Helicity-1/2 mode as a probe of interactions of a massive Rarita-Schwinger field

Rakibur Rahman

Physique Théorique et Mathématique and International Solvay Institutes, Université Libre de Bruxelles, Campus Plaine 231, B-1050 Bruxelles, Belgium

(Received 28 February 2013; published 28 March 2013)

We consider the electromagnetic and gravitational interactions of a massive Rarita-Schwinger field. Stückelberg analysis of the system, when coupled to electromagnetism in flat space or to gravity, reveals in either case that the effective field theory has a model-independent upper bound on its UV cutoff, which is finite but parametrically larger than the particle's mass. It is the helicity-1/2 mode that becomes strongly coupled at the cutoff scale. If the interactions are inconsistent, the same mode becomes a telltale sign of pathologies. Alternatively, consistent interactions are those that propagate this mode within the light cone. Studying its dynamics not only sheds light on the Velo-Zwanziger acausality, but also elucidates why supergravity and other known consistent models are pathology-free.

DOI: 10.1103/PhysRevD.87.065030 PACS numbers: 11.15.Bt, 11.15.−q, 04.65.+e

I. INTRODUCTION

The Rarita-Schwinger field carries a spin-3/2 representation of the Poincaré group, whose noninteracting massive theory is described by the following Lagrangian [1]:

$$\mathcal{L}_{\text{free}} = -i\bar{\psi}_\mu \gamma^{\mu\nu\rho}\partial_\nu \psi_\rho - im\bar{\psi}_\mu \gamma^{\mu\nu}\psi_\nu, \tag{1}$$

with m being the mass.[1] The Dirac equation, $(\partial\!\!\!/ - m)\psi_\mu = 0$, along with the correct constraints, $\partial^\mu \psi_\mu = \gamma^\mu \psi_\mu = 0$, can easily be reproduced from the Lagrangian equations of motion. The degrees-of-freedom count works as follows. In four dimensions the vector-spinor ψ_μ contains $4 \times 4 = 16$ components. The transversality and γ-tracelessness constraints each remove four of them, so that one is left with eight degrees of freedom (four field variables plus four conjugate momenta). Indeed, a massive spin-3/2 particle has four physical polarizations.

When interactions are turned on—as noticed by various authors [2–4]—the theory is generically fraught with inconsistencies even at the classical level,[2] despite the fact that one starts from a Lagrangian, as per suggestions made in Ref. [6]. The interacting theory may fail to reproduce the necessary constraints that forbid propagating unphysical modes, or it may give rise to the Velo-Zwanziger acausality [2], i.e., allow faster-than-light speeds for the physical modes. The addition of nonminimal terms and/or new dynamical fields may come to the rescue. For example, the Lagrangian proposed in Ref. [7] incorporates appropriate nonminimal terms that only causally propagate the physical modes of a massive spin-3/2 field in a constant external electromagnetic (EM) background. A more well-known example is $\mathcal{N} = 2$ (broken) supergravity [8,9], which contains a massive gravitino that propagates consistently—even when the cosmological constant is set to zero—given that it has a charge, $e = \frac{1}{\sqrt{2}}(m/M_{\text{P}})$, under the graviphoton [10]. Here causality is preserved by the presence of *both* EM and gravity, along with nonminimal terms.

The pathologies arising in an interacting theory are due to a simple fact: the kinetic part of the free theory (1) enjoys a gauge invariance, and the zero modes may acquire nonvanishing but noncanonical kinetic terms in the presence of interactions. The best way of understanding these issues is the Stückelberg formalism, which was employed in the context of massive spin-2 field, for example, in Refs. [11,12]. To understand this formalism, let us notice that in the *massive* theory (1) gauge invariance can be restored by introducing a spin-1/2 (Stückelberg) field χ through the field redefinition

$$\psi_\mu \to \psi'_\mu = \psi_\mu - \frac{1}{m}\partial_\mu \chi. \tag{2}$$

Now the Lagrangian is manifestly invariant under the Stückelberg symmetry,

$$\delta\psi_\mu = \partial_\mu \lambda, \qquad \delta\chi = m\lambda, \tag{3}$$

where λ is a fermionic gauge parameter. Note that when the field redefinition (2) is implemented, potentially bad higher-derivative terms in χ are killed by the antisymmetry of $\gamma^{\mu\nu}$. This is a slick way of understanding the structure of the mass term in Eq. (1).

The Stückelberg field is a mere redundancy since one can always choose a gauge in which $\chi = 0$, as in the Lagrangian (1). The unitary gauge, however, obscures the subtleties associated with an interacting theory, and is therefore not particularly illuminating when interactions are present. On the other hand, as we will see, the intricacies become rather transparent in a different, judiciously chosen gauge that instead renders the kinetic operators

[1]Our conventions are that the metric is mostly positive, the Clifford algebra is $\{\gamma^\mu, \gamma^\nu\} = +2g^{\mu\nu}$, $\gamma^{\mu\dagger} = \eta^{\mu\mu}\gamma^\mu$, $\gamma_5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$, $\gamma^{\mu_1 \dots \mu_n} = \frac{1}{n!}\gamma^{\mu_1}\gamma^{\mu_2}\dots\gamma^{\mu_n} +$ antisymmetrization. The Dirac adjoint is defined as $\bar{\psi}_\mu = \psi^\dagger_\mu \gamma^0$. The totally antisymmetric tensor $\epsilon_{\mu\nu\rho\sigma}$ is normalized as $\epsilon_{0123} = +1$.

[2]Pathologies at the quantum level were noticed much earlier in Ref. [5], where it was shown that canonical commutators may become ill-defined in an interacting theory.

diagonal. For an interacting theory, the latter gauge choice enables one to assign canonical dimensions to potential nonrenormalizable operators.

The organization of this paper is as follows. In the Sec. II we consider minimal EM and gravitational couplings of a massive Rarita-Schwinger field, and show that each theory possesses an intrinsic finite UV cutoff, which can be improved neither by field redefinitions nor by the addition of nonminimal terms. In Sec. III we perform a Stückelberg analysis of various (in)consistent Lagrangians that attempt to describe interactions of a massive spin-3/2 field. In particular, Sec. III A considers minimal EM coupling and reproduces the Velo-Zwanziger result [2], while Sec. III B sheds new light on why the nonminimal Lagrangian presented in Ref. [7] is consistent. Section III C reconfirms that minimal gravitational coupling is pathology-free in arbitrary Einstein spaces [13], and finally Sec. III D analyzes the consistency of $\mathcal{N}=2$ (broken) supergravity [8–10]. We conclude with some remarks in Sec. IV.

II. ULTRAVIOLET CUTOFF

Local Lagrangians describing the interactions of a massive spin-3/2 field do not have a smooth massless limit. Because the free part of the Lagrangian acquires a gauge invariance in this limit, propagators of the massive theory become singular, so that scattering amplitudes diverge. Notice, however, that if we introduce minimal coupling (to EM or gravity) in the Rarita-Schwinger action (1), no inverse powers of the mass appear in the resulting Lagrangian. Thus the massless singularity is not at all obvious in the unitary gauge.

The Stückelberg formalism, on the other hand, focuses precisely on the gauge modes responsible for bad high-energy behavior. One can "invent" the Stückelberg symmetry and then exploit it to make a judicious covariant gauge fixing such that the propagators acquire a smooth massless limit. In this gauge one will end up having an explicit dependence on inverse powers of the mass in the form of nonrenormalizable interaction terms that involve the Stückelberg field χ. The cutoff scale can be read off from the most divergent terms in the Lagrangian—the terms that survive in an appropriate scaling limit of zero mass and zero coupling.

A. EM coupling in flat space

First we consider EM coupling in flat space, and show that the theory has an upper bound on its UV cutoff.[3] When minimally coupled to a U(1) gauge field, the Stückelberg-invariant Lagrangian for a massive Rarita-Schwinger field reads

[3]This was originally considered in Ref. [14]. Here we reconsider it, with a more refined analysis, for the sake of completeness. The analysis will also be useful for the latter parts of the paper.

$$\mathcal{L}_{\text{em}} = -i\left(\bar{\psi}_\mu - \frac{1}{m}\bar{\chi}\overleftarrow{D}_\mu\right) \times (\gamma^{\mu\nu\rho}D_\nu + m\gamma^{\mu\rho})\left(\psi_\rho - \frac{1}{m}D_\rho\chi\right) - \frac{1}{4}F_{\mu\nu}^2, \quad (4)$$

which has the manifest gauged Stückelberg symmetry,

$$\delta\psi_\mu = D_\mu\lambda, \qquad \delta\chi = m\lambda, \quad (5)$$

where the covariant derivatives obey $[D_\mu, D_\nu] = ieF_{\mu\nu}$. More explicitly,

$$\mathcal{L}_{\text{em}} = \mathcal{L}_{3/2} + \mathcal{L}_{\text{mix}} + \mathcal{L}_{\text{int}} - \frac{1}{4}F_{\mu\nu}^2, \quad (6)$$

where $\mathcal{L}_{3/2}$ involves only the helicity-3/2 mode, $\mathcal{L}_{\text{mix}}$ is the kinetic mixing between the two modes, and $\mathcal{L}_{\text{int}}$ are nonrenormalizable interaction terms, respectively, given as

$$\mathcal{L}_{3/2} = -i\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho - im\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu, \quad (7)$$

$$\mathcal{L}_{\text{mix}} = i(\bar{\psi}_\mu\gamma^{\mu\nu}D_\nu\chi + \bar{\chi}\overleftarrow{D}_\mu\gamma^{\mu\nu}\psi_\nu), \quad (8)$$

$$\mathcal{L}_{\text{int}} = \frac{e}{2m}F_{\mu\nu}(\bar{\chi}\gamma^{\mu\nu\rho}\psi_\rho - \bar{\psi}_\rho\gamma^{\mu\nu\rho}\chi - \bar{\chi}\gamma^{\mu\nu}\chi) - \frac{e}{2m^2}F_{\mu\nu}\bar{\chi}\gamma^{\mu\nu\rho}D_\rho\chi. \quad (9)$$

The kinetic mixing can be removed by a field redefinition, namely

$$\psi_\mu \to \psi_\mu + \frac{1}{2}\gamma_\mu\chi, \quad (10)$$

which, at the same time, produces a kinetic term for χ as well as mass mixing. Now we can add the following gauge-fixing term to the Lagrangian:

$$\mathcal{L}_{\text{gf}} = i\bar{\psi}_\mu(\gamma^{\mu\nu}\gamma^\rho - \gamma^\mu\eta^{\nu\rho})D_\nu\psi_\rho + im\bar{\psi}_\mu\gamma^\mu\gamma^\nu\psi_\nu + \frac{3}{2}im(\bar{\psi}_\mu\gamma^\mu\chi - \bar{\chi}\gamma^\mu\psi_\mu - \bar{\chi}\chi), \quad (11)$$

which renders the propagators smooth in the massless limit, thanks to the identity

$$\gamma^{\mu\nu\rho} = \gamma^{\mu\nu}\gamma^\rho + \eta^{\mu\rho}\gamma^\nu - \eta^{\nu\rho}\gamma^\mu. \quad (12)$$

The same removes the mass mixing as well, finally giving

$$\mathcal{L}_{\text{em}} = -i\bar{\psi}_\mu(\not{D} - m)\psi^\mu - \frac{3}{2}i\bar{\chi}(\not{D} - m)\chi - \frac{1}{4}F_{\mu\nu}^2 + \frac{e}{2m}F_{\mu\nu}(\bar{\chi}\gamma^{\mu\nu\rho}\psi_\rho - \bar{\psi}_\rho\gamma^{\mu\nu\rho}\chi + \bar{\chi}\gamma^{\mu\nu}\chi) - \frac{e}{2m^2}F_{\mu\nu}(\bar{\chi}\gamma^{\mu\nu\rho}D_\rho\chi). \quad (13)$$

For $e \ll 1$, the most dangerous terms in the high-energy limit are the dimension-six operators. Note that the degree of divergence does not improve with the addition of nonminimal terms, since any such operator is necessarily irrelevant. Even a dipole term,

$$\mathcal{L}_{\text{dipole}} = \frac{ea}{m} F^{\mu\nu} \bar{\psi}_\mu \psi_\nu \\ \to \frac{ea}{m} F^{\mu\nu} \left(\bar{\psi}_\mu - \frac{1}{2} \bar{\chi} \gamma_\mu - \frac{1}{m} \bar{\chi} \overleftarrow{D}_\mu \right) \\ \times \left(\psi_\nu + \frac{1}{2} \gamma_\nu \chi - \frac{1}{m} D_\nu \chi \right), \tag{14}$$

introduces, among others, equally bad but new dimension-six operators that involve both the helicities. Clearly, higher-multipole operators will worsen the degree of divergence.[4] Now one can take the scaling limit $m \to 0$ and $e \to 0$, such that $m^2/e \equiv \Lambda_{\text{em}}^2 = \text{constant}$. The Lagrangian then reduces, after the rescaling $\chi \to \sqrt{\frac{2}{3}}\chi$, to

$$\mathcal{L}_{\text{em}} \to -i \bar{\psi}_\mu \not{\partial} \psi^\mu - i \bar{\chi} \not{\partial} \chi - \frac{1}{4} F_{\mu\nu}^2 \\ - \frac{1}{3\Lambda_{\text{em}}^2} F_{\mu\nu} (\bar{\chi} \gamma^{\mu\nu\rho} \partial_\rho \chi). \tag{15}$$

Notice, however, that the nonrenormalizable operators in Eq. (15) are all proportional to the equations of motion, up to total derivatives. Indeed, one can use the identity (12) to write

$$F_{\mu\nu}(\bar{\chi}\gamma^{\mu\nu\rho}\partial_\rho \chi) = \frac{1}{2} F_{\mu\nu} (\bar{\chi} \gamma^{\mu\nu} \not{\partial} \chi - \bar{\chi} \overleftarrow{\not{\partial}} \gamma^{\mu\nu} \chi) \\ - \partial_\mu F^{\mu\nu} (\bar{\chi} \gamma_\nu \chi). \tag{16}$$

Therefore, one can eliminate them by appropriate field redefinitions of χ and A_μ, namely

$$\chi \to \chi + \frac{i}{6\Lambda_{\text{em}}^2} F_{\mu\nu} \gamma^{\mu\nu} \chi, \qquad A_\mu \to A_\mu - \frac{1}{3\Lambda_{\text{em}}^2} \bar{\chi} \gamma_\mu \chi, \tag{17}$$

as canceling contributions come from the helicity-1/2 and photon kinetic terms. The price one has to pay is that new nonrenormalizable operators of dimensions eight, ten, and 12 show up, all with various negative powers of the scale Λ_{em}. Can we add local counter-terms to the original action which eliminate all these operators up to total derivatives, and introduce only new terms that vanish in the above scaling limit? A positive answer would mean that one may improve the degree of divergence of the minimally coupled theory by field redefinitions plus the addition of local counterterms. To see that this is not the case, let us consider the dimension-eight operator $(\bar{\chi}\gamma^\mu \chi) \Box (\bar{\chi} \gamma_\mu \chi)$, which comes from the photon-field redefinition acting on the last term of Eq. (16). It is neither proportional to the equations of motion nor does it contain the EM field strength. Without worsening the degree of divergence, such operators may only be produced by four-Fermi-like local counterterms, which in the unitary gauge look like $(e^2/m^2)\bar{\psi}\psi\bar{\psi}\psi$. More explicitly,

$$\mathcal{L}_{\text{c.t.}} \to b\left(\frac{e}{m}\right)^2 \left(\bar{\psi}_\mu - \sqrt{\frac{1}{6}} \bar{\chi} \gamma_\mu - \frac{1}{m} \sqrt{\frac{2}{3}} \bar{\chi} \overleftarrow{D}_\mu \right) \\ \times \gamma^{\mu\nu\rho\sigma} \left(\psi_\nu + \sqrt{\frac{1}{6}} \gamma_\nu \chi - \frac{1}{m} \sqrt{\frac{2}{3}} D_\nu \chi \right) \\ \times \left(\bar{\psi}_\rho - \sqrt{\frac{1}{6}} \bar{\chi} \gamma_\rho - \frac{1}{m} \sqrt{\frac{2}{3}} \bar{\chi} \overleftarrow{D}_\rho \right) \\ \times \left(\psi_\sigma + \sqrt{\frac{1}{6}} \gamma_\sigma \chi - \frac{1}{m} \sqrt{\frac{2}{3}} D_\sigma \chi \right) + \cdots, \tag{18}$$

where $\gamma^{\mu\nu\rho\sigma}$ plays the essential role of killing the more dangerous operators. However, such counterterms produce—on top of those that we want to eliminate—new dimension-eight operators involving *both* helicities that survive in the scaling limit.

Thus the effective field theory of a massive Rarita-Schwinger field interacting with EM in flat space has a finite intrinsic upper bound on its cutoff,

$$\Lambda_{\text{em}} = \frac{m}{\sqrt{e}}, \tag{19}$$

which is parametrically larger than m. As seen from Eq. (15), the breakdown of the effective action is due to the helicity-1/2 mode χ that becomes strongly coupled at high energies.

B. Gravitational coupling

The Stückelberg-invariant action for a massive spin-3/2 field minimally coupled to gravity is

$$\mathcal{L}_{\text{g}} = -i\sqrt{-g} \left(\bar{\psi}_\mu - \frac{1}{m} \bar{\chi} \overleftarrow{\nabla}_\mu \right) (\gamma^{\mu\nu\rho} \nabla_\nu + m \gamma^{\mu\rho}) \\ \times \left(\psi_\rho - \frac{1}{m} \nabla_\rho \chi \right) + \frac{1}{2} M_{\text{P}}^2 \sqrt{-g} R. \tag{20}$$

Here the commutator of the covariant derivatives acts on different modes as

$$[\nabla_\mu, \nabla_\nu] \psi_\rho = -R_{\mu\nu\rho}{}^\sigma \psi_\sigma + \frac{1}{4} R_{\mu\nu\alpha\beta} \gamma^{\alpha\beta} \psi_\rho, \tag{21}$$

$$[\nabla_\mu, \nabla_\nu] \chi = \frac{1}{4} R_{\mu\nu\alpha\beta} \gamma^{\alpha\beta} \chi. \tag{22}$$

One can work out the Lagrangian (20) to write

$$\mathcal{L}_{\text{g}} = \mathcal{L}_{3/2} + \mathcal{L}_{\text{mix}} + \mathcal{L}_{\text{int}} + \frac{1}{2} M_{\text{P}}^2 \sqrt{-g} R, \tag{23}$$

where $\mathcal{L}_{3/2}$ and $\mathcal{L}_{\text{mix}}$ are the gravitational counterparts of those given by Eqs. (7) and (8), respectively, while $\mathcal{L}_{\text{int}}$ are the nonrenormalizable interactions. The latter can be

[4]We emphasize that here we are only attempting to improve the degree of divergence, as we are looking for a theoretical upper bound on the cutoff scale that no theory can beat. In no way do we mean that nonminimal terms are forbidden. In fact, they do appear in consistent models, e.g., supergravity. But then the theory will have a cutoff that is simply lower than the upper bound we are trying to find.

computed explicitly using Eqs. (21) and (22), the Bianchi identity, $R_{[\mu\nu\alpha]\beta} = 0$, and various γ-matrix identities. The following ones are particularly useful:

$$\gamma^{\mu\nu\rho}\gamma^{\alpha\beta}R_{\mu\nu\alpha\beta}(\psi_\rho, \nabla_\rho\chi) = 4G^{\mu\nu}\gamma_\mu(\psi_\nu, \nabla_\nu\chi), \qquad \gamma^{\mu\nu}\gamma^{\alpha\beta}R_{\mu\nu\alpha\beta} = -2R, \tag{24}$$

where $G^{\mu\nu}$ is the Einstein tensor. The result is

$$\mathcal{L}_{\rm int} = -\frac{i}{2m}\sqrt{-g}\Big[G^{\mu\nu}(\bar\chi\gamma_\mu\psi_\nu - \bar\psi_\mu\gamma_\nu\chi) + \frac{1}{2}\bar\chi R\chi\Big] + \frac{i}{2m^2}\sqrt{-g}G^{\mu\nu}\bar\chi\gamma_\mu\nabla_\nu\chi. \tag{25}$$

The field redefinition that eliminates the kinetic mixing is the same as Eq. (10), while the desired gauge-fixing term is just the gravitational counterpart of Eq. (11). One is left with

$$\mathcal{L}_{\rm g} = -i\sqrt{-g}\Big[\bar\psi_\mu(\nabla\!\!\!/ - m)\psi^\mu + \frac{3}{2}\bar\chi(\nabla\!\!\!/ - m)\chi\Big] + \frac{1}{2}M_{\rm P}^2\sqrt{-g}R - \frac{i}{2m}\sqrt{-g}\Big[G^{\mu\nu}(\bar\chi\gamma_\mu\psi_\nu - \bar\psi_\mu\gamma_\nu\chi) - \frac{1}{2}\bar\chi R\chi - \frac{1}{m}G^{\mu\nu}\bar\chi\gamma_\mu\nabla_\nu\chi\Big]. \tag{26}$$

Before assigning canonical dimensions to various operators, we must canonically normalize the graviton field $h_{\mu\nu}$, so that it has mass dimension one,

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_{\rm P}}h_{\mu\nu}. \tag{27}$$

We take $m \ll M_{\rm P}$, which is essential for a sensible effective field theory to exist. We see that in the high-energy limit the most dangerous terms are the dimension-seven operators contained in $G^{\mu\nu}\bar\chi\gamma_\mu\nabla_\nu\chi$, which are $\chi - h - \chi$ vertices. Because nonminimal interactions show up with Planck-mass suppression in the unitary gauge, they can contribute only less divergent terms to the Lagrangian (26). Thus they are harmless, but they do not improve the degree of divergence either.

The high-energy regime we are interested in—characterized by the center-of-mass energy $m \ll \sqrt{s} \ll M_{\rm P}$—includes two parametrically disparate scales of interest,

$$\Lambda_* \equiv \sqrt[3]{m^2 M_{\rm P}}, \qquad \Lambda_{\rm g} \equiv \sqrt{m M_{\rm P}}, \tag{28}$$

where $\Lambda_* \ll \Lambda_{\rm g}$. Now, with the rescaling $\chi \to \sqrt{\tfrac{2}{3}}\chi$, our Lagrangian (26) reduces to

$$\mathcal{L}_{\rm g} \to -i\bar\psi_\mu(\partial\!\!\!/ - m)\psi^\mu - i\bar\chi(\partial\!\!\!/ - m)\chi + h_{\mu\nu}\mathcal{G}^{\mu\nu} + \frac{i}{3\Lambda_*^3}\mathcal{G}^{\mu\nu}\bar\chi\gamma_\mu\partial_\nu\chi + \cdots, \tag{29}$$

where the ellipses stand for less divergent terms that become important at scales $\Lambda_{\rm g}$ or higher. Here $\mathcal{G}^{\mu\nu} \equiv (\mathcal{E}\cdot h)^{\mu\nu}$ is the linearized Einstein tensor, and

$$\mathcal{E}^{\mu\nu\alpha\beta} = \frac{1}{2}[(\eta^{\mu\nu,\alpha\beta} - \eta^{\mu\nu}\eta^{\alpha\beta})\Box + \eta^{\mu\nu}\partial^\alpha\partial^\beta + \eta^{\alpha\beta}\partial^\mu\partial^\nu - \eta^{\mu(\alpha}\partial^{\beta)}\partial^\nu - \eta^{\nu(\alpha}\partial^{\beta)}\partial^\mu], \tag{30}$$

so that $h_{\mu\nu}\mathcal{G}^{\mu\nu}$ is the kinetic term for the canonically normalized graviton $h_{\mu\nu}$. It is clear that the dimension-seven operator in Eq. (29) can be eliminated by the field redefinition

$$h_{\mu\nu} \to h_{\mu\nu} - \frac{i}{6\Lambda_*^3}\bar\chi\gamma_{(\mu}\partial_{\nu)}\chi. \tag{31}$$

But this will leave us with the following dimension-ten operator, quartic in χ:

$$\mathcal{L}_{\rm dim\text{-}10} = -\frac{1}{36\Lambda_*^6}(\bar\chi\gamma_\mu\partial_\nu\chi)\mathcal{E}^{\mu\nu\alpha\beta}(\bar\chi\gamma_\alpha\partial_\beta\chi). \tag{32}$$

This is a contact term for four helicity-1/2 modes. Because we are interested in on-shell scattering amplitudes, some pieces contained in Eq. (32) may actually be less divergent, thanks to the equation of motion $\partial\!\!\!/\chi = m\chi + \cdots$. Indeed, all but the first term from the expression (30) for $\mathcal{E}^{\mu\nu\alpha\beta}$ give—up to total derivatives—dimension-eight operators that go like $1/\Lambda_{\rm g}^4$. This follows partly from the fact that, unlike in the electromagnetic case, here one is dealing with Majorana fermions, so that one has $\bar\chi\gamma^\mu\chi = 0$. Thus one is left with

$$\mathcal{L}_{\rm int} \to -\frac{1}{72\Lambda_*^6}(\bar\chi\gamma_\mu\partial_\nu\chi)\eta^{\mu\nu,\alpha\beta}\Box(\bar\chi\gamma_\alpha\partial_\beta\chi) + \cdots. \tag{33}$$

This operator does not reduce further for on-shell χ. However, as we will see, it can be canceled, up to total derivatives, by the addition of local counterterms.

In the unitary gauge, the potentially interesting counterterms are four-Fermi interactions,

$$\mathcal{L}_{\rm c.t.} = M_{\rm P}^{-2}(\bar\psi_\rho\gamma_\mu\psi_\nu)\mathcal{A}^{\mu\nu\alpha\beta\rho\sigma}(\bar\psi_\sigma\gamma_\alpha\psi_\beta), \tag{34}$$

where $\mathcal{A}^{\mu\nu\alpha\beta\rho\sigma}$ is a dimensionless tensor. The replacement $\psi_\mu \to \psi_\mu + \sqrt{\tfrac{1}{6}}\gamma_\mu\chi - \sqrt{\tfrac{2}{3}}\partial_\mu\chi/m$ will then give rise to dimension-ten operators, quartic in χ, which may cancel those of Eq. (33). It is easy to find that the required cancelation takes place for

$$\mathcal{A}^{\mu\nu\alpha\beta\rho\sigma} = \frac{1}{32}(\eta^{\mu\alpha}\eta^{\nu[\beta}\eta^{\sigma]\rho} + 2\eta^{\alpha[\nu}\eta^{\rho][\sigma}\eta^{\beta]\mu}). \tag{35}$$

Note that the antisymmetry in the indices (ρ, ν) and (σ, β) ensures that no new dimension-ten operators are generated. Thus no terms remain that become important at Λ_*: the counterterm (34) has improved the high-energy behavior of the system.

Next, one would like to consider the dimension-nine operators coming from this counterterm that blow up at the scale $\sqrt{\Lambda_*\Lambda_{\rm g}}$—higher than Λ_* but lower than $\Lambda_{\rm g}$.

A straightforward computation shows that all such contact terms are actually less divergent for on-shell χ. Therefore, the strong-coupling regime is pushed even higher, to the scale Λ_g.

Can we improve the cutoff scale any further? The answer is negative. To see this, let us take the scaling limit $m \to 0$ and $M_P \to \infty$, such that $\Lambda_g =$ constant. This gives

$$\mathcal{L}_g + \mathcal{L}_{c.t.}, \tag{36}$$

where the dimension-eight operators, which are $\mathcal{O}(1/\Lambda_g^4)$, contain quartic contact terms originating from the naive dimension-ten operator (32) as well as from the counterterm (34). Another field redefinition of the graviton, namely

$$h_{\mu\nu} \to h_{\mu\nu} + \frac{i}{2\Lambda_g^2}\left(\bar{\chi}\gamma_{(\mu}\psi_{\nu)} + \frac{1}{4}\eta_{\mu\nu}\bar{\chi}\chi\right), \tag{37}$$

will remove all the dimension-six operators in Eq. (36) and give rise to additional dimension-eight operators. For simplicity, let us look at all the dimension-eight quartic terms that involve two helicity-3/2 and two helicity-1/2 modes. They are

$$\begin{aligned}\Lambda_g^4\mathcal{L}_{\psi\psi\chi\chi} &= \frac{2}{3}(\bar{\psi}_\rho\gamma_\mu\psi_\nu)\mathcal{A}^{\mu\nu\alpha\beta\rho\sigma}(\partial_\sigma\bar{\chi}\gamma_\alpha\partial_\beta\chi)\\ &\quad + \frac{2}{3}(\partial_\rho\bar{\chi}\gamma_\mu\partial_\nu\chi)\mathcal{A}^{\mu\nu\alpha\beta\rho\sigma}(\bar{\psi}_\sigma\gamma_\alpha\psi_\beta)\\ &\quad + \frac{8}{3}(\bar{\psi}_\rho\gamma_\mu\partial_\nu\chi)\mathcal{A}^{\mu\nu\alpha\beta\rho\sigma}(\bar{\psi}_\sigma\gamma_\alpha\partial_\beta\chi)\\ &\quad - \frac{1}{4}(\bar{\chi}\gamma_\mu\psi_\nu)\mathcal{E}^{\mu\nu\alpha\beta}(\bar{\chi}\gamma_\alpha\psi_\beta).\end{aligned} \tag{38}$$

Notice that the last term that comes from the field redefinition (37) contains pieces that are nonvanishing on-shell. Can these be canceled by the first three terms? No, because of simple symmetry considerations. The latter set of terms enjoys the shift of χ by a constant spinor, while the former does not. At this point, we also have exhausted the possibility of local counterterms coming to the rescue.

Thus we have found an upper bound on the UV cutoff of the effective theory describing a gravitationally interacting massive spin-3/2 field,

$$\Lambda_g = \sqrt{mM_P}. \tag{39}$$

This is finite, but parametrically larger than the mass. Again, it is the helicity-1/2 mode that is responsible for the strong coupling around the cutoff scale.

Our result[5] is hardly a surprise given the existence of $\mathcal{N} = 1$ broken supergravity [16]. This theory possesses remarkably good properties in the high-energy limit, and its strong-coupling regime has been investigated in Ref. [17]. When the (pseudo)scalars are decoupled from the theory, with their masses sent to infinity, one ends up having only a massive gravitino coupled to gravity. This is the system we have considered in this section, and indeed the theory has a cutoff around the supersymmetry-breaking scale Λ_g [17].

III. INTERACTING THEORIES OF A RARITA-SCHWINGER FIELD

Now we will consider various (in)consistent models of an interacting massive spin-3/2 field, and analyze them through the Stückelberg formalism. As we already know, when interactions are present, the helicity-1/2 mode generally acquires nonstandard kinetic terms. In inconsistent theories this mode may move faster than light or even cease to propagate. The consistency of interacting theories crucially relies on having a pathology-free helicity-1/2 sector. Conversely, by ensuring that this mode does not exhibit pathological behavior, we can (re)derive conditions that render a theory consistent.

A. Minimal EM interaction

This has already been considered in Sec. II A, and we recall from Eq. (13) that the Lagrangian can be written as

$$\begin{aligned}\mathcal{L}_{em} = &-i\bar{\psi}_\mu(\not{D} - m)\psi^\mu - \frac{3}{2}i\bar{\chi}(\not{D} - m)\chi - \frac{1}{4}F_{\mu\nu}^2\\ &+ \frac{e}{2m}F_{\mu\nu}(\bar{\chi}\gamma^{\mu\nu\rho}\psi_\rho - \bar{\psi}_\rho\gamma^{\mu\nu\rho}\chi + \bar{\chi}\gamma^{\mu\nu}\chi)\\ &- \frac{e}{2m^2}F_{\mu\nu}(\bar{\chi}\gamma^{\mu\nu\rho}D_\rho\chi).\end{aligned} \tag{40}$$

It is manifest that the helicity-3/2 sector enjoys a healthy kinetic term. On the other hand, the χ sector is tricky, because in an external EM background the last term in Eq. (40) will act like a kinetic operator. Let us write down the equations of motion for χ,

$$\begin{aligned}&-i\not{\partial}\chi - \frac{1}{2}\alpha\gamma^{\mu\nu\rho}F_{\mu\nu}\partial_\rho\chi + (\text{lower-derivative terms}) = 0,\\ &\alpha \equiv \frac{2}{3}e/m^2.\end{aligned} \tag{41}$$

Now we will use the method of characteristic determinants [2] to investigate whether this system allows propagation outside the light cone. The method consists of determining the normal, $n_\mu = (n_0, \vec{n})$, to the characteristic hypersurfaces. We replace ∂_μ with $-in_\mu$ in the highest-derivative terms in Eq. (41), and then equate the determinant $\Delta(n)$ of the resulting coefficient matrix to zero. The system is hyperbolic if for any $\vec{n}$ the algebraic equation $\Delta(n) = 0$ has real solutions for n_0; then the ratio $n_0/|\vec{n}|$ gives the maximum wave speed. The required coefficient matrix is given by

[5]This result agrees with the conjecture $\Lambda_g = (m^{2s-2}M_P)^{1/(2s-1)}$ for generic spin s made in Ref. [15].

$$\mathcal{M} = -\gamma^\mu n_\mu + \frac{i}{2}\alpha\gamma^{\mu\nu\rho}F_{\mu\nu}n_\rho = -i\begin{pmatrix} \mathbf{0} & -\vec{\sigma}\cdot(\vec{n}+\alpha n_0\vec{B}) - (n_0+\alpha\vec{n}\cdot\vec{B}) \\ \vec{\sigma}\cdot(\vec{n}-\alpha n_0\vec{B}) - (n_0-\alpha\vec{n}\cdot\vec{B}) & \mathbf{0}\end{pmatrix}. \quad (42)$$

To compute its determinant let us assume, without loss of generality, that the magnetic field $\vec{B}$ points in the z direction, and that the three-vector $\vec{n}$ lies on the zx plane making an angle θ with $\vec{B}$. Thus we obtain

$$\Delta(n) \equiv \det\mathcal{M} = [n_0^2 - \vec{n}^2 - \alpha^2\vec{B}^2(n_0^2 - \vec{n}^2\cos^2\theta)]^2, \quad (43)$$

which vanishes for

$$\frac{n_0}{|\vec{n}|} = \sqrt{\frac{1-\alpha^2\vec{B}^2\cos^2\theta}{1-\alpha^2\vec{B}^2}}. \quad (44)$$

We see that the system ceases to be hyperbolic whenever $\alpha^2\vec{B}^2$ exceeds unity, i.e., when

$$\vec{B}^{\,2} \geq \left(\frac{3m^2}{2e}\right)^2. \quad (45)$$

In addition, even an infinitesimal magnetic field will cause superluminal propagation for generic θ. This is the so-called Velo-Zwanziger problem. The pathologies are serious in that they can very well arise when the EM field invariants $\vec{B}^2 - \vec{E}^2$ and $\vec{B}\cdot\vec{E}$ are nonvanishing but small (in the units of m^4/e^2), so that we are far away from the regime of instabilities [18] and the notion of long-lived propagating particles makes sense.

B. Consistent nonminimal EM couplings

The Velo-Zwanziger acausality shows up even for the simplest possible interaction setup of a constant external EM background. A wide class of nonminimal models [3] also exhibits the same pathological features. Porrati and Rahman [7] wrote down a nonminimal Lagrangian, which consistently describes a charged massive Rarita-Schwinger field exposed to a constant EM background in flat space. In the unitary gauge it reads [7]

$$\mathcal{L}_{\rm PR} = -i\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho - im\bar{\psi}_\mu b^{\mu\nu}\psi_\nu, \quad (46)$$

where the antisymmetric tensor $b_{\mu\nu}$ contains "corrections" to $\gamma_{\mu\nu}$ of the form

$$b_{\mu\nu} = \gamma_{\mu\nu} + B^+_{\mu\nu} + \gamma^\rho C_{\rho[\mu}\gamma_{\nu]}, \qquad B^\pm_{\mu\nu} \equiv B_{\mu\nu} \mp i\gamma_5\tilde{B}_{\mu\nu}, \quad (47)$$

with $\tilde{B}_{\mu\nu} \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}B^{\rho\sigma}$. The Lorentz tensor $B_{\mu\nu}$ is antisymmetric, while the Lorentz tensor $C_{\mu\nu}$ is symmetric and traceless. They are respectively imaginary and real, as implied by the Hermiticity condition, and they both vanish in the limit $F \longrightarrow 0$. They are related as [7]

$$C^{\mu\nu} = -\frac{1}{2}B^{+\mu\rho}B^{-\nu}_{\rho} = -\frac{1}{2}B^{-\mu\rho}B^{+\nu}_{\rho} = -[B^{\mu\rho}B_\rho{}^\nu - \frac{1}{4}\eta^{\mu\nu}\,\mathrm{Tr}(B^2)], \quad (48)$$

while $B_{\mu\nu}$ can be computed from the relation [7]

$$B_{\mu\nu} = i(e/m^2)F_{\mu\nu} + \frac{1}{4}\,\mathrm{Tr}(B^2)B_{\mu\nu} - \frac{1}{4}\,\mathrm{Tr}(B\tilde{B})\tilde{B}_{\mu\nu} \quad (49)$$

as a power series in the EM field strength $F_{\mu\nu}$, which is always possible in physically interesting situations, i.e., when the EM field invariants are small.

In what follows we perform a Stückelberg analysis of the Lagrangian (46) to reveal that the relations (48) and (49) are precisely those that ensure a healthy helicity-1/2 sector. We can render the Lagrangian Stückelberg invariant as usual, and work out the various terms to arrive at the nonminimal counterpart of Eq. (6),

$$\begin{aligned}\mathcal{L}_{\rm PR} = &-i\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho - im\bar{\psi}_\mu b^{\mu\nu}\psi_\nu \\ &+ i(\bar{\psi}_\mu b^{\mu\nu}D_\nu\chi + \bar{\chi}\overleftarrow{D}_\mu b^{\mu\nu}\psi_\nu) \\ &+ \frac{e}{2m}F_{\mu\nu}(\bar{\chi}\gamma^{\mu\nu\rho}\psi_\rho - \bar{\psi}_\rho\gamma^{\mu\nu\rho}\chi - \bar{\chi}b^{\mu\nu}\chi) \\ &- \frac{e}{2m^2}F_{\mu\nu}(\bar{\chi}\gamma^{\mu\nu\rho}D_\rho\chi). \end{aligned} \quad (50)$$

As we have already seen in the minimal theory, a redefinition of ψ_μ can eliminate the kinetic mixing. To find such a field redefinition in the present case, let us take note of the following identities that follow from elementary manipulations of γ-matrix algebra:

$$B^+_{\mu\nu} = -\frac{1}{4}\gamma^\rho\not{B}\gamma_{\rho\mu\nu} = -\frac{1}{4}\gamma_{\mu\nu\rho}\not{B}\gamma^\rho, \qquad \not{B} \equiv \gamma^{\mu\nu}B_{\mu\nu}, \quad (51)$$

$$\gamma^\alpha C_{\alpha[\mu}\gamma_{\nu]} = -\frac{1}{2}\gamma_\alpha C^{\alpha\rho}\gamma_{\rho\mu\nu} = -\frac{1}{2}\gamma_{\mu\nu\rho}C^{\rho\alpha}\gamma_\alpha. \quad (52)$$

Given this, it is not difficult to see that the required field redefinition is

$$\psi_\mu \rightarrow \psi_\mu + \frac{1}{2}\left(\gamma_\mu - \frac{1}{2}\not{B}\gamma_\mu - \gamma^\alpha C_{\alpha\mu}\right)\chi. \quad (53)$$

This, when implemented in Eq. (50), will also produce new noncanonical kinetic operators for χ, which add to the already existing troublesome operator $F_{\mu\nu}\bar{\chi}\gamma^{\mu\nu\rho}D_\rho\chi$. One can also add the gauge-fixing term (11) to make manifest that the helicity-3/2 sector is hyperbolic and causal. The result is the nonminimal counterpart of Eq. (13), given by

$$\mathcal{L}_{\rm PR} = -i\bar{\psi}_\mu(\not{D} - m)\psi^\mu - \frac{3}{2}i\bar{\chi}(\not{D} - m)\chi + \frac{e}{2m}F_{\mu\nu}(\bar{\chi}\gamma^{\mu\nu\rho}\psi_\rho - \bar{\psi}_\rho\gamma^{\mu\nu\rho}\chi + \bar{\chi}b^{\mu\nu}\chi) + \frac{1}{2}i\bar{\chi}\Big[i(e/m^2)\gamma^{\mu\nu\rho}F_{\mu\nu} + b^{\mu\rho}\Big(\gamma_\mu - \frac{1}{2}\not{B}\gamma_\mu - \gamma^\alpha C_{\alpha\mu}\Big) + 3\gamma^\rho\Big]D_\rho\chi. \tag{54}$$

The key point is that we have at our disposal two functions of the EM field strength, $B_{\mu\nu}$ and $C_{\mu\nu}$, which could be judiciously chosen so as to render the χ sector pathology-free. With this end in view, we make the rescaling $\chi \to \sqrt{\frac{2}{3}}\chi$, and look at the helicity-1/2 kinetic-like operators, which we symbolically write as

$$\mathcal{L}_{\chi,\rm kin} = -i\bar{\chi}\Gamma^\mu D_\mu\chi. \tag{55}$$

If Γ^μ is proportional to γ^μ with a positive coefficient, the χ sector will be ghost-free, hyperbolic, and causal. The expression for Γ^μ can be simplified to

$$\Gamma^\mu = \gamma^\mu + \frac{1}{3}\Big\{-i(e/m^2)\gamma^{\mu\nu\rho}F_{\nu\rho} + \gamma^{\mu\nu\rho}B_{\nu\rho} + \gamma^\rho C_{\rho\nu}\Big(B^{-\nu\mu} - \frac{1}{2}B^{+\nu\alpha}\gamma^\mu\gamma_\alpha\Big)\Big\} + \frac{1}{3}\gamma_\nu[2C^{\mu\nu} + B^{+\mu\rho}B_\rho^{-\nu}] + \frac{1}{6}\gamma^\nu C_{\nu\rho}C^{\rho\sigma}\gamma^\mu\gamma_\sigma, \tag{56}$$

thanks to the identities

$$\gamma^\mu B^+_{\mu\nu} = \frac{1}{2}\not{B}\gamma_\nu, \qquad B^+_{\mu\nu}\gamma^\nu = \frac{1}{2}\gamma_\mu\not{B}, \qquad \gamma_\mu\not{B}\gamma^\mu = 0, \qquad \frac{1}{2}(\gamma^\mu\not{B} + \not{B}\gamma^\mu) = \gamma^{\mu\nu\rho}B_{\nu\rho}. \tag{57}$$

In Eq. (56), if one sets the symmetric traceless tensor inside the brackets to zero—which is nothing but the choice of the relation (48)—the entire second line becomes proportional to γ^μ. This is because of the identity

$$B^\pm_{\mu\rho}B^{\pm\rho\nu} = \frac{1}{2}[{\rm Tr}(B^2) \pm i\gamma_5{\rm Tr}(B\tilde{B})]\delta^\nu_\mu, \tag{58}$$

which, along with Eq. (48), enables one to write

$$\gamma^\nu C_{\nu\rho}C^{\rho\sigma}\gamma^\mu\gamma_\sigma = \frac{1}{4}\gamma^\nu(B_\nu^{+\alpha}B^-_{\alpha\rho}B^{-\rho\lambda}B_\lambda^{+\sigma})\gamma^\mu\gamma_\sigma = -\frac{1}{8}\{[{\rm Tr}(B^2)]^2 + [{\rm Tr}(B\tilde{B})]^2\}\gamma^\mu. \tag{59}$$

Moreover, one can use Eqs. (48) and (58) and the definitions of $B^\pm_{\mu\nu}$ and $\tilde{B}_{\mu\nu}$ to write

$$\gamma^\rho C_{\rho\nu}\Big(B^{-\nu\mu} - \frac{1}{2}B^{+\nu\alpha}\gamma^\mu\gamma_\alpha\Big) = -\frac{1}{4}\gamma^{\mu\nu\rho}[{\rm Tr}(B^2)B_{\nu\rho} - {\rm Tr}(B\tilde{B})\tilde{B}_{\nu\rho}]. \tag{60}$$

Now in view of Eqs. (48), (59), and (60), the expression for Γ^μ reduces to

$$\Gamma^\mu = \Big\{1 - \frac{1}{48}[{\rm Tr}(B^2)]^2 - \frac{1}{48}[{\rm Tr}(B\tilde{B})]^2\Big\}\gamma^\mu + \frac{1}{3}\gamma^{\mu\nu\rho}\Big\{-i(e/m^2)F_{\nu\rho} + B_{\nu\rho} - \frac{1}{4}{\rm Tr}(B^2)B_{\nu\rho} + \frac{1}{4}{\rm Tr}(B\tilde{B})\tilde{B}_{\nu\rho}\Big\}. \tag{61}$$

This produces the same kind of helicity-1/2 kinetic terms as the minimally coupled theory. Clearly, the second line in the above expression will give rise to pathologies unless it is set to zero. Then, the consistent propagation of χ requires Eq. (49), and we are left with

$$\mathcal{L}_{\chi,\rm kin} = -i\Big\{1 - \frac{1}{48}[{\rm Tr}(B^2)]^2 - \frac{1}{48}[{\rm Tr}(B\tilde{B})]^2\Big\}\bar{\chi}\not{D}\chi. \tag{62}$$

The factor appearing in the kinetic term manifestly depends on the relativistic field invariants in such a way that it is always positive in the regimes of physical interest. Thus the mere requirement of a healthy helicity-1/2 sector recovers the model (46)–(49).

C. Minimal coupling to gravity

As already considered in Sec. II B, minimal gravitational coupling shows up, interestingly, as one tries to write down consistent models for a massive spin-3/2 field in Einstein space [13]. The Lagrangian found in Ref. [13] (by using the Becchi-Rouet-Stora-Tyutin approach to higher-spin fields) boils down to the minimal Lagrangian in the unitary gauge. It means that if we take the minimally coupled theory with the spin-3/2 field as a probe, the consistent propagation of the helicity-1/2 mode would require that the Einstein tensor be proportional to the metric.

The consistency of minimal gravitational coupling in Einstein spaces becomes rather obvious in the Stückelberg language. We recall from Eq. (26) that the minimally coupled theory, in $d = 4$ dimensions, can be cast into the following form:

$$\mathcal{L}_{\rm g} = -i\sqrt{-g}\Big[\bar{\psi}_\mu(\not{\nabla} - m)\psi^\mu + \frac{3}{2}\bar{\chi}(\not{\nabla} - m)\chi\Big] - \frac{i}{2m}\sqrt{-g}\Big[G^{\mu\nu}(\bar{\chi}\gamma_\mu\psi_\nu - \bar{\psi}_\mu\gamma_\nu\chi) - \frac{1}{2}\bar{\chi}R\chi - \frac{1}{m}G^{\mu\nu}\bar{\chi}\gamma_\mu\nabla_\nu\chi\Big]. \tag{63}$$

With the rescaling $\chi \to \sqrt{\frac{2}{d-1}}\chi$, the kinetic-like operators for χ become

$$\mathcal{L}_{\chi,\mathrm{kin}} = -i\sqrt{-g}\left[g^{\mu\nu} - \frac{1}{(d-1)m^2}G^{\mu\nu}\right]\bar{\chi}\gamma_\mu\nabla_\nu\chi. \tag{64}$$

The above expression actually holds true even when d is arbitrary. It is clear that, if $G^{\mu\nu}$ is proportional to $g^{\mu\nu}$, the system reduces to a manifestly hyperbolic and causal one. We must ensure, however, that the coefficient in front of $(\bar{\chi}\nabla\chi)$ is always non-negative. Otherwise, as χ becomes a propagating ghost, there will be loss of unitarity. The coefficient can be computed by noting that for Einstein spaces one has

$$G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = -\left(\frac{d-2}{2d}\right)g^{\mu\nu}R, \tag{65}$$

which enables one to rewrite Eq. (64) as

$$\mathcal{L}_{\chi,\mathrm{kin}} = -i\sqrt{-g}\left[1 + \frac{d-2}{2d(d-1)}\left(\frac{R}{m^2}\right)\right]\bar{\chi}\nabla\chi. \tag{66}$$

Therefore, everywhere in spacetime, the Ricci scalar must satisfy

$$\left(\frac{d-2}{2d}\right)R \geq -(d-1)m^2. \tag{67}$$

Of special interest are constant-curvature spaces, for which the left-hand side of Eq. (67) is nothing but the cosmological constant Λ. The unitarity bound then reduces to

$$\Lambda \geq -(d-1)m^2. \tag{68}$$

This is precisely the result of Ref. [19] for a neutral massive spin-3/2 field in cosmological backgrounds. The equality sign in Eq. (68) renders the field χ algebraic by setting its kinetic term to zero, and this corresponds to a genuinely massless spin-3/2 field in AdS [13,19].

D. Supergravity

It is well known that $\mathcal{N} = 2$ gauged supergravity [8] incorporates a consistently propagating Rarita-Schwinger field (gravitino), which is coupled to a U(1) field (graviphoton) as well as gravity with a cosmological constant. When the cosmological constant is detuned from its supersymmetric value, $\Lambda = -3m^2$, the resulting broken supergravity theory [9,10] still propagates the massive gravitino causally for any unitarily allowed Λ, provided the usual mass-charge relation holds [10], i.e., the gravitino charge e under the graviphoton is

$$e = \frac{1}{\sqrt{2}}\left(\frac{m}{M_{\mathrm{P}}}\right). \tag{69}$$

We consider the gravitino as a probe in the dynamical Maxwell-Einstein background; the latter satisfies the bosonic equations of motion of $\mathcal{N} = 2$ (broken) supergravity,

$$\nabla_\mu F^{\mu\nu} = 0, \qquad G^{\mu\nu} + \Lambda g^{\mu\nu} = \frac{1}{M_{\mathrm{P}}^2}T^{\mu\nu}, \tag{70}$$

where $T^{\mu\nu}$ is the stress-energy tensor of the Maxwell field, given by

$$\begin{aligned} T^{\mu\nu} &= -\frac{1}{2}F^{+\mu\rho}F^{-\nu}_{\rho} = -\frac{1}{2}F^{-\mu\rho}F^{+\nu}_{\rho} \\ &= -\left[F^{\mu\rho}F^{\nu}_{\rho} - \frac{1}{4}\eta^{\mu\nu}\mathrm{Tr}(F^2)\right]. \end{aligned} \tag{71}$$

In this combined background of EM and gravitational fields, the probe Rarita-Schwinger field is described in the unitary gauge by the following nonminimal Lagrangian:

$$\begin{aligned} &\mathcal{L}_{\mathrm{gravitino}} = -i\sqrt{-g}[\bar{\psi}_\mu\gamma^{\mu\nu\rho}\mathcal{D}_\nu\psi_\rho + m\bar{\psi}_\mu f^{\mu\nu}\psi_\nu], \\ &\quad f^{\mu\nu} \equiv \gamma^{\mu\nu} + i(e/m^2)F^{+\mu\nu}. \end{aligned} \tag{72}$$

The commutator of covariant derivatives is given by

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] = [\nabla_\mu, \nabla_\nu] + ieF_{\mu\nu}, \tag{73}$$

which, along with the relations (21) and (22), enables one to work out the Stückelberg-invariant Lagrangian. Thanks to the Bianchi identities and Eq. (24), the result is

$$\begin{aligned} \mathcal{L}_{\mathrm{gravitino}} = &-i\sqrt{-g}(\bar{\psi}_\mu\gamma^{\mu\nu\rho}\mathcal{D}_\nu\psi_\rho + m\bar{\psi}_\mu f^{\mu\nu}\psi_\nu) + i\sqrt{-g}(\bar{\psi}_\mu f^{\mu\nu}\mathcal{D}_\nu\chi + \bar{\chi}\mathcal{D}_\mu f^{\mu\nu}\psi_\nu) \\ &+ \frac{e}{2m}\sqrt{-g}\left[F_{\mu\nu}(\bar{\chi}\gamma^{\mu\nu\rho}\psi_\rho - \bar{\psi}_\rho\gamma^{\mu\nu\rho}\chi - \bar{\chi}f^{\mu\nu}\chi) - \frac{1}{m}F_{\mu\nu}\bar{\chi}\gamma^{\mu\nu\rho}\mathcal{D}_\rho\chi\right] \\ &- \frac{i}{2m}\sqrt{-g}\left[G^{\mu\nu}(\bar{\chi}\gamma_\mu\psi_\nu - \bar{\psi}_\mu\gamma_\nu\chi) + \frac{1}{2}\bar{\chi}R\chi - \frac{1}{m}G^{\mu\nu}\bar{\chi}\gamma_\mu\mathcal{D}_\nu\chi\right]. \end{aligned} \tag{74}$$

The field redefinition that will remove the kinetic mixing is

$$\psi_\mu \to \psi_\mu + \frac{1}{2}\left[\gamma_\mu - \frac{i}{2}(e/m^2)F\gamma_\mu\right]\chi, \tag{75}$$

which can simply be found upon comparison with the model in Sec. III B. The gauge-fixing term to be added is the appropriate version of Eq. (11). Thus we are left with

$$\mathcal{L}_{\text{gravitino}} = -i\sqrt{-g}\Big[\bar{\psi}_\mu(\not{\mathcal{D}} - m)\psi^\mu + \frac{3}{2}\bar{\chi}(\not{\mathcal{D}} - m)\chi\Big] + \frac{e}{2m}\sqrt{-g}\Big[F_{\mu\nu}(\bar{\chi}\gamma^{\mu\nu\rho}\psi_\rho - \bar{\psi}_\rho\gamma^{\mu\nu\rho}\chi + \bar{\chi}f^{\mu\nu}\chi) - \frac{1}{m}F_{\mu\nu}\bar{\chi}\gamma^{\mu\nu\rho}\mathcal{D}_\rho\chi\Big] - \frac{i}{2m}\sqrt{-g}\Big[G^{\mu\nu}(\bar{\chi}\gamma_\mu\psi_\nu - \bar{\psi}_\mu\gamma_\nu\chi) - \frac{1}{2}\bar{\chi}R\chi - \frac{1}{m}G^{\mu\nu}\bar{\chi}\gamma_\mu\mathcal{D}_\nu\chi\Big] + \frac{i}{2} \times \sqrt{-g}\chi\Big[(\gamma_{\mu\nu} + \frac{ie}{m^2}F^+_{\mu\nu})(\gamma^\mu - \frac{ie}{2m^2}F\gamma^\mu) + 3\gamma_\nu\Big]\mathcal{D}^\nu\chi. \quad (76)$$

The $\mathcal{O}(F)$ contributions coming from the last line exactly cancel the original offending operator $F_{\mu\nu}\bar{\chi}\gamma^{\mu\nu\rho}\mathcal{D}_\rho\chi$, and this can be seen by making use of identities like (57). Then, upon the rescaling $\chi \to \sqrt{\frac{2}{3}}\chi$, the kinetic-like operators for χ reduce to

$$\mathcal{L}_{\chi,\text{kin}} = -i\sqrt{-g}\bar{\chi}\Big[g^{\mu\nu} - \frac{1}{3m^2} \times \Big(G^{\mu\nu} + \frac{e^2}{m^2}F^{+\mu\rho}F_\rho^{-\nu}\Big)\Big]\gamma_\mu\mathcal{D}_\nu\chi. \quad (77)$$

Now one can use the equations of motion (70) of the background fields and the definition (71) of the EM stress-energy tensor $T^{\mu\nu}$ to rewrite the above expression as

$$\mathcal{L}_{\chi,\text{kin}} = -i\sqrt{-g}\bar{\chi}\Big[\Big(1 + \frac{\Lambda}{3m^2}\Big)g^{\mu\nu} - \frac{1}{3m^2}\Big(\frac{1}{M_P^2} - \frac{2e^2}{m^2}\Big)T^{\mu\nu}\Big]\gamma_\mu\mathcal{D}_\nu\chi. \quad (78)$$

If the symmetric tensor inside the brackets is proportional to the metric with a non-negative coefficient, the χ sector will be ghost-free, and manifestly hyperbolic and causal. This is possible if the factor in front of the stress-energy tensor is set to zero, which is nothing but imposing the charge-mass relation (69). Then, unitarity requires that the cosmological constant be bounded from below: $\Lambda \geq -3m^2$. In this unitarily allowed region, any value of Λ will be consistent, and in particular one can set $\Lambda = 0$.

This shows that the various parameters in $\mathcal{N} = 2$ (broken) supergravity [8–10] are tuned precisely in a way that ensures a pathology-free helicity-1/2 sector. When $m^2 = -\Lambda/3 = 2e^2M_P^2$, the kinetic term (78) vanishes, so that χ is relegated to a nondynamical field. Thus we recover the unbroken $\mathcal{N} = 2$ AdS supergravity [8], where the Rarita-Schwinger field is truly massless and enjoys null propagation.

Notice that arriving at Eq. (78) from Eq. (77) is a nontrivial step, and it crucially depends on the fact that both EM and gravity are dynamical, so that the Einstein equation is sourced by the Maxwell stress-energy tensor. This relates the two *a priori* different noncanonical kinetic terms in Eq. (77), and reduces their number to one. Then the charge-mass relation removes the sole dangerous kinetic-like operator in Eq. (78). Finally, one forbids propagating ghosts in the χ sector by restricting the cosmological constant.

IV. REMARKS

The purpose of this paper was to demonstrate the power of the Stückelberg formalism in making transparent the intricacies associated with interacting theories of a massive Rarita-Schwinger field. All the peculiarities—such as the onset of strong coupling, loss of (causal) propagation and unitarity, etc.—are essentially captured in the dynamics of the helicity-1/2 mode, and a study thereof elucidates why (in)consistent models are (in)consistent.

We have seen that EM or gravitational interactions of a massive spin-3/2 field can have a local effective field theory description up to energy scales parametrically larger than the mass. The finite UV cutoff signals the onset of a dynamical regime where the helicity-1/2 sector becomes strongly coupled. Causal propagation may call for nonminimal interactions, which could lower the intrinsic cutoff of the theory from the theoretical upper bound reported in this paper. For example, in the case of EM coupling the required unitary-gauge Pauli term, $i(e/m)\bar{\psi}_\mu F^{+\mu\nu}\psi_\nu$, gives rise to an $\mathcal{O}(e)$ dimension-seven operator in the helicity-1/2 sector, and this lowers the UV cutoff to the scale $m/\sqrt[3]{e} \ll m/\sqrt{e}$.

As pointed out in Ref. [14], the cutoff scale may also mean that there could be new interacting degrees of freedom lighter than that scale. These new fields may further improve the high-energy behavior of the theory. For the gravitational case this is exactly what happens in broken $\mathcal{N} = 1$ supergravity [16]. As was shown in Ref. [17], a scalar and a pseudoscalar with masses much lower than Λ_g (slightly above m) can push the strong-coupling regime all the way to the Planck scale M_P.

We have performed a Stückelberg analysis as a consistency check of a number of interacting theories known in the literature. The Velo-Zwanziger acausality [2] of a massive spin-3/2 field minimally coupled to EM indeed shows up as a pathology of the helicity-1/2 mode itself. "Appropriate" nonminimal EM interactions [7] are precisely those that ensure the light-cone propagation of this mode. In the case of minimal gravitational coupling, the noncanonical kinetic terms can be rendered harmless by requiring the spacetime to be an Einstein manifold, provided that the curvature has the well-known unitarity bound; this reconfirms the results of Refs. [13,19]. Finally, we have analyzed $\mathcal{N} = 2$ (broken) supergravity [8–10] to reveal that the helicity-1/2 sector acquires healthy kinetic terms in the presence of dynamical Maxwell-Einstein fields if the usual charge-mass relation holds.

The Stückelberg mode(s) can be used as a probe of the consistency of interactions for any massive particle with $s \geq 1$. While spin 2 was considered in Refs. [11,12], it remains to be seen what more we can learn from them about consistent interactions of massive higher spins.

ACKNOWLEDGMENTS

R. R. would like to thank R. Argurio, C. Petersson, and M. Porrati for useful discussions. He gratefully acknowledges a research grant from the Solvay Foundation. His work is also partially supported by IISN—Belgium (conventions 4.4511.06 and 4.4514.08), by the Belgian Federal Science Policy Office through the Interuniversity Attraction Pole P6/11 and by the "Communauté Française de Belgique" through the ARC program, as well as by Scuola Normale Superiore, by INFN and by the ERC Advanced Investigator Grant No. 226455 "Supersymmetry, Quantum Gravity and Gauge Fields" (SUPERFIELDS).

[1] W. Rarita and J. Schwinger, Phys. Rev. **60**, 61 (1941).
[2] G. Velo and D. Zwanziger, Phys. Rev. **186**, 1337 (1969).
[3] A. Shamaly and A. Z. Capri, Ann. Phys. (N.Y.) **74**, 503 (1972); Can. J. Phys. **52**, 919 (1974); M. Hortacsu, Phys. Rev. D **9**, 928 (1974); J. Prabhakaran, M. Seetharaman, and P. M. Mathews, Phys. Rev. D **12**, 3191 (1975); **12**, 458 (1975); J. Phys. A **8**, 560 (1975); S. Deser, V. Pascalutsa, and A. Waldron, Phys. Rev. D **62**, 105031 (2000).
[4] S. Deser and A. Waldron, Nucl. Phys. **B631**, 369 (2002).
[5] K. Johnson and E. C. G. Sudarshan, Ann. Phys. (N.Y.) **13**, 126 (1961).
[6] M. Fierz and W. Pauli, Proc. R. Soc. A **173**, 211 (1939); Helv. Phys. Acta **12**, 297 (1939).
[7] M. Porrati and R. Rahman, Phys. Rev. D **80**, 025009 (2009).
[8] S. Ferrara and P. van Nieuwenhuizen, Phys. Rev. Lett. **37**, 1669 (1976); D. Z. Freedman and A. K. Das, Nucl. Phys. **B120**, 221 (1977).
[9] J. Scherk and J. H. Schwarz, Phys. Lett. **82B**, 60 (1979); Nucl. Phys. **B153**, 61 (1979); B. de Wit, P. G. Lauwers, and A. Van Proeyen, Nucl. Phys. **B255**, 569 (1985).
[10] S. Deser and B. Zumino, Phys. Rev. Lett. **38**, 1433 (1977).
[11] N. Arkani-Hamed, H. Georgi, and M. D. Schwartz, Ann. Phys. (Amsterdam) **305**, 96 (2003).
[12] M. Porrati and R. Rahman, Nucl. Phys. **B801**, 174 (2008); arXiv:0809.2807.
[13] I. L. Buchbinder and V. A. Krykhtin, Mod. Phys. Lett. A **25**, 1667 (2010).
[14] M. Porrati and R. Rahman, Nucl. Phys. **B814**, 370 (2009).
[15] R. Rahman, Ph.D. thesis, New York University, 2009.
[16] E. Cremmer, B. Julia, J. Scherk, S. Ferrara, L. Girardello, and P. van Nieuwenhuizen, Nucl. Phys. **B147**, 105 (1979).
[17] R. Casalbuoni, S. De Curtis, D. Dominici, F. Feruglio, and R. Gatto, Phys. Lett. B **216**, 325 (1989); **229**, 439(E) (1989).
[18] J. S. Schwinger, Phys. Rev. **82**, 664 (1951); N. K. Nielsen and P. Olesen, Nucl. Phys. **B144**, 376 (1978).
[19] S. Deser and A. Waldron, Phys. Rev. Lett. **87**, 031601 (2001); Nucl. Phys. **B607**, 577 (2001).