Light front Casimir effect

T. Almeida,* Van Sérgio Alves,† Danilo T. Alves,‡ Silvana Perez,§ and P. L. M. Rodrigues‖
Faculdade de Física, Universidade Federal do Pará, Belém, Pará 66075-110, Brazil
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The description of the Casimir effect via the conventional light front frame gives the result that no finite Casimir energy density exists when the boundary conditions are related to the unphysical situation where the boundaries move at the speed of light. In the present paper, we investigate if the consideration of the oblique light front coordinates, more convenient for studies in thermal light front quantum field theories, can also be used to describe in a physically consistent manner the Casimir effect in the light front dynamics. Using these coordinates, we show that the correct prescription in the light front formalism to recover the standard Casimir effect involves two aspects: the presence of boundaries that are not at the speed of light and that impose conditions to the field taken at the same Minkowski time.

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I. INTRODUCTION

In 1949, Dirac considered the light front (LF) coordinates in the study of the relativistic dynamics of physical systems [1]. Using these coordinates—in the present paper named conventional LF coordinates—he proposed a new formalism, in which the quantization surface is the one formed by a plane wave front advancing with the velocity of light. The quantization of field theories on this plane has found applications in many branches of physics [2–4]. One of the distinct features of LF dynamics is the energy-momentum dispersion relation which is linear in the LF energy. As a consequence, the conservation of the total momentum in the longitudinal direction forbids the excitation of massive quanta by the LF vacuum with vanishing longitudinal momentum. In this sense, the structure of the interacting light front ground state is much simpler than the instant form counterpart [3].

The generalization of light front quantized field theories to finite temperature has been discussed in many publications, including Refs. [5–10]. The formalism was then used to study several problems [11–20]. Particularly, in Refs. [8,11], Weldon has discussed a collection of generalized LF coordinate systems, defined as

$$\bar{x}^0 = x^0 + x^3 \qquad \bar{x}^3 = Ax^0 + Bx^3$$
$$\bar{x}^\alpha = x^\alpha, \quad \alpha = 1, 2, \tag{1}$$

where A and B are real constants with the restriction $|A| \leq |B|$ (hereafter we assume $\hbar = k_B = c = 1$). One of the main aspects of these generalized coordinates is that they preserve the dynamics of the conventional LF frame ($B = -A = -1$), in the sense that the LF "time" variable is kept the same. Another aspect of these generalized LF coordinates is that a heat bath put at rest in the frame $\bar{x}$ is seen in the inertial frame x as having a velocity v given by

$$v = -A/B. \tag{2}$$

Particularly if we consider the conventional LF coordinates, the heat bath is viewed by the inertial frame x with the speed of light. For all other values of A and B, the heat bath is seen in the inertial frame x as having a velocity $|v| < 1$. Specifically, for $A = 0$, $B = 1$ (oblique LF frame), $v = 0$, and we have a more convenient frame for the statistical description of the LF theories.

The investigation of boundary condition problems, specifically the Casimir effect [21], in the conventional light front quantized field theory was carried out by Lenz and Steinbacher [22]. These authors considered the nonmassive scalar field in (3 + 1) dimensions described in the conventional light front frame, obtaining the standard result for the Casimir pressure by imposing periodic boundary conditions in the transverse directions. On the other hand, considering periodic boundary conditions in the longitudinal direction, they showed that no regularization of the quantum fluctuations yields a finite Casimir energy density. Physically, this absence of regularization is related to the fact that the boundary condition considered in the conventional LF frame leads to the plates moving at the speed of light. This problem resembles the one we have just summarized above in the context of thermal LF quantum field theories.

Our main goal is to investigate if the consideration of the oblique LF coordinates, more convenient for studies in thermal light front quantum field theories, can also be used to describe in a physically consistent manner the Casimir effect in the LF dynamics. In the present paper, considering the field model used in Ref. [22], we study the Casimir energy density related to periodic boundary conditions imposed in the transverse and longitudinal directions but adopting the general LF coordinate system [8,11]. Reducing the dimensions of the model to (1 + 1),

*tercio@ufpa.br
†vansergi@ufpa.br
‡danilo@ufpa.br
§silperez@ufpa.br
‖pennlee@ufpa.br

we study the regularization of the quantum fluctuations in a simplified model in the oblique LF frame. After that, we continue considering these coordinates and propose a new prescription, more closely related with the measurable Casimir force usually discussed in the instant form formalism (where the dynamics of the system is governed by x^0). In addition, we calculate the thermal correction to the Casimir energy.

The paper is organized as follows. In Sec. II we discuss the general aspects of the generalized LF coordinates. In Sec. III we investigate the Casimir effect in the generalized LF coordinates at zero temperature for the massless scalar theory in both (3 + 1) and (1 + 1) dimensions. In Sec. IV, we consider the oblique LF coordinates and propose an alternative boundary condition on the light front that enables us to recover the instant form Casimir energy density. In Sec. V we calculate the thermal correction to the Casimir energy. The summary of our conclusions is given in Sec. VI. Throughout this paper we use $\alpha = 1, 2$ and μ, ν, σ, $\xi = 0, \ldots, 3$.

II. GENERALIZED LF FRAME

As we have discussed in the Introduction, LF field theories do not admit a naive generalization to finite temperature [5,10,17]. One way of introducing thermal effects into the LF quantization is to consider the generalized light front coordinates [8], defined in Eq. (1). This new system of coordinates $\bar{x}$ is related to the inertial coordinate frame x (used in the instant form description of quantum field theories) through a linear transformation,

$$\bar{x}^\mu = L^\mu{}_\nu x^\nu, \tag{3}$$

where

$$L^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ A & 0 & 0 & B \end{pmatrix}. \tag{4}$$

Under (3) the metric tensor transforms as

$$\bar{g}^{\mu\nu} = L^\mu{}_\sigma L^\nu{}_\xi \eta^{\sigma\xi} = \begin{pmatrix} 0 & 0 & 0 & (A-B) \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ (A-B) & 0 & 0 & (A^2 - B^2) \end{pmatrix}, \tag{5}$$

where η is the Minkowski metric, namely, $\eta = \mathrm{diag}(1, -1, -1, -1)$.

To describe the statistical mechanics in the generalized coordinate system, we first use that, for a heat bath that is moving with a normalized velocity $\bar{u}^\mu$, the density matrix ρ is given by [8]

$$\rho(\beta) = e^{-\beta \bar{u}^\mu \bar{p}_\mu}, \tag{6}$$

where $\bar{p}_\mu$ is the energy-momentum tensor of the statistical system, and β is the inverse of the reservoir temperature. In the rest frame of the statistical system,

$$\bar{u}^\mu_{\text{rest}} = \left(\frac{1}{\sqrt{\bar{g}_{00}}}, 0, 0, 0 \right), \tag{7}$$

the density matrix has the form

$$\rho = e^{-\frac{\beta}{\sqrt{\bar{g}_{00}}} \bar{p}_0}. \tag{8}$$

Therefore, the generalized LF frame allows a statistical description with the heat bath at rest in the system $\bar{x}$ as long as $A \pm B \neq 0$, which excludes the conventional LF frame, for which $\bar{g}_{00}$ vanishes. On the other hand, for the oblique LF frame, $\bar{g}_{00} = 1$, leading to a convenient statistical description where the temperature can be identified with that of the original inertial coordinate frame x.

To discuss the presence of boundaries in the generalized LF coordinate system, let us consider, for instance, the presence of plates at rest (relative to the generalized time $\bar{x}^0$) at fixed positions $\bar{x}^3 = a$ and $\bar{x}^3 = b$. From the point of view of the instant form system, the plates are in movement with constant velocity v given in Eq. (2). For $B = -A = -1$ (conventional LF frame), we have the unphysical situation where the plates move at the speed of light ($v = c = 1$), which is related to the absence of regularization of the quantum fluctuations when a boundary condition is considered in the longitudinal direction. On the other hand, for $|v| < 1$ we get physically acceptable situations for v. Specifically for $A = 0$ and $B = 1$ (oblique LF frame), the plates are at rest ($v = 0$). Since the oblique LF frame leads to a consistent statistical description where the temperature can be identified with that of the instant form frame, next we investigate if, in the oblique LF frame, the Casimir effect can also be described consistently.

III. CASIMIR EFFECT AT ZERO TEMPERATURE

We consider the massless scalar field, with Lagrangian density given by

$$\mathcal{L} = \frac{1}{2}(\bar{g}^{\mu\nu}\bar{\partial}_\mu\phi\bar{\partial}_\nu\phi) = (A-B)\bar{\partial}_0\phi\bar{\partial}_3\phi - \frac{1}{2}(\bar{\partial}_\alpha\phi)^2 - \frac{1}{2}(B^2 - A^2)(\bar{\partial}_3\phi)^2. \tag{9}$$

The field decomposition takes the form

$$\phi(\bar{x}) = \frac{1}{(2\pi)^3} \int \frac{d^4\bar{k}}{\sqrt{\bar{g}}} \delta(\bar{k}^2) e^{-i\bar{k}\cdot\bar{x}} \hat{\phi}(\bar{k}), \tag{10}$$

which, after evaluating the $\bar{k}_0$ integral, is written as

$$\phi(\bar{x}) = \frac{1}{(2\pi)^{3/2}} \int d^2\bar{k} \int_0^\infty \frac{d\bar{k}_3}{2\bar{k}_3}[e^{-i\tilde{\bar{k}}\cdot\bar{x}}a(\bar{k}) + e^{+i\tilde{\bar{k}}\cdot\bar{x}}a^\dagger(\bar{k})], \tag{11}$$

with $\tilde{\bar{k}} \equiv (\bar{k}_0, \mathrm{sgn}(A-B)\bar{k}_\alpha, \mathrm{sgn}(A-B)\bar{k}_3)$ and

$$\bar{k}_0 \equiv \frac{\bar{k}_\alpha^2 + (B^2 - A^2)\bar{k}_3^2}{2|A-B|\bar{k}_3} > 0, \quad \alpha = 1, 2. \tag{12}$$

Furthermore, $a^\dagger$ and a satisfy

$$[a(\bar{k}), a^\dagger(\bar{q})] = 2\bar{k}_3\delta^3(\bar{k} - \bar{q}), \tag{13}$$

from which we see that they are the creation and annihilation operators, respectively. The canonical Hamiltonian is then given by

$$H = \frac{1}{2}\int d^2\bar{k}\int \frac{d\bar{k}_3}{2\bar{k}_3}\bar{k}_0(aa^\dagger + a^\dagger a), \tag{14}$$

and the canonical quantization on the light cone breaks the explicit rotational invariance of the system.

The boundary conditions may be applied in one of the transverse directions or in the $\bar{x}^3$ direction. We start with a transverse direction and impose a periodic boundary condition, namely

$$\phi(\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3) = \phi(\bar{x}^0, \bar{x}^1 + L, \bar{x}^2, \bar{x}^3), \tag{15}$$

for which the energies of the one-particle states are given by

$$\bar{k}_0 \equiv \frac{(2\pi n/L)^2 + \bar{k}_2^2 + (B^2 - A^2)\bar{k}_3^2}{2|A-B|\bar{k}_3}. \tag{16}$$

The regularized energy density is then given by

$$\langle H\rangle_\lambda = \frac{1}{2L}\frac{|A-B|}{(2\pi)^2}\sum_n \int d\bar{k}_2 \int_0^\infty d\bar{k}_3\bar{k}_0 e^{-\lambda^3\bar{k}_3 - \lambda^0\bar{k}_0}, \tag{17}$$

where the two regulators are required to suppress the divergences in both the light front energy and momenta [22]. The calculation of $\langle H\rangle_\lambda$ is straightforward, and after subtracting the free field contribution, the Casimir energy density is

$$\langle H\rangle_{\mathrm{finite}} = -\pi^2/90L^4, \tag{18}$$

recovering the instant form result for the Casimir energy, for any particular choice of the generalized LF coordinate system, including the conventional LF frame.

Now we consider a periodic boundary condition in the $\bar{x}^3$ direction,

$$\phi(\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3) = \phi(\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3 + L). \tag{19}$$

The regularized energy density for this case is given by

$$\langle H\rangle_\lambda = \frac{1}{2L}\frac{|A-B|}{(2\pi)^2}\sum_{n=0}^\infty \int_{-\infty}^{+\infty} d^2\bar{k}_\alpha\bar{k}_0 e^{-\lambda^3\bar{k}_3 - \lambda^0\bar{k}_0}. \tag{20}$$

We then find the following result for the energy density:

$$\langle H\rangle_\lambda = \frac{1}{8\pi^2(\lambda^0\lambda^3)^2} - \frac{1}{24(L\lambda^0)^2} + \frac{\pi^2}{120L^4}\left(\frac{\lambda^3}{\lambda^0}\right)^2 - \frac{\pi^2}{480L^4}(\sqrt{-\bar{g}})^2(B^2 - A^2)^2. \tag{21}$$

We analyze this result by first considering the particular case where $B = -A = -1$, namely, the conventional LF frame. In this situation the fourth term in the right-hand side of Eq. (21) is null, and we are left with

$$\langle H\rangle_\lambda = \frac{1}{8\pi^2(\lambda^0\lambda^3)^2} - \frac{1}{24(L\lambda^0)^2} + \frac{\pi^2}{120L^4}\left(\frac{\lambda^3}{\lambda^0}\right)^2, \tag{22}$$

consistently recovering the result obtained in Ref. [22] for the conventional LF frame, where no term exhibits independence of the regulators. As we already emphasized in the introduction, this result is related to the imposition of the velocity of light for the plates. What is surprising is that even though for a general LF frame with $A^2 \neq B^2$, the fourth term of Eq. (21) is now a finite term with no regulator, the complete expression for $\langle H\rangle_\lambda$ remains poorly defined. Therefore, more than a generalization of the result found in Ref. [22], our result shows that, even with the plates at rest ($A = 0$, $B = 1$), a situation analogous to a heat bath at rest with respect to the statistical system, the problem persists.

Trying to understand why the direct use of the oblique LF coordinates also fails to produce a regularization of the quantum vacuum fluctuations, we choose to investigate the problem in a simplified model in (1 + 1) dimensions [LF space-time coordinates labeled, for convenience, as $(\bar{x}^0, \bar{x}^3)$]. First of all, from the structure of Eq. (12), we can write the dispersion relation for the model in (1 + 1) dimensions as

$$\bar{k}_0 \equiv \frac{(B^2 - A^2)\bar{k}_3}{2|A-B|} > 0. \tag{23}$$

Particularly in the oblique light front frame, we have

$$\bar{k}_0 \equiv \frac{\bar{k}_3}{2} > 0, \tag{24}$$

and, from Eq. (11), the field is decomposed as

$$\phi(\bar{x}) = \frac{1}{(2\pi)^{1/2}}\int_0^\infty \frac{d\bar{k}_3}{2\bar{k}_3}[e^{-i\bar{k}\cdot\bar{x}}a(\bar{k}) + e^{+i\bar{k}\cdot\bar{x}}a^\dagger(\bar{k})]. \tag{25}$$

Rearranging the terms, we then write

$$\phi(\bar{x}) = \frac{1}{(2\pi)^{1/2}}\int_0^\infty \frac{d\bar{k}_3}{2\bar{k}_3}\Big[e^{-i(\frac{\bar{x}^0}{2} - \bar{x}^3)\bar{k}_3}a(\bar{k}) + \mathrm{H.c.}\Big]. \tag{26}$$

Imposing a periodic boundary condition in the longitudinal direction, namely,

$$\phi(\bar{x}^0, \bar{x}^3) = \phi(\bar{x}^0, \bar{x}^3 + L), \tag{27}$$

we then find

$$\bar{k}_3 = \frac{2\pi n}{L}, \quad n = 0, 1, 2, \ldots, \tag{28}$$

and the energy density is written as

$$\langle H \rangle = \frac{1}{4}\left(\frac{2\pi}{L^2}\right)\sum_{n=0}^{\infty} n. \tag{29}$$

The regularized energy density is then given by

$$\langle H \rangle_\lambda = \frac{1}{8\pi\lambda^2} - \frac{\pi}{24L^2} + O(\lambda^2), \tag{30}$$

where the regulator λ is required to suppress the divergence. The Casimir energy density for this case is

$$\langle H \rangle_{\text{finite}} = -\frac{\pi}{24L^2}. \tag{31}$$

From this calculation we show that the absence of regularization of the quantum vacuum fluctuations—detected in (3 + 1) dimensions—disappears when the calculations are carried out via the oblique light front frame in (1 + 1) dimensions, so that the renormalizability of the energy density depends on both the space-time dimension and the light front coordinate system adopted. From a mathematical point of view, this behavior is a consequence of the simplified dispersion relation (24) in (1 + 1) dimensions in the oblique LF frame, requesting the same number of regulators as the usual IF formalism.

We can go further and ask to which "experiment" this calculation would be related. To answer this question, let us go back to the usual Minkowsky Casimir effect, calculating the problem in instant form (IF) formalism (x^0, x^3), for which the field ϕ_{IF} is given by

$$\phi_{\text{IF}}(x) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \frac{dk}{\sqrt{2|k|}} [e^{-ik\cdot x} a(k) + \text{H.c.}], \tag{32}$$

with $k^\mu = (|k|, k)$. Imposing the periodic boundary condition

$$\phi_{\text{IF}}(x^0, x^3) = \phi_{\text{IF}}(x^0, x^3 + L), \tag{33}$$

the energy density obtained for this case is

$$\langle H \rangle^{\text{IF}} = \frac{2\pi}{L^2} \sum_{n=0}^{\infty} n, \tag{34}$$

which, after renormalization, is given by

$$\langle H \rangle^{\text{IF}}_{\text{finite}} = -\frac{\pi}{6L^2}. \tag{35}$$

Notice that the energy density in Eq. (29) does not match the one obtained in Eq. (34), so that we conclude that the situations (27) and (33) are not physically equivalent or, in other words, the two boundary conditions are not related to the same "experiment."

In fact, what would be in the IF formalism the equivalent to the boundary condition (27) is, according to Eq. (1), the imposition of boundary conditions at different times, i.e.,

$$\phi_{\text{IF}}(x^0, x^3) = \phi_{\text{IF}}(x^0 - L, x^3 + L). \tag{36}$$

This condition requires

$$k^3 + |k^3| = 2n\pi/L, \tag{37}$$

which is automatically satisfied by $k^3 < 0$ $(n = 0)$, and for $k^3 > 0$ we get

$$k^3 = n\pi/L, \quad n = 0, 1, 2, \ldots \tag{38}$$

In other words, the boundary condition (36) implies in discrete modes only for positive values of k^3, so that just the right propagating modes of ϕ_{IF} in Eq. (33) feel the compactification of the space, whereas the left propagating modes see no compactification. Therefore, we can say that the compactification at the same light front time (27), when mapped back to the instant form description (36), breaks down the isotropy in x^3 direction, leading to a physical situation with nontrivial interpretation. Considering only the contribution to the energy coming from $k^3 > 0$, we then get

$$\langle H \rangle^{\text{IF}}_{\text{finite}} = -\frac{\pi}{48L^2}. \tag{39}$$

It is worth emphasizing that the breaking of the isotropy also occurs in (3 + 1) dimensions if we consider the instant form field obeying the condition $\phi_{\text{IF}}(x^0, x^\alpha, x^3) = \phi_{\text{IF}}(x^0 - L, x^\alpha, x^3 + L)$, which corresponds to a naive mapping of Eq. (19) to the instant form description.

We therefore summarize this section, pointing out that, in the study of the Casimir effect in the LF formalism, one has to be careful with two aspects: first, the imposition of unphysical situations where the boundaries move at the speed of light and, second, the imposition of boundary conditions at different times in the original Minkowski frame stickling the physical interpretation.

In the next section we propose a new prescription that involves the oblique LF frame, in such way to solve the problem of boundaries moving at speed of light, and impose boundary conditions at different LF times corresponding to equal time boundary conditions in Minkowski frame.

IV. ALTERNATIVE PRESCRIPTION FOR THE LF CASIMIR EFFECT

In the oblique LF frame, we propose the following boundary condition:

$$\phi(\bar{x}^0, \bar{x}^3) = \phi(\bar{x}^0 + L, \bar{x}^3 + L). \tag{40}$$

The main feature of (40) is that its naive mapping to the instant form results in the (equal time) periodic boundary condition (33). With this choice we get

$$\bar{k} = \frac{4\pi n}{L}, \quad n = 0, 1, 2, \ldots, \tag{41}$$

and the energy density is given by

$$\langle H\rangle = \frac{2\pi}{L^2}\sum_{n=0}^{\infty} n. \quad (42)$$

This result—both divergent and finite parts—is exactly the one coming from the instant form formalism [Eq. (34)]. We emphasize that the origin of this result is associated with three points: the first one is the use of the oblique LF coordinates; the second point is the choice of the suited boundary condition (40); and the third one is related to the simplified dispersion relation in (1 + 1) dimensions, requesting the same number of regulators as the usual IF formalism.

Our goal now is to generalize this prescription to (3 + 1) dimensions, where the dispersion relation exhibits more complexity, involving also transverse directions and, as a consequence, affecting the divergence structure, specifically introducing two regulators instead of only one needed in the IF formalism.

We start writing the field decomposition, Eq. (11), in the oblique light front frame,

$$\phi(\bar{x}) = \frac{1}{(2\pi)^{3/2}}\int d^2\bar{k}_\alpha \int_0^\infty \frac{d\bar{k}_3}{2\bar{k}_3} \times\left[e^{-i\left(\frac{\bar{k}_\alpha^2\bar{x}^0}{2\bar{k}_3}+\bar{k}_3\left(\frac{\bar{x}^0}{2}-\bar{x}^3\right)-\bar{k}_\alpha\bar{x}^\alpha\right)}a(\bar{k})+\mathrm{H.c.}\right]. \quad (43)$$

Applying the compactification in the longitudinal direction $\bar{x}^3$ with x^0 fixed (again using oblique coordinates),

$$\phi(\bar{x}^0,\bar{x}^\alpha,\bar{x}^3) = \phi(\bar{x}^0+L,\bar{x}^\alpha,\bar{x}^3+L), \quad (44)$$

we then find the following constraint:

$$\frac{\bar{k}_\alpha^2}{2\bar{k}_3} - \frac{\bar{k}_3}{2} = \frac{2\pi n}{L}, \qquad n = 0, \pm 1, \pm 2, \ldots \quad (45)$$

Now, using (45), we write the regularized energy density as

$$\langle H\rangle_\lambda = \frac{1}{2L}\frac{1}{(2\pi)^2}\sum_{n=-\infty}^{+\infty}\int d^2\bar{k}_\alpha\int_0^\infty d\bar{k}_3\bar{k}_0 \times\delta\left(\frac{\bar{k}_\alpha^2-\bar{k}_3^2}{2\bar{k}_3}-\frac{2\pi n}{L}\right)e^{-\lambda^3\bar{k}_3-\lambda^0\bar{k}_0}. \quad (46)$$

Considering the change of variables

$$\bar{k}_3 = k_3 + E_k, \qquad E_k = \sqrt{k_\alpha^2+k_3^2}, \qquad \bar{k}_\alpha = k_\alpha, \quad (47)$$

such that the range of k_3 is $-\infty < k_3 < \infty$, the energy density is written as

$$\langle H\rangle_\lambda = \frac{1}{2L}\sum_{n=-\infty}^{+\infty}\frac{1}{(2\pi)^2}\int d^3kE_k \times\delta\left(k_3-\frac{2\pi n}{L}\right)e^{-\lambda^3(k_3+E_k)-\lambda^0E_k}, \quad (48)$$

where we have dropped the term linear in k_3 in the integrand, which vanishes by parity arguments after removing the regulators. Using the delta function to eliminate the k_3 integral, we are left with

$$\langle H\rangle_\lambda = \frac{1}{2L}\frac{1}{(2\pi)^2}\sum_{n=-\infty}^{+\infty}\int d^2k_\alpha E_k e^{\lambda^3\frac{2\pi n}{L}-(\lambda^0+\lambda^3)E_k}. \quad (49)$$

This nonrenormalized result disagrees with the well-known instant form one [see Eq. (2.1) in Ref. [22]] because of the $e^{\lambda^3\frac{2\pi n}{L}}$ term, which is a consequence of the extra regulator necessary for the light front calculation in (3 + 1) dimensions. What is surprising is that even though the two nonrenormalized results do not agree, the explicit calculation of Eq. (49) gives

$$\langle H\rangle_\lambda = \frac{1}{4\pi(\lambda^3\lambda^0)^2} - \frac{1}{32\pi^2(\lambda^3)^4} + \frac{3\lambda^0}{64\pi^2\lambda_3^5} - \frac{3(\lambda^0)^2}{64\pi^2(\lambda_3)^6} - \frac{\pi^2}{90L^4} + \mathcal{O}\left(\frac{\lambda^0\lambda^3}{L^6},\frac{(\lambda^0)^2}{L^6},\frac{(\lambda^3)^2}{L^6},\frac{(\lambda^0)^3}{(\lambda^3)^7}\right), \quad (50)$$

showing that there is no more ambiguity with the regulators and, further, the finite contribution matches exactly the standard Casimir energy.

We learn from this that the problem of the nonexistence of a finite Casimir energy in the LF formalism is not only related with the the nonphysical imposition of the speed of light for the plates, because in the oblique LF frame where this speed is reduced to zero, the problem persists. Our calculation shows that the mentioned problem is also related to the choice of boundary conditions imposed at different times in the original Minkowski frame.

V. CASIMIR EFFECT AT FINITE TEMPERATURE

Let us obtain the thermal corrections to the Casimir energy on the LF. For this study, we will follow the basic ideas given, for example, in Refs. [8,23]. We then start remembering that for the canonical ensemble, all the physical information about the system is contained in the partition function Z and in the Helmholtz free energy F, given, respectively, by

$$Z = \mathrm{Tr}\rho \quad (51)$$

and

$$F = -\frac{1}{\beta}\ln Z, \quad (52)$$

where $\rho = e^{-\beta\bar{k}_0}$ is the density matrix of the system in the oblique LF frame [see Eq. (8)]. Considering the scalar field Hamiltonian given in Eq. (14), the Helmholtz free energy is written as

$$\begin{aligned} F = -T\ln Z &= -T\ln\Pi_{n_k}\frac{e^{-\beta\bar{k}_0/2}}{1-e^{\beta\bar{k}_0}} \\ &= \langle 0|H|0\rangle + T\sum_{n_k}\ln(1-e^{-\beta\bar{k}_0}) \\ &\equiv \langle 0|H|0\rangle + \tilde{F}, \end{aligned} \quad (53)$$

where $\langle 0|H|0\rangle$ is the vacuum energy density. For example, considering the electromagnetic field inside a big and empty cube (free field configuration), we obtain the following explicit finite temperature contribution,

$$\tilde{F}_{(0)} = -\frac{2V\zeta(4)T^4}{\pi^2}, \tag{54}$$

where V is the volume of the cube and ζ is the Riemann zeta function. We see that for the oblique LF coordinates, it agrees with the standard Stefan-Boltzman law for the blackbody radiation.

Next, if we apply periodic boundary conditions in the $\bar{x}^1$ direction at fixed $\bar{x}^0$, we find in the low temperature limit ($LT \ll 1$),

$$\begin{aligned}\tilde{F} &= \frac{T\mathcal{A}}{(2\pi)^2}\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\bar{k}_2 \int_0^{\infty} d\bar{k}_3 \ln\left[1 - e^{-\beta\frac{\bar{k}_\alpha^2+\bar{k}_3}{2\bar{k}_3}}\right] \\ &= -\frac{T\mathcal{A}}{(2\pi)^2}\sum_{n=-\infty}^{+\infty}\sum_{\nu=1}^{+\infty}\frac{1}{\nu}\int_{-\infty}^{+\infty} d\bar{k}_2 \int_0^{\infty} d\bar{k}_3 \\ &\quad \times e^{-\frac{\beta\nu\bar{k}_2^2}{2\bar{k}_3}} e^{-\frac{\beta\nu}{2}\left(\frac{\bar{k}_1^2}{\bar{k}_3}+\bar{k}_3\right)},\end{aligned} \tag{55}$$

where $\mathcal{A}$ stands for certain area of the plates. The integrals are evaluated using

$$\int_0^{\infty} dy y^{n-1/2} e^{-py-q/y} = (-1)^n\sqrt{\pi}\frac{\partial^n}{\partial p^n}\left(\frac{e^{-2\sqrt{pq}}}{p^{1/2}}\right),$$

such that in the low temperature limit we find

$$\tilde{F} \sim \frac{T\mathcal{A}}{2\pi}\left[-T^2\zeta(3) - 2T^2 e^{-\frac{2\pi}{LT}}\left(1 + \frac{2\pi}{LT}\right)\right], \tag{56}$$

so that it recovers the result in the literature when the oblique light front frame is considered. The calculation of the high temperature limit, on the other hand, uses the Poisson's sum formula. Without going into technical details, one finds for $LT \gg 1$,

$$\begin{aligned}\tilde{F} &= -\frac{T^4\mathcal{A}L}{\pi^2}\zeta(4) - \frac{T\mathcal{A}}{2\pi L^2}\zeta(3) + \frac{\mathcal{A}}{\pi^2 L^3}\zeta(4) \\ &\quad - \frac{T\mathcal{A}}{\pi L^2}(1 + 2\pi LT)e^{-2\pi LT},\end{aligned} \tag{57}$$

consistent with instant form calculations. It is direct to check that these results satisfy the temperature inversion symmetry (TIS) given by [24],

$$\tilde{F}(\xi) = \xi^4\tilde{F}(1/\xi), \tag{58}$$

where $\xi = LT$ is a dimensionless variable.

For boundary conditions in the $\bar{x}^3$ direction in the oblique LF frame, we are again able to find a finite result in both limits, namely

$$\tilde{F} = -\frac{T^2\mathcal{A}}{L}\left(e^{-\frac{\pi}{LT}} + \frac{9}{4}e^{-\frac{2\pi}{LT}}\right), \tag{59}$$

in the low temperature limit, and

$$\begin{aligned}\tilde{F} &= -\frac{T^4\mathcal{A}L}{\pi^2}\zeta(4) + \frac{T^2\mathcal{A}}{2\pi^2 L}\zeta(2)^2 + \frac{3\mathcal{A}}{16\pi^2 L^3}\zeta(4) \\ &\quad - \frac{T\mathcal{A}}{4\pi L^2}\zeta(3) - \frac{T\mathcal{A}}{2\pi L^2}(1 + 2\pi LT)e^{-4\pi LT}\end{aligned} \tag{60}$$

in the high temperature limit.

These results, unlike the previous one, do not respect the TIS (58). We guess that this behavior is again a manifestation of the breakdown of the isotropy of the space-time for boundary conditions considered over constant LF time coordinate and compactification in the longitudinal direction. Of course, the Bose-Einstein distribution function is a natural regulator, making all the results finite, but the symmetries that depend on the temperature of the theory are broken for the considered boundary condition.

VI. CONCLUSIONS

In the present paper, considering the massless scalar field, we investigated the Casimir energy density related to periodic boundary conditions imposed in the transverse and longitudinal directions in the LF description of field theories, at zero and finite temperature.

Considering the generalized LF coordinates, we showed that no regularization of the vacuum fluctuations exists when the boundary condition is taken in the longitudinal direction at fixed LF time in (3 + 1) dimensions. Our first conclusion is that the problem is not only related to boundaries moving at the speed of light as we have in the conventional LF frame, once the problem remains even in the oblique LF frame where this speed vanishes.

Studying a simplified model involving the oblique LF frame in (1 + 1) dimensions, with the periodic boundary conditions taken at the same LF time $\bar{x}^0$ (27), we showed that the energy density is surprisingly renormalizable, so that the renormalizability of the quantum vacuum fluctuations depends on both the space-time dimension and the LF coordinate system adopted. We consider this equality as a consequence of the simplified dispersion relation in (1 + 1) dimensions in the oblique LF frame, which imposes the same number of regulators as the usual IF formalism. However, comparing the finite contribution (31) with the standard Casimir energy (35), we showed that the two results are not the same. This means that the boundary condition (27) in the oblique LF description is not physically equivalent to the IF boundary condition (33), used to obtain the standard Casimir effect.

Investigating what would be in the IF formalism the equivalent of the boundary condition (27), we found a boundary condition taken at different Minkowski times x^0 [given in Eq. (36)]. However, the use of this boundary condition forces a break down of the isotropy in the x^3 direction and, as a consequence, a problematic physical interpretation.

Looking for a boundary condition in the LF that avoids the unphysical situation where the boundaries move at the speed of light and also imposes boundary conditions at same x^0, we considered in the oblique LF frame the boundary condition given by Eqs. (40) and (44), for $(1+1)$ and $(3+1)$ dimensions, respectively. In both cases, although the divergence structures depend on the space-time dimensions, the renormalized energy densities match with the standard Casimir energy densities found in the literature.

In summary, we conclude that the problem of the nonexistence of a finite Casimir energy in the LF formalism is not only related to the nonphysical imposition of the speed of light for the plates but also to the choice of boundary conditions imposed at different times in the original Minkowski frame. When one uses a prescription that carefully takes into account both points, the oblique LF formalism is able to reproduce the standard Casimir effect.

In addition, we obtained the thermal corrections to the Casimir energy in the LF formalism. For a boundary condition in the longitudinal direction, we calculated the Helmholtz free energy and showed that it does not respect the temperature inversion symmetry, which can be an indirect consequence of the space-time isotropy broken for boundary conditions that keep constant the LF time. We obtained, for the oblique LF coordinates, the standard Stefan-Boltzman law for the blackbody radiation.

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