PHYSICAL REVIEW D **87,** 066004 (2013)

Nonequilibrium fluctuation-dissipation relation from holography

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(Received 25 June 2012; revised manuscript received 13 December 2012; published 12 March 2013)

We derive a nonequilibrium fluctuation-dissipation relation for bosonic correlation functions from holography in the classical gravity approximation at strong coupling. This generalizes the familiar thermal fluctuation-dissipation relation in the absence of external sources. This also holds universally for any nonequilibrium state which can be obtained from a stable thermal equilibrium state in perturbative derivative (hydrodynamic) and amplitude (nonhydrodynamic) expansions. Therefore, this can provide a strong experimental test for the applicability of the holographic framework. We discuss how it can be tested in heavy-ion collisions. We also make a conjecture regarding multipoint holographic nonequilibrium Green's functions.

DOI: 10.1103/PhysRevD.87.066004 PACS numbers: 11.25.Tq, 04.20.Cv, 25.75.Ld, 25.75.Gz

I. INTRODUCTION

Nature challenges us to understand phenomena in real time. This is very difficult to do in the microscopic formulation of the laws of nature given in the framework of quantum field theory.

Though quantum field theory is experimentally successful in understanding microscopic processes when perturbation theory works, it has very limited success in describing macroscopic phenomena in real time. The best we can typically do is to calculate rate and cross sections using perturbation theory, and use the results as inputs for kinetic or phenomenological equations. For example, for dilute quantum gases we can use the Boltzmann equation, where the two-body scattering cross section is an input. However, for few systems do we know how to formulate systematic corrections to the Boltzmann equations, which take into account the uncertainty principles of quantum dynamics.

The main difficulty is that time-dependent perturbation theory gives us the behavior in time only in a Taylor series. This fails to give a uniform approximation in time away from the initial phase of evolution, or to describe decoherence, thermalization and hydrodynamics in a unified framework. Even when the coupling constants are small, we need an essentially nonperturbative approach to understand the origin of irreversibility from the microscopic quantum field theory.

In recent decades, there has been impressive progress in nonequilibrium quantum field theory, which attempts to bridge this gap using the two-particle irreducible action formalism.[1] Still, a lot of progress is needed to develop approximation schemes which take into account unitarity and conservation laws in a controlled manner. As these methods are nonperturbative, some progress is also needed in understanding renormalizability in these formalisms.

Holography in the form of gauge/gravity duality is a new tool which gives a nonperturbative reformulation of quantum field theory [7]. This is particularly tractable when the quantum field theory is a strongly coupled gauge theory and the rank of the gauge group is large. In such a case, gauge/gravity duality maps a quantum field theory to a familiar classical theory of gravity in one higher dimension. At present, this is the best nonperturbative reformulation of quantum gauge theories at hand which can describe phenomena in real time.

In this paper, we will try to unearth a special feature of holographic duality at strong coupling and large rank of the gauge group (i.e., large N), which gives a genuinely nonequilibrium result and is also a model-independent (i.e., universal) feature.[2] By a genuinely nonequilibrium result, we imply one which cannot be obtained readily from equilibrium correlation functions. We also try to focus on a feature which gives fundamental insight on thermalization and the nature of quantum kinetics. This leads us to study the nonequilibrium fluctuation-dissipation relation, as described below.

It is known from nonequilibrium quantum field theory that in order to get the quantum formulation of kinetic equations, we need the nonequilibrium spectral and statistical functions. We define them below.

The spectral function can be understood as an off-shell generalization of the density of states. Here we will define

[1]This started with the closed time-path formalism of Schwinger and Keldysh [1]. The next advancement came with the two-particle irreducible (2PI) action formalism due to Cornwall, Jackiw and Tomboulis [2], and the incorporation of the Schwinger-Keldysh closed time contour into it by Calzetta and Hu [3]. This will be briefly reviewed in Sec. II of this paper. For a review of recent progress of systematic approximations of the two-particle irreducible action and its applications, please see Refs. [4–6].

[2]At this stage we can make a comparison to the famous universal holographic result at strong coupling and large N that the viscosity-to-entropy-density ratio η/s equals $1/4\pi$ [8]. However, this is not a genuinely nonequilibrium result by our definition, because it can be computed via holographic prescriptions from thermal correlators and applying the Kubo formula [9].

it for a bosonic operator $O(\mathbf{x}, t)$. It can be obtained as a Wigner transform of the commutator, i.e., a Fourier transform of the relative coordinate as below:

$$\mathcal{A}(\omega, \mathbf{k}, \mathbf{x}, t) = \int d^3 r dt_r e^{i(\omega t_r - \mathbf{k}\cdot\mathbf{r})} \left\langle \left[O\left(\mathbf{x} + \frac{\mathbf{r}}{2}, t + \frac{t_r}{2}\right), O\left(\mathbf{x} - \frac{\mathbf{r}}{2}, t - \frac{t_r}{2}\right) \right] \right\rangle. \tag{1}$$

The expectation value above is taken in a nonequilibrium state. In the case of fermionic fields, we need to take the anticommutator above.

It can be proved (see Appendix A) that the spectral function is proportional to the imaginary part of the Wigner-transformed retarded Green's function as below:

$$\mathcal{A}(\omega, \mathbf{k}, \mathbf{x}, t) = -2\,\mathrm{Im} G_R(\omega, \mathbf{k}, \mathbf{x}, t). \tag{2}$$

Therefore, the spectral function is real. In the case of fermionic Hermitian operators, it is also non-negative.

Physically, $\mathcal{A}(\omega, \mathbf{k}, \mathbf{x}, t)$ encodes the probability that a quasiparticle with momentum $\mathbf{k}$ at a given point in space-time given by $\mathbf{x}$ and t will have energy ω. We note that the uncertainty principle forces us to consider the quasiparticle off shell. At strong coupling, we expect the broadening of peaks and typically no sharply defined quasiparticles.

At equilibrium, due to translational invariance, the spectral function $\mathcal{A}(\omega, \mathbf{k}, \mathbf{x}, t)$ does not depend on the center-of-mass coordinates $\mathbf{x}$ and t. It is measurable by angle-resolved photoemission spectroscopy of electrons. Away from equilibrium, a time-resolved version of the same angle-resolved photoemission spectroscopy is necessary (see, for instance, Ref. [10]).

The statistical function can be understood as an off-shell generalization of the quasiparticle distribution function in phase space. It is the Wigner transform of the anticommutator for bosonic fields (the commutator for fermionic fields), as below:

$$\begin{aligned} G_{\mathcal{K}}(\omega, \mathbf{k}, \mathbf{x}, t) &= -\frac{i}{2} \int d^3 r dt_r e^{i(\omega t_r - \mathbf{k}\cdot\mathbf{r})} \\ &\quad \times \left\langle \left\{ O\left(\mathbf{x} + \frac{\mathbf{r}}{2}, t + \frac{t_r}{2}\right), O\left(\mathbf{x} - \frac{\mathbf{r}}{2}, t - \frac{t_r}{2}\right) \right\} \right\rangle. \end{aligned} \tag{3}$$

We will show in the next section that the statistical function is related to the imaginary part of the Wigner-transformed Feynman propagator as follows:

$$G_{\mathcal{K}}(\omega, \mathbf{k}, \mathbf{x}, t) = i\,\mathrm{Im} G_F(\omega, \mathbf{k}, \mathbf{x}, t). \tag{4}$$

Thus, the statistical function is purely imaginary. In literature, this is often defined without a $-i$ factor in front as in Eq. (3). In that case, it is real, and when integrated over ω at fixed $\mathbf{k}$, $\mathbf{x}$ and t, it gives the time-dependent phase space distribution of quasiparticles in the semiclassical limit at weak coupling. Here, we will keep the $-i$ factor—in this case, it is also known as the Keldysh propagator.

The statistical function/Keldysh propagator can be indirectly measured by following the space-time evolution of expectation values of operators, like conserved currents and the energy-momentum tensor. We note that both the nonequilibrium statistical and spectral functions need regularization for removing ultraviolet divergences, which are, however, expected to be independent of the state. What is relevant for physical observations is their dependence on the nonequilibrium hydrodynamic and nonhydrodynamic variables characterizing the state.

At equilibrium, the statistical function is again independent of $\mathbf{x}$ and t due to translational invariance. Also, it is determined from the spectral function using analyticity. The relation between the spectral function and the statistical function at equilibrium is known as the fluctuation-dissipation relation, which at thermal equilibrium is as follows in the bosonic case:

$$G_{\mathcal{K}}(\omega, \mathbf{k}) = -i\left(n_{\mathrm{BE}}(\omega) + \frac{1}{2}\right)\mathcal{A}(\omega, \mathbf{k}). \tag{5}$$

It is called the fluctuation-dissipation relation because it implies the fluctuation-dissipation theorems [6], like that relating the thermal electrical noise (current fluctuations) to the electric resistance.

Away from equilibrium, analyticity is insufficient to relate the spectral and statistical functions. They evolve in a coupled way given by the so-called Kadanoff-Baym equations, which can be derived from the two-particle irreducible action (see Sec. II). In the semiclassical limit at weak coupling, the Kadanoff-Baym equations reduce to the Boltzmann equation with quantum corrections. Formally, these equations relating off-shell quantities are valid at strong coupling also. Even at weak coupling, however, nonperturbative methods are needed to make complete sense of these equations and for making systematic expansions, as mentioned earlier. In general, we do not expect any generalization of the flucutation-dissipation relation away from equilibrium which can be stated in a universal way independent of the details of the nonequilibrium state and the nature of external perturbations.

Numerical simulations of Kadanoff-Baym equations based on the 2PI effective action at weak coupling indeed show apparent thermalization—at long time, the spectral and statistical functions satisfy Eq. (5). Such simulations as, for instance, $O(N)$ scalar field theories at large N and weak coupling [11] indeed show thermalization, but far away from equilibrium, there seems to be no simple fluctuation-dissipation relation between spectral and statistical functions.

Summary of results: In this paper, we study the nonequilibrium spectral and statistical functions in states which are perturbatively connected to thermal equilibrium with external sources absent. These are nonequilibrium states undergoing strongly coupled hydrodynamic and nonhydrodynamic relaxation.

At strong coupling, we expect a few operators to suffice in describing typical states. This is so because most of the operators except the relevant order parameters, conserved currents and the energy-momentum tensor will be expected to have large anomalous dimensions. This will be more generic in nonsupersymmetric theories [12,13], where we do not have chiral primary operators,[3] unless there are emergent symmetries at large N (as in Ref. [14]).

Thus, we can describe the nonequilibrium states holographically at strong coupling and large N using solutions that use Einstein's gravity coupled to a few gauge fields (dual to conserved currents) and scalar fields (dual to relevant order parameters). Here we will further specialize to states where conserved currents and order parameters vanish in the course of evolution. Thus, we will be able to describe these states using the hydrodynamic and nonhydrodynamic quasinormal modes of pure gravity along with their nonlinear evolution.

The holographic prescription for obtaining the spectral function in this class of states systematically in the derivative (hydrodynamic) and amplitude (nonhydrodynamic) expansions has been obtained before. In this paper, we will generalize this methodology to obtain the statistical function in the same expansions systematically. We will find that the nonequilibrium statistical function, like the nonequilibrium spectral function, individually carries a detailed imprint of the nonequilibrium state—in particular, detailed information about the dispersion relation of the relaxation modes and their nonlinear interactions. However, there is a simple, generalized nonequilibrium fluctuation-dissipation relation between the spectral and statistical functions, again set exactly by the final temperature of thermal equilibrium.

This nonequilibrium fluctuation-dissipation relation, being independent of the nonequilibrium state concerned, is universal and is as follows:

$$G_{\mathcal{K}}(\omega, \mathbf{k}, \mathbf{x}, t) = -i\left(n_{\rm BE}(\omega) + \frac{1}{2}\right)\mathcal{A}(\omega, \mathbf{k}, \mathbf{x}, t), \quad (6)$$

where $n_{\rm BE}(\omega)$ is set by the final equilibrium temperature. We note that the definitions of both the spectral and statistical functions involve the Wigner transform, so this relation is nonlocal in both space and time. However, we recover the equilibrium fluctuation-dissipation relation at long times when the spectral and statistical functions are independent of $\mathbf{x}$ and t. Also, this relation is strictly true in the absence of external sources, so conservation of energy implies that the final equilibrium temperature is set by the total energy of the initial state. We elaborate later in Sec. V that the relation is not in contradiction with the locality and causality of the underlying field theory.

Furthermore, at strong coupling, we expect the Kadanoff-Baym equations to have strong quantum memory; thus, a Boltzmann-like limit where the spectral and statistical functions will behave quasilocally is likely to be absent. Therefore, the nonequilibrium fluctuation-dissipation relation need not be specified by local data. What is surprising is that the only data required is the total conserved energy of the state in the large N and strong coupling limit.

This relation has been obtained in the probe-brane limit in Refs. [15,16].

Methodology in brief and organization of this paper: We describe briefly the methodology we will follow here along with the organization of this paper.

We start from nonequilibrium field theory in Sec. II. We review briefly the Schwinger-Keldysh formalism. We then obtain new parametrizations of the nonequilibrium fluctuation-dissipation relation which will be necessary for having consistent derivative/amplitude expansions of the spectral and statistical functions. We show that the parameters must satisfy additional field-theoretic constraints. Based on this nonequilibrium fluctuation-dissipation relation, we find that any nonequilibrium Green's function can be written as an appropriate weighted sum of the nonequilibrium retarded and advanced Green's functions. The weights are also subject to field-theoretic constraints. Additionally, we obtain new formal understanding of some important features of the effective action for nonequilibrium Green's functions.

In Sec. III, we review the construction of gravity duals of nonequilibrium states and introduce the derivative/amplitude expansions. We then study the dynamics of the scalar field dual to the bosonic order parameter in the same expansions. We find the unique boundary conditions for the nonequilibrium modes consistent with the derivative/amplitude expansions which give regularity at the horizon. The only new element here will be the study of conjugation properties of the boundary conditions necessary for later analysis of field-theoretic constraints.

In Sec. IV, we map boundary conditions of the nonequilibrium modes to the parametrization of nonequilibrium Green's functions as an appropriate sum of the nonequilibrium retarded and advanced Green's functions obtained in Sec. II.

This map is possible because of our previous work, in which the boundary conditions for the nonequilibrium modes which give solutions regular at the horizon have been identified and utilized to obtain the causal response function which gives the holographic nonequilibrium retarded Green's function [13]. This builds on the holographic prescription for the thermal retarded Green's function proposed by Son and Starinets [17]. We show that the advanced response function obtained from the regular nonequilibrium solution gives the holographic

[3]The chiral primary operators are special, because they are typically protected from receiving large quantum corrections to their anomalous dimensions.

nonequilibrium advanced Green's function, which passes field-theoretic consistency tests.

We then show that the sum of causal and advanced gravitational response functions evaluated using appropriate boundary conditions maps to the parametrization of the nonequilibrium Feynman propagator required for it to have consistent derivative/amplitude expansions. The boundary conditions which depart from regularity are responsible for the nonequilibrium shifts in the weights of the nonequilibrium retarded and advanced Green's functions in the weighted sum. We then study reality constraints on the parameters obtained from field theory and show that the only boundary conditions which are consistent are those which give regularity at the horizon. Thus, we find that there are no nonequilibrium shifts in the weights of the nonequilibrium retarded and advanced Green's functions in the weighted sum which gives the nonequilibrium Feynman propagator.

We also give intuitive arguments why regularity at the horizon in combination with linear response theory should determine all nonequilibrium propagators holographically.

In Sec. V, we obtain the nonequilibrium statistical function from the nonequilibrium Feynman propagator, and hence the nonequilibrium fluctuation-dissipation relation. We argue why its form is consistent with the locality and causality of the underlying field theory. We also give a conjecture on higher point nonequilibrium correlation functions in the absence of external sources.

In Sec. VI, we show that although our holographic prescriptions generalize to some extent in higher-derivative gravity, we cannot expect our results to hold beyond strong coupling even when the classical gravity approximation is valid. The main reason for this will be that the bulk scalar will not be minimally coupled to gravity generically.

In Sec. VII, we conclude with a summary of the key results and a discussion on how the holographic nonequilibrium fluctuation-dissipation result can be tested in heavy-ion collisions particularly.

The appendixes give relevant supporting material.

Before proceeding further, we would like to mention that an interesting formalism has been proposed by van Rees and Skenderis in Ref. [18] for doing real-time holography by construction of a gravitational analogue of the Schwinger-Keldysh closed time contour. Though this formalism seems ideally suited for doing thermal field theory in real time, it is very difficult to adapt this to nonequilibrium states. The main difficulty lies in the construction of smooth Euclidean caps for nonequilibrium geometries, which is necessary in this formalism to set nontrivial initial boundary conditions, i.e., to represent the creation of nontrivial states holographically.

Furthermore, there is an interesting approach [19] due to Son and Herzog for obtaining the thermal Schwinger-Keldysh propagators. It is difficult to generalize this to nonequilibrium context as well, as this will involve the construction of Penrose diagrams of nonequilibrium geometries. Indeed, it is not known how to do this for nonequilibrium geometries in perturbation theory.

However, elements of these approaches may be concretely realized in special nonequilibrium geometries. It will also be necessary to understand how to construct the nonequilibrium spectral and statistical functions holographically in the presence of external sources using systematic expansions. We leave such investigations to future work.

Notations: We will denote the position and momentum four-vectors without bold fonts, as for instance by x and k, respectively. The spatial position and momentum vectors will be denoted in bold fonts, as for instance by $\mathbf{x}$ and $\mathbf{k}$, respectively.

II. ASPECTS OF NONEQUILIBRIUM QUANTUM FIELD THEORY

In this section, we will first briefly review the Schwinger-Keldysh formalism of nonequilibrium field theory. In particular, we will define the effective action, which gives the equation of motion of the expectation value of operators and their Green's functions. Then we will show formally that this effective action contains no new information from the usual effective action, which is the Legendre transform of the generating functional of connected vacuum correlation functions. Finally, we will establish some new relations between the nonequilibrium Green's functions of bosonic Hermitian local operators.

A. The nonequilibrium propagators from an effective action

The basic tool of nonequilibrium field theory is the construction of an effective action, which is a functional of the operator and correlation functions. The evolution of the expectation value of the operator and the Green's functions in all states, including those which are away from equilibrium, can be obtained by extremizing the effective action $\Gamma[\mathcal{O}(x), G(x, y)]$. By construction, this effective action does not depend on any equilibrium or nonequilibrium variables like temperature, velocity or shear-stress tensor. These nonequilibrium variables parametrize the solutions, which can be obtained from definite initial conditions.[4]

The effective action $\Gamma[\mathcal{O}(x), G(x, y)]$ can be defined as the double Legendre transform of a generalization of the generating functional of vacuum correlation functions: $Z[J(x), K(x, y)]$. This generalized partition function

[4]Typically, we need infinite variables to parametrize nonequilibrium states, hence nonequilibrium expectation values of operators and their correlation functions. However, as we will see in the next section, we may expect that, at least in field theories admitting holographic gravity duals, there are nonequilibrium states which can be parametrized by hydrodynamic variables and shear-stress tensor alone.

includes not only a source $J(x)$, but also an arbitrary bilocal interaction term $K(x, y)$, and is defined as below:

$$\begin{aligned} Z[J, K] &= e^{iW[J,K]} \\ &= \int \mathcal{D}\Phi_s \exp\left[i\left(S[\Phi] + \int d^4x J(x) O(x) \right.\right. \\ &\quad \left.\left. + \frac{1}{2}\int d^4x d^4y O(x) K(x, y) O(y) \right)\right]. \end{aligned} \tag{7}$$

Above, Φ collectively denotes all the elementary fields. The operator O, which couples to J locally and K bilocally, is a polynomial of these elementary fields Φ and their derivatives.

We then define the expectation value of the operator $\mathcal{O}(x)$ and its Green's function $G(x, y)$ through the following:

$$\begin{aligned} \frac{\delta W[J, K]}{\delta J(x)} &= \mathcal{O}(x), \\ \frac{\delta W[J, K]}{\delta K(x, y)} &= \frac{1}{2}(\mathcal{O}(x)\mathcal{O}(y) + G(x, y)). \end{aligned} \tag{8}$$

Eliminating $J(x)$ and $K(x, y)$ in favor of $\mathcal{O}(x)$ and $G(x, y)$, we can now perform a double Legendre transform to define the effective action as below:

$$\begin{aligned} \Gamma[\mathcal{O}(x), G(x, y)] = W[J, K] &- \int d^4x J(x)\mathcal{O}(x) \\ &- \frac{1}{2}\int d^4x d^4y K(x, y)(\mathcal{O}(x)\mathcal{O}(y) \\ &+ G(x, y)). \end{aligned} \tag{9}$$

Clearly,

$$\begin{aligned} \frac{\delta\Gamma[\mathcal{O}, G]}{\delta\mathcal{O}(x)} &= -J(x) - \int d^4y K(x, y)\mathcal{O}(y), \\ \frac{\delta\Gamma[O, G]}{\delta G(x, y)} &= -\frac{1}{2}K(x, y). \end{aligned} \tag{10}$$

Therefore, in the absence of the sources $J(x)$ and $K(x, y)$, extremizing the generalized effective action $\Gamma[\mathcal{O}(x), G(x, y)]$ gives the dynamics of both the expectation value of the operator and their Green's functions.

There is one important point in the above construction which we mention now. As stated in the Introduction, for nonequilibrium states we need to know separately the spectral function (the commutator/anticommutator) and the statistical function (the anticommutator/commutator) for both bosonic/fermionic operators.

To get complete information about the nonequilibrium Green's functions, we need to construct the partition function $W[J(x), K(x, y)]$, and consequently the effective action $\Gamma[\mathcal{O}(x), G(x, y)]$, over the so-called Schwinger-Keldysh closed real time contour $\mathcal{C}_s$, as shown in Fig. 1. This time contour travels from $-\infty$ to ∞ and then back from ∞ to $-\infty$, thus transversing the entire real line first forward and then backward.

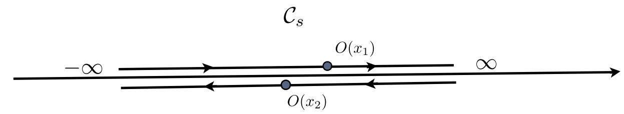

FIG. 1 (color online). The Schwinger-Keldysh closed time contour $\mathcal{C}_s$. The forward- and backward-directed parts of the contour have been displaced slightly above and below the real axis, respectively, just to distinguish them clearly. The operator insertions can be in either the forward or backward leg of the contour.

In fact, the full closed-time-contour ordered Green's function $G_{\mathcal{C}_s}(x, y)$ can be written as a combination of the commutator and the anticommutator, and hence as a sum over the spectral function and the statistical function as below:

$$G_{\mathcal{C}_s}(x, y) = G_{\mathcal{K}}(x, y) - \frac{i}{2}\mathcal{A}(x, y), \mathrm{sign}_{\mathcal{C}_s}(x^0 - y^0), \tag{11}$$

where $\mathrm{sign}_{\mathcal{C}_s}(x^0 - y^0)$ is defined according to the contour ordering over the full closed time contour $\mathcal{C}_s$.[5]

The usefulness of the effective action $\Gamma[\mathcal{O}(x), G(x, y)]$ is realized only when we can reformulate it diagrammatically. This has been done first in Ref. [2] via a 2PI effective action when the operator O is an elementary field. A full discussion of this is outside the scope of the present paper. We do mention, however, that though there is a diagrammatic formulation, this is nonperturbative, as the lines of the diagrams are the full propagator $G(x, y)$. Therefore, the effective action $\Gamma[\mathcal{O}(x), G(x, y)]$ typically admits no perturbative expansion in a small parameter.

Despite this drawback, in happy circumstances, we do get good approximations to experimental results or simulations, particularly when the coupling is weak and we can have a good intuition about the relevant diagrams to sum over in the effective action [6]. We need to resum over infinite diagrams to remove infrared divergences. There is no systematic understanding about how we can improve a given approximation while removing the infrared divergences in order to better the description. Nevertheless, this is so far the best field-theoretic tool at hand which gives a fundamental description of a physical process in an approximation that is uniform in time, i.e., which not only works at the initial time but also far in the future. Furthermore, this also reproduces quantum corrections to the Boltzmann equation.

[5]When both the operator insertions are on the upper contour, we get the Feynman propagator, for instance. In general, both the commutator and anticommutator are needed to get the propagator over the full contour. Thus, the contour is convenient for obtaining the independent dynamics of the nonequilibrium spectral and statistical functions.

B. Redundancy of information in the effective action

We will demonstrate here that the effective action $\Gamma[\mathcal{O}(x), G(x, y)]$ formally does not contain any more information than the usual effective action $\Gamma[\mathcal{O}(x)]$. We recall that $\Gamma[\mathcal{O}(x), G(x, y)]$ is obtained from $Z[J(x), K(x, y)]$ by definition as in Eqs. (8) and (9), while $\Gamma[\mathcal{O}(x)]$ is obtained from the usual partition function $Z[J(x)]$, the generating functional of vacuum correlation functions. $\Gamma[\mathcal{O}(x)]$ is the usual one-particle irreducible (1PI) effective action when $O(x)$ is an elementary field. The effective action $\Gamma[O(x), G(x, y)]$ does not admit any perturbative expansion in a small parameter, but $\Gamma[O(x)]$ does.

The effective action $\Gamma[\mathcal{O}(x), G(x, y)]$ is necessary to obtain a description which is a uniform approximation in time, but our current understanding of such approximations seems to be inadequate. It is not known how to have an approximation scheme which simultaneously takes care of unitarity in a controlled manner, is renormalizable by standard counterterms and satisfies all Ward identities (for recent progress, see Ref. [4]). Therefore, it is useful to have an argument for why, in principle, all the information is there in the usual effective action $\Gamma[\mathcal{O}(x)]$, for which such issues have been resolved. This will be useful for us later.

To show this, it is sufficient to prove that $W[J(x), K(x, y)]$ can be constructed from $W[J(x)]$, as $\Gamma[\mathcal{O}(x), G(x, y)]$ and $\Gamma[\mathcal{O}(x)]$ are obtained from $W[J(x), K(x, y)]$ and $W[J(x)]$, respectively, via Legendre transforms. From Eq. (8), we readily observe that

$$\frac{\delta W[J, K]}{\delta K(x, y)} = \frac{1}{2}\left(\frac{\delta W[J, K]}{\delta J(x)}\frac{\delta W[J, K]}{\delta J(y)} - \frac{\delta^2 W[J, K]}{\delta J(x)\delta J(y)}\right), \quad (12)$$

where we have used $\mathcal{O}(x) = \delta W[J, K]/\delta J(x)$ and $G(x, y) = \delta \mathcal{O}(x)/\delta J(y)$. Thus, all functional K- derivatives of $W[J, K]$ can be converted to functional J- derivatives of $W[J, K]$. Therefore, $W[J, K]$ can be constructed readily from $W[J, K = 0] = W[J]$. This is also expected, because a quantum field theory is indeed uniquely defined once we know all connected vacuum correlation functions, hence $W[J]$.

C. New relations between nonequilibrium Green's functions

We will now investigate the general structure of nonequilibrium Green's functions of local bosonic operators in nonequilibrium states. Here the operators will be taken in the Heisenberg representation.

In particular, we will establish some useful relations between the retarded propagator and the Feynman propagator, based on the assumption that the nonequilibrium parts of both admit systematic perturbative expansions.

The definition of the Feynman propagator is

$$G_F(x_1, x_2) = -i\langle T(O(x_1)O(x_2))\rangle, \quad (13)$$

where T denotes time ordering.

We begin with the known relation (for proof, see Appendix A)[6]

$$G_F(x_1, x_2) = G_{\mathcal{K}}(x_1, x_2) - \frac{i}{2}\mathcal{A}(x_1, x_2)\mathrm{sign}(t_1 - t_2). \quad (14)$$

Note that this is just a special case of Eq. (11) when the operator insertions are in the forward part of the closed time contour.

However,

$$\begin{aligned}-i\mathcal{A}(x_1, x_2)\mathrm{sign}(t_1 - t_2) &= -i\mathcal{A}(x_1, x_2)\Theta(t_1 - t_2)\\ &\quad + i\mathcal{A}(x_1, x_2)\Theta(t_2 - t_1)\\ &= G_R(x_1, x_2) + G_A(x_1, x_2). \quad (15)\end{aligned}$$

Therefore,

$$G_F(x_1, x_2) = G_{\mathcal{K}}(x_1, x_2) + \frac{1}{2}(G_R(x_1, x_2) + G_A(x_1, x_2)), \quad (16)$$

or in the Wigner-transformed form,

$$G_F(k, x) = G_{\mathcal{K}}(k, x) + \mathrm{Re}G_R(k, x), \quad (17)$$

using $G_R * (k, x) = G_A(k, x)$, which has been proved in Appendix A.

We further note that $G_{\mathcal{K}}(k, x)$ is purely imaginary, which follows from its definition as proved in Appendix A. Therefore, we conclude from Eq. (17) that

$$\mathrm{Re}G_F(k, x) = \mathrm{Re}G_R(k, x), \qquad \mathrm{Im}G_F(k, x) = -iG_{\mathcal{K}}(k, x). \quad (18)$$

It is to be noted that unlike the retarded Green's function $G_R(k, x)$ or the advanced Green's function $G_A(k, x)$, the Feynman propagator is not analytic in ω in either the upper half plane or the lower half plane. This can be seen from the instance of the vacuum Feynman propagator itself, in which positive frequency modes are propagated in the future and negative frequency modes are propagated in the past; i.e., the vacuum Feynman propagator is $\Theta(\omega)G_R(k) - \Theta(-\omega)G_A(k)$. The term $\Theta(\omega)G_R(k)$ implies lack of analyticity in ω in the lower half plane, as the positive frequency poles are shifted there with the standard $i\epsilon$ prescription, and similarly $\Theta(-\omega)G_A(k)$ implies there is no analyticity in the upper half plane, as the negative frequency poles are shifted there.

[6]From now on, we will denote the commutator as $\mathcal{A}(x_1, x_2)$, where it is implicitly implied that the inverse Wigner transform has been done on the spectral function $\mathcal{A}(k, x)$. Similarly, we will denote $-i/2$ times the anticommutator as $G_{\mathcal{K}}(x_1, x_2)$.

Generally speaking, if a function is analytic in a variable in the upper half plane or lower half plane, with other variables held fixed, it can be fully constructed from its imaginary part or its real part. For instance, as $\mathrm{Im}G_R(k,x) = -(1/2)\mathcal{A}(k,x)$ (for proof, see Appendix A), and as it is analytic in the upper half plane in ω,[7] the retarded Green's function can be constructed from the spectral function using

$$G_R(\omega, \mathbf{k}, \mathbf{x}, t) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\mathcal{A}(\omega', \mathbf{k}, \mathbf{x}, t)}{\omega - \omega' + i\epsilon}. \qquad (19)$$

In the case of the Feynman propagator, its real part $\mathrm{Re}G_R(k,x)$ and its imaginary part $G_{\mathcal{K}}(k,x)$ hold independent information in nonequilibrium states; one cannot be kinematically constructed from the other because of nonanalyticity in ω in the complex upper half plane or the lower half plane.

We also note that as the Feynman propagator is symmetric, i.e., $G_F(x_1,x_2) = G_F(x_2,x_1)$, it follows that after Wigner transform, $G_F(k,x) = G_F(-k,x)$, as proved in Appendix A.

The thermal Feynman propagator is a linear combination of the retarded or causal response with weight $n_{\mathrm{BE}}(\omega)+1$ and the advanced or anticausal response with weight $-n_{\mathrm{BE}}(\omega)$ for a given frequency, with $n_{\mathrm{BE}}(\omega) = \frac{1}{e^{\frac{\omega}{T}}-1}$ being the Bose-Einstein distribution at temperature T. Thus,

$$G_F(k) = (n_{\mathrm{BE}}(\omega)+1)G_R(k) - n_{\mathrm{BE}}(\omega)G_A(k). \qquad (20)$$

As $T \to 0$,

$$n_{\mathrm{BE}}(\omega)+1 \to \Theta(\omega), \qquad -n_{\mathrm{BE}}(\omega) \to \Theta(-\omega), \qquad (21)$$

so we recover the vacuum Feynman propagator in this limit.

It is not hard to see that Eq. (20) satisfies $G_F(k) = G_F(-k)$, because

$$\begin{gathered} n_{\mathrm{BE}}(-\omega)+1 = -n_{\mathrm{BE}}(\omega) \quad \text{or} \\ -n_{\mathrm{BE}}(-\omega) = n_{\mathrm{BE}}(\omega)+1, \end{gathered} \qquad (22)$$

and also $G_A(-\omega,-\mathbf{k}) = G_R(\omega,\mathbf{k})$.

We can rewrite the thermal Feynman propagator [Eq. (20)] as

$$G_F(k) = \mathrm{Re}G_R(k) + i(2n_{\mathrm{BE}}(\omega)+1)\mathrm{Im}G_R(k). \qquad (23)$$

So, comparing Eq. (17) with the above, we get

$$G_{\mathcal{K}}(k) = i(2n_{\mathrm{BE}}(\omega)+1)\mathrm{Im}G_R(k). \qquad (24)$$

However, $\mathrm{Im}G_R(k) = -(1/2)\mathcal{A}(k)$. So,

$$G_{\mathcal{K}}(k) = -i\left(n_{\mathrm{BE}}(\omega)+\frac{1}{2}\right)\mathcal{A}(k). \qquad (25)$$

The above is the fluctuation-dissipation relation in thermal equilibrium. It is so called because it implies the fluctuation-dissipation theorems [6], like those relating the thermal-electrical noise (current fluctuations) to the electric resistance.

We will now try to parameterize the nonequilibrium fluctuation-dissipation relation. Using this, we will get a parameterization of nonequilibrium Green's functions.

Let us assume we are near equilibrium. Then the nonequilibrium state can be parametrized in terms of hydrodynamic and relaxational modes. This will allow us to do a perturbative expansion. Let us, for instance, linearize the state in terms of hydrodynamic fluctuations $\delta\mathbf{u}$ and δT. In this approximation, both the spectral and the statistical functions will depend linearly on these variables. Thus, the nonequilibrium fluctuation-dissipation relation will be linear in this approximation.

The exact nature of the dependence of the spectral and statistical functions on the hydrodynamic and nonhydrodynamic variables will be determined later in the holographic classical gravity approximation. As of now, we will take this dependence to be implicit.

It is also clear that we can improve the linear approximation by including quadratic dependence of the spectral and statistical functions on the hydrodynamic and nonhydrodynamic variables, which we will do systematically later. The general structure will be as follows:

$$\begin{aligned} G_{\mathcal{K}}(k,x) &= G_{\mathcal{K}}^{(\mathrm{eq})}(k) + G_{\mathcal{K}}^{(\mathrm{neq})}(k,x), \\ \mathcal{A}(k,x) &= \mathcal{A}^{(\mathrm{eq})}(k) + \mathcal{A}^{(\mathrm{neq})}(k,x), \\ G_{\mathcal{K}}^{(\mathrm{neq})}(k,x) &= G_{\mathcal{K}}^{(1)}(k,x) + G_{\mathcal{K}}^{(2)}(k,x) + \cdots + G_{\mathcal{K}}^{(\mathrm{n})}(k,x) + \cdots, \\ \mathcal{A}^{(\mathrm{neq})}(k,x) &= \mathcal{A}^{(1)}(k,x) + \mathcal{A}^{(2)}(k,x) + \cdots + \mathcal{A}^{(\mathrm{n})}(k,x) + \cdots. \end{aligned} \qquad (26)$$

Above, the superscript (n) denotes the order of multilinear dependence of the term on the nonequilibrium variables.

[7] Analyticity of $G_R(k,x)$ in the lower half ω plane is guaranteed by the $\Theta(t_1-t_2)$ term in $G_R(x_1,x_2)$.

The nonequilibrium fluctuation-dissipation relation can be parametrized as follows:

$$\begin{aligned} G_{\mathcal{K}}^{(\mathrm{eq})}(k) &= -i\left(n_{\mathrm{BE}}(\omega) + \frac{1}{2}\right)\mathcal{A}^{(\mathrm{eq})}(k), \\ G_{\mathcal{K}}^{(1)}(k,x) &= -i\int d^4k_1 f^{(1,0)}(k,k_1,x)\mathcal{A}^{(\mathrm{eq})}(k_1) \\ &\quad - i\left(n_{\mathrm{BE}}(\omega) + \frac{1}{2}\right)\mathcal{A}^{(1)}(k,x), \ldots . \end{aligned} \tag{27}$$

Above, $f^{(1,0)}(k,k_1,x)$ is real because each term in the expansion of $\mathcal{A}(k,x)$ is real, while each term in the $G_{\mathcal{K}}(k,x)$ expansion is purely imaginary. Also, the Bose-Einstein distribution $n_{\mathrm{BE}}(\omega)$ appearing above is always determined by the final equilibrium temperature. Furthermore, as we have shown in Appendix A, because $G_{\mathcal{K}}(x_1,x_2)$ is symmetric in x_1 and x_2, after Wigner transform it should satisfy $G_{\mathcal{K}}(k,x) = G_{\mathcal{K}}(-k,x)$. Similarly, as shown in Appendix A, $\mathcal{A}(k,x) = -\mathcal{A}(-k,x)$. Therefore, these imply that

$$\begin{aligned} &f^{(1,0)}(k,k_1,x) = f_{\mathrm{S}}^{(1,0)}(k,k_1,x) + f_{\mathrm{A}}^{(1,0)}(k,k_1,x), \quad \text{such that } f_{\mathrm{S}}^{(1,0)}(k,k_1,x) \text{ and } f_{\mathrm{A}}^{(1,0)}(k,k_1,x) \text{ are real, and} \\ &f_{\mathrm{S}}^{(1,0)}(k,k_1,x) = f_{\mathrm{S}}^{(1,0)}(-k,k_1,x), \qquad f_{\mathrm{A}}^{(1,0)}(k,k_1,x) = -f_{\mathrm{A}}^{(1,0)}(-k,-k_1,x). \end{aligned} \tag{28}$$

Similarly, at the next order,

$$\begin{aligned} G_{\mathcal{K}}^{(2)}(k,x) &= -i\int d^4k_1 d^4k_2 f^{(2,0,0)}(k,k_1,k_2,x)\mathcal{A}^{(\mathrm{eq})}(k_1)\mathcal{A}^{(\mathrm{eq})}(k_2) \\ &\quad - i\int d^4k_1 d^4k_2 d^4x_1 f^{(2,1,0)}(k,k_1,k_2,x,x_1)\mathcal{A}^{(1)}(k_1,x_1)\mathcal{A}^{(\mathrm{eq})}(k_2) \\ &\quad - i\int d^4k_1 d^4x_1 f^{(2,1)}(k,k_1,x,x_1)\mathcal{A}^{(1)}(k_1,x_1) - i\left(n_{\mathrm{BE}}(\omega)+\frac{1}{2}\right)\mathcal{A}^{(2)}(k,x), \text{ etc.} \end{aligned} \tag{29}$$

Again, by definition $f^{(2,0,0)}(k,k_1,k_2,x)$, $f^{(2,1,0)}(k,k_1,k_2,x,x_1)$ and $f^{(2,1)}(k,k_1,x,x_1)$ have to be real, and the Bose-Einstein distribution $n_{\mathrm{BE}}(\omega)$ appearing above is always determined by the final equilibrium temperature. Similarly, the symmetry of the statistical function and the antisymmmetry of the spectral function in x_1 and x_2 prior to Wigner transform give further restrictions on $f^{(2,0,0)}(k,k_1,k_2,x)$, $f^{(2,1,0)}(k,k_1,k_2,x,x_1)$ and $f^{(2,1)}(k,k_1,x,x_1)$ as in Eq. (28). The last two terms of Eq. (29) are linear, while the remaining terms are nonlinear corrections to the equilibrium fluctuation-dissipation relation. Clearly, nonlinear corrections are expected, because the Kadanoff-Baym equations which give equations of motion of the spectral and statistical functions are nonlinear. We easily note that as $G_{\mathcal{K}}^{(2)}$ and $\mathcal{A}^{(2)}$ are quadratic in nonequilibrium variables while $G_{\mathcal{K}}^{(1)}$ and $\mathcal{A}^{(1)}$ are linear, $f^{(2,0,0)}$ depends quadratically while $f^{(2,1,0)}$ and $f^{(2,1)}$ depend linearly on nonequilibrium variables.

We will find the above parametrization of the nonequilibrium fluctuation-dissipation relation useful in determining it holographically. It can be also useful for experimental determination.

It then follows from Eqs. (18) and (27) that the general structure of the nonequilibrium Feynman propagator for small perturbations away from equilibrium should be

$$\begin{aligned} G_F(k,x) &= G_F^{(\mathrm{eq})}(k) + G_F^{(\mathrm{neq})}(k,x), \qquad G_F^{(\mathrm{neq})}(k,x) = G_F^{(1)}(k,x) + G_F^{(2)}(k,x) + \cdots + G_F^{(\mathrm{n})}(k,x) + \cdots, \\ G_F^{(\mathrm{eq})}(k) &= (n_{\mathrm{BE}}(\omega)+1)G_R^{(\mathrm{eq})}(k) - n_{\mathrm{BE}}(\omega)G_A^{(\mathrm{eq})}(k), \\ G_F^{(1)}(k,x) &= iG_{\mathcal{K}}^{(1)}(k,x) + \mathrm{Re}G_R^{(1)}(k,x), \\ &= \int d^4k_1 f^{(1,0)}(k,k_1,x)(G_R^{(\mathrm{eq})}(k_1) - G_A^{(\mathrm{eq})}(k_1)) + (n_{\mathrm{BE}}(\omega)+1)G_R^{(1)}(k,x) \\ &\quad - n_{\mathrm{BE}}(\omega)G_A^{(1)}(k,x). \end{aligned} \tag{30}$$

We have also used $\mathcal{A}(k,x) = -2\,\mathrm{Im}G_R(k,x)$ and $G_R * (k,x) = G_A(k,x)$ above. Similarly, from Eq. (29), we can derive that

$$
\begin{aligned}
G_F^{(2)}(k,x) = &-\int d^4k_1 d^4k_2 f^{(2,0,0)}(k,k_1,k_2,x)(G_R^{(\mathrm{eq})}(k_1)-G_A^{(\mathrm{eq})}(k_1))(G_R^{(\mathrm{eq})}(k_2)-G_A^{(\mathrm{eq})}(k_2)) \\
&-\int d^4k_1 d^4k_2 d^4x_1 f^{(2,1,0)}(k,k_1,k_2,x,x_1)(G_R^{(1)}(k_1,x_1)-G_A^{(1)}(k_1,x_1))(G_R^{(\mathrm{eq})}(k_2)-G_A^{(\mathrm{eq})}(k_2)) \\
&+\int d^4k_1 d^4x_1 f^{(2,1)}(k,k_1,x,x_1)(G_R^{(1)}(k_1,x_1)-G_A^{(1)}(k_1,x_1))+(n_{\mathrm{BE}}(\omega)+1)G_R^{(2)}(k,x)-n_{\mathrm{BE}}(\omega)G_A^{(2)}(k,x), \text{ etc.}
\end{aligned}
\tag{31}
$$

We note that with the explicit form of the fluctuation-dissipation relation in Eq. (27), we can write any nonequilibrium Green's function as a functional of the nonequilibrium retarded Green's function. As the Wigner-transformed advanced Green's function can be obtained from the Wigner-transformed retarded Green's function simply by complex conjugation, an alternative representation is a weighted sum of the nonequilibrium retarded and advanced Green's functions, where the weights are specified by the nonequilibrium fluctuation-dissipation relation.

Thus, any nonequilibrium Green's function $G(k,x)$ should have the following structure in the linearized approximation:

$$
\begin{aligned}
&G(k,x) = G^{(\mathrm{eq})}(k)+G^{(\mathrm{neq})}(k,x), \qquad G^{(\mathrm{neq})}(k,x)=G^{(1)}(k,x)+G^{(2)}(k,x)+\cdots+G^{(\mathrm{n})}(k,x)+\cdots, \\
&G^{(\mathrm{eq})}(k) = f_R^{(\mathrm{eq})}(\omega)G_R^{(\mathrm{eq})}(k)+f_A^{(\mathrm{eq})}(k)G_A^{(\mathrm{eq})}(k), \\
&G^{(1)}(k,x) = \int d^4k_1(f_R^{(1,0)}(k,k_1,x)G_R^{(\mathrm{eq})}(k_1)+f_A^{(1,0)}(k,k_1,x)G_A^{(\mathrm{eq})}(k_1))+f_R^{(\mathrm{eq})}(\omega)G_R^{(1)}(k,x)+f_A^{(1)}(k)G_A^{(1)}(k,x),
\end{aligned}
\tag{32}
$$

where $f_{R,A}^{(\mathrm{eq})}$ should be determined by the equilibrium fluctuation-dissipation relation and $f_{R,A}^{(1,0)}$ should be determined by $f^{(1,0)}$ in the nonequilibrium fluctuation-dissipation relation [Eq. (27)]. Similarly,

$$
\begin{aligned}
G^{(2)}(k,x) = &\int d^4k_1 d^4k_2(f_{RR}^{(2,0,0)}(k,k_1,k_2,x)G_R^{(\mathrm{eq})}(k_1)G_R^{(\mathrm{eq})}(k_2)+f_{RA}^{(2,0,0)}(k,k_1,k_2,x)G_R^{(\mathrm{eq})}(k_1)G_A^{(\mathrm{eq})}(k_2) \\
&+f_{AR}^{(2,0,0)}(k,k_1,k_2,x)G_A^{(\mathrm{eq})}(k_1)G_R^{(\mathrm{eq})}(k_2)+f_{AA}^{(2,0,0)}(k,k_1,k_2,x)G_A^{(\mathrm{eq})}(k_1)G_A^{(\mathrm{eq})}(k_2)) \\
&+\int d^4k_1 d^4k_2 d^4x_1(f_{RR}^{(2,1,0)}(k,k_1,k_2,x,x_1)G_R^{(1)}(k_1,x_1)G_R^{(\mathrm{eq})}(k_2)+f_{RA}^{(2,1,0)}(k,k_1,k_2,x,x_1)G_R^{(1)}(k_1,x_1)G_A^{(\mathrm{eq})}(k_2) \\
&+f_{AR}^{(2,1,0)}(k,k_1,k_2,x,x_1)G_A^{(1)}(k_1,x_1)G_R^{(\mathrm{eq})}(k_2)+f_{AA}^{(2,1,0)}(k,k_1,k_2,x,x_1)G_A^{(1)}(k_1,x_1)G_A^{(\mathrm{eq})}(k_2)) \\
&+\int d^4k_1 d^4x_1(f_R^{(2,1,0)}(k,k_1,x,x_1)G_R^{(1)}(k_1,x_1)+f_A^{(2,1,0)}(k,k_1,x,x_1)G_A^{(1)}(k_1,x_1))+f_R^{(\mathrm{eq})}(\omega)G_R^{(2)}(k,x) \\
&+f_A^{(\mathrm{eq})}(k)G_A^{(2)}(k,x)), \text{ etc.}
\end{aligned}
\tag{33}
$$

Above, "$f_{RR}^{(2,0,0)}$, etc." should be determined from "$f^{(2,0,0)}$, etc." in Eq. (29). We will find the above form of the nonequilibrium Green's functions useful for their holographic determination.

III. NONEQUILIBRIUM GEOMETRIES AND THE REAL SCALAR FIELD

In this section, we will first review the geometries which capture basic nonequilibrium processes like hydrodynamics and relaxation in the dual theory. Then we will study how the solution of a free massive real scalar field can be perturbatively obtained in these geometries. This real scalar field will be dual to a condensate—for instance, the chiral condensate of QCD. Finally, we will determine which solutions of the bulk scalar field are regular at the horizon. We will show that the nonequilibrium corrections to the equilibrium solutions are unique and determined by boundary conditions at the horizon which can be stated in a background independent manner, i.e., independently of the details of the dual nonequilibrium state, in the regime of validity of perturbation theory.

Most of this section is a review of the results of our previous works [13,20,21]. The only new element that we introduce is a careful study of complex conjugation properties of various terms, leading to the background metric and solution of the bulk scalar field being real.

A. Holographic description of nonequilibrium states

The solutions of Einstein's vacuum gravity with a negative cosmological constant, which settle down to black holes with regular future horizons, holographically depict many fundamental nonequilibrium processes. Each such solution maps to a nonequilibrium state via gauge/gravity duality at strong coupling and large N. Each such nonequilibrium state can be characterized by the expectation value of the energy-momentum tensor alone, because this completely determines the dual solution in gravity, which is a regular perturbation of anti–de Sitter black brane [22].

Conceptually, these states are similar to certain special solutions of the Boltzmann equation called conservative solutions, which are completely determined by the energy-momentum tensor [20]. This is only a conceptual similarity, because the Boltzmann equation is valid only in the weak coupling regime. Nevertheless, it is seen that one can systematically construct phenomenological equations for the energy-momentum tensor, and any solution of these phenomenological equations can be lifted to a full solution of the Boltzmann equation. Thus, such conservative solutions give a systematic approach to obtaining the phenomenology of irreversible processes in terms of energy-momentum tensor alone. At long times, such solutions can be approximated by purely hydrodynamic solutions of the Boltzmann equation, known as normal solutions [23] in literature. These normal solutions are thus special conservative solutions and allow us to compute linear and nonlinear hydrodynamic transport coefficients at weak coupling and dilute densities when the Boltzmann approximation is valid.[8]

In the case of gravity, one can also systematically construct phenomenological equations involving the energy-momentum tensor alone, which when satisfied implies that the corresponding solution in gravity should have a regular future horizon [20,21,25]. Just like in the case of the Boltzmann equation, these phenomenological equations can be expanded perturbatively in the derivative (hydrodynamic) and amplitude (nonhydrodynamic) expansion parameters as reviewed in Appendix B. These equations have the most general structure valid in any field theory when the quantum corrections to the Boltzmann equation are considered. Also, the values of phenomenological parameters are different at strong coupling.

At long times, all regular perturbations of anti–de Sitter black branes become purely hydrodynamic. In such cases, the metric can be constructed by the now well-known fluid/gravity correspondence [9,26–28].

This hydrodynamic limit is also a natural consequence of the phenomenological equations which admit purely hydrodynamic solutions [20,21,25]. The phenomenological equations, however, also describe other nonequilibrium processes like decoherence and relaxation.

Another special class of solutions to these phenomenological equations describes homogeneous relaxation [21]. In this case, it has been explicitly proved that the future horizon in corresponding solutions in gravity is indeed regular, as outlined in Appendix B.

In this paper, for illustrative purposes we will use two basic examples. The first example is that of the five-dimensional anti–de Sitter black brane perturbed by a hydrodynamic shear mode. The corresponding metric is[9]

$$\begin{aligned} ds^2 = {}& \frac{l^2}{r^2}\frac{dr^2}{f(\frac{rr_0}{l^2})} + \frac{l^2}{r^2}\left(-f\left(\frac{rr_0}{l^2}\right)dt^2 + dx^2 + dy^2 + dz^2\right) \\ & - \frac{2l^2}{r^2}\left(1 - f\left(\frac{rr_0}{l^2}\right)\right)\delta u_i(\mathbf{k}_{(\mathrm{h})})e^{i\mathbf{k}_{(\mathrm{h})}\cdot\mathbf{x}}e^{-\frac{\mathbf{k}^2_{(\mathrm{h})}}{4\pi T}t}dtdx^i \\ & + \frac{2l^2}{r^2}\left(-i\frac{l^2}{4r_0}k_{(\mathrm{h})i}\delta u_j(\mathbf{k}_{(\mathrm{h})})e^{i\mathbf{k}_{(\mathrm{h})}\cdot\mathbf{x}}e^{-\frac{\mathbf{k}^2_{(\mathrm{h})}}{4\pi T}t}h\right. \\ & \left.\times\left(\frac{rr_0}{l^2}\right)dx^i dx^j\right) + O(\epsilon^2), \end{aligned} \tag{34}$$

where

$$f(s) = 1 - s^4, \qquad h(s) = -\ln(1 - s^4). \tag{35}$$

The first line in Eq. (34) corresponds to the AdS_5 black brane at temperature $T = r_0/(\pi l^2)$, with l being the asymptotic radius of curvature and l^2/r_0 being the radius of the horizon. The second and third lines describe the hydrodynamic perturbations. Here ϵ is the derivative expansion parameter, which is $\mathbf{k}_{(\mathrm{h})}/T$.

The corresponding energy-momentum tensor is

$$\begin{aligned} t_{\mu\nu} = {}& \mathrm{diag}(\epsilon, p, p, p) - i\eta(k_{(\mathrm{h})i}\delta\mathbf{u}(\mathbf{k}_{(\mathrm{h})})_j \\ & + k_{(\mathrm{h})j}\delta\mathbf{u}(\mathbf{k}_{(\mathrm{h})})_i)e^{i\mathbf{k}_{(\mathrm{h})}\cdot\mathbf{x}}e^{-\frac{\mathbf{k}^2_{(\mathrm{h})}}{4\pi T}t} + O(\epsilon^2), \end{aligned} \tag{36}$$

where

$$\epsilon = 3p = \frac{3r_0^4}{2\kappa^2 l^5}, \qquad \eta = \frac{r_0^3}{2\kappa^2 l^3}. \tag{37}$$

Above, ϵ and p are the thermal energy density and pressure, and η is the shear viscosity satisfying $\eta/s = 1/4\pi$ (applying standard thermodynamic relations to determine the entropy density s). Also, κ^2 is given by $\kappa^2 = (8\pi G_N)^{-1}$, where G_N is the five-dimensional Newton's constant.

The conservation of the energy-momentum tensor is required for Eq. (34) to be a solution of Einstein's vacuum equations. In fact, this follows from the vector constraint of Einstein's equations. The conservation of the energy-momentum tensor yields the Navier-Stokes equation. The latter implies that the velocity perturbation $\delta\mathbf{u}(\mathbf{k}_{(\mathrm{h})})$ is transverse, i.e.,

$$\delta\mathbf{u}(\mathbf{k}_{(\mathrm{h})})\cdot\mathbf{k}_{(\mathrm{h})} = 0. \tag{38}$$

The above procedure can be extended nonlinearly to yield higher-derivative [27,28] and nonlinear corrections [28] to the Navier-Stokes equations.

The second example will be that of homogeneous relaxation. The energy-momentum tensor for homogeneous relaxation takes the following form:

[8]Indeed, such an approach is also valid for non-Abelian gauge theories, like quantum chromodynamics (QCD) at temperatures higher then the confining scale Λ_{QCD} and low baryon densities [24].

[9]One can readily construct this metric using the methods of Ref. [22].

$$t_{\mu\nu} = \mathrm{diag}(\epsilon, p, p, p) + \pi_{ij}^{(\mathrm{nh})}(t). \tag{39}$$

It is easy to see that the above energy-momentum tensor is conserved for any $\pi_{ij}^{(\mathrm{nh})}(t)$. In the case of a conformal field theory, we further require that

$$\pi_{ij}^{(\mathrm{nh})}(t)\delta_{ij} = 0. \tag{40}$$

The above is also implied by the scalar constraint of Einstein's equations.

It is convenient to do a Fourier transform over time to define $\pi_{ij}^{(\mathrm{nh})}(\omega_{(\mathrm{nh})})$. The corresponding metric up to linear order in $\pi_{ij}^{(\mathrm{nh})}(\omega_{(\mathrm{nh})})$ is [21]

$$\begin{aligned} ds^2 = {} & \frac{l^2}{r^2}\frac{dr^2}{f(\frac{rr_0}{l^2})} + \frac{l^2}{r^2}\left(-f\left(\frac{rr_0}{l^2}\right)dt^2 + dx^2 + dy^2\right) \\ & + \frac{2l^2}{r^2}\left(\pi_{ij}^{(\mathrm{nh})}(\omega_{(\mathrm{nh})})\tilde{h}\left(\frac{rr_0}{l^2}, \omega_{(\mathrm{nh})}\right)dx^i dx^j\right) + O(\delta^2), \end{aligned} \tag{41}$$

with δ being the nonhydrodynamic amplitude expansion parameter, $\pi_{ij}^{(\mathrm{nh})}/p$. Furthermore, $\tilde{h}(s, \omega_{(\mathrm{nh})})$ follows the equation of motion:

$$\begin{aligned} & \frac{d^2\tilde{h}(s, \omega_{(\mathrm{nh})})}{ds^2} - \frac{(2 + (1 + 3\frac{r_*^4}{r_0^4})s^3 - 6\frac{r_*^4}{r_0^4}s^4)}{sf(s)}\frac{d\tilde{h}(s, \omega_{(\mathrm{nh})})}{ds} \\ & + \frac{\omega_{(\mathrm{nh})}^2 l^4}{r_0^2}\left(\frac{1}{f^2(s)}\right)\tilde{h}(s, \omega_{(\mathrm{nh})}) = 0, \end{aligned} \tag{42}$$

such that

$$\tilde{h}(s, \omega_{(\mathrm{nh})}) = s^3 + O(s^4) \quad \text{as } s \to 0. \tag{43}$$

This uniquely defines $\tilde{h}$.

It can be shown that Eq. (41) is regular at the horizon, provided $\pi_{ij}^{(\mathrm{nh})}(\omega_{(\mathrm{nh})})$ has simple poles in $\omega_{(\mathrm{nh})}$ exactly at the location of the homogeneous quasinormal modes [21]. These quasinormal frequencies have both real and imaginary parts, and unlike hydrodynamic modes, both of these are large, i.e., $O(T)$. In case of the AdS_5 black brane, these are known to be [29]

$$\begin{aligned} \omega_{(n)\pm} = {} & \pi T[\pm 1.2139 - 0.7775i \pm 2n(1 \mp i)], \\ & \text{for large } n. \end{aligned} \tag{44}$$

This follows from the general phenomenological equations (see Appendix B).

The above analysis can be readily extended nonlinearly in the amplitude parameter $\pi_{ij}^{(\mathrm{nh})}/p$, but we need to sum over all time derivatives at each order in the amplitude expansion to see manifest regularity (see Appendix B). This is because the time derivatives of these modes are large, i.e., $O(T)$. This is usual for nonhydrodynamic relaxation modes even close to equilibrium.

The general phenomenological equations as discussed in Appendix B have covariant form. In our examples, we have written them in the laboratory frame, where the final equilibrium configuration is at rest. This is useful from the point of view of comparing our results for Green's functions with experiments.

The derivative and amplitude expansions in the laboratory frame are as follows: The derivative expansion will count powers of $\mathbf{k}_{(\mathrm{b})}/T$, where $\mathbf{k}_{(\mathrm{b})}$ will be the momentum of the background quasinormal mode. It will be useful also to consider the quasinormal modes off shell (and impose the on-shell condition later). In that case, the derivative expansion will also count powers of $\omega_{(\mathrm{h})}/T$, with $\omega_{(\mathrm{h})}$ being real and the frequency of the hydrodynamic quasinormal modes without the dispersion relation imposed. For a generic quasinormal mode, the off-shell frequency will be denoted by $\omega_{(\mathrm{b})}$. The amplitude expansion will count powers of $\delta\mathbf{u}$, $\delta T/T^{(\mathrm{eq})}$ and $\pi_{ij}^{(\mathrm{nh})}/p^{(\mathrm{eq})}$, with all powers of $\omega_{(\mathrm{b})}/T$ summed up at each order for nonhydrodynamic modes.

B. The scalar field in nonequilibrium geometries

Here we will study the free real scalar field in the background of nonequilibrium geometries as described above. This real scalar field will be dual to an appropriate-order parameter like the chiral condensate in the field theory. The typical temperature of the nonequilibrium state will be above that of symmetry restoration. Thus, the chiral condensate will have no expectation value. Furthermore, the chemical potentials will be assumed to be small, so the flavor and baryon currents can be neglected. These conditions are indeed valid for the quark-gluon plasma produced by heavy-ion collisions at the Relativistic Heavy-Ion Collider (RHIC) in Brookhaven and at A Large Ion Collider Experiment (ALICE) at CERN.

Thus, our nonequilibrium backgrounds can be modeled by the regular perturbations of AdS black branes in Einstein's vacuum gravity. As we are interested in the two-point correlations in these nonequilibrium geometries, we can ignore the backreaction of the scalar field, which will contribute only to higher-point correlations.

It is convenient to Fourier transform the dependence on the boundary coordinates. We first need to do this for the background metric.

Let us take the example of the hydrodynamic shear mode perturbation discussed before for concreteness. As is evident from Eq. (34), to achieve this, we need to Fourier transform the velocity perturbation $\delta\mathbf{u}$. We have already done the Fourier transform in the boundary spatial coordinates, but we have not done the Fourier transform of the time dependence yet.

The Fourier transform of $\delta\mathbf{u}(\mathbf{x}, t)$ can be defined as [13]

$$\delta\mathbf{u}(\mathbf{k}_{(\mathrm{h})}, \omega_{(\mathrm{h})}) = -\left(\frac{1}{2\pi i}\right)\frac{\delta\mathbf{u}(\mathbf{k}_{(\mathrm{h})})}{\omega_{(\mathrm{h})} + i\frac{\mathbf{k}_{(\mathrm{h})}^2}{4\pi T}}, \tag{45}$$

such that $\delta\mathbf{u}^*(\mathbf{k}_{(\mathrm{h})}) = \delta\mathbf{u}(-\mathbf{k}_{(\mathrm{h})})$. If we integrate $\omega_{(\mathrm{h})}$ over the real line and then close it in with the semicircle at infinity in the lower half plane, as shown in Fig. 2, we find that

$$\int_{\mathcal{C}} d\omega_{(\mathrm{h})}\delta\mathbf{u}(\mathbf{k}_{(\mathrm{h})}, \omega_{(\mathrm{h})})e^{i\mathbf{k}_{(\mathrm{h})}\cdot\mathbf{x}}e^{-i\omega_{(\mathrm{h})}t} = \delta\mathbf{u}(\mathbf{k}_{(\mathrm{h})})e^{i\mathbf{k}_{(\mathrm{h})}\cdot\mathbf{x}}e^{-\frac{\mathbf{k}^2_{(\mathrm{h})}}{4\pi T}t}. \tag{46}$$

Similarly, in the case of the homogeneous perturbation given by Eq. (41), we can write $\pi^{(\mathrm{nh})}_{ij}(\omega_{(\mathrm{nh})})$ as

$$\pi^{(\mathrm{nh})}_{ij}(\omega_{(\mathrm{nh})}) = -\sum_{n=0}^{\infty}\sum_{+,-}\left(\frac{1}{2\pi i}\right)\frac{a_{(\mathrm{n})ij}}{\omega_{(\mathrm{nh})} - \omega_{(\mathrm{n})\pm}}, \tag{47}$$

where $a_{(\mathrm{n})ij}$ are constants satisfying $a_{(\mathrm{n})ij}\delta_{ij} = 0$, and $\omega_{(\mathrm{n})\pm}$ satisfies Eq. (44).

This can be generalized to any quasinormal mode, which can also be represented as a solution of the linearized phenomenological equations. The dispersion relation of any quasinormal mode can be written as

$$\omega_{(\mathrm{b})\pm}(\mathbf{k}_{(\mathrm{b})}) = \pm\mathrm{Re}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})}) + i\,\mathrm{Im}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})}), \quad \mathrm{Im}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})}) < 0. \tag{48}$$

This gives rise to a complex pole in the Fourier-transformed $\delta\mathbf{u}(\omega_{(\mathrm{b})}, \mathbf{k}_{(\mathrm{b})})$, $\delta T(\omega_{(\mathrm{b})}, \mathbf{k}_{(\mathrm{b})})$, $\pi^{(\mathrm{nh})}_{ij}(\omega_{(\mathrm{b})}, \mathbf{k}_{(\mathrm{b})})$, as below:

$$\begin{aligned}
\delta\mathbf{u}(\mathbf{k}_{(\mathrm{b})}, \omega_{(\mathrm{b})}) &= -\left(\frac{1}{2\pi i}\right)\frac{\delta\mathbf{u}_{\pm}(\mathbf{k}_{(\mathrm{b})})}{\omega_{(\mathrm{b})} \pm \mathrm{Re}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})}) + i\mathrm{Im}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})},\\
\delta T(\mathbf{k}_{(\mathrm{b})}, \omega_{(\mathrm{b})}) &= -\left(\frac{1}{2\pi i}\right)\frac{\delta T_{\pm}(\mathbf{k}_{(\mathrm{b})})}{\omega_{(\mathrm{b})} \pm \mathrm{Re}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})}) + i\mathrm{Im}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})},\\
\pi^{(\mathrm{nh})}_{ij}(\mathbf{k}_{(\mathrm{b})}, \omega_{(\mathrm{b})}) &= -\left(\frac{1}{2\pi i}\right)\frac{a_{ij\pm}(\mathbf{k}_{(\mathrm{b})})}{\omega_{(\mathrm{b})} \pm \mathrm{Re}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})}) + i\mathrm{Im}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})},\\
a_{ij\pm}(\mathbf{k}_{(\mathrm{b})})\delta_{ij} &= 0.
\end{aligned} \tag{49}$$

We will denote the frequency and momentum of the background perturbation as $\omega_{(\mathrm{b})}$ and $\mathbf{k}_{(\mathrm{b})}$, respectively, to distinguish them clearly from those of the scalar field, which will be denoted by ω and $\mathbf{k}$. Note that both $\omega_{(\mathrm{b})}$ and ω are real by definition.

The metric is real, and so are $\delta\mathbf{u}(\mathbf{x}, t)$, $\delta T(\mathbf{x}, t)$ and $\pi^{(\mathrm{nh})}_{ij}(\mathbf{x}, t)$. So we need to consider a pair of modes which are complex conjugates to each other. The complex conjugation involves a subtlety which has not been mentioned in our earlier work. It involves

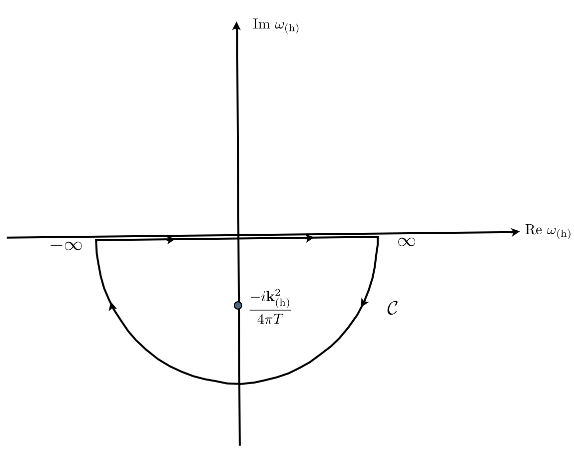

FIG. 2 (color online). Contour for integration of $\omega_{(\mathrm{h})}$ over $\mathcal{C}$, which picks the contribution from the pole in the lower half plane.

(1) Reversing the sign of $\mathbf{k}_{(\mathrm{b})}$ and $\omega_{(\mathrm{b})}$, as (for example) we can see readily from Eq. (45) that

$$\begin{aligned}
\delta\mathbf{u}(-\mathbf{k}_{(\mathrm{h})}, -\omega_{(\mathrm{h})}) &= \left(\frac{1}{2\pi i}\right)\frac{\delta\mathbf{u}(-\mathbf{k}_{(\mathrm{h})})}{\omega_{(\mathrm{h})} - i\frac{\mathbf{k}^2_{(\mathrm{h})}}{4\pi T}}\\
&= \left(\frac{1}{2\pi i}\right)\frac{\delta\mathbf{u}*(\mathbf{k}_{(\mathrm{h})})}{\omega_{(\mathrm{h})} + i\frac{\mathbf{k}^2_{(\mathrm{h})}}{4\pi T}}\\
&= \delta\mathbf{u}*(\mathbf{k}_{(\mathrm{h})}, \omega_{(\mathrm{h})}).
\end{aligned} \tag{50}$$

(2) Integration over $\omega_{(b)}$, which makes the mode on shell, over the contour $\mathcal{C}*$ as shown in Fig. 3, which goes over the real line and closes in the semicircle over the upper half plane instead of the lower half plane, as for example

$$\begin{aligned}
&\int_{\mathcal{C}*} d\omega_{(\mathrm{h})}\delta\mathbf{u}(-\mathbf{k}_{(\mathrm{h})}, -\omega_{(\mathrm{h})})e^{-i\mathbf{k}_{(\mathrm{h})}\cdot\mathbf{x}}e^{i\omega_{(\mathrm{h})}t}\\
&\quad = \delta\mathbf{u}(-\mathbf{k}_{(\mathrm{h})})e^{-i\mathbf{k}_{(\mathrm{h})}\cdot\mathbf{x}}e^{-\frac{\mathbf{k}^2_{(\mathrm{h})}}{4\pi T}t}\\
&\quad = \delta\mathbf{u}*(\mathbf{k}_{(\mathrm{h})})e^{-i\mathbf{k}_{(\mathrm{h})}\cdot\mathbf{x}}e^{-\frac{\mathbf{k}^2_{(\mathrm{h})}}{4\pi T}t}.
\end{aligned} \tag{51}$$

We note that the contour $\mathcal{C}*$ is a reflection of the original contour $\mathcal{C}$ over the real line. It is necessary to integrate the complex conjugates of Eq. (48) over $\mathcal{C}*$ so that the complex conjugate modes also decay in the future. We readily see

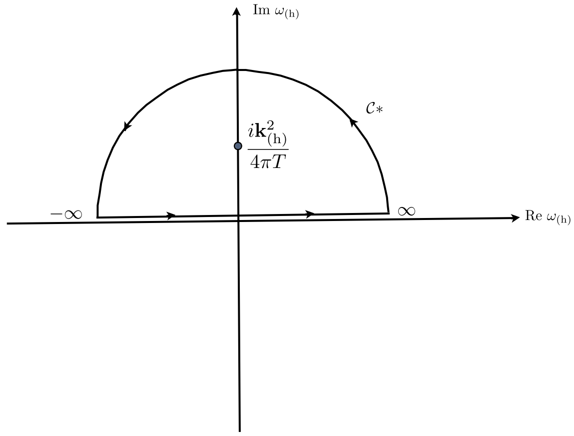

FIG. 3 (color online). Contour for integration of $\omega_{(h)}$ over $\mathcal{C}*$, which picks the contribution from the pole in the upper half plane.

that we need to combine the two modes to produce a real perturbation, as for example

$$\int_{\mathcal{C}} d\omega_{(b)}\delta\mathbf{u}(\omega_{(b)}, \mathbf{k}_{(b)})e^{i\mathbf{k}_{(b)}\cdot\mathbf{x}}e^{-i\omega_{(b)}t} + \int_{\mathcal{C}*} d\omega_{(b)}\delta\mathbf{u}(-\omega_{(b)}, -\mathbf{k}_{(b)})e^{-i\mathbf{k}_{(b)}\cdot\mathbf{x}}e^{i\omega_{(b)}t}$$
$$= (\delta\mathbf{u}(\mathbf{k}_{(b)})e^{i\mathbf{k}_{(b)}\cdot\mathbf{x}}e^{-i\mathrm{Re}\,\omega_{(b)}(\mathbf{k}_{(b)})t} + \delta\mathbf{u}(-\mathbf{k}_{(b)})e^{-i\mathbf{k}_{(b)}\cdot\mathbf{x}}e^{i\mathrm{Re}\,\omega_{(b)}(\mathbf{k}_{(b)})t})e^{\mathrm{Im}\,\omega_{(b)}(\mathbf{k}_{(b)})t} \tag{52}$$

for a general velocity perturbation $\delta\mathbf{u}(\mathbf{x}, t)$.

The free scalar field satisfies the usual equation of motion in the nonequilibrium geometry, which is

$$(\Box + m^2)\phi(r, \mathbf{x}, t) = 0. \tag{53}$$

The d'Alembertian $\Box$, however, is that of the nonequilibrium geometry.

Let us first consider the case in which the perturbation of the geometry away from the AdS black brane is linearized in $\delta\mathbf{u}$, δT and $\pi^{(\mathrm{nh})}_{ij}$. Let these background perturbations be a sum of the specific mode given by the frequency $\omega_{(b)}$ and momentum $\mathbf{k}_{(b)}$ as in Eq. (48) and its complex conjugate at frequency $-\omega_{(b)}$ and momentum $-\mathbf{k}_{(b)}$.

The profile of the scalar field can be constructed as follows: Let us consider a specific mode of the scalar field at a frequency ω and momentum $\mathbf{k}$. Interacting with the quasinormal modes in the background, new modes at frequencies $\omega \pm \omega_{(b)}$ and momenta $\mathbf{k} \pm \mathbf{k}_{(b)}$ will be generated. So, in the linearized nonequilibrium background, the general solution of the scalar field will take the form

$$\Phi(\mathbf{x}, t, r) = A(\omega, \mathbf{k})\Big(\Phi^{(\mathrm{eq})}(\omega, \mathbf{k}, r)e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})} + \int_{\mathcal{C}} d\omega_{(b)}\Phi^{(\mathrm{neq})}(\omega, \mathbf{k}, \omega_{(b)}, \mathbf{k}_{(b)}, r) \times e^{-i((\omega+\omega_{(b)})t - (\mathbf{k}+\mathbf{k}_{(b)})\cdot\mathbf{x})} + \int_{\mathcal{C}*} d\omega_{(b)}\Phi^{(\mathrm{neq})}(\omega, \mathbf{k}, -\omega_{(b)}, -\mathbf{k}_{(b)}, r) \times e^{-i((\omega-\omega_{(b)})t - (\mathbf{k}-\mathbf{k}_{(b)})\cdot\mathbf{x})}\Big). \tag{54}$$

We note the following points regarding the above equation:

(1) The nonequilibrium corrections given by $\Phi^{(\mathrm{neq})}(\omega, \mathbf{k}, \omega_{(b)}, \mathbf{k}_{(b)}, r)$ and $\Phi^{(\mathrm{neq})}(\omega, \mathbf{k}, -\omega_{(b)}, -\mathbf{k}_{(b)}, r)$ can be evaluated systematically in the derivative and amplitude expansions.

(2) Both $\Phi^{(\mathrm{neq})}(\omega, \mathbf{k}, \omega_{(b)}, \mathbf{k}_{(b)}, r)$ and $\Phi^{(\mathrm{neq})}(\omega, \mathbf{k}, -\omega_{(b)}, -\mathbf{k}_{(b)}, r)$ will be determined uniquely by the boundary conditions at the horizon.

(3) We need to perform integrations over $\omega_{(b)}$ using the contour prescriptions described above so that the background is on shell.

(4) The most general solution can be obtained as above by simply superposing over different equilibrium modes at ω, $\mathbf{k}$ along with their unique nonequilibrium corrections, with an overall arbitrary coefficient $A(\omega, \mathbf{k})$.

As an explicit illustration, let us examine the profile of the scalar field in the background metric [Eq. (34)] with a hydrodynamic shear wave perturbation.

The equation of motion of the equilibrium part $\Phi^{(\mathrm{eq})}(\omega, \mathbf{k})$ is

$$(\Box^{ABB}_{\omega',\mathbf{k}'} + m^2)\delta(\omega' - \omega)\delta^2(\mathbf{k}' - \mathbf{k})\Phi^{(\mathrm{eq})}(\omega, \mathbf{k}, r) = 0, \tag{55}$$

where $\Box^{ABB}_{\omega,\mathbf{k}}$ is the Laplacian in the unperturbed AdS black brane metric given by

$$l^2\Box^{ABB}_{\omega,\mathbf{k}} = r^2 f\Big(\frac{rr_0}{l^2}\Big)\partial_r^2 + r\Big[-2f\Big(\frac{rr_0}{l^2}\Big) + \frac{rr_0}{l^2}f'\Big(\frac{rr_0}{l^2}\Big)\Big]\partial_r + r^2\Big[\frac{\omega^2}{f(\frac{rr_0}{l^2})} - \mathbf{k}^2\Big]. \tag{56}$$

Above, f is the blackening function of the AdS Reissner-Nordstorm black brane, which vanishes at the horizon located at $r = l^2/r_0$, and is as given in Eq. (35).

The equations of motion of the nonequilibrium parts of the solution up to the first order in the derivative expansion are

$$(\Box^{ABB}_{\omega',\mathbf{k}'} + m^2)\delta(\omega' - \omega - \omega_{(h)})\delta^3(\mathbf{k}' - \mathbf{k} - \mathbf{k}_{(h)})\Phi^{(\mathrm{neq})}(\omega, \mathbf{k}, \omega_{(h)}, \mathbf{k}_{(h)}, r) = V(\omega, \mathbf{k}, \omega_{(h)}, \mathbf{k}_{(h)}, r)\Phi^{(\mathrm{eq})}(\omega, \mathbf{k}, r),$$
$$(\Box^{ABB}_{\omega',\mathbf{k}'} + m^2)\delta(\omega' - \omega + \omega_{(h)})\delta^3(\mathbf{k}' - \mathbf{k} + \mathbf{k}_{(h)})\Phi^{(\mathrm{neq})}(\omega, \mathbf{k}, -\omega_{(h)}, -\mathbf{k}_{(h)}, r) = V(\omega, \mathbf{k}, -\omega_{(h)}, -\mathbf{k}_{(h)}, r), \quad \Phi^{(\mathrm{eq})}(\omega, \mathbf{k}, r), \tag{57}$$

with

$$V(\omega, \mathbf{k}, \omega_{(h)}, \mathbf{k}_{(h)}, r) = V_1(\omega, \mathbf{k}, \omega_{(h)}, \mathbf{k}_{(h)}, r) + V_2(\omega, \mathbf{k}, \omega_{(h)}, \mathbf{k}_{(h)}, r) + O(\epsilon^2),$$
$$V_1(\omega, \mathbf{k}, \omega_{(h)}, \mathbf{k}_{(h)}, r) = \frac{2r^2}{l^2 f(\frac{rr_0}{l^2})}\omega\left(1 - f\left(\frac{rr_0}{l^2}\right)\right)\delta\mathbf{u}(\omega_{(h)}, \mathbf{k}_{(h)})\cdot\mathbf{k}, \tag{58}$$
$$V_2(\omega, \mathbf{k}, \omega_{(h)}, \mathbf{k}_{(h)}, r) = i\frac{r^2}{2r_0}h\left(\frac{rr_0}{l^2}\right)k_i k_j k_{(h)i}\delta u_j(\omega_{(h)}, \mathbf{k}_{(h)}).$$

Above, $h(rr_0/l^2)$ gives the hydrodynamic correction to the background metric which is proportional to $k_{(h)i}\delta u_j + (i \leftrightarrow j)$, and is as given in Eq. (35). Also, V has been expanded up to first order in the derivative expansion only, with the first term V_1 depicting the contribution at the zeroth order in the derivative expansion, and V_2 depicting the contribution at the first order in the derivative expansion, respectively. Both are linear in $\delta\mathbf{u}$, as we have treated the background in the quasinormal mode approximation. Note that the derivative expansion counts only the order of appearance of hydrodynamic frequency $\omega_{(h)}$ and $\mathbf{k}_{(h)}$ through derivatives of $\delta\mathbf{u}$, while dependence in ω and $\mathbf{k}$ should be treated exactly at each order.

As indicated in Eq. (54), we need to integrate $\omega_{(b)}$ so that the background metric is on shell. After this integration, the profile of the scalar field is

$$\Phi(\mathbf{x}, t, r) = \Phi^{(eq)}(\omega, \mathbf{k}, r)e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})} + (\Phi^{(neq)}(\omega, \mathbf{k}, \mathbf{k}_{(h)}, r)e^{-i\omega t}e^{i(\mathbf{k}+\mathbf{k}_{(h)})\cdot\mathbf{x}} + \Phi^{(neq)}(\omega, \mathbf{k}, -\mathbf{k}_{(h)}, r)e^{-i\omega t}e^{i(\mathbf{k}-\mathbf{k}_{(h)})\cdot\mathbf{x}})e^{-\frac{\mathbf{k}_{(h)}^2}{4\pi T}t}. \tag{59}$$

Above, $\Phi^{(neq)}(\omega, \mathbf{k}, \mathbf{k}_{(h)}, r)$ and $\Phi^{(neq)}(\omega, \mathbf{k}, -\mathbf{k}_{(h)}, r)$ are the residues of $\Phi^{(neq)}(\omega, \mathbf{k}, \omega_{(h)}, \mathbf{k}_{(h)}, r)$ and $\Phi^{(neq)}(\omega, \mathbf{k}, -\omega_{(h)}, -\mathbf{k}_{(h)}, r)$ at the poles $\omega_{(h)} = \mp i\mathbf{k}_{(h)}^2/(4\pi T)$, respectively. The functions $\Phi^{(neq)}(\omega, \mathbf{k}, \omega_{(h)}, \mathbf{k}_{(h)}, r)$ and $\Phi^{(neq)}(\omega, \mathbf{k}, -\omega_{(h)}, -\mathbf{k}_{(h)}, r)$ depend linearly on $\delta\mathbf{u}(\omega_{(h)}, \mathbf{k}_{(h)})$ and $\delta\mathbf{u}(-\omega_{(h)}, -\mathbf{k}_{(h)})$, respectively, through the V terms in Eq. (57). The contours $\mathcal{C}$ and $\mathcal{C}*$ pick up the residues of the poles in $\delta\mathbf{u}(\omega_{(h)}, \mathbf{k}_{(h)})$ and $\delta\mathbf{u}(-\omega_{(h)}, -\mathbf{k}_{(h)})$ in the upper and lower half planes, respectively.

It follows that $\Phi^{(neq)}(\omega, \mathbf{k}, \mathbf{k}_{(h)}, r)$ takes the following form up to first order in the derivative expansion:

$$\Phi^{(neq)}(\omega, \mathbf{k}, \mathbf{k}_{(h)}, r) = \Phi_1^{(neq)}(\omega, \mathbf{k}, r)\delta\mathbf{u}(\mathbf{k}_{(h)})\cdot\mathbf{k} + \Phi_2^{(neq)}(\omega, \mathbf{k}, r)k_i k_j k_{(h)i}\delta u_j(\mathbf{k}_{(h)}) + O(\epsilon^2). \tag{60}$$

The similar expansion for $\Phi^{(neq)}(\omega, \mathbf{k}, -\mathbf{k}_{(h)}, r)$ is obtained simply by reversing the sign of $\mathbf{k}_{(h)}$ above.

The above procedure can be readily generalized when the background metric is expanded nonlinearly in the perturbations parametrized by $\delta\mathbf{u}$, δT and $\pi_{ij}^{(nh)}$.

Let us focus on the case of the nonlinear hydrodynamic background whose explicit forms can be found in the literature. Let us consider the Laplacian when we take into account quadratic dependence on two distinct velocity perturbations $\delta\mathbf{u}(k_{(h)})$ and $\delta\mathbf{u}(k'_{(h)})$, for instance, at a given order in the derivative expansion m (i.e., at the mth order in the hydrodynamic momentum). The solution for Φ will receive a correction quadratic in the amplitude of velocity perturbation, and at mth order in the derivative expansion, this takes the form

$$\Phi^{(2,m)}(r, k, k_{(h)}, k'_{(h)})e^{i(k+k_{(h)}+k'_{(h)})\cdot x} + (k_{(h)} \to -k_{(h)}) + (k'_{(h)} \to -k'_{(h)}) + (k_{(h)} \to -k_{(h)}, k'_{(h)} \to -k'_{(h)}). \tag{61}$$

The radial dependence above can be determined by the equation of motion:

$$\Box_{k'}^{ABB}\delta^3(k' - k - k_{(h)} - k'_{(h)})\Phi^{(2,m)}(r, k', k_{(h)}, k'_{(h)}) = S^{(2,m)}(r, k, k_{(h)}, k'_{(h)}), \tag{62}$$

where $\Box_k^{ABB}$ is the Laplacian for a scalar with four-momentum k in the unperturbed AdS_5 black brane, and $S^{(2,m)}$ is the source term determined by the background perturbation. For $m = 1$, i.e., at first order in the derivative expansion, the source term $S^{(2,1)}$ can contain terms like $(\mathbf{k}\cdot\delta\mathbf{u}(k_{(h)}))(\mathbf{k}_{(h)}\cdot\delta\mathbf{u}(k'_{(h)}))\Phi^{(eq)}$, $(\mathbf{k}_{(h)}\cdot\delta\mathbf{u}(k'_{(h)}))\Phi^{(1,0)}$, etc. Finally, we need to integrate $\omega_{(h)}$ and $\omega'_{(h)}$ over $\mathcal{C}$ or $\mathcal{C}'$ so that the background metric is taken on shell.

Thus, perturbatively, the solution of the scalar field in an arbitrary background can be systematically expanded in both the derivative and amplitude expansions.

C. The boundary conditions for regularity

The behavior of the general solution $\Phi^{(eq)}(\omega, \mathbf{k}, r)$ in the equilibrium black brane background can be split into one that is incoming at the horizon and another that is outgoing at the horizon. The incoming mode $\Phi^{in(eq)}(\omega, \mathbf{k}, r)$ can be defined uniquely by its behavior near the horizon $r = \frac{l^2}{r_0}$ as below:

$$\Phi^{\mathrm{in(eq)}}(\omega,\mathbf{k},r)\approx\left(1-\frac{rr_0}{l^2}\right)^{-i\frac{\omega}{4\pi T}}. \tag{63}$$

The outgoing mode $\Phi^{\mathrm{out(eq)}}(\omega,\mathbf{k})$ can be similarly defined uniquely by its behavior near the horizon, given by

$$\Phi^{\mathrm{out(eq)}}(\omega,\mathbf{k},r)\approx\left(1-\frac{rr_0}{l^2}\right)^{i\frac{\omega}{4\pi T}}. \tag{64}$$

Also,

$$\begin{aligned}\Phi^{\mathrm{in(eq)}}*(\omega,\mathbf{k},r)&=\Phi^{\mathrm{in(eq)}}(-\omega,-\mathbf{k},r)\\&=\Phi^{\mathrm{out(eq)}}(\omega,\mathbf{k},r)\\&=\Phi^{\mathrm{out(eq)}}*(-\omega,-\mathbf{k},r).\end{aligned} \tag{65}$$

In fact, the effect of reversing the sign of $\mathbf{k}$ is trivial, as the solution can only depend on its modulus due to the rotational symmetry of the black brane background.

It is natural to include only the incoming solution when $\omega>0$, because it respects the causal structure of the black brane, which forbids anything coming out of the horizon classically. We can also follow Ref. [30] to argue that if we keep both the incoming and outgoing modes, it will cause a singular backreaction at the horizon. As we will be considering quasinormal mode perturbations of the background metric which are incoming at the horizon, we should exclude the outgoing solution of the scalar field to prevent singular backreaction at the horizon. Thus, the general solution at the zeroth order is

$$\Phi^{\mathrm{(eq)}}(\omega,\mathbf{k},r)=A^{\mathrm{in(eq)}}(\omega,\mathbf{k})\Phi^{\mathrm{in(eq)}}(\omega,\mathbf{k},r). \tag{66}$$

Until the end of this subsection, we will assume $\omega>0$. Later, we will add a complex conjugate term necessary to make the full solution real. This complex conjugate will take care of the case $\omega<0$.

In order to study the regularity of the nonequilibrium solution, let us first examine the approximation where the background metric perturbation is linearized in $\delta\mathbf{u}$, δT and $\pi_{ij}^{(\mathrm{nh})}$. In particular, let us study the case of the hydrodynamic shear wave perturbation.

It is easy to see from Eq. (57) that the general form of the nonequilibrium part of the solution can be written as

$$\begin{aligned}&\Phi^{\mathrm{(neq)}}(\omega,\mathbf{k},\omega_{(\mathrm{h})},\mathbf{k}_{(\mathrm{h})},r)\\&=A^{\mathrm{in(neq)}}(\omega,\mathbf{k},\omega_{(\mathrm{h})},\mathbf{k}_{(\mathrm{h})})\Phi^{\mathrm{in(eq)}}(\omega+\omega_{(\mathrm{h})},\mathbf{k}+\mathbf{k}_{(\mathrm{h})},r)\\&\quad+A^{\mathrm{out(neq)}}(\omega,\mathbf{k},\omega_{(\mathrm{h})},\mathbf{k}_{(\mathrm{h})})\Phi^{\mathrm{in(eq)}}(\omega+\omega_{(\mathrm{h})},\mathbf{k}+\mathbf{k}_{(\mathrm{h})},r)\\&\quad+A^{\mathrm{in(eq)}}(\omega,\mathbf{k})\Phi^{\mathrm{in(neq)}}(\omega,\mathbf{k},\omega_{(\mathrm{h})},\mathbf{k}_{(\mathrm{h})},r)\\&\quad+(\omega_{(\mathrm{h})}\to-\omega_{(\mathrm{h})},\mathbf{k}_{(\mathrm{h})}\to-\omega_{(\mathrm{h})})\end{aligned} \tag{67}$$

up to first order in the derivative expansion. The first two lines above indicate the homogeneous solutions of Eq. (57), but with ω now replaced by $\omega+\omega_{(\mathrm{h})}$, and $\mathbf{k}$ replaced by $\mathbf{k}+\mathbf{k}_{(\mathrm{h})}$. The coefficients of these homogeneous solutions, $A^{\mathrm{in(neq)}}(\omega,\omega_{(\mathrm{h})},\mathbf{k},\mathbf{k}_{(\mathrm{h})})$ and $A^{\mathrm{out(neq)}}(\omega,\omega_{(\mathrm{h})},\mathbf{k},\mathbf{k}_{(\mathrm{h})})$, respectively, have to be linear in $\delta\mathbf{u}$ and have a consistent derivative expansion also. Their dependence on $\omega_{(\mathrm{h})}$ and $\mathbf{k}_{(\mathrm{h})}$ can be expanded systematically in terms of rotationally invariant scalars like $\delta\mathbf{u}\cdot\mathbf{k}$, $k_ik_jk_{(\mathrm{h})i}\delta u_j$, $\omega_{(\mathrm{h})}k_ik_jk_{(\mathrm{h})i}\delta u_j$, etc. Up to first order in the derivative expansion, only the first two scalars will appear. The coefficients of these scalars should be functions of ω and $\mathbf{k}$ only, as the dependence on $\omega_{(\mathrm{h})}$ and $\mathbf{k}_{(\mathrm{h})}$ can be absorbed in coefficients of the scalars appearing at higher orders in the derivative expansion. Thus, up to first order in the derivative expansion, we should have

$$\begin{aligned}A^{\mathrm{in(neq)}}(\omega,\omega_{(\mathrm{h})},\mathbf{k},\mathbf{k}_{(\mathrm{h})})&=A_1^{\mathrm{in(neq)}}(\omega,\mathbf{k})\delta\mathbf{u}(\omega_{(\mathrm{h})},\mathbf{k}_{(\mathrm{h})})\cdot\mathbf{k}+A_2^{\mathrm{in(neq)}}(\omega,\mathbf{k})k_ik_jk_{(\mathrm{h})i}\delta u_j(\omega_{(\mathrm{h})},\mathbf{k}_{(\mathrm{h})})+O(\epsilon^2),\\A^{\mathrm{out(neq)}}(\omega,\omega_{(\mathrm{h})},\mathbf{k},\mathbf{k}_{(\mathrm{h})})&=A_1^{\mathrm{out(neq)}}(\omega,\mathbf{k})\delta\mathbf{u}(\omega_{(\mathrm{h})},\mathbf{k}_{(\mathrm{h})})\cdot\mathbf{k}+A_2^{\mathrm{out(neq)}}(\omega,\mathbf{k})k_ik_jk_{(\mathrm{h})i}\delta u_j(\omega_{(\mathrm{h})},\mathbf{k}_{(\mathrm{h})})+O(\epsilon^2).\end{aligned} \tag{68}$$

We recall that for the hydrodynamic shear mode, $\delta\mathbf{u}\cdot\mathbf{k}_{(\mathrm{h})}=0$, so there are no more possible terms up to first order in $\mathbf{k}_{(\mathrm{h})}$. In Eq. (68) above, $A_1^{\mathrm{in(neq)}}(\omega,\mathbf{k})$, $A_2^{\mathrm{in(neq)}}(\omega,\mathbf{k})$, $A_1^{\mathrm{out(neq)}}(\omega,\mathbf{k})$ and $A_2^{\mathrm{out(neq)}}(\omega,\mathbf{k})$ are arbitrary.

The last line of Eq. (67) denotes the particular solution determined completely by the perturbation of the background [Eq. (58)] and the zeroth-order solution [Eq. (66)] as appearing on the right hand side of Eq. (57). There is no new constant appearing here. $\Phi^{\mathrm{in(neq)}}(\omega,\mathbf{k},\omega_{(\mathrm{h})},\mathbf{k}_{(\mathrm{h})},r)$ is the particular solution sourced by $\Phi^{\mathrm{in(eq)}}(\omega,\mathbf{k},r)$ and can be thought of as the nonequilibrium correction to the incoming mode. Its behavior at the horizon is given by

$$\begin{aligned}&\Phi^{\mathrm{in(neq)}}(\omega,\mathbf{k},\omega_{(\mathrm{h})},\mathbf{k}_{(\mathrm{h})},r)\\&\quad\approx i2\left(\frac{\pi Tl^2}{r_0}\right)^2\frac{\omega\,\delta\mathbf{u}(\omega,\mathbf{k}_{(\mathrm{h})})\cdot\mathbf{k}}{(2\omega+\omega_{(\mathrm{h})})\omega_{(\mathrm{h})}}\left(1-\frac{rr_0}{l^2}\right)^{-i\frac{\omega}{4\pi T}}.\end{aligned} \tag{69}$$

We observe that this particular solution is simply proportional to the equilibrium incoming mode at the leading order, and hence is regular at the horizon.

Just like in the case of equilibrium, we should discard the outgoing solution to prevent potentially harmful backreaction. Therefore, we should choose

$$\begin{aligned}&A^{\mathrm{out(neq)}}(\omega,\mathbf{k},\omega_{(\mathrm{h})},\mathbf{k}_{(\mathrm{h})})=0,\\&\text{i.e.,}\quad A_i^{\mathrm{out(neq)}}(\omega,\mathbf{k})=0,\quad\text{for } i=1,2,\ldots.\end{aligned} \tag{70}$$

We recall that, as in Eq. (54), we need to integrate over $\omega_{(\mathrm{h})}$ on the contour $\mathcal{C}$ given in Fig. 2. It is clear from the form of $A^{\mathrm{in(neq)}}(\omega, \mathbf{k}, \omega_{(\mathrm{h})}, \mathbf{k}_{(\mathrm{h})})$ in Eq. (68) that the pole in the lower half plane of $\delta\mathbf{u}(\omega_{(\mathrm{h})}, \mathbf{k}_{(\mathrm{h})})$ as given by Eq. (45) will produce a divergence at the horizon whose leading term is

$$\left(1 - \frac{rr_0}{l^2}\right)^{-i\frac{\omega}{4\pi T} - \frac{\mathbf{k}_{(\mathrm{h})}^2}{16\pi^2 T^2}}. \tag{71}$$

This can be readily seen from the behavior of the incoming homogeneous solution $\Phi^{\mathrm{in(eq)}}(\omega + \omega_{(\mathrm{h})}, \mathbf{k} + \mathbf{k}_{(\mathrm{h})}, r)$ near the horizon as given by Eq. (63), substituting $\omega_{(\mathrm{h})}$ with its on-shell value. To get rid of this divergence, we also need to set

$$\begin{aligned} &A^{\mathrm{in(neq)}}(\omega, \mathbf{k}, \omega_{(\mathrm{h})}, \mathbf{k}_{(\mathrm{h})}) = 0, \\ &\text{i.e.,} \quad A_i^{\mathrm{in(neq)}}(\omega, \mathbf{k}) = 0, \quad \text{for } i = 1, 2, \ldots. \end{aligned} \tag{72}$$

Similarly, we can write the general solution for $\Phi^{(\mathrm{neq})}(\omega, \mathbf{k}, -\omega_{(\mathrm{h})}, -\mathbf{k}_{(\mathrm{h})}, r)$ in Eq. (57) as a sum of homogeneous solutions given by $\Phi^{\mathrm{in(eq)}}(\omega - \omega_{(\mathrm{h})}, \mathbf{k} - \mathbf{k}_{(\mathrm{h})}, r)$ and $\Phi^{\mathrm{out(eq)}}(\omega - \omega_{(\mathrm{h})}, \mathbf{k} - \mathbf{k}_{(\mathrm{h})}, r)$ with coefficients $A^{\mathrm{in(neq)}}(\omega, -\omega_{(\mathrm{h})}, \mathbf{k}, -\mathbf{k}_{(\mathrm{h})})$ and $A^{\mathrm{out(neq)}}(\omega, -\omega_{(\mathrm{h})}, \mathbf{k}, -\mathbf{k}_{(\mathrm{h})})$, respectively. These coefficients are arbitrary but have a consistent derivative expansion as in Eq. (68). The particular solution is $A^{\mathrm{in(eq)}}(\omega, \mathbf{k})\Phi^{\mathrm{in(neq)}}(\omega, \mathbf{k}, -\omega_{(\mathrm{h})}, -\mathbf{k}_{(\mathrm{h})}, r)$ with coefficient $A^{\mathrm{in(eq)}}(\omega, \mathbf{k})$ and $\Phi^{\mathrm{out(neq)}}(\omega, \mathbf{k}, -\omega_{(\mathrm{h})}, -\mathbf{k}_{(\mathrm{h})}, r)$, which is completely determined by the equilibrium solution and is regular at the horizon.

Once again, we should set $A^{\mathrm{out(neq)}}$ to zero to prevent harmful backreaction. We should also set $A^{\mathrm{in(neq)}}$ to zero, as the integration over the contour $\mathcal{C}*$ (see Fig. 3), as in Eq. (54), will produce a leading divergence at the horizon of the form shown in Eq. (71).

In a way, these boundary conditions for regularity are also a consequence of reality constraints [Eq. (78)] which relate $A^{\mathrm{in(neq)}}$ to $A^{\mathrm{out(neq)}}$. We note that as the background metric is real, for every background $(\omega_{(\mathrm{b})}, \mathbf{k}_{(\mathrm{b})})$ quasinormal mode, we need to include the complex conjugate $(-\omega_{(\mathrm{b})}, -\mathbf{k}_{(\mathrm{b})})$ quasinormal mode. Thus, Eq. (78) tells us that if the coefficients of the outgoing homogeneous solutions are set to zero, so should be the coefficients of the ingoing homogeneous solutions.

The particular solutions which remain are regular at the horizon. Therefore, regularity at the horizon uniquely determines the nonequilibrium correction to the profile of the scalar field in the unperturbed black brane background. The full solution is

$$\begin{aligned} &A(\omega, \mathbf{k})\Big(\Phi^{(\mathrm{eq})}\Big(\omega, \mathbf{k}, r)e^{-i(\omega t - \mathbf{k}\cdot\mathbf{x})} \\ &\quad + \int_{\mathcal{C}} d\omega_{(\mathrm{h})}\Phi^{\mathrm{in(neq)}}(\omega, \mathbf{k}, \omega_{(\mathrm{h})}, \mathbf{k}_{(\mathrm{h})}, r) \\ &\quad \times e^{-i((\omega+\omega_{(\mathrm{h})})t - (\mathbf{k}+\mathbf{k}_{(\mathrm{h})})\cdot\mathbf{x})} \\ &\quad + \int_{\mathcal{C}*} d\omega_{(\mathrm{h})}\Phi^{\mathrm{in(neq)}}(\omega, \mathbf{k}, -\omega_{(\mathrm{h})}, -\mathbf{k}_{(\mathrm{h})}, r) \\ &\quad \times e^{-i((\omega-\omega_{(\mathrm{h})})t - (\mathbf{k}-\mathbf{k}_{(\mathrm{h})})\cdot\mathbf{x})}\Big). \end{aligned} \tag{73}$$

The above boundary conditions apply to all quasinormal mode backgrounds. In each case, we need to set the coefficient of the outgoing mode at the horizon to zero to prevent harmful backreaction. Also, the coefficient of the incoming mode should be set to zero; otherwise the $\omega_{(\mathrm{b})}$ integration will produce divergence at the horizon whose leading term will take the form

$$\left(1 - \frac{rr_0}{l^2}\right)^{-i\frac{(\omega + \mathrm{Re}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})}))}{4\pi T}}\left(1 - \frac{rr_0}{l^2}\right)^{\frac{\mathrm{Im}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})}{4\pi T}}. \tag{74}$$

The above is divergent, because for any quasinormal mode, $\mathrm{Im}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})}) < 0$.

The particular solution is regular at the horizons in all the instances checked so far. For instance, in the case of the metric perturbed by the homogeneous relaxation mode in Eq. (41), the particular solution vanishes at the horizon due to factors like

$$\left(1 - \frac{rr_0}{l^2}\right)^n\left(\ln\left(1 - \frac{rr_0}{l^2}\right)\right)^m.$$

This particular solution gives the unique nonequilibrium correction to the equilibrium solution. The full solution, as in Eq. (73), has only one arbitrary constant, which is $A(\omega, \mathbf{k})$.

The above boundary conditions are also valid when we take into account the nonlinear dependence of the background metric on $\delta\mathbf{u}$, δT and $\pi_{ij}^{(\mathrm{nh})}$.

Let us consider the case of the scalar field in the nonlinear hydrodynamic background. In particular, let us consider quadratic dependence on $\delta\mathbf{u}$ at mth order in the derivative expansion. This nonequilibrium correction is $\Phi^{(2,m)}$, with the equation of motion given in Eq. (62). Clearly, the general solution of $\Phi^{(2,m)}$ near the horizon can again be separated into two homogeneous pieces—the incoming and the outgoing modes—and a particular piece which has no arbitrary integration constant and is completely determined by the source term $S^{(2,m)}$. In order to prevent harmful backreaction, we should set the coefficient of the outgoing mode to zero. Also, as discussed before, the integration over the hydrodynamic frequencies $\omega_{(\mathrm{h})}$ and $\omega'_{(\mathrm{h})}$ will produce a divergence at the horizon for the incoming mode, as (for instance) in the case above, with dependence on two hydrodynamic shear wave background modes like

$$\left(1-\frac{rr_0}{l^2}\right)^{-i\frac{\omega}{4\pi T}-\frac{\mathbf{k}^2_{(\mathrm{h})}}{16\pi^2T^2}-\frac{\mathbf{k}'^2_{(\mathrm{h})}}{16\pi^2T^2}-\cdots}. \tag{75}$$

Obviously, the coefficient of the incoming mode has to depend on $\delta\mathbf{u}(\mathbf{k}_{(\mathrm{h})})$ and $\delta\mathbf{u}(\mathbf{k}'_{(\mathrm{h})})$ as required by the order in the perturbation expansion, as in Eq. (68). The poles in $\omega_{(\mathrm{h})}$ and $\omega'_{(\mathrm{h})}$, as in Eq. (45), produce the above divergent behavior after integration over the contours $\mathcal{C}$ or $\mathcal{C}*$. In a consistent holographic setup, the particular solution should be regular at the horizon.

Thus, the full solution of the scalar field in the nonequilibrium background is undetermined only up to an overall constant $A(\omega, \mathbf{k})$. This is because the equilibrium solution $\Phi^{\mathrm{in}}(\omega, \mathbf{k})$ has a unique nonequilibrium correction which is regular at the horizon.

We also note that for the full solution of $\Phi(r, \mathbf{x}, t)$ to be real, we also need to superimpose $\Phi^{\mathrm{in}}(r, -\omega, -\mathbf{k}) = \Phi^{\mathrm{out}}(r, \omega, \mathbf{k})$ with $\Phi^{\mathrm{in}}(r, \omega, \mathbf{k})$. Note that the outgoing boundary condition is indeed natural for the negative-frequency modes. As long as there is CPT invariance in the dual theory, the outgoing mode with negative frequency is the natural CPT conjugate of the incoming mode with positive frequency.

We also note that the nonequilibrium correction to $\Phi^{\mathrm{in}}(r, -\omega, -\mathbf{k})$ is also the complex conjugate to the nonequilibrium correction to $\Phi^{\mathrm{in}}(r, \omega, \mathbf{k})$. In particular, the corrections to $\Phi^{\mathrm{in}}(r, -\omega, -\mathbf{k})$ due to the background quasinormal modes are complex conjugates to the corrections to $\Phi^{\mathrm{in}}(r, -\omega, -\mathbf{k})$ due to the complex conjugated quasinormal modes, i.e.,

$$\begin{aligned}&\Phi^{\mathrm{in(neq)}}*(r, \omega, \mathbf{k}, \omega_{(\mathrm{b})}, \mathbf{k}_{(\mathrm{b})}, \omega'_{(\mathrm{b})}, \mathbf{k}'_{(\mathrm{b})}, \ldots)\\&\quad=\Phi^{\mathrm{in(neq)}}(r, -\omega, -\mathbf{k}, -\omega_{(\mathrm{b})}, -\mathbf{k}_{(\mathrm{b})}, -\omega'_{(\mathrm{b})}, -\mathbf{k}'_{(\mathrm{b})}, \ldots).\end{aligned} \tag{76}$$

As in the background perturbations, for every $(\omega_{(\mathrm{b})}, \mathbf{k}_{(\mathrm{b})})$ mode there is the complex conjugate $(-\omega_{(\mathrm{b})}, -\mathbf{k}_{(\mathrm{b})})$ mode; the full nonequilibrium corrections of $\Phi^{\mathrm{in}}(\omega, \mathbf{k})$ and $\Phi^{\mathrm{in}}(-\omega, -\mathbf{k})$ are complex conjugates of each other. We can readily see that the complex conjugate is also regular at the horizon.

Similarly,

$$\begin{aligned}&\Phi^{\mathrm{out(neq)}}*(r, \omega, \mathbf{k}, \omega_{(\mathrm{b})}, \mathbf{k}_{(\mathrm{b})}, \omega'_{(\mathrm{b})}, \mathbf{k}'_{(\mathrm{b})}, \ldots)\\&\quad=\Phi^{\mathrm{out(neq)}}(r, -\omega, -\mathbf{k}, -\omega_{(\mathrm{b})}, -\mathbf{k}_{(\mathrm{b})}, -\omega'_{(\mathrm{b})}, -\mathbf{k}'_{(\mathrm{b})}, \ldots).\end{aligned} \tag{77}$$

For the more general solution, which is not necessarily regular at the horizon, we also need

$$\begin{aligned}&A^{\mathrm{in(neq)}}(\omega, \mathbf{k}, \omega_{(\mathrm{b})}, \mathbf{k}_{(\mathrm{b})}, \omega'_{(\mathrm{b})}, \mathbf{k}'_{(\mathrm{b})}, \ldots)\\&\quad=A^{\mathrm{out(neq)}}(\omega, \mathbf{k}, -\omega_{(\mathrm{b})}, -\mathbf{k}_{(\mathrm{b})}, -\omega'_{(\mathrm{b})}, -\mathbf{k}'_{(\mathrm{b})}, \ldots)\end{aligned} \tag{78}$$

for the solution to be real. We can readily see that these imply specific relations between coefficients of the derivative expansion of both sides of the above equation. We can also derive similar requirements at higher orders in the amplitude expansion.

IV. BOUNDARY CONDITIONS AND NONEQUILIBRIUM RESPONSE FUNCTIONS

In the previous section, we have studied the solution of a free scalar field in a nonequilibrium geometry. These geometries represent nonequilibrium states which are perturbatively connected to thermal equilibrium in the derivative and amplitude expansions. Utilizing these expansions, we will now build a framework in which we can determine the nonequilibrium corrections to the Green's functions holographically.

A. On the physical significance of the boundary conditions at the horizon

We have seen in the previous section that the nonequilibrium modification of the solution of the scalar field is determined uniquely by regularity. We required the nonequilibrium modes generated by the background quasinormal modes to be regular at the horizon. This amounted to putting both the boundary conditions at the horizon—namely, setting the coefficient of the incoming and outgoing homogeneous solutions to zero.

We may wonder what is the physical significance of putting both boundary conditions at the horizon in terms of the dual field theory. This has been partly explored in Ref. [13].

Firstly, we indeed expect that the external source will receive nonequilibrium medium modifications just like the expectation value of the operator. This is because the external source as seen by the quasiparticles should be screened by the nonequilibrium collective excitations of the medium. Thus, the Dirichlet boundary condition is the wrong thing to use for the nonequilibrium bulk modes, as this would imply no such screening happens [13].

Secondly, the quasiparticle dispersion relation is modified by the nonequilibrium collective excitations of the medium. We know, for instance, that the mass of the quasiparticles receives thermal contributions. Taking this thermal mass into account is necessary for curing infrared divergences in thermal field theory. In nonequilibrium field theory, the effective mass is space-time dependent, which is natural, given that the temperature is space-time dependent as well. Furthermore, the effective mass can depend on the velocity and shear-stress perturbations as well. There is no systematic way to obtain the dependence of the effective mass as a function of the background temperature, velocity and shear-stress perturbations in nonequilibrium field theory.

Our holographic prescription gives a way to achieve this. The quasiparticle dispersion relation in equilibrium is obtained by solving for $\omega^{(\mathrm{eq})}(\mathbf{k})$ at a given $\mathbf{k}$, such that the external source satisfies $\mathcal{J}^{(\mathrm{eq})}(\omega^{(\mathrm{eq})}(\mathbf{k}), \mathbf{k}) = 0$, the latter being determined by the incoming boundary condition at the horizon. This can be readily generalized to nonequilibrium to obtain the space-time-dependent shifts of the quasiparticle dispersion relations as parametrized by the background velocity, temperature and shear-stress perturbations [13]. This is reviewed in Appendix C.

The shift in the effective mass should come from the resummation of infrared divergences. As the infrared physics is given by the dynamics of the horizon holographically, this indeed justifies why both the boundary conditions for the nonequilibrium modes should be applied at the horizon. Thus, we determine the nonequilibrium modification of the source uniquely, and hence the space-time shifts of the quasiparticle dispersion relations.

Building on this intuition, we can claim that the dependence of the nonequilibrium contributions of the Green's functions on the temperature, velocity and shear-stress perturbations characterizing the collective excitations should also come from resumming infrared divergences. Hence, we can build on the hypothesis that the boundary conditions which will determine the nonequilibrium contributions to the propagators should be applied at the horizon.

Thus, we will examine general boundary conditions at the horizon which need not lead to a regular solution in the bulk. In the next subsection, we will study how the nonequilibrium corrections to the external source and the expectation value of the operator are parametrized by these boundary conditions. Then, in the following subsection, we will use linear response theory and field theoretic consistency to determine the relevant boundary conditions for a given nonequilibrium Green's function.

We will find eventually that regularity at the horizon determines all the nonequilibrium Green's functions in the holographic classical gravity approximation. In the final subsection, we will return to nonequilibrium field theory to intuitively understand this.

B. Nonequilibrium modifications of external source and expectation value

For the moment, let us assume we have specified a pair of boundary conditions at the horizon which need not give a solution which is regular at the horizon. This means we have made specific choices of $A^{\mathrm{in(neq)}}(\omega, \mathbf{k}, \omega_{(\mathrm{b})}, \mathbf{k}_{(\mathrm{b})})$ and $A^{\mathrm{in(neq)}}(\omega, \mathbf{k}, \omega_{(\mathrm{b})}, \mathbf{k}_{(\mathrm{b})})$, the coefficients of the two homogeneous solutions. Any such choice has to be consistent with derivative and amplitude expansions. As, for instance, in Eq. (68), this amounts to making arbitrary choices for $A_i^{\mathrm{in(neq)}}(\omega, \mathbf{k})$ and $A_i^{\mathrm{out(neq)}}(\omega, \mathbf{k})$.

After such choices are made, the entire solution is determined uniquely. We can readily determine the normalizable and non-normalizable modes from the solution by following its behavior at the boundary. The full non-normalizable mode, which is the sum of the equilibrium and nonequilibrium parts, is identified with the external source coupling to the dual operator. Let us denote this as $\mathcal{J}(x)$. The full normalizable mode is identified with the expectation value of the dual operator in the dual state, namely $\mathcal{O}(x)$.

Thus, the equilibrium part of the source and the expectation value of the operator can be obtained from the behavior of $\Phi^{(\mathrm{eq})}(k, r)$ near $r = 0$, as given below:

$$\Phi^{(\mathrm{eq})}(k, r) \approx \mathcal{J}^{(\mathrm{eq})}(k) r^{4-\Delta} + \mathcal{O}^{(\mathrm{eq})}(k) r^{\Delta}. \tag{79}$$

In the above equation, Δ is the anomalous dimension of the dual operator and is related to the mass of the bulk scalar field by

$$\Delta = 2 + \sqrt{4 + m^2 l^2}. \tag{80}$$

The nonequilibrium corrections to the source and the expectation value of the operator in the dual nonequilibrium state can be obtained from the behavior of $\Phi^{(\mathrm{neq})}(k, k_{(\mathrm{b})}, r)$ and $\Phi^{(\mathrm{neq})}(k, -k_{(\mathrm{b})}, r)$ near $r = 0$,

$$\Phi^{(\mathrm{neq})}(k, \pm k_{(\mathrm{b})}, r) \approx \mathcal{J}^{(\mathrm{neq})}(k, \pm k_{(\mathrm{b})}) r^{4-\Delta} + \mathcal{O}^{(\mathrm{neq})}(k, \pm k_{(\mathrm{b})}) r^{\Delta}. \tag{81}$$

In the case of the hydrodynamic shear mode background, for example, the nonequilibrium source and expectation value take the following form, as can be seen from Eq. (67):

$$\begin{aligned}(\mathcal{J}, \mathcal{O})^{(\mathrm{neq})}(\omega, \mathbf{k}, \pm\omega_{(\mathrm{h})}, \pm\mathbf{k}_{(\mathrm{h})}) = {} & A^{\mathrm{in(neq)}}(\omega, \mathbf{k}, \pm\omega_{(\mathrm{h})}, \pm\mathbf{k}_{(\mathrm{h})})(\mathcal{J}, \mathcal{O})^{\mathrm{in(eq)}}(\omega \pm \omega_{(\mathrm{h})}, \mathbf{k} \pm \mathbf{k}_{(\mathrm{h})}) \\ & + A^{\mathrm{out(neq)}}(\omega, \mathbf{k}, \pm\omega_{(\mathrm{h})}, \pm\mathbf{k}_{(\mathrm{h})})(\mathcal{J}, \mathcal{O})^{\mathrm{in(eq)}}(\omega \pm \omega_{(\mathrm{h})}, \mathbf{k} \pm \mathbf{k}_{(\mathrm{h})}) \\ & + A^{\mathrm{in(eq)}}(\omega, \mathbf{k})(\mathcal{J}, \mathcal{O})^{\mathrm{in(neq)}}(\omega, \mathbf{k}, \pm\omega_{(\mathrm{h})}, \pm\mathbf{k}_{(\mathrm{h})}).\end{aligned} \tag{82}$$

In the above equation, $(\mathcal{J}, \mathcal{O})^{\mathrm{in(eq),out(eq)}}(\omega \pm \omega_{(\mathrm{h})}, \mathbf{k} \pm \mathbf{k}_{(\mathrm{h})})$ are obtained from asymptotic expansions of $\Phi^{\mathrm{in(eq),out(eq)}}(\omega \pm \omega_{(\mathrm{h})}, \mathbf{k} \pm \mathbf{k}_{(\mathrm{h})}, r)$. Similarly, $(\mathcal{J}, \mathcal{O})^{\mathrm{in(neq)}} \times (\omega, \mathbf{k}, \pm\omega_{(\mathrm{h})}, \pm\mathbf{k}_{(\mathrm{h})})$ are obtained from the asymptotic expansions of $\Phi^{\mathrm{in(neq)}}(\omega, \mathbf{k}, \pm\omega_{(\mathrm{h})}, \pm\mathbf{k}_{(\mathrm{h})}, r)$. We note that $\Phi^{\mathrm{in(eq),out(eq)}}(\omega \pm \omega_{(\mathrm{h})}, \mathbf{k} \pm \mathbf{k}_{(\mathrm{h})}, r)$ and $\Phi^{\mathrm{in(neq)}}(\omega, \mathbf{k}, \pm\omega_{(\mathrm{h})}, \pm\mathbf{k}_{(\mathrm{h})}, r)$ are each solutions of the radial equation, and therefore have the same asymptotic expansions.

Furthermore, using the specific form of the coefficients $A^{\mathrm{in(neq),out(neq)}}$ as given in Eq. (68), it is easy to see that the nonequilibrium modifications of the external source and the expectation value take the form

$$(\mathcal{J},\mathcal{O})^{(\mathrm{neq})}(\omega,\pm\omega_{(\mathrm{h})},\mathbf{k},\pm\mathbf{k}_{(\mathrm{h})}) = (\mathcal{J},\mathcal{O})_1^{(\mathrm{neq})}(\omega,\mathbf{k})\delta\mathbf{u}(\pm\omega_{(\mathrm{h})},\pm\mathbf{k}_{(\mathrm{h})})\cdot\mathbf{k} \pm(\mathcal{J},\mathcal{O})_2^{(\mathrm{neq})}(\omega,\mathbf{k})k_ik_jk_{(\mathrm{h})i}\delta u_j(\pm\omega_{(\mathrm{h})},\pm\mathbf{k}_{(\mathrm{h})})+O(\epsilon^2), \tag{83}$$

with

$$(\mathcal{J},\mathcal{O})_i^{(\mathrm{neq})}(\omega,\mathbf{k}) = A_i^{\mathrm{in(neq)}}(\omega,\mathbf{k})(\mathcal{J},\mathcal{O})^{\mathrm{in(eq)}}(\omega,\mathbf{k}) + A_i^{\mathrm{out(neq)}}(\omega,\mathbf{k})(\mathcal{J},\mathcal{O})^{\mathrm{out(eq)}}(\omega,\mathbf{k}) + (\mathcal{J},\mathcal{O})_i^{\mathrm{in(neq)}}(\omega,\mathbf{k}). \tag{84}$$

In this form, it is explicitly clear how the nonequilibrium modifications of the source and the expectation value depend on the boundary conditions at the horizon, namely in the choice of $A_i^{\mathrm{in(neq),out(neq)}}(\omega,\mathbf{k})$.

The full space-time dependence of the source $\mathcal{J}(x)$ and the expectation value of the operator $\mathcal{O}(x)$ thus take the form

$$(\mathcal{J},\mathcal{O})(\omega,\mathbf{k},\mathbf{x},t) = (\mathcal{J},\mathcal{O})^{(\mathrm{eq})}(\omega,\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + \int_{\mathcal{C}} d\omega_{(\mathrm{b})}(\mathcal{J},\mathcal{O})^{(\mathrm{neq})}(\omega,\mathbf{k},\omega_{(\mathrm{b})},\mathbf{k}_{(\mathrm{b})}) \times e^{i((\mathbf{k}+\mathbf{k}_{(\mathrm{b})})\cdot\mathbf{x}-(\omega+\omega_{(\mathrm{b})})t)} + \int_{\mathcal{C}*} d\omega_{(\mathrm{b})}(\mathcal{J},\mathcal{O})^{(\mathrm{neq})} \times(\omega,\mathbf{k},-\omega_{(\mathrm{b})},-\mathbf{k}_{(\mathrm{b})})e^{i((\mathbf{k}-\mathbf{k}_{(\mathrm{b})})\cdot\mathbf{x}-(\omega-\omega_{(\mathrm{b})})t)}. \tag{85}$$

After the integrations over $\omega_{(\mathrm{b})}$ required to take the background on shell are done as above, we are left with the residues at the poles of the quasinormal modes. Therefore,

$$(\mathcal{J},\mathcal{O})(\omega,\mathbf{k},\mathbf{x},t) = (\mathcal{J},\mathcal{O})^{(\mathrm{eq})}(\omega,\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + (\mathcal{J},\mathcal{O})^{(\mathrm{neq})}(\omega,\mathbf{k},\mathbf{k}_{(\mathrm{b})})e^{i((\mathbf{k}+\mathbf{k}_{(\mathrm{b})})\cdot\mathbf{x}))}e^{-i(\omega+\mathrm{Re}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})}))t} \times e^{-\mathrm{Im}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})t} + (\mathcal{J},\mathcal{O})^{(\mathrm{neq})}(\omega,\mathbf{k},-\mathbf{k}_{(\mathrm{b})})e^{i((\mathbf{k}-\mathbf{k}_{(\mathrm{b})})\cdot\mathbf{x}))} \times e^{-i(\omega-\mathrm{Re}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})}))t}e^{-\mathrm{Im}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})t}. \tag{86}$$

Thus, in the case of the hydrodynamic shear mode background, we get

$$(\mathcal{J},\mathcal{O})(\omega,\mathbf{k},\mathbf{x},t) = (\mathcal{J},\mathcal{O})^{(\mathrm{eq})}(\omega,\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + (\mathcal{J},\mathcal{O})^{(\mathrm{neq})} \times(\omega,\mathbf{k},\mathbf{k}_{(\mathrm{h})})e^{i((\mathbf{k}+\mathbf{k}_{(\mathrm{h})})\cdot\mathbf{x}))}e^{-i\omega t}e^{-\frac{\mathbf{k}_{(\mathrm{h})}^2}{4\pi T}t} + (\mathcal{J},\mathcal{O})^{(\mathrm{neq})}(\omega,\mathbf{k},-\mathbf{k}_{(\mathrm{h})})e^{i((\mathbf{k}-\mathbf{k}_{(\mathrm{h})})\cdot\mathbf{x}))}e^{-i\omega t}e^{-\frac{\mathbf{k}_{(\mathrm{h})}^2}{4\pi T}t}, \tag{87}$$

where

$$(\mathcal{J},\mathcal{O})^{(\mathrm{neq})}(\omega,\mathbf{k},\pm\mathbf{k}_{(\mathrm{h})}) = (\mathcal{J},\mathcal{O})_1^{(\mathrm{neq})}(\omega,\mathbf{k})\delta\mathbf{u}(\pm\mathbf{k}_{(\mathrm{h})})\cdot\mathbf{k}\pm(\mathcal{J},\mathcal{O})_2^{(\mathrm{neq})} \times(\omega,\mathbf{k})k_ik_jk_{(\mathrm{h})i}\delta u_j(\pm\mathbf{k}_{(\mathrm{h})})+O(\epsilon^2), \tag{88}$$

with $(\mathcal{J},\mathcal{O})_i^{(\mathrm{neq})}$ given by Eq. (84). Comparing Eqs. (83) and (88), we see that we have extracted the residue of $\delta\mathbf{u}(\pm\omega_{(\mathrm{h})},\pm\mathbf{k}_{(\mathrm{h})})$ as given in Eq. (45) at the poles $\omega_{(\mathrm{h})}=\pm i\mathbf{k}_{(\mathrm{h})}^2/(4\pi T)$. In general, $(\mathcal{J},\mathcal{O})^{(\mathrm{neq})}(\omega,\mathbf{k},\pm\mathbf{k}_{(\mathrm{b})})$ will take the form

$$(\mathcal{J},\mathcal{O})^{(\mathrm{neq})}(\omega,\mathbf{k},\pm\mathbf{k}_{(\mathrm{b})}) = A^{\mathrm{in(neq)}}(\omega,\mathbf{k},\pm\mathbf{k}_{(\mathrm{b})})(\mathcal{J},\mathcal{O})^{\mathrm{in(eq)}}(\omega,\mathbf{k}) + A^{\mathrm{out(neq)}}(\omega,\mathbf{k},\pm\mathbf{k}_{(\mathrm{b})})(\mathcal{J},\mathcal{O})^{\mathrm{out(eq)}}(\omega,\mathbf{k}) + (\mathcal{J},\mathcal{O})^{\mathrm{in(neq)}}(\omega,\mathbf{k},\pm\mathbf{k}_{(\mathrm{b})}), \tag{89}$$

with $A^{\mathrm{in(neq),out(neq)}}$ and $(\mathcal{J},\mathcal{O})^{\mathrm{in(neq)}}$ having consistent derivative and amplitude expansions. For instance, $A^{\mathrm{in(neq)}}(\omega,\mathbf{k},\mathbf{k}_{(\mathrm{b})}) = A_3^{\mathrm{in(neq)}}(\omega,\mathbf{k})\pi_{ij}(\mathbf{k}_{(\mathrm{b})})k_ik_j+\cdots$.

The above discussion is readily generalized by including nonlinearities in the dynamics of the nonequilibrium perturbations of the background. For instance, if we take into account the quadratic dependence on two hydrodynamic shear modes $\delta\mathbf{u}(\mathbf{k}_{(\mathrm{h})})$ and $\delta\mathbf{u}(\mathbf{k}'_{(\mathrm{h})})$, the corrections will take the form

$$(\mathcal{J},\mathcal{O})^{(\mathrm{neq})}(\omega,\mathbf{k},\mathbf{k}_{(\mathrm{h})},\mathbf{k}'_{(\mathrm{h})})e^{i(\mathbf{k}_{(\mathrm{h})}+\mathbf{k}'_{(\mathrm{h})})\cdot\mathbf{x}}e^{-i\omega t}e^{-\left(\frac{\mathbf{k}_{(\mathrm{h})}^2+\mathbf{k}'^2_{(\mathrm{h})}}{4\pi T}\right)t} + (\mathbf{k}_{(\mathrm{h})}\to-\mathbf{k}_{(\mathrm{h})})+(\mathbf{k}'_{(\mathrm{h})}\to-\mathbf{k}'_{(\mathrm{h})}) \tag{90}$$

after doing integrations over $\omega_{(\mathrm{h})}$ and $\omega'_{(\mathrm{h})}$. Furthermore,

$$(\mathcal{J},\mathcal{O})^{(\mathrm{neq})}(\omega,\mathbf{k},\mathbf{k}_{(\mathrm{h})},\mathbf{k}'_{(\mathrm{h})}) = (\mathcal{J},\mathcal{O})_3^{(neq)}(\omega,\mathbf{k})(\delta\mathbf{u}(\mathbf{k}_{(\mathrm{h})})\cdot\mathbf{k})(\delta\mathbf{u}(\mathbf{k}'_{(\mathrm{h})})\cdot\mathbf{k}_{(\mathrm{h})})+\cdots \tag{91}$$

at the first order in the derivative expansion. All these terms are fixed uniquely in terms of the boundary conditions at the horizon.

The general form of the nonequilibrium corrections to source and expectation value as in Eq. (89) after including nonlinear dynamics of the background involves

$$(\mathcal{J},\mathcal{O})^{(\mathrm{neq})}(\omega,\mathbf{k},\pm\mathbf{k}_{(\mathrm{b})1},\ldots,\pm\mathbf{k}_{(\mathrm{b})n}) = A^{\mathrm{in(neq)}}(\omega,\mathbf{k},\pm\mathbf{k}_{(\mathrm{b})1},\ldots,\pm\mathbf{k}_{(\mathrm{b})n})(\mathcal{J},\mathcal{O})^{\mathrm{in(eq)}} \times(\omega,\mathbf{k})+A^{\mathrm{out(neq)}}(\omega,\mathbf{k},\pm\mathbf{k}_{(\mathrm{b})1},\ldots,\pm\mathbf{k}_{(\mathrm{b})n}) \times(\mathcal{J},\mathcal{O})^{\mathrm{out(eq)}}(\omega,\mathbf{k})+(\mathcal{J},\mathcal{O})^{\mathrm{in(neq)}} \times(\omega,\mathbf{k},\pm\mathbf{k}_{(\mathrm{b})1},\ldots,\pm\mathbf{k}_{(\mathrm{b})n}), \tag{92}$$

which will appear with a spatiotemporal factor

$$e^{i(\pm\mathbf{k}_{(b)1}\pm\cdots\pm\mathbf{k}_{(b)n})\cdot\mathbf{x}}e^{-i(\pm\mathrm{Re}\omega_{(b)1}(\mathbf{k}_{(b)1})\pm\cdots\pm\mathrm{Re}\omega_{(b)n}(\mathbf{k}_{(b)n}))t}\times e^{(\pm\mathrm{Im}\omega_{(b)1}(\mathbf{k}_{(b)1})\pm\cdots\pm\mathrm{Im}\omega_{(b)n}(\mathbf{k}_{(b)n}))t}.$$

C. Mapping linear response to nonequilibrium Green's functions

There is a basic linear response to an external perturbation in any physical system: the causal response. This causal response gives the retarded Green's function.

Holographically, the equilibrium causal response maps to the incoming boundary condition at the horizon [17]. This is expected to be the case because it is the only boundary condition that is consistent with the causal structure of the black hole, which forbids anything propagating out of the horizon classically.

We have seen that the incoming equilibrium solution of the bulk scalar field has a unique nonequilibrium correction which is regular at the horizon and is free of potentially harmful backreaction at the future horizon. We need regularity at the horizon to preserve the global causal structure of the black brane. We recall that this nonequilibrium solution which is regular at the horizon requires very specific boundary conditions at the horizon. These boundary conditions have been determined even after including nonlinearities in the dynamics of the perturbations of the background quasinormal modes.

Furthermore, this nonequilibrium solution which is regular at the future horizon uniquely determines the nonequilibrium modifications to the external source and the expectation value of the dual operator. It has been proposed that it determines the nonequilibrium corrections to the causal response, and therefore the nonequilibrium retarded Green's function [13].

The retarded Green's function in any state is given by the causal response in that state. Thus, the holographic nonequilibrium retarded Green's function is

$$G_R(\mathbf{x}_1, t_1; \mathbf{x}_2 t_2) = \int d\omega d^3k \frac{\mathcal{O}(\omega, \mathbf{k}, \mathbf{x}_1, t_1)}{\mathcal{J}(\omega, \mathbf{k}, \mathbf{x}_2, t_2)}. \quad (93)$$

Let us go back to the linearized approximation when the nonequilibrium perturbation in the background is given by a single quasinormal mode, or a single relaxation mode in the dual theory. In that case, the full spatiotemporal forms of $\mathcal{J}$ and $\mathcal{O}$ above are as in Eq. (86). The boundary conditions required for regularity at the horizon are that the coefficients of the homogeneous solutions $A^{\mathrm{in(neq),out(neq)}}$ should be set to zero, so that the nonequilibrium corrections to external source/expectation values are given fully by $(\mathcal{J}, \mathcal{O})^{\mathrm{in(neq)}}(\omega, \mathbf{k}, \pm\mathbf{k}_{(b)})$, respectively.

One can see that after performing the Wigner transform (for intermediate steps, see Appendix D), the nonequilibrium retarded Green's function takes the following form [13]:

$$\begin{aligned}
G_R(\omega, \mathbf{k}, \mathbf{x}, t) = \int d\omega_1 \int d^3k_1 G_R^{(\mathrm{eq})}(\omega_1, \mathbf{k}_1)\Bigg[&\delta(\omega-\omega_1)\delta^3(\mathbf{k}-\mathbf{k}_1)\\
&-\frac{1}{2\pi i}\Bigg(\mathcal{M}(\omega_1, \mathbf{k}_1, \mathbf{k}_{(b)})\delta^3\left(\mathbf{k}-\mathbf{k}_1-\frac{\mathbf{k}_{(b)}}{2}\right)\frac{1}{(\omega-\omega_1-\frac{\mathrm{Re}\omega_{(b)}(\mathbf{k}_{(b)})}{2}-i\frac{\mathrm{Im}\omega_{(b)}(\mathbf{k}_{(b)})}{2})}\\
&-\mathcal{N}(\omega_1, \mathbf{k}_1, \mathbf{k}_{(b)})\delta^3\left(\mathbf{k}-\mathbf{k}_1+\frac{\mathbf{k}_{(b)}}{2}\right)\frac{1}{(\omega-\omega_1+\frac{\mathrm{Re}\omega_{(b)}(\mathbf{k}_{(b)})}{2}+i\frac{\mathrm{Im}\omega_{(b)}(\mathbf{k}_{(b)}))}{2})}\Bigg)e^{i\mathbf{k}_{(b)}\cdot\mathbf{x}}e^{-i\,\mathrm{Re}\omega_{(b)}(\mathbf{k}_{(b)})t}e^{\mathrm{Im}\omega_{(b)}(\mathbf{k}_{(b)})t}\\
&+\frac{1}{2\pi i}\Bigg(\mathcal{M}(\omega_1, \mathbf{k}_1, -\mathbf{k}_{(b)})\delta^3\left(\mathbf{k}-\mathbf{k}_1+\frac{\mathbf{k}_{(b)}}{2}\right)\frac{1}{(\omega-\omega_1+\frac{\mathrm{Re}\omega_{(b)}(\mathbf{k}_{(b)})}{2}+i\frac{\mathrm{Im}\omega_{(b)}(\mathbf{k}_{(b)})}{2})}\\
&-\mathcal{N}(\omega_1, \mathbf{k}_1, -\mathbf{k}_{(b)})\delta^3\left(\mathbf{k}-\mathbf{k}_1-\frac{\mathbf{k}_{(b)}}{2}\right)\frac{1}{(\omega-\omega_1-\frac{\mathrm{Re}\omega_{(b)}(\mathbf{k}_{(b)})}{2}-i\frac{\mathrm{Im}\omega_{(b)}(\mathbf{k}_{(b)})}{2})}\Bigg)\\
&\times e^{-i\mathbf{k}_{(b)}\cdot\mathbf{x}}e^{i\,\mathrm{Re}\omega_{(b)}(\mathbf{k}_{(b)})t}e^{\mathrm{Im}\omega_{(b)}(\mathbf{k}_{(b)})t}\Bigg],
\end{aligned} \quad (94)$$

with

$$G_R^{(\mathrm{eq})}(\omega, \mathbf{k}) = \frac{\mathcal{O}^{(\mathrm{eq})}(\omega, \mathbf{k})}{\mathcal{J}^{(\mathrm{eq})}(\omega, \mathbf{k})} \quad (95)$$

being the equilibrium retarded Green's function and

$$\mathcal{M}(\omega_1, \mathbf{k}_1, \pm\mathbf{k}_{(b)}) = \frac{\mathcal{O}^{\mathrm{in(neq)}}(\omega_1, \mathbf{k}_1, \pm\mathbf{k}_{(b)})}{\mathcal{O}^{(\mathrm{eq})}(\omega_1, \mathbf{k}_1)}, \qquad \mathcal{N}(\omega_1, \mathbf{k}_1, \mathbf{k}_{(b)}) = \frac{\mathcal{J}^{\mathrm{in(neq)}}(\omega_1, \mathbf{k}_1, \pm\mathbf{k}_{(b)})}{\mathcal{J}^{(\mathrm{eq})}(\omega_1, \mathbf{k}_1)}. \quad (96)$$

The complex conjugate mode corrections included above needed for complete field-theoretic consistency (see below) were not included in our earlier work [13].

The crucial elements to note are as follows:

(1) The nonequilibrium contributions involve two pieces, spatiotemporally comoving with the background quasinormal modes at momenta $\pm\mathbf{k}_{(\mathrm{b})}$, respectively.

(2) The nonequilibrium contributions to the equilibrium retarded propagator at frequency ω and momentum $\mathbf{k}$ involves a convolution with support at momenta $\mathbf{k}\pm\mathbf{k}_{(\mathrm{b})}/2$ and poles at frequencies $\omega\pm\mathrm{Re}\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})/2\pm i\,\mathrm{Im}\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})/2$.

These features reflect the classical gravity approximation. We will see how these generalize beyond the quasinormal-mode approximation of the background.

Also note that as the nonequilibrium correction to the equilibrium solution which is regular at the future horizon is unique, the overall solution is undetermined up to an overall multiplicative constant $A(\omega,\mathbf{k})$, as in Eq. (54). This multiplicative constant $A(\omega,\mathbf{k})$ cancels between the numerator and denominator in Eq. (93). Therefore, the nonequilibrium retarded Green's function is uniquely determined by this holographic prescription.

The advanced Green's function should be naturally related to the advanced response. Thus,

$$G_A(\mathbf{x}_1,t_1;\mathbf{x}_2,t_2)=\int d\omega d^3k\,\frac{\mathcal{O}(\omega,\mathbf{k},\mathbf{x}_2,t_2)}{\mathcal{J}(\omega,\mathbf{k},\mathbf{x}_1,t_1)}. \tag{97}$$

The above also follows from the definitions that $G_A(\mathbf{x}_1,t_1;\mathbf{x}_2,t_2)=G_R(\mathbf{x}_2,t_2;\mathbf{x}_1,t_1)$ (for proof of the latter, see Appendix A). For the holographic definition of advanced response, both $\mathcal{O}$ and $\mathcal{J}$ above must be evaluated in the same nonequilibrium solution as in Eq. (93) in the case of the retarded propagator, i.e., in the solution involving the unique correction to the equilibrium incoming mode which is regular at the future horizon. Therefore, $\mathcal{O}$ and $\mathcal{J}$ take the form, as in Eq. (86), with $(\mathcal{J},\mathcal{O})^{(\mathrm{neq})}=(\mathcal{J},\mathcal{O})^{\mathrm{in(neq)}}$.

From Appendix A, it follows that after the Wigner transform, $G_A(\omega,\mathbf{k},\mathbf{x},t)=G_R(-\omega,-\mathbf{k},\mathbf{x},t)$. Therefore, $G_A(\omega,\mathbf{k},\mathbf{x},t)$ is given by Eq. (94) with $(\omega,\mathbf{k})$ replaced by $(-\omega,-\mathbf{k})$.

Also, $G_R*(k,x)=G_A(k,x)$, as proved in Appendix A. This is also satisfied, as can be seen from Eq. (94), using $G_A(k,x)=G_R(-k,x)$. To do this, one first reverses the sign of the integrated variables ω_1 and $\mathbf{k}_1$ and then uses results from Sec. III C which imply that $(\mathcal{J},\mathcal{O})^{(\mathrm{eq})}*(k)=(\mathcal{J},\mathcal{O})^{(\mathrm{eq})}(-k)$ and $(\mathcal{J},\mathcal{O})^{\mathrm{in(neq)}}*(k,\mathbf{k}_{(\mathrm{b})})=(\mathcal{J},\mathcal{O})^{\mathrm{in(neq)}}(-k,-\mathbf{k}_{(\mathrm{b})})$. Thus, our holographic prescriptions for the nonequilibrium retarded and advanced propagators pass important field-theoretic consistency tests.

At the beginning of this section, we have already made the hypothesis that nonequilibrium Green's functions in the holographic classical gravity approximation should be determined by appropriate boundary conditions at the horizon. These boundary conditions should be determined by the consistency of the map with field-theoretic requirements.

In order to map a nonequilibrium Green's function to boundary conditions at the horizon, we start from the gravity side. Consider an arbitrary solution of the bulk scalar field in the nonequilibrium geometry which is consistent with the derivative/amplitude expansions but not necessarily regular at the horizon. This means that the space-time profile of the source and expectation value takes the form of Eq. (86), with the nonequilibrium parts taking the general form of Eq. (89). We should make a choice of both $A^{\mathrm{in(neq)}}(\omega,\mathbf{k},\mathbf{k}_{(\mathrm{b})})$ and $A^{\mathrm{out(neq)}}(\omega,\mathbf{k},\mathbf{k}_{(\mathrm{b})})$ to specify the solution uniquely. This amounts to making choices of coefficients of $A_i^{\mathrm{in(neq)}}(\omega,\mathbf{k})$ and $A_i^{\mathrm{out(neq)}}(\omega,\mathbf{k})$ of terms like $\delta\mathbf{u}(\mathbf{k}_{(\mathrm{b})})\cdot\mathbf{k}$, $\pi_{ij}^{(\mathrm{nh})}(\mathbf{k}_{(\mathrm{b})})k_ik_j$ in their derivative/amplitude expansions, as in Eq. (68).

The most well-defined response functions on the gravity side are the causal and advanced response functions. Let us consider the following response function evaluated in an arbitrary nonequilibrium solution:

$$G(x_1,x_2)=\int d\omega d^3k\left(f_R^{(\mathrm{eq})}(\omega)\frac{O(\omega,\mathbf{k},\mathbf{x}_1,t_1)}{J(\omega,\mathbf{k},\mathbf{x}_2,t_2)}+f_A^{(\mathrm{eq})}(\omega)\frac{O(\omega,\mathbf{k},\mathbf{x}_2,t_2)}{J(\omega,\mathbf{k},\mathbf{x}_1,t_1)}\right). \tag{98}$$

Physically, the boundary conditions given by the choice of $A^{\mathrm{in(neq)}}(\omega,\mathbf{k},\mathbf{k}_{(\mathrm{b})})$ and $A^{\mathrm{out(neq)}}(\omega,\mathbf{k},\mathbf{k}_{(\mathrm{b})})$ can be interpreted as follows: The full response function above at equilibrium is the sum of the causal response with weight $f_R^{(\mathrm{eq})}(\omega)$ and the advanced response with weight $f_A^{(\mathrm{eq})}(\omega)$. The boundary conditions $A^{\mathrm{in(neq)}}(\omega,\mathbf{k},\mathbf{k}_{(\mathrm{b})})$ and $A^{\mathrm{out(neq)}}(\omega,\mathbf{k},\mathbf{k}_{(\mathrm{b})})$ thus shift the weight of the nonequilibrium causal response and the nonequilibrium advanced response in the total weighted sum, because individually they are given by the boundary conditions $A^{\mathrm{in(neq)}}(\omega,\mathbf{k},\mathbf{k}_{(\mathrm{b})})$ and $A^{\mathrm{out(neq)}}(\omega,\mathbf{k},\mathbf{k}_{(\mathrm{b})})$, which are set to zero.

It is then easy to repeat the steps which lead from Eqs. (93) to (94), as explicitly shown in Appendix D. We find that Eq. (98) reproduces our general parametrization of any nonequilibrium Green's function as a weighted sum of the retarded and advanced Green's functions [Eq. (32)], which is valid at the leading order in amplitude expansion, with the identification

$$
\begin{aligned}
f_R^{(1,0)}(\omega, \mathbf{k}, \omega_1, \mathbf{k}_1, \mathbf{x}, t) = & -\frac{f_R^{(\mathrm{eq})}(\omega)}{2\pi i}\bigg(\tilde{\mathcal{M}}(\omega_1, \mathbf{k}_1, \mathbf{k}_{(\mathrm{b})})\delta^3\bigg(\mathbf{k} - \mathbf{k}_1 - \frac{\mathbf{k}_{(\mathrm{b})}}{2}\bigg)\frac{1}{(\omega - \omega_1 - \frac{\mathrm{Re}\omega_{(\mathrm{b})}(\mathbf{k}_{(b)})}{2} - i\frac{\mathrm{Im}\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})}{2})} \\
& + \tilde{\mathcal{N}}(\omega_1, \mathbf{k}_1, \mathbf{k}_{(\mathrm{b})})\delta^3\bigg(\mathbf{k} - \mathbf{k}_1 + \frac{\mathbf{k}_{(\mathrm{b})}}{2}\bigg)\frac{1}{(\omega - \omega_1 + \frac{\mathrm{Re}\omega_{(\mathrm{b})}(\mathbf{k}_{(b)})}{2} + i\frac{\mathrm{Im}\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})}))}{2})}\bigg) \\
& \times e^{i\mathbf{k}_{(\mathrm{b})}\cdot\mathbf{x}}e^{-i\,\mathrm{Re}\omega_{(\mathrm{b})}(\mathbf{k}_{(b)})t}e^{\mathrm{Im}\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})t} + \frac{f_R^{(\mathrm{eq})}(\omega)}{2\pi i}\bigg(\tilde{\mathcal{M}}(\omega_1, \mathbf{k}_1, -\mathbf{k}_{(\mathrm{b})})\delta^3\bigg(\mathbf{k} - \mathbf{k}_1 + \frac{\mathbf{k}_{(\mathrm{b})}}{2}\bigg) \\
& \times \frac{1}{(\omega - \omega_1 + \frac{\mathrm{Re}\omega_{(\mathrm{b})}(\mathbf{k}_{(b)})}{2} + i\frac{\mathrm{Im}\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})}{2})} + \tilde{\mathcal{N}}(\omega_1, \mathbf{k}_1, -\mathbf{k}_{(\mathrm{b})})\delta^3\bigg(\mathbf{k} - \mathbf{k}_1 - \frac{\mathbf{k}_{(\mathrm{b})}}{2}\bigg) \\
& \times \frac{1}{(\omega - \omega_1 - \frac{\mathrm{Re}\omega_{(\mathrm{b})}(\mathbf{k}_{(b)})}{2} - i\frac{\mathrm{Im}\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})}{2})}\bigg)e^{-i\mathbf{k}_{(\mathrm{b})}\cdot\mathbf{x}}e^{i\,\mathrm{Re}\omega_{(\mathrm{b})}(\mathbf{k}_{(b)})t}e^{\mathrm{Im}\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})t},
\end{aligned}
\tag{99}
$$

where

$$
\begin{aligned}
\tilde{\mathcal{M}}(\omega_1, \mathbf{k}_1, \pm\mathbf{k}_{(\mathrm{b})}) &= \frac{\mathcal{O}^{(\mathrm{neq})}(\omega_1, \mathbf{k}_1, \pm\mathbf{k}_{(\mathrm{b})}) - \mathcal{O}^{\mathrm{in(neq)}}(\omega_1, \mathbf{k}_1, \pm\mathbf{k}_{(\mathrm{b})})}{\mathcal{O}^{(\mathrm{eq})}(\omega_1, \mathbf{k}_1)} \\
&= \frac{A^{\mathrm{in(neq)}}(\omega_1, \mathbf{k}_1, \pm\mathbf{k}_{(\mathrm{b})})\mathcal{O}^{\mathrm{in(eq)}}(\omega_1, \mathbf{k}_1) + A^{\mathrm{out(neq)}}(\omega_1, \mathbf{k}_1, \pm\mathbf{k}_{(\mathrm{b})})\mathcal{O}^{\mathrm{out(eq)}}(\omega_1, \mathbf{k}_1)}{\mathcal{O}^{(\mathrm{eq})}(\omega_1, \mathbf{k}_1)}, \\
\tilde{\mathcal{N}}(\omega_1, \mathbf{k}_1, \pm\mathbf{k}_{(\mathrm{b})}) &= \frac{\mathcal{J}^{(\mathrm{neq})}(\omega_1, \mathbf{k}_1, \pm\mathbf{k}_{(\mathrm{b})}) - \mathcal{J}^{\mathrm{in(neq)}}(\omega_1, \mathbf{k}_1, \pm\mathbf{k}_{(\mathrm{b})})}{\mathcal{J}^{(\mathrm{eq})}(\omega_1, \mathbf{k}_1)} \\
&= \frac{A^{\mathrm{in(neq)}}(\omega_1, \mathbf{k}_1, \pm\mathbf{k}_{(\mathrm{b})})\mathcal{J}^{\mathrm{in(eq)}}(\omega_1, \mathbf{k}_1) + A^{\mathrm{out(neq)}}(\omega_1, \mathbf{k}_1, \pm\mathbf{k}_{(\mathrm{b})})\mathcal{J}^{\mathrm{out(eq)}}(\omega_1, \mathbf{k}_1)}{\mathcal{J}^{(\mathrm{eq})}(\omega_1, \mathbf{k}_1)}
\end{aligned}
\tag{100}
$$

Also, $f_A^{(1,0)}(k, k_1, x)$ is given by the above Eqs. (99) and (100) with $f_R^{(\mathrm{eq})}(\omega)$ replaced by $f_A^{(\mathrm{eq})}(\omega)$ and $(\omega, \mathbf{k})$ replaced by $(-\omega, -\mathbf{k})$.

Thus, there is a map between between the parameters $f_{R,A}^{(1,0)}$ in the leading-order parametrization [Eq. (32)] of nonequilibrium Green's functions required for consistent perturbation expansions, and the leading-order boundary conditions $A^{\mathrm{in(neq),out(neq)}}$ for the nonequilibrium modes set at the horizon. However, $f_{R,A}^{(1,0)}$ should also satisfy other field-theoretic requirements, aside from producing consistent derivative and amplitude expansions.

Let us work this out explicitly for the nonequilibrium Feynman propagator. In this case, we should set in the gravity response function [Eq. (98)] $f_R^{(\mathrm{eq})}(\omega) = n_{\mathrm{BE}}(\omega) + 1$ and $f_A^{(\mathrm{eq})}(\omega) = f_R^{(\mathrm{eq})}(-\omega) = -n_{\mathrm{BE}}(\omega)$ to reproduce the equilibrium Feynman propagator.

The holographic nonequilibrium Feynman propagator $G_F(x_1, x_2)$ is therefore given by

$$
\begin{aligned}
G_F(x_1, x_2) = \int d\omega d^3k\bigg(&(n_{\mathrm{BE}}(\omega) + 1)\frac{O(\omega, \mathbf{k}, \mathbf{x}_1, t_1)}{J(\omega, \mathbf{k}, \mathbf{x}_2, t_2)} \\
&- n_{\mathrm{BE}}(\omega)\frac{O(\omega, \mathbf{k}, \mathbf{x}_2, t_2)}{J(\omega, \mathbf{k}, \mathbf{x}_1, t_1)}\bigg),
\end{aligned}
\tag{101}
$$

with both $\mathcal{J}$ and $\mathcal{O}$ evaluated in a nonequilibrium solution with boundary conditions given by $A^{\mathrm{in(neq)}}(\omega, \mathbf{k}, \mathbf{k}_{(\mathrm{b})})$ and $A^{\mathrm{out(neq)}}(\omega, \mathbf{k}, \mathbf{k}_{(\mathrm{b})})$, both of which we should determine now using field-theoretic consistency conditions.

Note that the holographic Feynman propagator [Eq. (101)] is symmetric in x_1 and x_2 as required. This can be seen by exchanging x_1 and x_2 while reversing the signs of the integrated variables ω and $\mathbf{k}$.

Let us now compare the holographic form of the nonequilibrium Feynman propagator [Eq. (101)] with the parametrization [Eq. (30)] that yields consistent perturbative expansions obtained earlier. Our comparison yields

$$
\begin{aligned}
& f^{(1,0)}(k, k_1, x) \\
&\quad = -i\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}P\bigg(\frac{f_R^{(1,0)}(k, \omega', \mathbf{k}_1, x) + f_A^{(1,0)}(k, \omega', \mathbf{k}_1, x)}{\omega' - \omega_1}\bigg) \\
&\qquad -\frac{1}{2}(f_R^{(1,0)}(k, k_1, x) - f_A^{(1,0)}(k, k_1, x)),
\end{aligned}
\tag{102}
$$

where $f_R^{(1,0)}(k, k_1, x)$ is given by Eq. (99) with $f_R^{(\mathrm{eq})}(\omega) = n_{\mathrm{BE}}(\omega) + 1$, and $f_A^{(1,0)}(k, k_1 x)$ is also given by Eq. (99) with $f_R^{(\mathrm{eq})}(\omega)$ replaced by $f_A^{(\mathrm{eq})}(\omega) = f_R^{(\mathrm{eq})}(-\omega) = -n_{\mathrm{BE}}(\omega)$. Also, P denotes the principal value.

For the case of the Feynman propagator, using Eq. (78) and $f_A^{(\mathrm{eq})}(\omega) = f_R^{(\mathrm{eq})}(-\omega)$, we find that

$$
f_R^{(1,0)}(k, k_1, x) = f_A^{(1,0)}(-k, -k_1, x). \tag{103}
$$

We can now compare with the requirements in Eq. (28) to obtain

$$f_S^{(1,0)}(k,k_1,x)=0, \qquad f_A^{(1,0)}(k,k_1,x)=f^{(1,0)}(k,k_1,x). \tag{104}$$

We recall that the above implies that the constraint of symmetry in x_1 and x_2 is satisfied for the holographic $G_F(x_1,x_2)$, which is also evident in the defining form [Eq. (101)].

There are two crucial field-theoretic requirements in Eq. (28) which should be additionally satisfied in the above identifications. These are that $f_S^{(1,0)}(k,k_1,x)$ and $f_A^{(1,0)}(k,k_1,x)$ should be real. One can readily see from Eq. (99) that it is impossible to satisfy these for all x unless both $f_R^{(1,0)}(k,k_1,x)$ and $f_A^{(1,0)}(k,k_1,x)$ vanish identically. The latter is possible only when $A^{\mathrm{in(neq)}}$ and $A^{\mathrm{out(neq)}}$ are both set to zero, giving the solution which is regular at the horizon. This implies there are no nonequilibrium shifts in the weights of the causal and advanced responses in Eq. (101).

Thus, we have proved when the quasinormal approximation of the background is valid:

$$G_F(k,x)=(n_{\mathrm{BE}}(\omega)+1)G_R(k,x)-n_{\mathrm{BE}}(\omega)G_A(k,x). \tag{105}$$

This follows simply from Eq. (101), using boundary conditions where $A^{\mathrm{in(neq)}}$ and $A^{\mathrm{out(neq)}}$ are set to zero in Eq. (101) and then performing a Wigner transform.

We can now go beyond the quasinormal mode approximation of the background. This means taking into account nonlinearities in the dynamics of the hydrodynamic and relaxational modes constituting the nonequilibrium state.

We note that Eq. (93) still gives the holographic retarded Green's function because it follows from linear response theory—the expectation value and source are obtained from the full nonequilibrium solution which depends nonlinearly on the nonequilibrium perturbations $\delta\mathbf{u}$, δT and $\pi_{ij}^{(\mathrm{nh})}$ and is regular at the future horizon. We have discussed how we obtain this full nonequilibrium solution in Sec. III C—namely we apply the same boundary conditions which cut down both the homogeneous incoming and outgoing solutions at each order in the amplitude perturbation expansion.

The retarded response function depends nonlinearly on each term in the perturbation expansion, though. Let us denote (n) as the order in the amplitude expansion; i.e., it counts the total number of pertubation parameters $\delta\mathbf{u}$, δT and $\pi_{ij}^{(\mathrm{nh})}$ as in Sec. III C. Then the retarded Green's function up to nth order in the amplitude expansion is obtained from

$$\frac{\mathcal{O}^{(\mathrm{eq})}+\mathcal{O}^{(1)}+\cdots+\mathcal{O}^{(n)}}{\mathcal{J}^{(\mathrm{eq})}+\mathcal{J}^{(1)}+\cdots+\mathcal{J}^{(n)}}.$$

At the nth order we get not only $\mathcal{O}^{(n)}/\mathcal{J}^{(n)}$ but also $(\mathcal{O}^{(1)}/\mathcal{J}^{(1)})^n$. Above, we have suppressed the derivative expansion, but it clear we can do a double expansion in the derivatives and amplitudes of the background perturbation close to equilibrium.

Similarly, the advanced Green's function is given by Eq. (97), but the expectation value and the source are obtained from the full nonequilibrium solution which is regular at the horizon. One can readily check that the full holographic retarded and advanced propagators also satisfy the relation $G_R*(k,x)=G_A(k,x)$ by simply repeating the same steps discussed earlier to all orders in the perturbation expansion.

One can now map the holographic linear response as given by Eq. (98) to a given nonequilibrium propagator. In Eq. (98), we need to use appropriate boundary conditions at the horizon instead of those fixed by regularity at each order in the amplitude expansion. These boundary conditions give the nonequilibrium shifts of the weight of the retarded and advanced Green's functions which constitute the given nonequilibrium propagator.

It is also easy to see that, at the second order in the amplitude expansion, we will obtain the same structure [Eq. (33)] as we argued from field theory from the holographic response [Eq. (98)]. Namely, $f_{RR,RA,AR,AA}^{(2,0,0)}(k,k_1,k_2,x)$ will depend on the boundary conditions on the solutions $\Phi^{(2,n,m)}$, where n and m denote the order of the derivative expansion acting on each of the two amplitude expansions. These are reflected by the boundary conditions given by $A^{(2,n,m)\mathrm{in(neq)}}(k,\pm k_{(\mathrm{b})1},\pm k_{(\mathrm{b})2})$ and $A^{(2,n,m)\mathrm{out(neq)}}(k,\pm k_{(\mathrm{b})}$. We note that the reality constraints [Eq. (78)] tell us there are exactly four independent boundary conditions for the four independent parameters $f_{RR,RA,AR,AA}^{(2,0,0)}(k,k_1,k_2,x)$. Without writing the detailed expression for $f_{RR,RA,AR,AA}^{(2,0,0)}(k,k_1,k_2,x)$ we can readily see that, taking all orders in derivative expansion into account,

(1) The dependence on $\mathbf{k}$ is given by a sum of four terms which get supported by $\delta^3(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2\pm\mathbf{k}_{(\mathrm{b})1}\pm\mathbf{k}_{(\mathrm{b})2})$, respectively.

(2) As functions of ω, these terms have poles in $\omega-\omega_1-\omega_2\pm\omega(k_{(\mathrm{b})1})\pm\omega(k_{(\mathrm{b})2})$, respectively, where $\omega(k_{(\mathrm{b})1})$ and $\omega(k_{(\mathrm{b})2})$ are the complex dispersion relations of the two quasinormal modes appearing at the quadratic order.

(3) The space-time dependences are given by $e^{i(\pm\mathbf{k}_{(\mathrm{b})1}\pm\mathbf{k}_{(\mathrm{b})2})\cdot\mathbf{x})}e^{-i(\pm\omega(\mathbf{k}_{(\mathrm{b})1})\pm\omega(\mathbf{k}_{(\mathrm{b})2}))t}$, respectively.

One can similarly work out that $f^{(2,1,0)}(k,k_1,k_2,x,x_1)$ are determined by $A^{(1)\mathrm{in(neq),out(neq)}}(k,k_{(\mathrm{b})})$, the boundary conditions appearing at the linear order in the amplitude expansion.

We can now specialize to the holographic Feynman propagator as given by Eq. (101). We can see that we reproduce the general form [Eq. (31)] at the second order in the amplitude expansion. We get $f^{(2,0,0)}(k,k_1,k_2,x)$ similarly from the boundary conditions at the quadratic order and satisfy the properties listed above. Once again, these satisfy the symmetry requirements—but not the reality constraints—at all **x** and t, unless both $A^{\mathrm{in(neq)}}(k, k_{(\mathrm{b})1}, k_{(\mathrm{b})2})$ and $A^{\mathrm{out(neq)}}(k, k_{(\mathrm{b})1}, k_{(\mathrm{b})2})$ are set to zero; thus, the solution is taken to be exactly the same as given by regularity at the horizon. Similarly, one can show that $f^{(2,1,0)}(k, k_1, k_2, x, x_1)$ in Eq. (31) should also vanish. This result generalizes to higher orders in the amplitude expansion.

Thus, we show that the holographic Feynman propagator is given by Eq. (105) to all orders in the amplitude expansion. In other words, only the solution which is regular at the horizon contributes in the linear response [Eq. (101)] holographically in determining the nonequilibrium Feynman propagator.

Given that both the nonequilibrium retarded Green's function and the Feynman propagator are determined by the regular solution, all nonequilibrium propagators are also determined by the regular solution in the gravitational response [Eq. (98)]. Thus, the boundary conditions which determine the nonequilibrium corrections to the Green's functions are determined by regularity alone.

D. On why regularity at the horizon is sufficient

Our main result is that the regularity at the horizon and the linear response theory together determine all the nonequilibrum Green's functions holographically. We can try to understand the physical significance of this result in terms of the dual field theory.

Clearly, the nonequilibrium dynamics of the expectation value of the operator, as (for instance) the energy-momentum tensor, is determined by regularity at the horizon. Indeed, this is how hydrodynamic transport coefficients are determined. In field theory, this is given by the effective action $\Gamma[\mathcal{O}]$, as discussed in Sec. II.

We recall from Sec. II A that the nonequilibrium evolution of the propagators is given by the effective action $\Gamma[\mathcal{O}, G]$ evaluated over the Schwinger-Keldysh contour. In Sec. II A we also showed that the effective action $\Gamma[\mathcal{O}, G]$ can be obtained from $\Gamma[\mathcal{O}]$ applying functional identities. Thus, we should require no new information to determine the dynamics of nonequilibrium Green's functions other than those required to determine the nonequilibrium evolution of the expectation value of the operator.

Holographically, this should imply that, just as regularity at the horizon is a sufficient principle to determine the nonequilibrium evolution of the expectation value of the operator, it should be a sufficient principle to determine the nonequilibrium Green's functions also. Therefore, the results of Sec. II A give a field-theoretic justification of the holographic arguments advanced here.

V. THE HOLOGRAPHIC NONEQUILIBRIUM FLUCTUATION-DISSIPATION RELATION

In the previous section, we have proved that the holographic nonequilibrium Feynman propagator is given by Eq. (105). This result is true in nonequilibrium states close to thermal equilibrium, or more precisely, in states perturbatively connected to equilibrium in derivative and amplitude expansions. This result holds in the absence of any external forces. Also, this result is valid even when the nonlinearities in the dynamics of hydrodynamic and nonhydrodynamic variables characterizing the nonequilibrium state are taken into account.

From the holographic Feynman propagator [Eq. (105)], one can readily obtain the nonequilibrium statistical function using Eq. (18). Thus, we obtain the holographic nonequilibrium fluctuation-dissipation relation:

$$\begin{aligned} G_{\mathcal{K}}(k, x) &= i(2n_{\mathrm{BE}}(\omega) + 1)\mathrm{Im}G_R(k, x) \\ &= -i\left(n_{\mathrm{BE}}(\omega) + \frac{1}{2}\right)\mathcal{A}(k, x). \end{aligned} \tag{106}$$

We recall again that the Bose-Einstein distribution, $n_{\mathrm{BE}}(\omega)$ above, is determined by the temperature of final thermal equilibrium. Thus, at long times we recover the usual equilibrium fluctuation-dissipation relation.

We may wonder if this result is in contradiction with the locality and causality of the dual field theory, because it depends on the temperature of thermal equilibrium to be attained in the future. This is not the case, for the following reasons:

(1) The fluctuation-dissipation relation [Eq. (106)] relates the Wigner-transformed statistical function $G_{\mathcal{K}}(k, x)$ and $\mathcal{A}(k, x)$. The Wigner transform involves Fourier transform in the relative coordinate $x_1 - x_2$ in both $G_{\mathcal{K}}(x_1, x_2)$ and $\mathcal{A}(x_1, x_2)$. The Fourier transform receives contributions from all values of the relative coordinate $x_1 - x_2$; thus, both $G_{\mathcal{K}}(k, x)$ and $\mathcal{A}(k, x)$ are nonlocal in space and time by construction.

(2) The result [Eq. (106)] is true strictly in the absence of external forces. In such situations, no energy is being pumped in externally; therefore, the final equilibrium temperature is fixed by the total energy of the initial state. So there is no teleological adjustment involved, as may naively appear to be the case.

Crucially, $G_{\mathcal{K}}(k, x)$ and $\mathcal{A}(k, x)$ behave as local objects in space-time typically at weak coupling. In this case, the Kadanoff-Baym equations governing their evolution reduce approximately to a Boltzmann equation which is local in space and time. At strong coupling, we do not expect this to be the case; therefore, we have no reason to expect that $G_{\mathcal{K}}(k, x)$ and $\mathcal{A}(k, x)$ will be related by local data. What we find is that the total conserved energy in the laboratory frame is sufficient to relate them—we do not need any more details of the nonequilibrium state, provided the field theory can be described holographically and the classical gravity approximation with a minimally coupled bulk scalar is valid. Individually, though, $G_{\mathcal{K}}(k, x)$ and $\mathcal{A}(k, x)$ carry detailed information of the relaxational modes of the system and their nonlinear dynamics.

We also note that in the late equilibrium state where our derivative and amplitude expansions are valid, the background hydrodynamic and nonhydrodynamic variables satisfy generic phenomenological equations which can be derived from classical gravity (for details, see Appendix B). These phenomenological equations, just like the Navier-Stokes equation, hold irrespectively of initial conditions. The dual geometries have regular future horizons. Nevertheless, we can expect that not all solutions of these generic equations can be lifted to solutions in the field theory. Only a class of dual nonequilibrium geometries will also be regular in the past, and these will be genuine solutions in field theory. However, regularity in the past cannot be analyzed in perturbative derivative and amplitude expansions. Thus, there will indeed be hidden constraints in the initial conditions for $\delta\mathbf{u}$, δT and $\pi_{ij}^{(\mathrm{nh})}$ coming from the requirement of regularity in the past. Nevertheless, these do not affect our main conclusions, at least for the large class of nonequilibrium states which admit permit perturbative derivative and amplitude expansions in the future.

We may also ask how this result may generalize to higher-point nonequilibrium Green's functions. To address this, we have to study the backreaction of the scalar field on the background geometry with an arbitrary external source at the boundary. Regarding the generalization, we make the following conjectures:

(1) In the classical gravity approximation, the fully causal higher-point Green's function $\Theta(t_n - t_{n-1}) \ldots \Theta(t_2 - t_1)\langle[O(x_n), [O(x_{n-1}), \ldots [O(x_2), O(x_1)] \ldots]]\rangle$ can be determined from the regular solution alone.

(2) The other higher-point Green's functions can be obtained from the causal higher-point function using their general field-theoretic properties and the consistency of their allowed forms with the classical gravity approximation.

The first point above simply says that a regular future horizon determines the full causal structure, and hence the causal Green's functions. The second point above generalizes our derivation of the holographic Feynman propagator. This conjecture says that we do not need to construct any partition function to obtain nonequilibrium correlation functions—general dynamical principles should suffice to determine them as in field theory. In future work, we would like to test or prove this conjecture.

VI. DO OUR RESULTS SURVIVE STRINGY CORRECTIONS TO CLASSICAL GRAVITY?

We will find here that we cannot certainly say that our results will survive the stringy corrections which typically give $1/\sqrt{\lambda}$ corrections in the strong coupling limit as a function of the dimensionless coupling λ. In gauge theories, λ is the 't Hooft coupling.

Nevertheless, all of our conclusions generalize even if there are stringy corrections in gravity dynamics (involving higher orders in curvatures and their derivatives), provided the scalar field dual to the operator is minimally coupled to gravity. The latter is certainly not guaranteed, but could be true in supersymmetric contexts.

Firstly, we will show that our prescriptions which give solutions which are regular at the future horizon in the nonequilibrium geometry indeed remain valid, as long as the bulk scalar field is minimally coupled to gravity.

We observe that the structure of the homogeneous incoming solution near the horizon can be determined from geometric optics, if the bulk scalar field is minimally coupled to gravity. We can certainly construct an appropriate function of r, which we denote as $r_*(r)$, such that the incoming radial null geodesic at the horizon is

$$v = t - r_*(r).$$

Clearly, $r_*(r)$ has to increase monotonically as r moves towards the horizon because of blueshifting. The homogeneous incoming wave solution in the thermal background by a quasinormal mode perturbation will always behave as

$$\approx e^{-i(\omega + \omega_{(\mathrm{b})})v}$$

near the horizon at the leading order, as the geometrical optics approximation is always good at the horizon due to the blueshifting. Therefore, as long as thermal geometry is stable, the integration over $\omega_{(b)}$, which we need to do to put the metric on shell, will produce a divergent factor:

$$e^{-\mathrm{Im}\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})r*(r)}.$$

If the thermal background is stable, all its quasinormal fluctuations should satisfy $\mathrm{Im}\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})}) < 0$. Therefore, the homogeneous incoming solution will always be divergent at the horizon.

On the other hand, the argument that the outgoing modes will cause divergent backreaction at the horizon is based on analyticity in ω—this remains true as long as the thermal background is stable.

In any consistent classical gravity, the black brane dual to the thermal background should be stable; thus, we need to cut down both the homogeneous incoming and the outgoing modes for regularity at the horizon.

The chain of arguments determining nonequilibrium propagators will similarly carry through all the way to the nonequilibrium fluctuation-dissipation relation [Eq. (106)]. This will be true, provided the black brane does not support quasinormal modes where both the momentum and frequency could be imaginary, or where the frequency/momentum is imaginary while the momentum/frequency vanishes, respectively. One can check, if this happens, the reality arguments which cut down other boundary conditions to determine that nonequilibirum propagators other than those required by regularity will no longer hold.

In fact, a quasinormal mode with imaginary momentum at zero frequency will signify the existence of hair. Also, a quasinormal mode with imaginary frequency at zero momentum can be interpreted as a thermal tensorlike Goldstone mode. The first situation is unlikely, as usually black branes do not have hair. The second situation is unlikely, as usually in field theories we do not have spontaneous breaking of Lorentz symmetry.

Even if the bulk scalar remains minimally coupled taking stringy corrections into account, we still need some care. Though the equations of motion may admit systematic perturbative expansion in $1/\sqrt{\lambda}$, all solutions which thermalize to black branes need not admit such expansion. Also, we cannot take the limit $\lambda \rightarrow 0$ smoothly, as in this limit curvature diverges asymptotically. Therefore, we still need to be on the strong coupling side for our results to generalize.

We may conclude that the nonequilibrium fluctuation-dissipation relation [Eq. (106)] requires not only the validity of the classical gravity approximation, but also for the classical theory of gravity to be nearly Einstein's theory, and the bulk scalar field to be minimally coupled to gravity.

VII. CONCLUDING REMARKS

The key points of this paper may be summarized as follows:

(1) The holographic nonequilibrium spectral function carries systematic information about the relaxation modes of the system and their collective nonlinear dynamics.

(2) The holographic prescription for obtaining the nonequilibrium spectral function is independent of the nonequilibrium state (i.e., the gravitational background) and is obtained by requiring regularity at the horizon at strong coupling and large N.

(3) This holographic prescription should also hold in any theory of classical gravity which has a stable thermal background as a solution as long as the bulk scalar dual to the bosonic operator is minimally coupled to gravity.

(4) It is possible to map the parametrization of nonequilibrium Green's functions required for their consistent derivative/amplitude expansions to linear response functions in gravity using arbitrary boundary conditions for the nonequilibrium modes.

(5) Field-theoretic consistency determines the holographic nonequilibrium Feynman propagator.

(6) From the holographic nonequilibrium Feynman propagator, we obtain the nonequilibrium fluctuation-dissipation relation.

(7) The holographic nonequilibrium fluctuation-dissipation relation holds universally for any nonequilibrium state at strong coupling in the classical gravity approximation and thus provides a potentially strong test of the applicability of holographic duality.

Except for the first two points, all these have been derived in this paper.

We may now discuss how the holographic scenario can be tested at RHIC or ALICE. Our discussion will assume that the plasma produced by the heavy-ion collision undergoes three stages of evolution: (i) the initial phase, (ii) a strongly coupled nearly conformal phase, and (iii) the final hadron gas phase. The classical gravity approximation studied here can only apply to the middle phase of evolution. This picture is supported by lattice studies, as in Ref. [31], where it was found that QCD is indeed close to a strongly coupled fixed point at a scale of about 175 MeV (which is nearly the temperature of the quark-gluon plasma produced at RHIC/ALICE) for small chemical potentials. This is also consistent with the observed incredibly short time of thermalization (≈ 1 fm/c) [32,33].

We note that in the early stage of the ultrarelativistic collisions, we may use perturbative QCD, which gives a quantum kinetic theory involving quarks and gluons. In this case, we may use models like that of Ref. [34]. We may also use color glass condensate models where the 2PI effective action technique has been used to obtain the growth of fluctuations and correlations of various operators [35]. In the late stage of expansion of the fireball, we can use the ultrarelativistic quantum molecular dynamics (UrQMD) simulations, which model the quantum kinetics of the hadron gas [36]. The holographic fluctuation and correlation of the chiral condensate needs to be matched with the quantum kinetics of the early and late phases of the expansion to obtain a complete picture that can be tested experimentally.

In the holographic phase, we may use the gravitational backgrounds provided by Einstein's gravity with anti–de Sitter boundary conditions to calculate the nonequilibrium spectral function. The nonequilibrium spectral function gives us a way to determine the collective nonlinear dynamics of the relaxation modes and thus check if the phenomenological equations obtained from gravity really describe the space-time evolution of the fireball in the strongly coupled phase. The latter can also be tested independently by detailed measurement of the transverse momentum spectra and elliptic flow coefficient of the hadron gas. A mutual confirmation will be a strong test for holography.

The nonequilibrium spectral function combined with the nonequilibrium fluctuation-dissipation relation gives sufficient data to match with correlations of the emission of pions and their resonances in the later stage of expansion, as pions and their resonances have the same quantum numbers as the chiral condensate. Thus, we can build a more sophisticated theory of pion interferometry to reconstruct the expansion of the fireball and test the holographic nonequilibrium fluctuation-dissipation relation. As the fluctuation-dissipation relation holds universally for any nonequilibrium state in the classical gravity approximation, its validation will indeed be a very strong test for holography.

In fact, it has been observed that statistical models using pure thermal estimates work very well for predicting the spectrum of hadrons produced from the fireball with the temperature being set to the value at which the fireball thermalizes [32,37]. This is surprising, given that the phase transition does not occur adiabatically. This may already hint that the holographic nonequilibrium fluctuation dissipation relation is valid to a good accuracy.

The statistical function of the chiral condensate carries direct information about the production of pions and their resonances. Thus, a time-resolved study of the emission of pions and their resonances will allow us to determine the nonequilibrium statistical function once we know how to match its evolution to the early and final phases.

A more refined time-resolved study of the correlations of emitted pions and their resonances will allow determination of the nonequilibrium spectral function. This also needs to be matched with the early and late stages of evolution. Having thus determined the nonequilibrium statistical and spectral functions individually, we can check the nonequilibrium fluctuation-dissipation relation.

The matching can be done by noting that in the holographic phase, the nonequilibrium state, and also the nonequilibrium spectral and statistical functions, can be parametrized by a few nonequilibrium variables like the hydryodynamic variables and the shear-stress tensor. These variables represent the expectation value of a handful of operators like the energy-momentum tensor. Thus, the matching can be realized by tracking the evolution of these operators. Some interesting progress has already been made in this direction [38].

It thus looks possible that the holographic nonequilibrium fluctuation-dissipation relation can be tested in the future in heavy-ion collisions. This may also lead to an accurate experimental determination of the temperature at which the plasma thermalizes.

It will be also interesting to understand how the nonequilibrium fluctuation-dissipation relation can be measured in order parameter correlations in quantum critical systems.

ACKNOWLEDGMENTS

The author would like to thank Giuseppe Policastro, Umut Gursoy, Kostas Skenderis and Jan de Boer for useful discussions. The author would like to thank Giuseppe Policastro for comments on the manuscript. The author thanks Costas Bachas, Atish Dabholkar and Marios Petropoulos for encouragement in pursuing holographic nonequilibrium physics. The author thanks Souvik Banerjee for checking some equations and also for comments on the manuscript. The author would also like to thank the CERN Theory Division, the Albert Einstein Institute, Golm, and ITF, University of Amsterdam for opportunities to present this work prior to publication. The research of the author is presently supported by Grant No. ANR-07-CEXC-006 of L'Agence Nationale de La Recherche. The paper has been rewritten in its present form while the author has been visiting the CERN Theory Division.

APPENDIX A: USEFUL IDENTITIES

1. Consequences of symmetry and transposition

Consider a Hermitian scalar operator $O(x)$. The retarded Green's function for this operator is defined as

$$G_R(x_1, x_2) = -i\Theta(t_1 - t_2)\langle[O(x_1), O(x_2)]\rangle. \tag{A1}$$

The advanced Green's function is defined as

$$G_A(x_1, x_2) = i\Theta(t_2 - t_1)\langle[O(x_1), O(x_2)]\rangle. \tag{A2}$$

It follows from the definitions that

$$G_A(x_1, x_2) = G_R(x_2, x_1). \tag{A3}$$

We may also see that

$$G_R * (x_1, x_2) = G_R(x_1, x_2), \qquad G_A * (x_1, x_2) = G_A(x_1, x_2). \tag{A4}$$

Let us define the center-of-mass coordinate as $x = (x_1 + x_2)/2$ and the relative coordinate as $r = x_1 - x_2$. The Wigner-transformed retarded and advanced Green's functions are

$$G_{R,A}(k, x) = \int d^4 r e^{ik\cdot r} G_{R,A}(x, r). \tag{A5}$$

It follows from Eq. (A3) that $G_A(x, r) = G_R(x, -r)$. We thus see that

$$\begin{aligned} G_R(-k, x) &= \int d^4 r e^{-ik\cdot r} G_R(x, r) \\ &= \int d^4 r e^{ik\cdot r} G_R(x, -r) \\ &= \int d^4 r e^{ik\cdot r} G_A(x, r) \\ &= G_A(k, x). \end{aligned} \tag{A6}$$

In the second line above, we have changed variables from r to $-r$.

Similarly, Eq. (A4) implies that $G_R * (x, r) = G_R(x, r)$ and $G_A * (x, r) = G_A(x, r)$. It follows then that

$$\begin{aligned} G_R * (k, x) &= \int d^4 r e^{-ik\cdot r} G_R * (x, r) \\ &= \int d^4 r e^{-ik\cdot r} G_R(x, r) = G_A(k, x). \end{aligned} \tag{A7}$$

In the second line above, we have used $G_R * (x, r) = G_R(x, r)$. Observing that the second line coincides with the first equality in Eq. (A6), our final conclusion follows. We can similarly prove that $G_A * (k, x) = G_R(k, x)$.

We now turn to the Feynman propagator $G_F(x_1, x_2)$, which is symmetric. Therefore, $G_F(x, r) = G_F(x, -r)$. Therefore, it follows that the Wigner transform of the Feynman propagator should satisfy

$$G_F(-k, x) = \int d^4 r e^{-ik\cdot r} G_F(x, r) = \int d^4 r e^{ik\cdot r} G_F(x, -r) \\ = \int d^4 r e^{ik\cdot r} G_F(x, r) = G_F(k, x). \tag{A8}$$

In the second line above, we have changed variables from r to $-r$, and in the third line we have used $G_F(x, r) = G_F(x, -r)$.

Similarly, it follows from symmetry in x_1 and x_2 prior to Wigner transform that $G_{\mathcal{K}}(k, x) = G_{\mathcal{K}}(-k, x)$.

Using Eq. (A6), we also note that $\mathrm{Re}G_R(k, x) = (1/2) \times (G_R(k, x) + G_A(k, x)) = (1/2)(G_A(-k, x) + G_R(-k, x)) = \mathrm{Re}G_R(-k, x)$.

2. The statistical function is purely imaginary

According to Eq. (3), the inverse Wigner transform of the statistical function is given by the anticommutator as below:

$$G_{\mathcal{K}}(x, r) = -\frac{i}{2}\left\langle\left\{O\left(x + \frac{r}{2}\right), O\left(x - \frac{r}{2}\right)\right\}\right\rangle. \tag{A9}$$

Clearly,

$$G_{\mathcal{K}}(x, r) = G_{\mathcal{K}}(x, -r), \qquad G_{\mathcal{K}} * (x, r) = -G_{\mathcal{K}}(x, r). \tag{A10}$$

Therefore,

$$\begin{aligned} G_{\mathcal{K}} * (k, x) &= \int d^4 r e^{-ik\cdot r} G_{\mathcal{K}} * (x, r) \\ &= -\int d^4 r e^{-ik\cdot r} G_{\mathcal{K}}(x, r) \\ &= -\int d^4 r e^{ik\cdot r} G_{\mathcal{K}}(x, -r) \\ &= -\int d^4 r e^{ik\cdot r} G_{\mathcal{K}}(x, r) \\ &= -G_{\mathcal{K}}(k, x). \end{aligned} \tag{A11}$$

In the second line above, we have used $G_{\mathcal{K}} * (x, r) = -G_{\mathcal{K}}(x, r)$; in the third line, we have changed variables from r to $-r$; and in the fourth line, we have used $G_{\mathcal{K}}(x, r) = G_{\mathcal{K}}(x, -r)$.

Clearly, $G_{\mathcal{K}} * (k, x) = -G_{\mathcal{K}}(k, x)$ implies that $G_{\mathcal{K}}(k, x)$ is purely imaginary.

3. The spectral function and the retarded Green's function

According to Eq. (1), the inverse Wigner transform of the spectral function is given by the commutator as below:

$$\mathcal{A}(x, r) = \left\langle\left[O\left(x + \frac{r}{2}\right), O\left(x - \frac{r}{2}\right)\right]\right\rangle. \tag{A12}$$

It follows from Eqs. (A1) and (A2) and that

$$\mathcal{A}(x, r) = i(G_R(x, r) - G_A(x, r)) = -2\mathrm{Im}G_R(x, r), \tag{A13}$$

using $\Theta(t_1 - t_2) + \Theta(t_2 - t_1) = 1$. Clearly, then, after Wigner transform,

$$\mathcal{A}(k, x) = i(G_R(k, x) - G_A(k, x)) = -2\mathrm{Im}G_R(k, x). \tag{A14}$$

Using Eq. (A6), we readily see from the above that

$$\mathcal{A}(k, x) = -\mathcal{A}(-k, x). \tag{A15}$$

4. The Feynman propagator as a sum of the statistical and spectral functions

The definition of the Feynman propagator is

$$G_F(x_1, x_2) = -i\langle T(O(x_1)O(x_2))\rangle, \tag{A16}$$

where T denotes time ordering.

It follows from the definition of the Feynman propagator that

$$\begin{aligned} G_F(x_1, x_2) &= -i\langle O(x_1)O(x_2)\rangle \\ \text{if } t_1 > t_2 &= -\frac{i}{2}\langle O(x_1)O(x_2) + O(x_2)O(x_1)\rangle \\ &\quad -\frac{i}{2}\langle O(x_1)O(x_2) - O(x_2)O(x_1)\rangle \\ \text{if } t_1 > t_2 &= G_{\mathcal{K}}(x_1, x_2) - \frac{i}{2}\mathcal{A}(x_1, x_2) \quad \text{if } t_1 > t_2. \end{aligned} \tag{A17}$$

Similarly,

$$\begin{aligned} G_F(x_1, x_2) &= -i\langle O(x_2)O(x_1)\rangle \\ \text{if } t_2 > t_1 &= -\frac{i}{2}\langle O(x_1)O(x_2) + O(x_2)O(x_1)\rangle \\ &\quad +\frac{i}{2}\langle O(x_1)O(x_2) - O(x_2)O(x_1)\rangle \\ \text{if } t_2 > t_1 &= G_{\mathcal{K}}(x_1, x_2) + \frac{i}{2}\mathcal{A}(x_1, x_2) \quad \text{if } t_2 > t_1. \end{aligned} \tag{A18}$$

Combining these, we obtain

$$G_F(x_1, x_2) = G_{\mathcal{K}}(x_1, x_2) - \frac{i}{2}\mathcal{A}(x_1, x_2)\mathrm{sign}(t_1 - t_2). \tag{A19}$$

APPENDIX B: THE GENERAL PHENOMENOLOGICAL EQUATIONS FOR IRREVERSIBLE PROCESSES

The phenomenological equations generalizing hydrodynamics which can be obtained from gravity can be argued to be as follows. We can perform a kinematic Landau-Lifshitz decomposition of the energy-momentum tensor as below:

$$t_{\mu\nu} = \epsilon u_\mu u_\nu + pP_{\mu\nu} + \pi_{\mu\nu}, \tag{B1}$$

with u^μ being a timelike vector of unit norm (so $u_\mu u^\nu = -1$), and $P_{\mu\nu}$ being the projection tensor in the spatial plane orthogonal to u^μ given by $P_{\mu\nu} = u_\mu u_\nu + \eta_{\mu\nu}$.

We can define a local temperature using the equation of state $\epsilon(T)$ locally. We can also set p by using the local equation of state $p(T)$. Then $\pi_{\mu\nu}$ denotes the local nonequilirbium part of the energy-momentum tensor.

Also, u^μ is defined to be the local velocity of energy transport; therefore

$$u^\mu \pi_{\mu\nu} = 0. \tag{B2}$$

Thus, $\pi_{\mu\nu}$ has six independent components only. Together with u^μ and T, it gives ten independent variables to parametrize ten independent components of $t_{\mu\nu}$.

Furthermore, conformal invariance requires

$$\epsilon = 3p, \qquad \pi_{\mu\nu}\eta^{\mu\nu} = 0. \tag{B3}$$

This reduces the number of variables to nine, cutting down the independent components of $\pi_{\mu\nu}$ from six to five.

To proceed further, we need to decompose $\pi_{\mu\nu}$ into two parts, a purely hydrodynamic part $\pi^{(\mathrm{h})}_{\mu\nu}$ and a nonhydordynamic part $\pi^{(\mathrm{nh})}_{\mu\nu}$ which is dynamically independent of the hydrodynamic variables, though they do couple to them [20,21]. Thus,

$$\pi_{\mu\nu} = \pi^{(\mathrm{h})}_{\mu\nu} + \pi^{(\mathrm{nh})}_{\mu\nu}. \tag{B4}$$

The fluid/gravity correspondence tells us that at strong coupling, purely hydrodynamic states exist in the dual field theory in the derivative expansion. Thus, we can consistently set the nonhydrodynamic part $\pi^{(\mathrm{nh})}_{\mu\nu}$ to zero.

The general form of the purely hydrodynamic part $\pi^{(\mathrm{h})}_{\mu\nu}$ can be readily obtained order by order in the derivative expansion by constructing the conformally covariant tensors which are algebraic functionals of the derivatives of the hydrodynamic variables. The derivative expansion parameter is a typical length scale of variation with respect to the thermal wavelength. The transport coefficients are determined via regularity at the future horizon [28]. Thus,

$$\pi^{(\mathrm{h})}_{\mu\nu} = -\eta P^\rho_\mu P^\sigma_\nu\Big(\partial_\rho u_\sigma + \partial_\sigma u_\rho - \frac{2}{3}P_{\rho\sigma}(\partial\cdot u)\Big) + O(\epsilon^2). \tag{B5}$$

Setting $\pi^{(\mathrm{nh})}_{\mu\nu} = 0$, plugging $\pi_{\mu\nu} = \pi^{(\mathrm{h})}_{\mu\nu}$ in the form of $t_{\mu\nu}$ into Eq. (B1), and demanding that $\partial^\mu t_{\mu\nu} = 0$ results in the hydrodynamic equations of motion.

Energy and momentum are always conserved, and this follows directly from the constraints of Einstein's equations. This implies that

$$\partial^\mu t_{\mu\nu} = \partial^\mu(\epsilon(T)u_\mu u_\nu + p(T)P_{\mu\nu} + \pi^{(\mathrm{h})}_{\mu\nu} + \pi^{(\mathrm{nh})}_{\mu\nu}) = 0. \tag{B6}$$

The above are sufficient to determine the evolution of u^μ and T, but not that of the nonhydrodynamic shear-stress tensor $\pi^{(\mathrm{nh})}_{\mu\nu}$. We require five more equations to determine the latter. These should be provided by the regularity of the horizon in gravity, as in the case of homogeneous relaxation. We need a separate expansion parameter—the amplitude parameter $\pi^{(\mathrm{nh})}_{ij}/p$. As in the case of homogeneous relaxation, though the spatial derivatives of $\pi^{(\mathrm{nh})}_{ij}$ can be expected to be small near equilibrium, the time derivatives will be $O(T)$. The Lorentz and Weyl covariant generalization of the time derivative is $\mathcal{D} = (u\cdot\partial) + \cdots$.

The equation of motion for $\pi^{(\mathrm{nh})}_{ij}$ should be such that (i) it can be consistently set to zero, (ii) it is Lorentz and Weyl covariant, and (iii) all time derivatives are summed over at each order in the amplitude expansion. With these requirements, it follows that this equation should take the form [21,25]

$$\begin{aligned}\Big(\sum_{n=0}^{\infty} D_R^{(1,n)}\mathcal{D}^n\Big)\pi^{(\mathrm{nh})}_{\mu\nu} = {}& \frac{1}{2}\sum_{n=0}^{\infty}\Big((\lambda_1^{(n)}\mathcal{D}^n\pi^{(\mathrm{nh})\alpha}_{\mu})\partial_{(\alpha}u_{\nu)} + (\lambda_1^{(n)}\mathcal{D}^n\pi^{(\mathrm{nh})\alpha}_{\nu})\partial_{(\alpha}u_{\mu)} - \frac{2}{3}P_{\mu\nu}(\lambda_1^{(n)}\mathcal{D}^n\pi^{(\mathrm{nh})}_{\alpha\beta})\partial^\alpha u^\beta \\ & + (\lambda_2^{(n)}\mathcal{D}^n\pi^{(\mathrm{nh})\alpha}_{\mu})\partial_{[\alpha}u_{\nu]} + (\lambda_2^{(n)}\mathcal{D}^n\pi^{(\mathrm{nh})\alpha}_{\nu})\partial_{[\alpha}u_{\mu]}\Big) \\ & - \sum_{n=0}^{\infty}\sum_{m=0}^{n} D_R^{(2,n,m)}\Big[\mathcal{D}^m\pi^{(\mathrm{nh})\alpha}_{\mu}\mathcal{D}^n\pi^{(\mathrm{nh})}_{\alpha\nu} - \frac{1}{3}P_{\mu\nu}\mathcal{D}^m\pi^{(\mathrm{nh})}_{\alpha\beta}\mathcal{D}^n\pi^{(\mathrm{nh})\alpha\beta}\Big] + O(\epsilon^2\delta, \epsilon\delta^2, \delta^3)\end{aligned} \tag{B7}$$

up to given orders in the derivative/amplitude expansion. Above, (...) and [...] denote symmetrization and antisymmetrization of the enclosed indices, respectively.

Let us see how we can derive these equations from gravity for the special case of homogeneous relaxation. In this case, the flow is at rest, so we can go to an inertial frame where $u^\mu = (1, 0, 0, 0)$. Also, the temperature T is constant in space and time. Furthermore, Eq. (B2) implies that the nonzero components of $\pi_{\mu\nu}$ are the spatial components π_{ij}. The conservation of energy and momentum implies that $\pi_{ij}(t)$ is an arbitrary function of time. Nevertheless, regularity at the horizon can be expected to be guaranteed only when $\pi_{ij}(t)$ follows a definite equation of motion.

One can construct the dual metric in such cases perturbatively in the so-called amplitude expansion $\pi^{(\mathrm{nh})}_{ij}(t)/p$. Omitting the details, the metric up to second order in the amplitude expansion takes the form below [21]:

$$\begin{aligned} ds^2 = {} & \frac{l^2}{r^2}\frac{dr^2}{f(\frac{rr_0}{l^2})} + \frac{l^2}{r^2}\bigg(-f\Big(\frac{rr_0}{l^2}\Big)dt^2 + d\mathbf{x}^2 + \sum_{n=1}^{\infty} r_0^{4+n} f^{(1,n)}\Big(\frac{rr_0}{l^2}\Big)\Big(\frac{d}{dt}\Big)^n \pi^{(\mathrm{nh})}_{ij} dx^i dx^j \\ & + \sum_{n=1}^{\infty}\sum_{m=1}^{n} r_0^{8+n+m} f_1^{(2,n,m)}\Big(\frac{rr_0}{l^2}\Big)\Big(\frac{d}{dt}\Big)^n \pi^{(\mathrm{nh})}_{kl}\Big(\frac{d}{dt}\Big)^m \pi^{(\mathrm{nh})}_{kl} dt^2 + \sum_{n=1}^{\infty}\sum_{m=1}^{n} r_0^{8+n+m} f_2^{(2,n,m)}\Big(\frac{rr_0}{l^2}\Big)\Big(\frac{d}{dt}\Big)^n \pi^{(\mathrm{nh})}_{kl}\Big(\frac{d}{dt}\Big)^m \pi^{(\mathrm{nh})}_{kl} d\mathbf{x}^2 \\ & + \sum_{n=1}^{\infty}\sum_{m=1}^{n} r_0^{8+n+m} f_3^{(2,n,m)}\Big(\frac{rr_0}{l^2}\Big)\bigg(\Big(\frac{d}{dt}\Big)^n \pi^{(\mathrm{nh})}_{ik}\Big(\frac{d}{dt}\Big)^m \pi^{(\mathrm{nh})}_{kj} + (i \leftrightarrow j) - \frac{2}{3}\delta_{ij}\Big(\frac{d}{dt}\Big)^n \pi^{(\mathrm{nh})}_{kl}\Big(\frac{d}{dt}\Big)^m \pi^{(\mathrm{nh})}_{kl}\bigg) dx^i dx^j + O(\delta^3)\bigg). \end{aligned} \tag{B8}$$

We note that at each order in the amplitude expansion, we have summed over all time derivatives of $\pi^{(\mathrm{nh})}_{ij}(t)$. Thus, at the first order in amplitude expansion, we get an infinite number of radial functions $f^{(1,n)}$ corresponding to the nth time derivative. These can be determined uniquely by requiring $f^{(1,1)}(s) = s^4 + O(s^8)$ and $f^{(1,n)}(s) = O(s^{4+n})$ for $n > 1$. In these Schwarzchild coordinates it turns out that $f^{(1,n)}$ vanishes for odd n. Similarly, at the second order in the amplitude expansion, we get an infinite number of radial functions $f_1^{(2,n,m)}$, $f_2^{(2,n,m)}$ and $f_3^{(2,n,m)}$, all of which can be determined uniquely from their boundary behavior, $f_i^{(2,n,m)} = O(s^{8+n+m})$ for $i = 1,\ 2,\ 3$. Also, in these Schwarzchild coordinates, $f_i^{(2,n,m)}$ vanishes when $n + m$ is odd.

The regularity analysis of the metric [Eq. (B8)] is subtle and involves its translation to Eddington-Finkelstein coordinates. One finds that the metric is regular if, up to second order in the amplitude expansion, $\pi^{(\mathrm{nh})}_{ij}(t)$ satisfies the following equation of motion [21]:

$$\begin{aligned} & \sum_{n=0}^{\infty} D^{(1,n)}\Big(\frac{d}{dt}\Big)^n \pi^{(\mathrm{nh})}_{ij} + \sum_{n=0}^{\infty}\sum_{m=0}^{n} D^{(2,m,n)}\bigg(\Big(\frac{d}{dt}\Big)^n \pi^{(\mathrm{nh})}_{ik}\Big(\frac{d}{dt}\Big)^m \\ & \quad \times \pi^{(\mathrm{nh})}_{kj} - \frac{1}{3}\delta_{ij}\Big(\frac{d}{dt}\Big)^n \pi^{(\mathrm{nh})}_{kl}\Big(\frac{d}{dt}\Big)^m \pi^{(\mathrm{nh})}_{kl}\bigg) + O(\delta^3) = 0. \end{aligned} \tag{B9}$$

In order to see regularity of the metric at a future horizon, order by order in the amplitude expansion, it is necessary to sum over all time derivatives at each order. One can determine the nonhydrodynamic phenomenological coefficients $D^{(1,n)}$ and $D^{(2,n,m)}$ in terms of complicated recursion relations. The first few terms are

$$\begin{aligned} & D_R^{(1,0)} = -\pi T; \qquad D_R^{(1,1)} = -\Big(\frac{\pi}{2} - \frac{1}{4}\ln 2\Big)(\pi T)^2, \text{ etc.}; \\ & D_R^{(2,0,0)} = \frac{1}{2(\pi T)^4}, \text{ etc.} \end{aligned} \tag{B10}$$

That it is necessary to sum over all time derivatives at each order in the amplitude expansion to obtain regularity at the horizon can be understood from the first order in the amplitude expansion itself. It can be shown that the series $\sum D_R^{(1,n)}(-i\omega)^n$ has simple zeroes at the location of the discrete homogenous quasinormal modes (with $\mathbf{k} = 0$) in the complex lower half plane. The locations of such quasinormal modes are approximately given by [29]

$$\omega_{(\mathrm{n})} \approx \pi T[\pm 1.2139 - 0.775 i \pm 2n(1 \mp i)]. \tag{B11}$$

We observe that both the real and imaginary parts are of $O(T)$, signifying that the time derivatives of $\pi^{(\mathrm{nh})}_{ij}$ are of $O(T)$. Thus, we do need to sum over all time derivatives, as we cannot do a small-frequency expansion even in the linearized approximation.

The Eq. (B7) reduces to Eq. (B9) in the special case in which T and u^μ are constant is space and time, and furthermore we go to the laboratory frame where $u^\mu = (1, 0, 0, 0)$. Thus, $D_R^{(1,n)}$ and $D_R^{(2,n,m)}$ should be as given in Eq. (B10). The coefficients $\lambda_1^{(n)}$ and $\lambda_2^{(n)}$ denote the coupling of $\pi^{(\mathrm{nh})}_{\mu\nu}$ and its local time derivatives to hydrodynamic variables. To determine these, one needs to construct metrics with regular future horizons for configurations where u^μ and T vary spatially and temporally while $\pi^{(\mathrm{nh})}_{\mu\nu}$ is nonzero.

APPENDIX C: NONEQUILIBRIUM SHIFTS IN QUASIPARTICLE DISPERSION RELATIONS

Holographically, the vanishing of the source gives us the quasiparticle poles (which can be so broad that we may not call them quasiparticles). At equilibrium, $\mathcal{J}^{\mathrm{in(eq)}}(\omega, \mathbf{k})$ vanishes only when ω takes certain discrete values at a given $\mathbf{k}$. We may select a branch of quasiparticle poles given by

$$\omega = \omega^{(\mathrm{eq})}(\mathbf{k}) \quad \text{such that } \mathcal{J}^{\mathrm{in(eq)}}(\omega^{(\mathrm{eq})}(\mathbf{k}), \mathbf{k}) = 0. \tag{C1}$$

These correspond to the poles which can be complex generally. The nonequilibrium modification depends on space and time, and takes the form

$$\omega = \omega^{(\mathrm{eq})}(\mathbf{k}) + \delta\omega(\mathbf{k}, \mathbf{x}, t). \tag{C2}$$

The nonequilibrium shift in the pole $\delta\omega(\mathbf{k}, \mathbf{x}, t)$ can be obtained by solving $\mathcal{J}(\mathbf{x}, t) = 0$ perturbatively [13]. This amounts to solving the linear complex equation

$$\begin{aligned}\delta\omega(\mathbf{k}, \mathbf{x}, t)\partial_\omega \mathcal{J}^{\mathrm{in(eq)}}(\omega = \omega^{(\mathrm{eq})}(\mathbf{k}), \mathbf{k})\\ = -\mathcal{J}^{\mathrm{in(neq)}}(\omega = \omega^{(\mathrm{eq})}(\mathbf{k}), \mathbf{k}, \mathbf{k}_{(\mathrm{b})})e^{-i\mathrm{Re}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})t}\\ \times e^{\mathrm{Im}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})t}e^{i\mathbf{k}_{(\mathrm{b})}\cdot\mathbf{x}}\\ - \mathcal{J}^{\mathrm{in(neq)}}(\omega = \omega^{(\mathrm{eq})}(\mathbf{k}), \mathbf{k}, -\mathbf{k}_{(\mathrm{b})})e^{i\mathrm{Re}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})}))t}\\ \times e^{\mathrm{Im}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})t}e^{-i\mathbf{k}_{(\mathrm{b})}\cdot\mathbf{x}}.\end{aligned} \tag{C3}$$

The nonequilibrium parts of the source, or equivalently, the right-hand side of the above, is determined completely by the equilibrium source by our prescription at the horizon. This is precisely what we expect in the dual field theory. The nonequilibrium shift in the pole should be determined solely by the infrared behavior. For example, if the temperature fluctuates, so should the thermal mass. This gives a good justification of our prescription. Our general expression [Eq. (C3)] shows that the pole of the resummed nonequilibrium propagator can depend on the velocity and shear-stress perturbations also.

It is evident from Eq. (C3) that $\delta\omega(\mathbf{k}, \mathbf{x}, t)$ will be comoving, with the relaxational mode constituting the nonequilibrium state (or the dual quasinormal mode) as below:

$$\begin{aligned}\delta\omega(\mathbf{k}, \mathbf{x}, t) = \delta\omega(\mathbf{k}, \mathbf{k}_{(\mathrm{b})})e^{-i\mathrm{Re}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})t}e^{\mathrm{Im}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})t}e^{i\mathbf{k}_{(\mathrm{b})}\cdot\mathbf{x}}\\ + \delta\omega(\mathbf{k}, -\mathbf{k}_{(\mathrm{b})})e^{i\mathrm{Re}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})}))t}e^{\mathrm{Im}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})t}\\ \times e^{-i\mathbf{k}_{(\mathrm{b})}\cdot\mathbf{x}}.\end{aligned} \tag{C4}$$

Furthermore, $\delta\omega(\mathbf{k}, \mathbf{k}_{(\mathrm{b})})$ will have a consistent expansion in the derivative/amplitude expansion which follows from that of $\mathcal{J}^{\mathrm{in(neq)}}(\omega, \mathbf{k}, \mathbf{k}_{(\mathrm{b})})$. For example, in the case of the hydrodynamic shear wave background, $\delta\omega(\mathbf{k}, \mathbf{k}_{(\mathrm{h})})$ takes the form

$$\delta\omega(\mathbf{k}, \mathbf{k}_{(\mathrm{h})}) = \delta\omega_1(\mathbf{k})\delta\mathbf{u}\cdot\mathbf{k} + \delta\omega_2(\mathbf{k})k_i k_j k_{(\mathrm{h})i}\delta u_j \tag{C5}$$

up to first order in the derivative expansion. We thus explicitly see in the above example how our holographic prescription gives the nonequilibrium shift in the quasiparticle pole parametrized in terms of the background velocity perturbation.

APPENDIX D: WIGNER TRANSFORM OF THE HOLOGRAPHIC RETARDED GREEN'S FUNCTION

Writing explicitly, we get

$$\begin{aligned}G_R(\mathbf{x}_1, t_1, \mathbf{x}_2, t_2) &= \int d\omega d^3k \frac{\mathcal{O}(\omega, \mathbf{k}, \mathbf{x}_1, t_1)}{\mathcal{J}(\omega, \mathbf{k}, \mathbf{x}_2, t_2)}\\ &= \int d\omega \int d^3k(\mathcal{O}^{(\mathrm{eq})}(\omega, \mathbf{k})e^{-i\omega t_1}e^{i\mathbf{k}\cdot\mathbf{x}_1} + \mathcal{O}^{\mathrm{in(neq)}}(\omega, \mathbf{k}, \mathbf{k}_{(\mathrm{b})})e^{-i(\omega+\mathrm{Re}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})}))t_1}e^{\mathrm{Im}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})t_1}e^{i(\mathbf{k}+\mathbf{k}_{(\mathrm{b})})\cdot\mathbf{x}_1}\\ &\quad + \mathcal{O}^{\mathrm{in(neq)}}(\omega, \mathbf{k}, -\mathbf{k}_{(\mathrm{b})})e^{-i(\omega-\mathrm{Re}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})}))t_1}e^{\mathrm{Im}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})t_1}e^{i(\mathbf{k}-\mathbf{k}_{(\mathrm{b})})\cdot\mathbf{x}_1})/(\mathcal{J}^{(\mathrm{eq})}(\omega, \mathbf{k})e^{-i\omega t_2}e^{i\mathbf{k}\cdot\mathbf{x}_2}\\ &\quad + \mathcal{J}^{\mathrm{in(neq)}}(\omega, \mathbf{k}, \mathbf{k}_{(\mathrm{b})})e^{-i(\omega+\mathrm{Re}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})}))t_2}e^{\mathrm{Im}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})t_2}e^{i(\mathbf{k}+\mathbf{k}_{(\mathrm{b})})\cdot\mathbf{x}_2}\\ &\quad + \mathcal{J}^{\mathrm{in(neq)}}(\omega, \mathbf{k}, -\mathbf{k}_{(\mathrm{b})})e^{-i(\omega-\mathrm{Re}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})}))t_2}e^{\mathrm{Im}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})t_2}e^{i(\mathbf{k}-\mathbf{k}_{(\mathrm{b})})\cdot\mathbf{x}_2}).\end{aligned} \tag{D1}$$

The above can be approximated in the derivative/amplitude expansion as below:

$$\begin{aligned}G_R(\mathbf{x}_1, t_1, \mathbf{x}_2, t_2) = \int d\omega \int d^3k e^{-i\omega(t_1-t_2)}e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}\frac{\mathcal{O}^{(\mathrm{eq})}(\omega, \mathbf{k})}{\mathcal{J}^{(\mathrm{eq})}(\omega, \mathbf{k})}\bigg(1 + \frac{\mathcal{O}^{\mathrm{in(neq)}}(\omega, \mathbf{k}, \mathbf{k}_{(\mathrm{b})})}{\mathcal{O}^{(\mathrm{eq})}(\omega, \mathbf{k})}e^{i\mathbf{k}_{(\mathrm{b})}\cdot\mathbf{x}_1}e^{-i\mathrm{Re}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})t_1}e^{\mathrm{Im}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})t_1}\\ + \frac{\mathcal{O}^{\mathrm{in(neq)}}(\omega, \mathbf{k}, -\mathbf{k}_{(\mathrm{b})})}{\mathcal{O}^{(\mathrm{eq})}(\omega, \mathbf{k})}e^{-i\mathbf{k}_{(\mathrm{b})}\cdot\mathbf{x}_1}e^{i\mathrm{Re}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})t_1}e^{\mathrm{Im}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})t_1}\\ - \frac{\mathcal{J}^{\mathrm{in(neq)}}(\omega, \mathbf{k}, \mathbf{k}_{(\mathrm{b})})}{\mathcal{J}^{(\mathrm{eq})}(\omega, \mathbf{k})}e^{i\mathbf{k}_{(\mathrm{b})}\cdot\mathbf{x}_2}e^{-i\mathrm{Re}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})t_2}e^{\mathrm{Im}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})t_2}\\ - \frac{\mathcal{J}^{\mathrm{in(neq)}}(\omega, \mathbf{k}, -\mathbf{k}_{(\mathrm{b})})}{\mathcal{J}^{(\mathrm{eq})}(\omega, \mathbf{k})}e^{-i\mathbf{k}_{(\mathrm{b})}\cdot\mathbf{x}_2}e^{i\mathrm{Re}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})t_2}e^{\mathrm{Im}\,\omega_{(\mathrm{b})}(\mathbf{k}_{(\mathrm{b})})t_2}\bigg).\end{aligned} \tag{D2}$$

It is easy to take the Wigner transform of the above to obtain Eq. (94).

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