory. If the worksheet lattice can reliably reproduce string theory diagrams, its $\alpha' \rightarrow 0$ limit should just as reliably reproduce gauge theory [6]. With this possibility in mind, we have recently embarked on a program [7] to critically evaluate the accuracy of this lattice in reproducing the continuum perturbative diagrams. In particular, it is very important that lattice artifacts be shown to be benign: they should either vanish in the continuum limit or be absorbed in redefinitions of parameters in the theory. This is a necessary prerequisite to applying these lattice methods to nonperturbative calculations of QCD.

In Ref. [7] we studied the one loop self-energy diagram for the bosonic closed string in enough detail to see that the lattice accurately reproduced the ultraviolet behavior of the diagram. This is as much as we should expect, since the open bosonic string tachyon should and does ruin the infrared behavior of the diagram.¹ The analysis in Ref. [7] employed what might be called a string field theory approach (see, for example, Ref. [8]): the diagram was built up from open and closed string propagators. While this approach was manageable at one loop, it quickly gets unwieldy for multiloop diagrams. Even for the one loop open string self-energy diagram, attaining enough accuracy to make definitive conclusions proved to be problematic. To improve on this situation we develop, in this article, a more powerful “worksheet” approach based on the techniques of worksheet quantum field theory defined on the lightcone lattice. The key to this approach is an

almost closed form expression for the worksheet propagator on the lattice [see Eq. (18)].

The perturbative string field theory approach of Ref. [7] keeps manifest the contribution of all the intermediate string mass eigenstates contributing to the diagram. While the ensuing formulas for the self-energy shifts were exact at finite lattice spacing, we had to resort to numerical analysis to analyze the continuum limit. The extrapolation of our numerical results to the continuum was sufficiently accurate to make rather convincing consistency checks, such as a vanishing graviton self-energy. These checks were nonetheless subject to numerical error. In contrast, the methods of the present paper are powerful enough to analyze the continuum limit exactly in the ultraviolet and to confirm rigorously such consistency requirements.

The Giles-Thorn (GT) discretization of the worksheet [4] begins with a representation of the free closed or open string propagator as a lightcone worksheet path integral defined on a lattice. The lattice replaces the transverse coordinates of the string $\mathbf{x}(\sigma, \tau)$, living on a rectangular $P^+ \times T$ domain, with discretely labeled coordinates $\mathbf{x}_k^j = \mathbf{x}(kaT_0, ja)$, living on an $M \times N$ grid with spacing a , where $P^+ = MaT_0$ and $T = a(N + 1)$. The free string propagator is then simply a Gaussian integral

$$\begin{aligned} \mathcal{D}_0 &= \int \prod_{kj} d\mathbf{x}_k^j e^{-S}, \\ S &= \frac{T_0}{2} \sum_{kj} [(\mathbf{x}_k^{j+1} - \mathbf{x}_k^j)^2 + (\mathbf{x}_{k+1}^j - \mathbf{x}_k^j)^2] \\ &\equiv \frac{T_0}{2} \mathbf{x}^T \cdot \Delta^{-1} \mathbf{x}, \end{aligned} \quad (1)$$

where the $MN \times MN$ matrix Δ is the lattice worksheet propagator that will be the central focus of this article. Then up to an overall normalization factor $\mathcal{D}_0 = \det^{-(D-2)} \Delta^{-1}$, where D is the spacetime dimension ($D = 26$ for the bosonic string).

On this lattice the sum of all open string multiloop planar diagrams can be obtained by summing over all patterns of missing spatial bonds. Formally, this is achieved by introducing Ising-like variables $S_k^j = 0, 1$ and taking the worksheet action to be

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¹It is logically possible that summing the bosonic string diagrams with an infrared cutoff stabilizes the vacuum in a way to produce QCD physics, but it is more likely that another string model, such as the tachyon-free Neveu-Schwarz sector of the superstring, is required to truly reproduce QCD.

$$\begin{aligned}
 S_{\text{Planar}} &= \frac{T_0}{2} \sum_{ij} [(\mathbf{x}_i^{j+1} - \mathbf{x}_i^j)^2 + S_i^j (\mathbf{x}_{i+1}^j - \mathbf{x}_i^j)^2] \\
 &\quad + (D-2)B \sum_{kj} (1 - S_k^j) - \sum_{ij} [S_i^j (1 - S_i^{j+1}) \\
 &\quad + S_i^{j+1} (1 - S_i^j)] \ln g \quad (2) \\
 &\equiv \frac{T_0}{2} \mathbf{x}^T \cdot [\Delta^{-1} + V(S)] \mathbf{x} + A(\{S\}). \quad (3)
 \end{aligned}$$

The terms in $A(\{S\})$ insert the coupling constant g in the appropriate way and allow for an open string self-energy counterterm B . Then we have

$$\mathcal{D} = \mathcal{D}_0 \sum_{\{S\}} \det^{-12}(I + V\Delta) e^{-A(\{S\})}. \quad (4)$$

When V is a sparse matrix, i.e., when there are a relatively small number of missing bonds [$\sum_{kj} (1 - S_{kj}) \ll M$, which can be arranged by taking $B \gg 1$], this will be a particularly efficient way to evaluate the terms of perturbation theory. Holding B sufficiently large serves as a physical and convenient infrared regulator in our studies of the properties of the planar diagrams.

The paper is organized as follows. In Sec. II we construct the worldsheet propagator on the GT worldsheet lattice. It is remarkable that, in spite of the discretization of time (ix^+), the result is explicit and not much more complicated than the well known continuum worldsheet propagator. In Sec. III we apply this expression to the calculation of the tachyon one loop closed string self-energy. Our results, being exact, can be carried out to arbitrary precision, agreeing with the numerical results of Ref. [7] to the precision achieved in Ref. [7] (only three significant figures for some of the subleading contributions). The self-energy of the closed string graviton and selected higher mass states is similarly analyzed in Sec. IV. We conclude with discussion in Sec. V of the significance of our results and their promise for analyzing the open string self-energy as well as higher loop diagrams.

II. LATTICE WORLDSHEET PROPAGATORS

We develop the tools of quantum field theory for the worldsheet lattice. Of central interest are the worldsheet correlators of the coordinates on the $M \times N$ lattice corresponding to the free closed or open string,

$$\Delta_{ij,kl} = T_0 \langle x_i^j x_k^l \rangle = T_0 \frac{\int \mathcal{D}x x_i^j x_k^l e^{-S}}{\int \mathcal{D}x e^{-S}}. \quad (5)$$

Because the expectations are taken with Gaussian weight, the two point correlator in a single dimension captures all of the relevant information in arbitrary multipoint correlators in any number of dimensions. For the bosonic string we should of course take 26 space-time dimensions or 24 transverse dimensions.

A straightforward evaluation is to use closure to write the numerator as the product of three string field propagators (see Appendix B): one from time $-(N-j)$ to j , one from time j to l , and the last from time l to $+(N+l)$. We can resolve x_i^j, x_k^l into normal modes q_m^j, q_n^l , respectively. Then because each normal mode path integral is independent, $\langle q_m^j q_n^l \rangle = \delta_{mn} \langle q_m^j q_m^l \rangle$, one ends up with a simple two variable Gaussian integral to do,

$$\begin{aligned}
 &\int dq_m^j dq_n^l q_m^j q_n^l \exp \left\{ -\frac{1}{2} [A(q_m^{j2} + q_n^{l2}) + 2Bq_m^j q_n^l] \right\} \\
 &= -\frac{B}{A^2 - B^2} \det^{-1/2} \begin{pmatrix} A & B \\ B & A \end{pmatrix}, \\
 \langle q_m^j q_n^l \rangle &= -\frac{B}{A^2 - B^2} \delta_{mn}. \quad (6)
 \end{aligned}$$

Here A and B are read off from the formulas of Appendix B. For simplicity we set the q 's at the initial and final times to zero.

Then for nonzero modes they are

$$\begin{aligned}
 A &= T_0 \sinh \lambda [\coth N\lambda + \coth (l-j)\lambda], \\
 B &= \frac{-T_0 \sinh \lambda}{\sinh (l-j)\lambda}, \quad (7)
 \end{aligned}$$

where λ is $\lambda_m^o = 2 \sinh^{-1} \sin(m\pi/2M)$ or $\lambda_m^c = 2 \sinh^{-1} \sin(m\pi/M)$ for the open or closed string, respectively. The nonzero mode contribution has a well defined $N \rightarrow \infty$ limit:

$$\begin{aligned}
 A &\rightarrow T_0 \sinh \lambda [1 + \coth (l-j)\lambda], \quad B = \frac{-T_0 \sinh \lambda}{\sinh (l-j)\lambda}, \\
 \frac{-B}{A^2 - B^2} &\rightarrow \frac{1}{2T_0 \sinh \lambda} \left(\frac{1}{(\cosh (l-j)\lambda + \sinh (l-j)\lambda)} \right) \\
 &= \frac{e^{-|l-j|\lambda}}{2T_0 \sinh \lambda}. \quad (8)
 \end{aligned}$$

For the zero modes

$$\begin{aligned}
 A_0 &= T_0 \frac{N+l-j}{N(l-j)}, \quad B_0 = -\frac{T_0}{l-j}, \\
 \frac{-B_0}{A_0^2 - B_0^2} &= \frac{N}{2T_0(1 + (l-j)/2N)} \rightarrow \frac{N}{2T_0} - \frac{l-j}{4T_0}, \quad (9)
 \end{aligned}$$

where we have taken N large on the right side. To properly isolate the diverging term, we must remember that $2N$ is not the total time length of the string propagator. Rather the total length is $2N_T = 2N + l - j$. N_T is the quantity that should be regarded as independent of j, l . In other words we should write

$$\langle q_0^j q_0^l \rangle \sim \frac{N}{2T_0} - \frac{l-j}{4T_0} + \mathcal{O}\left(\frac{1}{N}\right) = \frac{N_T}{2T_0} - \frac{l-j}{2T_0} + \mathcal{O}\left(\frac{1}{N_T}\right). \quad (10)$$

The zero mode contribution grows linearly with N_T . So it will be important that the zero mode be suppressed in the

physical quantities that require the input of worldsheet propagators. It is helpful to appreciate though that the divergent term is a constant independent of j, l . To define the inverse lattice Laplacian, it is consistent to drop it, in effect modifying the boundary conditions on the Green function.

In these derivations we have assumed $l > j$. For $l < j$ the roles of the two indices are switched. When $N \rightarrow \infty$, we simply replace $l - j \rightarrow |l - j|$ in the formulas:

$$\begin{aligned} \langle q_m^j q_n^l \rangle &= \delta_{mn} \frac{e^{-|l-j|\lambda_m}}{2T_0 \sinh \lambda_m}, & N \rightarrow \infty, \\ \langle q_0^j q_0^l \rangle &\sim \frac{N_T}{2T_0} - \frac{|l-j|}{2T_0}, & N \rightarrow \infty. \end{aligned} \quad (11)$$

From their physical interpretation these are inverses of the lattice Laplacian²

$$\begin{aligned} (-\Delta + 4\sinh^2 \lambda/2) f_j \\ \equiv 2f_j - f_{j+1} - f_{j-1} + 4f_j \sinh^2 \lambda/2. \end{aligned} \quad (12)$$

It is remarkable that this can be checked directly:

$$\begin{aligned} 2e^{-|l-j|\lambda} - e^{-|l+1-j|\lambda} - e^{-|l-1-j|\lambda} \\ = \begin{cases} e^{-(l-j)\lambda}(2 - 2\cosh \lambda) & l > j \\ e^{-(j-l)\lambda}(2 - 2\cosh \lambda) & l < j \\ 2 - e^{-\lambda} - e^{-\lambda} = (2 - 2\cosh \lambda) + 2\sinh \lambda & l = j \end{cases} \\ = -4e^{-|l-j|\lambda} \sinh^2 \frac{\lambda}{2} + 2\delta_{lj} \sinh \lambda, \end{aligned} \quad (13)$$

$$\left(-\Delta + 4\sinh^2 \frac{\lambda}{2} \right) \frac{e^{-|l-j|\lambda}}{2 \sinh \lambda} = \delta_{lj}, \quad (14)$$

which shows that $\langle q_m^j q_n^l \rangle$ is the inverse of the lattice Laplacian on the nonzero modes. The proof for zero modes is even simpler

$$\begin{aligned} -\Delta |l-j| &= 2|l-j| - |l+1-j| - |l-1-j| \\ &= -2\delta_{lj}, \end{aligned} \quad (15)$$

which confirms the same property for the zero modes.

Finally we return to the correlators on the spatial lattice by expanding in normal modes. The mode functions differ for the various types of string. For the Neumann open string

²Similar results for the inverse of the one-dimensional discrete Laplacian (discrete Green function) have actually appeared some time ago [9].

$$\begin{aligned} \Delta_{hj,kl}^o &= T_0 \langle x_h^j x_k^l \rangle \\ &= \frac{T_0}{M} \langle q_0^j q_0^l \rangle + \frac{2T_0}{M} \sum_{m=1}^{M-1} \langle q_m^j q_m^l \rangle \cos \frac{m(h-1/2)\pi}{M} \\ &\quad \times \cos \frac{m(k-1/2)\pi}{M} \\ &= \frac{N_T - |l-j|}{2M} + \frac{1}{M} \sum_{m=1}^{M-1} \frac{e^{-|l-j|\lambda_m^o}}{\sinh \lambda_m^o} \\ &\quad \times \cos \frac{m(h-1/2)\pi}{M} \cos \frac{m(k-1/2)\pi}{M}. \end{aligned} \quad (16)$$

For the Dirichlet open string

$$\begin{aligned} \Delta_{hj,kl}^D &= T_0 \langle y_h^j y_k^l \rangle = \frac{2T_0}{M} \sum_{m=1}^{M-1} \langle q_m^j q_m^l \rangle \sin \frac{mh\pi}{M} \sin \frac{mk\pi}{M} \\ &= \frac{1}{M} \sum_{m=1}^{M-1} \frac{e^{-|l-j|\lambda_m^D}}{\sinh \lambda_m^D} \sin \frac{mh\pi}{M} \sin \frac{mk\pi}{M}, \quad h, k \neq M, \end{aligned}$$

$$\Delta_{Mj,Ml}^D = T_0 \langle y_M^j y_M^l \rangle = \frac{e^{-|l-j|\lambda_M^D}}{2 \sinh \lambda_M^D},$$

$$\Delta_{Mj,kl}^D = 0, \quad k \neq M. \quad (17)$$

For the closed string

$$\begin{aligned} \Delta_{hj,kl}^c &= T_0 \langle x_h^j x_k^l \rangle \\ &= \frac{T_0}{M} \langle q_0^j q_0^l \rangle + \frac{T_0}{M} \sum_{m=1}^{M-1} \langle A_m^j A_{M-m}^l \rangle \exp \frac{2m(h-k)i\pi}{M} \\ &= \frac{N_T - |l-j|}{2M} + \frac{1}{2M} \sum_{m=1}^{M-1} \frac{e^{-|l-j|\lambda_m^c}}{\sinh \lambda_m^c} \exp \frac{2m(h-k)i\pi}{M}. \end{aligned} \quad (18)$$

III. CLOSED STRING SELF-ENERGY: TACHYON

For the rest of the paper, we will apply our new approach in order to assess its calculational efficiency. In particular, we will use it to obtain one loop self-energy corrections to low-lying states of the closed string, for which we have a measure of comparison from our previous treatment [7]. In this section, we will particularly focus on the tachyon ground state.

A. A single missing link

The matrix V has indices that are lattice locations; i.e., they are specified by two integers $V_{kj;ml}$. For a single missing link, at time j and linking spatial site k to site $k+1$, the term $(T_0/2)(\mathbf{x}_{j+1}^k - \mathbf{x}_k^j)^2$ is missing from S . That means that

$$\begin{aligned} \sum_{ml;l'} \mathbf{x}_m^l \cdot V_{m'l';ml} \mathbf{x}_{m'}^{l'} \\ = -(\mathbf{x}_{k+1}^j - \mathbf{x}_k^j)^2 = -\mathbf{x}_{k+1}^{j2} - \mathbf{x}_k^{j2} + 2\mathbf{x}_{k+1}^j \cdot \mathbf{x}_k^j, \end{aligned} \quad (19)$$

from which we see

$$V_{ml;m'l'} = -\delta_{lj}\delta_{l'j}(\delta_{m,k+1}\delta_{m',k+1} + \delta_{m,k}\delta_{m',k} - \delta_{m,k+1}\delta_{m',k} - \delta_{m',k+1}\delta_{mk}). \quad (20)$$

This matrix has entries only in rows and columns with labels kj and $k+1, j$, in other words, a 2×2 submatrix. However, the product matrix $V\Delta$ has nonzero entries only in rows with labels kj and $k+1, j$, but in general any column entry in these two rows can be nonzero: there are $2MN$ entries. But in calculating the determinant of $I + V\Delta$ by expanding in minors, one quickly sees that it is only the 2×2 subblock of $I + V\Delta$ that contributes. Similarly if

there are several missing links, the only part of $I + V\Delta$ that contributes to the determinant is a correspondingly sized subblock.

Let us work out $V\Delta$ and the determinant for a single missing link,

$$\begin{aligned} (V\Delta)_{ml,qp} &= -\delta_{lj}(\delta_{m,k+1}\Delta_{(k+1)j,qp} + \delta_{m,k}\Delta_{kj,qp} \\ &\quad - \delta_{m,k+1}\Delta_{kj,qp} - \delta_{mk}\Delta_{(k+1)j,qp}) \\ &= -\delta_{lj}((\delta_{m,k+1} - \delta_{mk})(\Delta_{(k+1)j,qp} - \Delta_{kj,qp})). \end{aligned} \quad (21)$$

Then the desired determinant is

$$\begin{aligned} \det(I + V\Delta) &= \det \begin{pmatrix} 1 + \Delta_{(k+1)j,kj} - \Delta_{kj,kj} & \Delta_{(k+1)j,(k+1)j} - \Delta_{kj,(k+1)j} \\ -\Delta_{(k+1)j,kj} + \Delta_{kj,kj} & 1 - \Delta_{(k+1)j,(k+1)j} + \Delta_{kj,(k+1)j} \end{pmatrix} \\ &= 1 - \Delta_{(k+1)j,(k+1)j} + \Delta_{kj,(k+1)j} + \Delta_{(k+1)j,kj} - \Delta_{kj,kj}. \end{aligned} \quad (22)$$

From (18)

$$\begin{aligned} &\Delta_{(k+1)j,kj} - \Delta_{kj,kj} \\ &= \frac{1}{2M} \sum_{m=1}^{M-1} \frac{1}{\sinh \lambda_m^c} \left(\exp \frac{2mi\pi}{M} - 1 \right) \\ &= -\frac{1}{M} \sum_{m=1}^{M-1} \frac{\sin^2(m\pi/M)}{\sinh \lambda_m^c} \\ &= -\frac{1}{2M} \sum_{m=1}^{M-1} \frac{\sin(m\pi/M)}{\sqrt{1 + \sin^2(m\pi/M)}}. \end{aligned} \quad (23)$$

Evidently the same result is obtained for the difference $\Delta_{kj,(k+1)j} - \Delta_{(k+1)j,(k+1)j}$. For large M we can apply the Euler-Maclaurin series

$$\begin{aligned} &\frac{1}{M} \sum_{m=1}^{M-1} f\left(\frac{m}{M}\right) \\ &= \int_0^1 dx f(x) - \frac{1}{2M} (f(0) + f(1)) \\ &\quad + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \frac{1}{M^{2k}} (f^{(2k-1)}(1) - f^{(2k-1)}(0)), \end{aligned} \quad (24)$$

where B_k are the Bernoulli numbers, to get

$$\begin{aligned} \Delta_{(k+1)j,kj} - \Delta_{kj,kj} &= -\frac{1}{2} \left[\int_0^1 dx \frac{\sin \pi x}{\sqrt{1 + \sin^2 \pi x}} - 2\pi \frac{B_2}{2M^2} \right. \\ &\quad \left. + 8\pi^3 \frac{B_4}{24M^4} + \mathcal{O}\left(\frac{1}{M^6}\right) \right] \\ &= -\frac{1}{2} \left[\frac{1}{2} - \frac{\pi}{6M^2} - \frac{\pi^3}{90M^4} + \mathcal{O}\left(\frac{1}{M^6}\right) \right]. \end{aligned} \quad (25)$$

So finally

$$\det(I + V\Delta) = \frac{1}{2} + \frac{\pi}{6M^2} + \frac{\pi^3}{90M^4} + \mathcal{O}\left(\frac{1}{M^6}\right). \quad (26)$$

B. Single slit with $K - 1$ missing links

The case of one missing link describes a one loop diagram with the loop occupying two time steps. A single loop occupying K time steps has $K - 1$ consecutive missing links, as we depict in Fig. 1. Proceeding with the case of $K - 1$ missing links, again between spatial positions k and $k + 1$, but this time for the time interval between instants $J + 1$ to $J + K - 1$, it is evident that we will simply have to sum the right-hand side of (20) over $j \in [J + 1, J + K - 1]$. The sum will also carry over to (21), whose nontrivial subblock will now have size $2(K - 1)$.

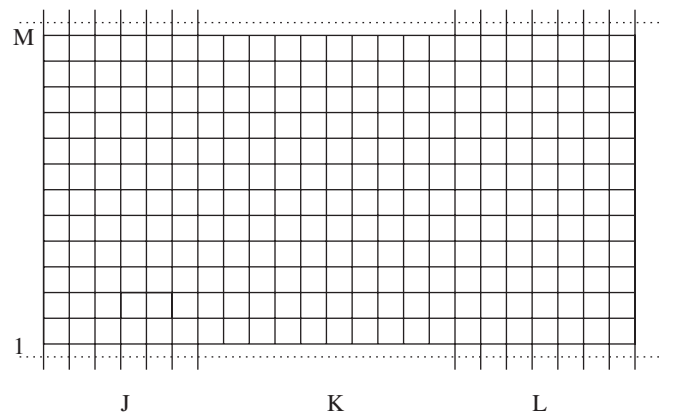


FIG. 1. GT worldsheet lattice for the closed string self-energy. The dotted lines are identified. There are $K - 1$ missing links, chosen for concreteness between the spatial positions $k = 1$ and $k = M$.

Because of the difference of delta functions on the latter relation, clearly the matrix rows with $m = k + 1$ will have the opposite values of the rows with $m = k$. So sorting our rows such that

$$(m, l) = \{(k, J + 1), \dots, (k, J + K - 1), \dots, (k + 1, J + 1), \dots, (k + 1, J + K - 1)\}, \quad (27)$$

and similarly for the columns, we can write in $(K - 1) \times (K - 1)$ block form

$$\begin{aligned} \det(I + V\Delta) &= \det \begin{pmatrix} I + A & B \\ -A & I - B \end{pmatrix} = \det \begin{pmatrix} I & I \\ -A & I - B \end{pmatrix} \\ &= \det \begin{pmatrix} I & 0 \\ -A & I + A - B \end{pmatrix} = \det(I + A - B), \end{aligned} \quad (28)$$

where we employed elementary row and column manipulations that leave the determinant invariant, together with the block matrix identity

$$\det \begin{pmatrix} Q & 0 \\ R & S \end{pmatrix} = \det(Q) \det(S). \quad (29)$$

In formula (28) above,

$$\begin{aligned} A_{lp} &= \Delta_{(k+1)l, kp} - \Delta_{kl, kp}, \\ B_{lp} &= \Delta_{(k+1)l, (k+1)p} - \Delta_{kl, (k+1)p}, \end{aligned} \quad (30)$$

and since these quantities depend on l, p only through $|l - p|$, the value of $J + 1$ will be immaterial and we can set it to zero. Hence our final expression for the determinant will be

$$\det(I + V\Delta) = \det(h_{lp}), \quad l, p = 1, 2, \dots, K - 1, \quad (31)$$

where

$$h_{lp} = \delta_{lp} + \Delta_{(k+1)l, kp} - \Delta_{kl, kp} + \Delta_{kl, (k+1)p} - \Delta_{(k+1)l, (k+1)p}. \quad (32)$$

Notice in particular that we now have the determinant of a $(K - 1)$ -dimensional matrix, whose elements depend on differences of propagators, such that the zero modes in (16)–(18) always cancel out. Specializing to the case of the closed string propagator (18), it is easy to show that³

$$\begin{aligned} h_{lp} &= \delta_{lp} - \frac{1}{M} \sum_{m=1}^{M-1} \frac{\sin(m\pi/M)}{\sqrt{1 + \sin^2(m\pi/M)}} \\ &\quad \times (\sin(m\pi/M) + \sqrt{1 + \sin^2(m\pi/M)})^{-2|l-p|}, \end{aligned} \quad (33)$$

³Evidently, $h_{lp} = h(|l - p|)$, namely $\det(h_{lp})$, is the determinant of a symmetric Toeplitz matrix. It can be shown [10] that it further reduces to a product of two determinants of approximately half the size.

and applying again the Euler-Maclaurin formula (24), we obtain

$$h_{lp} = \delta_{lp} - I_{|l-p|} + \frac{\pi}{6M^2} - \frac{(-1 + 3|l-p|^2)\pi^3}{90M^4} + \mathcal{O}\left(\frac{1}{M^6}\right), \quad (34)$$

where we have reduced the integral $I_{|l-p|}$ to a simple finite sum in Appendix C. Reexpressing

$$h_{lp}(x) = h_{lp}(0) + x + \mathcal{O}(x^2), \quad x = \frac{\pi}{6M^2}, \quad (35)$$

we can separate the M dependence of the determinant,

$$\det(h_{lp}) = \det(h_{lp}(0)) + \frac{\pi}{6M^2} \frac{\partial}{\partial x} \det(h_{lp}(x))|_{x=0} + \mathcal{O}\left(\frac{1}{M^4}\right). \quad (36)$$

Finally, the summand of the energy shift will be given by

$$\begin{aligned} \delta P_{G, \text{closed}}^- &= -\frac{e^{-24(K-1)B_0}}{\det^{12}(I + V\Delta)} = -\frac{e^{-24(K-1)B_0}}{\det^{12}(h_{lp})} \\ &= -\frac{e^{-24(K-1)B_0}}{\det^{12}(h_{lp}(0))} \left(1 - \frac{2\pi}{M^2} \frac{\partial}{\partial x} \frac{\det(h_{lp}(x))|_{x=0}}{\det(h_{lp}(0))}\right) \\ &\quad + \mathcal{O}\left(\frac{1}{M^4}\right), \end{aligned} \quad (37, 38)$$

where h_{lp} is given exactly in (33) and asymptotically in (34), and similarly we have expressed the summand in its exact (37) and asymptotic (38) form.

Let us now discuss how our current, worldsheet-based approach compares to the string field theory-related approach we employed in Ref. [7], when it comes to computing the tachyon energy shift. Clearly, the formulas we have derived here involve determinants of roughly size K , so that they are advantageous for analyzing the summand in the ultraviolet region $K \ll M$. Conversely, the approach [7] yields determinants of size M , more suitable for the infrared $K \gg M$ regime.

TABLE I. Asymptotic expansion up to $\mathcal{O}(x)$, $x = \frac{\pi}{6M^2}$, of the determinant (36), entering the tachyon energy shift (38), for a slit of length $K - 1$, $K = 2, \dots, 6$. Evidently, the coefficients of the expansion can be calculated exactly.

K	$\det(h_{lp}(x))$
2	$\frac{1}{2} + x$
3	$-\frac{4}{\pi^2} + \frac{2}{\pi} + \frac{4x}{\pi}$
4	$-2 - \frac{64}{\pi^3} + \frac{16}{\pi^2} + \frac{8}{\pi} + (-4 + \frac{16}{\pi})x$
5	$-16 - \frac{8192}{9\pi^4} - \frac{2048}{9\pi^3} + \frac{256}{\pi^2} + \frac{64}{3\pi} + (-64 - \frac{16384}{9\pi^3} + \frac{2048}{3\pi^2} + \frac{512}{3\pi})x$
6	$-128 - \frac{1441792}{81\pi^5} + \frac{45056}{27\pi^4} + \frac{3072}{\pi^3} - \frac{512}{3\pi^2} + (-768 - \frac{8388608}{81\pi^4} + \frac{262144}{27\pi^3} + \frac{163840}{9\pi^2} - \frac{1024}{\pi})x$

TABLE II. Asymptotic expansion up to $\mathcal{O}(1/M^2)$, of the tachyon energy shift summand (38) for $K = 2, \dots, 6$, where $K - 1$ is the slit length. The left-hand side (LHS) coefficients were determined by fitting the M dependence, as in Ref. [7], with an error estimate at the order of the last digit. The right-hand side (RHS) coefficients have been calculated exactly with the methods of the present paper and evaluated up to the desired precision.

K	$-\delta P_{G,\text{closed}}^-$ fit	$-\delta P_{G,\text{closed}}^-$ actual
2	$0.1044844648 - 1.31291/M^2$	$0.104484465146 - 1.31299/M^2$
3	$0.027700432 - 0.9578/M^2$	$0.0277004334342 - 0.957933/M^2$
4	$0.010959556 - 0.7268/M^2$	$0.0109595576932 - 0.727031/M^2$
5	$0.005388196 - 0.5811/M^2$	$0.00538819758183 - 0.581471/M^2$
6	$0.003032942 - 0.4828/M^2$	$0.00303294412639 - 0.483277/M^2$

The fact that each approach is more suitable for one of the two domains is also evident in our ability to derive asymptotic formulas there. Because in our earlier paper we had no such formula for analyzing the $K \ll M$ behavior of the integrand, we had to use the exact formula for the summand, evaluate it for a range of M and K , and perform fits in both variables in order to find the respective dependence. In contrast, here we obtain explicitly the form of the asymptotic expansion in M , and we only need to fit for the dependence of the coefficients on K .

Furthermore, it is possible to compute these coefficients exactly for specific K , or evaluate them at arbitrary precision. As an example, we present the exact values for the first two coefficients of $\det(h_{lp}(x))$ for $K = 2, \dots, 6$ in Table I. In Table II, we also present the respective asymptotic expansion for the summand (38), and compare the values for the coefficients, on the one hand obtained with our current method, and on the other hand by fitting the M dependence along the lines of Ref. [7].⁴

Clearly, the two results agree within our margins of error; notice, however, that their difference increases with K . This is a result of the systematic error coming from not taking into account the $\mathcal{O}(1/M^4)$ term in the fits, whose relative size also increases with K . Apart from the asymptotic expansion, we also confirmed that the exact formulas for the summand of the tachyon energy shift in the two approaches, (37) here and (51) in Ref. [7], agree for a large set of M, K values.

IV. CLOSED STRING SELF-ENERGY: GRAVITON

To extract information, e.g., energy shifts, about excited closed string states, we will need to consider the propagator on a worldsheet that includes interactions. If we denote the matrix describing a particular configuration of missing links by V , as we did in (3), then the propagator in question will be given by

$$\begin{aligned} \Delta^V &= (\Delta^{-1} + V)^{-1} = \Delta(I + V\Delta)^{-1} \\ &= \Delta - \Delta(I + V\Delta)^{-1}V\Delta \equiv \Delta - \Delta\mathcal{V}\Delta. \end{aligned} \quad (39)$$

⁴See also Fig. 5 in the Ref. [7].

The final form on the right is useful when V is sparse, because then the inverse matrix appearing in the second term can be evaluated as the inverse of the submatrix obtained by projecting onto the sparse subspace.

Using index notation for the propagator, $\Delta_{kj,pq}^V$, and choosing the times q and j much earlier and much later than all of the times occupied by V , respectively, we can also write

$$\begin{aligned} \Delta_{kj,pq}^V &= \sum_{m,m'} \exp\left\{-j\lambda_m + q\lambda_{m'} + \frac{2\pi i}{M}(mk - m'p)\right\} \\ &\times \frac{\tilde{\Delta}_{mm'}^V}{2M\sqrt{\sinh\lambda_m \sinh\lambda_{m'}}}, \end{aligned} \quad (40)$$

$$\tilde{\Delta}_{mm'}^V = \delta_{mm'} - \frac{\tilde{\mathcal{V}}_{mm'}}{2M\sqrt{\sinh\lambda_m \sinh\lambda_{m'}}}, \quad (41)$$

$$\tilde{\mathcal{V}}_{mm'} = \sum_{kl,rs} \exp\left\{l\lambda_m - s\lambda_{m'} - \frac{2\pi i}{M}(mk - m'r)\right\} \mathcal{V}_{kl,rs}. \quad (42)$$

We have normalized $\tilde{\Delta}_{mm'}^V$ so that it is $\delta_{mm'}$ for $\mathcal{V} = 0$. Thus $\tilde{\Delta}_{mm'}^V \det^{-12}(I + V\Delta)$ gives the probability amplitude that the mode m' at early times evolves to mode m at late times. For the graviton self-energy, the relevant process is modes $m = 1, M - 1$ at early times evolving to the same modes at late times. Thus this contribution to the graviton self-energy is

$$-(\tilde{\Delta}_{11}^V \tilde{\Delta}_{(M-1)(M-1)}^V + \tilde{\Delta}_{1(M-1)}^V \tilde{\Delta}_{(M-1)1}^V) \det^{-12}(I + V\Delta). \quad (43)$$

A. A single missing link

For starters, let us take V with a single missing link. Its nonvanishing 2×2 subblock is the matrix

$$V = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}. \quad (44)$$

Putting $A = \Delta_{(k+1)j,kj} - \Delta_{kj,kj} = \Delta_{(k+1)j,kj} - \Delta_{(k+1)j,(k+1)j}$ the matrix $I + V\Delta$ projected onto the subspace of V , and its inverse times V are

$$\begin{aligned}
 I + V\Delta &= \begin{pmatrix} 1 + A & -A \\ -A & 1 + A \end{pmatrix}, \\
 \mathcal{V} &= (I + V\Delta)^{-1}V \\
 &= \frac{1}{1 + 2A} \begin{pmatrix} 1 + A & A \\ A & 1 + A \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \\
 &= \frac{V}{1 + 2A}.
 \end{aligned} \tag{45}$$

Then we easily compute

$$\begin{aligned}
 \tilde{\mathcal{V}}_{mm'} &= -4e^{j(\lambda_m - \lambda_{m'}) + \pi i(m' - m)(2k+1)/M} \\
 &\quad \times \frac{\sin(\pi m/M) \sin(\pi m'/M)}{1 + 2A},
 \end{aligned} \tag{46}$$

$$\tilde{\mathcal{V}}_{mm} = -4 \frac{\sin^2(\pi m/M)}{1 + 2A}, \tag{47}$$

$$\tilde{\mathcal{V}}_{m(M-m)} = 4e^{-2\pi i m(2k+1)/M} \frac{\sin^2(\pi m/M)}{1 + 2A}.$$

Then we find

$$\begin{aligned}
 \tilde{\Delta}_{mm}^V &= \tilde{\Delta}_{(M-m)(M-m)}^V \\
 &= 1 + 4 \frac{\sin^2(\pi m/M)}{2M(1 + 2A) \sinh \lambda_m} \\
 &= 1 + \frac{2\pi m}{M^2} - \frac{2m\pi^2(1 + 2m^2\pi)}{3M^4} + \mathcal{O}\left(\frac{1}{M^6}\right),
 \end{aligned} \tag{48}$$

$$\begin{aligned}
 \tilde{\Delta}_{m(M-m)}^V &= -4 \frac{\sin^2(\pi m/M)}{2M(1 + 2A) \sinh \lambda_m} e^{-2\pi i m(2k+1)/M} \\
 &= -\frac{2\pi m}{M^2} e^{-2\pi i m(2k+1)/M} + \mathcal{O}\left(\frac{1}{M^4}\right).
 \end{aligned} \tag{49}$$

The one missing link contribution to the self-energy of the closed string state $|m, M - m\rangle \pm |M - m, m\rangle$ is up to $\mathcal{O}(1/M^4)$

$$\begin{aligned}
 &-\frac{\tilde{\Delta}_{mm}^V \tilde{\Delta}_{(M-m)(M-m)}^V \pm \tilde{\Delta}_{m(M-m)}^V \tilde{\Delta}_{(M-m)m}^V}{\det^{12}(1 + V\Delta)} \sim -\frac{(1 + 4\pi m/M^2 - 4m\pi^2(1 - 3(1 \pm 1)m + 2m^2\pi))/3M^4}{(1/2 + \pi/6M^2 + \pi^3/90M^4)^{12}} \\
 &\sim -2^{12} \left(1 + \frac{4\pi(m-1)}{M^2} - \frac{2\pi^2(-65 + 130m - 30(1 \pm 1)m^2 + 2\pi + 20m^3\pi)}{15M^4} \right).
 \end{aligned} \tag{50}$$

This formula includes the shift for tachyon ($m = 0$) and the graviton ($m = 1$). The latter receives no $1/M^2$ correction, consistent with zero shift in the continuum limit. Note also that we have assumed in these formulas that the polarizations of the first and second entries of $|m, m'\rangle$ are different, so they do not properly describe the dilaton self-energy shift.

Furthermore, for the graviton ($m = 1$ and plus sign), we can also compare the above formula with the fits we obtained for the value of the graviton energy shift in Ref. [7]. Multiplying (50) with the boundary counterterm $\exp(-24B_0)$, and writing the result in the notation of the latter paper, we have

$$\begin{aligned}
 &-\frac{1}{2}(1 + C_G^K) \\
 &= -\left(\frac{2}{1 + \sqrt{2}}\right)^{12} \left(1 - \frac{2\pi^2(5 + 22\pi)}{15M^4}\right) + \mathcal{O}\left(\frac{1}{M^6}\right) \\
 &\simeq -0.104484 + \frac{10.1905}{M^4} + \mathcal{O}\left(\frac{1}{M^6}\right).
 \end{aligned} \tag{51}$$

The result above is in excellent agreement with the fit presented in Fig. 11 of Ref. [7].

B. Single slit with $K - 1$ missing links

Let us now generalize the discussion of the previous section, for the worldsheet configuration where a link is

missing between the same spatial sites k and $k + 1$, but for a time interval $K - 1$ links long. Using the same reasoning as for the tachyon in the same configuration, it is possible to show that the matrix \mathcal{V} defined in (39) has the special structure

$$\begin{aligned}
 \mathcal{V}_{kl,ks} &= \mathcal{V}_{(k+1)l,(k+1)s} = -\mathcal{V}_{(k+1)l,ks} \\
 &= -\mathcal{V}_{kl,(k+1)s} = -h_{ls}^{-1},
 \end{aligned} \tag{52}$$

where h is the same $(K - 1)$ -dimensional matrix appearing in (31). With the help of these relations, we do the k, r summation in (42)

$$\begin{aligned}
 \tilde{\mathcal{V}}_{mm'} &= -\sum_{ls} e^{l\lambda_m - s\lambda_{m'}} (e^{\frac{2\pi i}{M}(m' - m)k} + e^{\frac{2\pi i}{M}(m' - m)(k+1)}) \\
 &\quad - e^{\frac{2\pi i}{M}(m' + m'k - mk)} - e^{\frac{2\pi i}{M}(m' - mk - m)} h_{ls}^{-1},
 \end{aligned}$$

and if we take out an overall factor $\exp[\pi i(m' - m)(2k + 1)]$, this simplifies to

$$\begin{aligned}
 \tilde{\mathcal{V}}_{mm'} &= -4e^{\pi i(m' - m)(2k+1)} \sin \frac{m\pi}{M} \\
 &\quad \times \sin \frac{m'\pi}{M} \sum_{ls} e^{l\lambda_m - s\lambda_{m'}} h_{ls}^{-1}.
 \end{aligned} \tag{53}$$

It is evident that the above relation implies

$$\begin{aligned}
 \tilde{\mathcal{V}}_{m(M-m)} \tilde{\mathcal{V}}_{(M-m)m} &= \tilde{\mathcal{V}}_{mm}^2, \\
 \tilde{\mathcal{V}}_{(M-m)(M-m)} &= \tilde{\mathcal{V}}_{mm},
 \end{aligned} \tag{54}$$

TABLE III. Asymptotic expansion up to $\mathcal{O}(1/M^4)$, of the graviton energy shift summand for $K = 2, \dots, 6$, where $K - 1$ is the slit length. The LHS coefficients have been determined by fitting M , as in Ref. [7], with an error estimate at the order of the last digit. The RHS coefficients have been calculated exactly with the methods of the present paper and evaluated with two additional digits of precision.

K	$-\delta P_{\text{Graviton}}^-$ fit	$-\delta P_{\text{Graviton}}^-$ actual
2	$0.104484465145 - 10.19/M^4$	$0.10448446514630 - 10.1905/M^4$
3	$0.027700433434 - 3.85/M^4$	$0.02770043343416 - 3.8499/M^4$
4	$0.010959557693 + 1.87/M^4$	$0.01095955769317 + 1.8837/M^4$
5	$0.0053881975 + 6.82/M^4$	$0.00538819758183 + 6.8571/M^4$
6	$0.003032944127 + 11.28/M^4$	$0.00303294412639 + 11.3355/M^4$

so that our final working formula for the graviton summand, also including the required boundary counterterm, will be

$$\begin{aligned}
 \delta P_{\text{Graviton}}^- &= -\frac{e^{-24(K-1)B_0}}{\det^{12}(I+V\Delta)} (\tilde{\Delta}_{11}^V \tilde{\Delta}_{(M-1)(M-1)}^V + \tilde{\Delta}_{1(M-1)}^V \tilde{\Delta}_{(M-1)1}^V) \\
 &= -\frac{e^{-24(K-1)B_0}}{\det^{12}(I+V\Delta)} \left[\left(1 + \frac{\tilde{V}_{11}}{2M \sinh \lambda_1}\right)^2 + \left(\frac{\tilde{V}_{11}}{2M \sinh \lambda_1}\right)^2 \right] \\
 &= -\frac{e^{-24(K-1)B_0}}{\det^{12}(I+V\Delta)} (1 + 2\tilde{U} + 2\tilde{U}^2), \tag{55}
 \end{aligned}$$

where

$$\tilde{U} = \frac{\sin \frac{\pi}{M}}{M \sqrt{1 + \sin^2 \frac{\pi}{M}}} \sum_{l,s=1}^{K-1} \left(\sin \frac{\pi}{M} + \sqrt{1 + \sin^2 \frac{\pi}{M}} \right)^{2(l-s)} h_{ls}^{-1}, \tag{56}$$

and h^{-1} is the inverse of the $(K-1)$ -dimensional matrix with elements (32). The same interesting phenomenon that we encountered for the tachyon also appears here; namely, we can reduce the size of the matrices entering the energy shift by a half. The asymptotic expansion of (55) in M readily follows from the respective expansion of h_{ls} (34), and in particular it is easy to show that

$$h_{ls}^{-1} = h_{ls}^{-1}(0) - \frac{\pi}{6M^2} \left(\sum_{i=1}^{K-1} h_{ii}^{-1}(0) \right) \left(\sum_{i=1}^{K-1} h_{is}^{-1}(0) \right) + \mathcal{O}\left(\frac{1}{M^4}\right), \tag{57}$$

where $h(0)$ is the M -independent part of the matrix h . Because the overall factor in (56) starts as $\mathcal{O}(1/M^2)$, we do not need additional terms in order to obtain (55) at $\mathcal{O}(1/M^4)$. In fact, if we only focus at $\mathcal{O}(1/M^2)$ for a moment, the term on the right-hand side of (55) simplifies to

$$\begin{aligned}
 1 + 2\tilde{U} + 2\tilde{U}^2 &\approx 1 + \frac{2\pi}{M^2} \sum_{l,s=1}^{K-1} h_{ls}^{-1}(0) \\
 &= 1 + \frac{2\pi}{M^2} \frac{\frac{\partial}{\partial x} \det(h_{lp}(x))|_{x=0}}{\det(h_{lp}(0))}, \tag{58}
 \end{aligned}$$

where for the last equality we used the identity

$$\frac{\partial}{\partial x} \det(h) = \det(h) \text{Tr}\left(h^{-1} \frac{\partial h}{\partial x}\right), \tag{59}$$

and also the fact that in our case the derivative matrix has all entries equal to one.

Comparing (58) and (55) with (38), we observe that we have rigorously proven two important facts: that the leading order of the asymptotic M expansion for the graviton is equal to the tachyon one, and that the subleading term is always $\mathcal{O}(1/M^4)$ for any $K \ll M$. Of course, these properties were expected to hold on physical grounds; however, in the approach of our previous paper, we could only obtain empirical indications about them from the fits. Finally, for sample slit lengths, we compare the coefficients of the aforementioned fits with the exact values obtained with our new method, and evaluated at higher precision, in Table III.

V. DISCUSSION AND CONCLUSION

In this paper, we continued our investigation of lattice-regularized string theory in the lightcone gauge, by introducing a new approach for evaluating the corresponding path integral. Whereas in our earlier work [7] we built the path integral by integrating products of free string propagators over the interaction points, here we treated it as a quantum field theory on the worldsheet. Given that free string propagators are the two-point functions of string field theory, we could call the former approach string field theory based, and the latter approach worldsheet based.

The key idea for treating string interactions in this framework was to examine how the path integral is modified as we start removing links from the free worldsheet (3). An essential ingredient for describing this departure is the worldsheet correlation function Δ of two target space coordinates (5). We consider as the main result of this paper the determination of this quantity explicitly in Fourier mode space (11), and as a simple sum in coordinate space (16)–(18).

We then moved on to assess the efficiency of the worldsheet approach, by calculating the one loop self-energy corrections for the closed string tachyon and graviton, and

performing a comparison with the results of Ref. [7]. The self-energy corrections involve determinants of size M and K for the string field theory and worldsheet approach, respectively, and hence the first one is more convenient for analyzing the $K \gg M$ (infrared) regime, whereas the second one is for the $K \ll M$ (ultraviolet) regime.⁵

Indeed, with our current approach we were able not only to find the structure of the asymptotic expansion in M of the self-energy summand but also to calculate its coefficients exactly, for each value of K . In this manner, we were able to rigorously prove two important facts, for which we only had strong indications up to now, namely, that the leading term in the expansion is the same for the tachyon and the graviton, and that the $\mathcal{O}(1/M^2)$ subleading term for the graviton is zero (i.e., the graviton is massless in the ultraviolet region). In contradistinction, analyzing the $K \ll M$ regime in Ref. [7] had to rely on fits for both variables M and K , which introduced larger numerical errors and made conclusions less definitive.

Apart from its calculational virtues, our new approach adopts the point of view, implicit in our representation of the planar sum as a sum over Ising spin variables (2), which is much closer to the treatment of more general lattice systems: Each string diagram is like a lattice state, and we build all states by gradually adding more and more “excitations,” namely missing links, to the “vacuum,” or free worldsheet. It would be very interesting to explore this point of view further, as it seems to suggest that string diagrams of different loop order but the same excitation number may be similar to each other. Indeed, generalizing the considerations of Secs. III B and IV B for arbitrary positions of the $K - 1$ missing links $(m, l) = \{(k_1, j_1), \dots, (k_{K-1}, j_{K-1})\}$ yields again a $(K - 1)$ -dimensional determinant, this time with elements

$$h_{m,l,m'l'} = \delta_{mm'} \delta_{mm'} + \Delta_{(m+1)l,m'l'} - \Delta_{m,l,m'l'} + \Delta_{m,l,(m'+1)l'} - \Delta_{(m+1)l,(m'+1)l'}. \quad (60)$$

It may be more advantageous to organize the sum over diagrams not by loop order, as dictated by the conventional wisdom of string perturbation theory, but by the number of missing link excitations.

With this more efficient method now in place, a primary objective will be its application for the study of the one loop self-energy corrections to the low-lying states of open string theory, which though more intricate, is of main interest because of its relation to large N gauge theory. Once we have similarly established the compatibility of the lattice regularization with Lorentz invariance in this case as well, then the next natural step will be the numerical evaluation of the full path integral with the help of Monte Carlo methods.

⁵We remind the reader that M is the spatial size of the lattice and K is the temporal length of the slit representing the loop.

In this respect, it will be very interesting to examine whether efficiency can be further improved by performing the sums in (16)–(18) analytically, in order to obtain explicit expressions for the worldsheet propagators in coordinate space as well. In the most probable scenario, that the summation of all bosonic string diagrams does not succeed in stabilizing the vacuum, we will, of course, be aiming to develop a similar treatment for the superstring as well.

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APPENDIX A: NORMAL MODES

A string with $P^+ = MaT_0$ is described at a fixed time by M coordinates x_i or y_i , $i = 1, \dots, M$. In this article we require several normal mode decompositions depending on the boundary conditions.

Neumann Open String

$$x_i = \frac{1}{\sqrt{M}} q_0 + \sqrt{\frac{2}{M}} \sum_{m=1}^{M-1} q_{om} \cos \frac{m\pi(i-1/2)}{M}, \quad (A1)$$

$$q_0 = \sqrt{\frac{1}{M}} \sum_{i=1}^M x_i, \quad q_{om} = \sqrt{\frac{2}{M}} \sum_i x_i \cos \frac{m\pi(i-1/2)}{M}. \quad (A2)$$

Dirichlet Open String

$$y_k = \sqrt{\frac{2}{M}} \sum_{m=1}^{M-1} q_{Dm} \sin \frac{m\pi k}{M} \quad \text{for } k = 1, \dots, M-1, \quad (A3)$$

$$y_M = q_{DM},$$

$$q_{Dm} = \sqrt{\frac{2}{M}} \sum_{k=1}^{M-1} y_k \sin \frac{m\pi k}{M}, \quad 0 < m < M, \quad (A4)$$

$$q_{DM} = y_M.$$

Closed String

$$x_k = \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} A_m \exp \frac{2mk\pi i}{M}, \quad (A5)$$

$$A_m = \frac{1}{\sqrt{M}} \sum_k x_k \exp \frac{2(M-m)k\pi i}{M}.$$

This goes to the normal mode expansion with trigonometric functions with the substitutions $A_m = A_{M-m}^* = (q_{cm} - iq_{sm})/\sqrt{2}$, with $0 < m < M/2$, $A_0 = q_0$, and $A_{M/2} = q_{cM/2}$ (if M is even). From this dictionary the nonzero correlators are

$$\begin{aligned} \langle A_m A_{M-m} \rangle &= \frac{1}{2} (\langle q_{cm} q_{cm} \rangle + \langle q_{sm} q_{sm} \rangle), \quad m \neq 0, \frac{M}{2}, \\ \langle A_0 A_0 \rangle &= \langle q_0 q_0 \rangle, \quad \langle A_{M/2} A_{M/2} \rangle = \langle q_{M/2} q_{M/2} \rangle. \end{aligned} \quad (\text{A6})$$

APPENDIX B: STRING PROPAGATORS

1. Neumann open string propagator

$$\langle N+1, x^f | 0, x^i \rangle^{\text{open}} = \mathcal{D}^{\text{open}}(N+1) e^{iW_{\text{open}}}, \quad (\text{B1})$$

$$\begin{aligned} iW_{\text{open}} &= -\frac{T_0}{2} \left[\frac{(q_{0,f} - q_{0,i})^2}{N+1} \right. \\ &+ \sum_{m=1}^{M-1} \sinh \lambda_m^o \left((q_{m,i}^2 + q_{m,f}^2) \coth(N+1) \lambda_m^o \right. \\ &\left. \left. - 2 \frac{q_{m,i} q_{m,f}}{\sinh(N+1) \lambda_m^o} \right) \right], \end{aligned} \quad (\text{B2})$$

$$\lambda_m^o = 2 \sinh^{-1} \left(\sin \frac{m\pi}{2M} \right). \quad (\text{B3})$$

Where the q_m 's are the normal mode coordinates for the x 's. The right side is the result of doing the integrations over all the x_i^j with $i = 1, \dots, M$ and $j = 1, \dots, N$. The propagator spans $N+1$ time steps, and this result corresponds to assigning half the potential energy $T_0 \sum_{i=1}^{M-1} (x_{i+1}^j - x_i^j)^2 / 2$ to time $j = 0$ and half to $j = N+1$.

2. Dirichlet open string propagator

The Dirichlet open string propagator over a time of $K = N+1$ steps is evaluated to be

$$\langle q^f, N+1 | q^i, 0 \rangle^D = \mathcal{D}^D(N+1) e^{iW^D}, \quad (\text{B4})$$

where

$$\begin{aligned} iW^D &= -\frac{T_0}{2} \left[\sum_{m=1}^M \left((q_{Dm}^f)^2 + q_{Dm}^i{}^2 \right) \sinh \lambda_m^D \coth K \lambda_m^D \right. \\ &\left. - 2q_{Dm}^f q_{Dm}^i \frac{\sinh \lambda_m^D}{\sinh K \lambda_m^D} \right], \end{aligned} \quad (\text{B5})$$

$$\mathcal{D}^D(N+1) = \left(\frac{T_0}{2\pi} \right)^{M/2} \prod_{m=1}^M \left[\frac{\sinh(N+1) \lambda_m^D}{\sinh \lambda_m^D} \right]^{-1/2}, \quad (\text{B6})$$

$$\begin{aligned} \lambda_M^D &= 2 \sinh^{-1} \frac{1}{\sqrt{2}}, \quad \lambda_m^D = \lambda_m^o = 2 \sinh^{-1} \sin \frac{m\pi}{2M}, \\ m &= 1, \dots, M-1. \end{aligned} \quad (\text{B7})$$

We recall that the above expressions give the result of integrating over all the variables y_i^j , for $j = 1, \dots, N$, with half the potential energy assigned to $j = 0, N+1$, which is consistent with the closure requirement.

3. Closed string propagator

$$\langle N+1, x^f | 0, x^i \rangle^{\text{closed}} = \mathcal{D}^{\text{closed}}(N+1) e^{iW_{\text{closed}}}, \quad (\text{B8})$$

$$\begin{aligned} iW_{\text{closed}} &= -\frac{T_0}{2} \left[\frac{(q_{0,f} - q_{0,i})^2}{N+1} \right. \\ &+ \sum_{m=1}^{M-1} \sinh \lambda_m^c \left((q_{m,i}^2 + q_{m,f}^2) \coth(N+1) \lambda_m^c \right. \\ &\left. \left. - 2 \frac{q_{m,i} q_{m,f}}{\sinh(N+1) \lambda_m^c} \right) \right], \end{aligned} \quad (\text{B9})$$

$$\lambda_m^c = 2 \sinh^{-1} \left(\sin \frac{m\pi}{M} \right), \quad (\text{B10})$$

where the q_m 's are the normal mode coordinates for the x 's. When we divide the closed string normal modes into sine and cosine modes, we arbitrarily call the $m > M/2$ modes sine modes and the $m < M/2$ modes cosine modes. When M is even, the $M/2$ mode is not doubles. The right side is the result of doing the integrations over all the x_i^j with $i = 1, \dots, M$ and $j = 1, \dots, N$. The propagator spans $N+1$ time steps and this result corresponds to assigning half the potential energy $T_0 \sum_{i=1}^M (x_{i+1}^j - x_i^j)^2 / 2$ to time $j = 0$ and half to $j = N+1$. In sums like these it is understood that $x_{M+1}^j \equiv x_1^j$. Whenever we concatenate at a time j propagators with different numbers of missing links, we will understand that we *add* terms $T_0(\Delta x)^2 / 4$ in the exponent so that the potential assigned to time j is that of the system with the least number of missing links. For example, the concatenation of an open string propagator with a closed string propagator entails the addition of $T_0(x_M^j - x_1^j)^2 / 4$ to the exponent.

APPENDIX C: A USEFUL EULER-MACLAURIN EXPANSION

As we saw in Sec. III B for the closed string propagator, when many links are missing, the leading term in the Euler-Maclaurin expansion of the elements of the corresponding determinant involves an integral of the form

$$\begin{aligned} I_n &\equiv \int_0^1 dx \frac{\sin \pi x}{\sqrt{1 + \sin^2 \pi x}} (\sin \pi x + \sqrt{1 + \sin^2 \pi x})^{-2n} \\ &= \int_0^1 dx \frac{\sin \frac{\pi x}{2}}{\sqrt{1 + \sin^2 \frac{\pi x}{2}}} \left(\sin \frac{\pi x}{2} + \sqrt{1 + \sin^2 \frac{\pi x}{2}} \right)^{-2n} \\ &= \int_0^1 dx \frac{\sin \frac{\pi x}{2}}{\sqrt{1 + \sin^2 \frac{\pi x}{2}}} \left(-\sin \frac{\pi x}{2} + \sqrt{1 + \sin^2 \frac{\pi x}{2}} \right)^{2n} \\ &= \frac{2}{\pi} \int_0^1 dz \frac{z(-z + \sqrt{1+z^2})^{2n}}{\sqrt{1-z^2} \sqrt{1+z^2}}. \end{aligned} \quad (\text{C1})$$

We can evaluate this with the help of the identity [10]

$$(z + \sqrt{1+z^2})^{2n} = \sum_{r=0}^n \lambda_{nr} z^{2r} + \sqrt{1+z^2} \sum_{r=1}^n \mu_{nr} z^{2r-1}, \quad (C2)$$

where

$$\lambda_{nr} = \frac{n}{n+r} \binom{n+r}{2r} 2^{2r}, \quad \mu_{nr} = \frac{r\lambda_{nr}}{n}, \quad (C3)$$

so that the integral can be rewritten as

$$\begin{aligned} I_n &= \frac{2}{\pi} \sum_{r=0}^n \lambda_{nr} \int_0^1 dz \frac{z^{2r+1}}{\sqrt{1-z^2}\sqrt{1+z^2}} \\ &\quad - \frac{2}{\pi} \sum_{r=1}^n \mu_{nr} \int_0^1 dz \frac{z^{2r}}{\sqrt{1-z^2}} \\ &= \sum_{r=0}^n \lambda_{nr} \frac{\Gamma(\frac{1}{2} + \frac{r}{2})}{2\sqrt{\pi}\Gamma(1 + \frac{r}{2})} - \sum_{r=1}^n \mu_{nr} \frac{\Gamma(\frac{1}{2} + r)}{\sqrt{\pi}\Gamma(1+r)}. \end{aligned} \quad (C4)$$

If desired, we can formally express these finite sums in terms of hypergeometric functions, for example,

$$\sum_{r=1}^n \mu_{nr} \frac{\Gamma(\frac{1}{2} + r)}{\sqrt{\pi}\Gamma(1+r)} = {}_2F_1(1-n, 1+n; 2; -1). \quad (C5)$$

TABLE IV. Values of integral I_n (C1), for $n = 0, 1, \dots, 6$.

n	I_n
0	$1/2 \approx 0.5$
1	$-1/2 + 2/\pi \approx 0.1366$
2	$-5/2 + 8/\pi \approx 0.04648$
3	$-25/2 + 118/(3\pi) \approx 0.02019$
4	$-129/2 + 608/(3\pi) \approx 0.01080$
5	$-681/2 + 16046/(15\pi) \approx 0.006696$
6	$-3653/2 + 86072/(15\pi) \approx 0.004568$

In any case, the sums can be readily evaluated for specific values of n , and for the reader's convenience we have tabulated the first few cases in Table IV.

Summarizing, the sums that are relevant for the computation of the closed string propagator when many missing links are present have an Euler-Maclaurin expansion of the form

$$\begin{aligned} &\frac{1}{M} \sum_{m=0}^{M-1} \sin \frac{\pi m}{M} \frac{(\sin \frac{\pi m}{M} + \sqrt{1 + \sin^2 \frac{\pi m}{M}})^{-2n}}{\sqrt{1 + \sin^2 \frac{\pi m}{M}}} \\ &= I_n - \frac{\pi}{6M^2} + \frac{(-1 + 3n^2)\pi^3}{90M^4} + \mathcal{O}\left(\frac{1}{M^6}\right). \end{aligned} \quad (C6)$$

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