

**Theories with finite Green's functions**

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The addition of certain nonrenormalizable terms to the usual action density of a free scalar field leads to nonrenormalizable theories whose exact Euclidian and Minkowskian Green's functions are less singular than those of the free theory. In some cases, they are finite. One may use lattice methods to extract physical information from these less singular, nonrenormalizable theories.

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**I. INFINITIES**

Infinite terms have been an awkward aspect of quantum field theory for more than 80 years. This paper will show that by adding certain nonrenormalizable terms to the usual action density of a free scalar field, one can construct nonrenormalizable theories whose exact Euclidian and Minkowskian Green's functions are less singular than those of the free theory. In some cases, they are finite. One may use lattice methods to extract physical information from these less singular, nonrenormalizable theories. The perturbative expansions of these nonrenormalizable theories are, of course, singular.

The history of attempts to cope with the infinities of quantum field theory is too vast to review here, but it may be useful for me to say what this paper is *not* about. Most of the early work on infinities described ways to cancel the infinities of the perturbative expansion of a theory against other infinite terms present in the original Lagrangian of the same theory. Dimensional regularization [1] was a high point of this work. This paper is not about renormalization [2]. Somewhat more recent work used space-time lattices [3] or strong-coupling expansions [4]. This paper has nothing to do with these techniques, but one may use them to extract physical information from the nonrenormalizable theories to which this paper points. Over the past three decades, string theorists have constructed theories that are intrinsically finite because their basic objects are extended in at least one dimension [5]. This paper is much more modest. Its main point is that some nonrenormalizable theories are less singular than theories that are free or renormalizable. Its only antecedent, as far as I know, is a very interesting paper by Boettcher and Bender [6].

I will discuss Green's functions first in Euclidian space and then in Minkowski space.

**II. EUCLIDIAN GREEN'S FUNCTIONS**

The mean value in the ground state of a Euclidian-time-ordered product of fields is a ratio of path integrals [7]

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$$G_e(x_1, \dots, x_n) \equiv \langle 0 | \mathcal{T}[\phi_e(x_1) \dots \phi_e(x_n)] | 0 \rangle \\ = \frac{\int \phi(x_1) \dots \phi(x_n) \exp[-\int L_e(\phi) d^4x] D\phi}{\int \exp[-\int L_e(\phi) d^4x] D\phi} \quad (1)$$

in which  $L_e$  is the Euclidian action density and the time dependence of the field is  $\phi_e(t, \vec{x}) = e^{iH} \phi(t, \vec{x}) e^{-iH}$  where  $H$  is the Hamiltonian. If the action density is quadratic,

$$L_e = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2 \phi^2 = \frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}(\nabla \phi)^2 + \frac{1}{2}m^2 \phi^2, \quad (2)$$

we can compute the Green's functions by doing Gaussian integrals. The 2-point function is

$$G_e(x_1, x_2) = \langle 0 | \mathcal{T}[\phi_e(x_1)\phi_e(x_2)] | 0 \rangle \\ = \frac{\int \phi(x_1)\phi(x_2) \exp[-\int L_e(\phi) d^4x] D\phi}{\int \exp[-\int L_e(\phi) d^4x] D\phi} \\ = \Delta_e(x_1 - x_2) = \int \frac{e^{ip(x_1-x_2)}}{p^2 + m^2} \frac{d^4p}{(2\pi)^4}. \quad (3)$$

It diverges quadratically as  $\epsilon \equiv |x_1 - x_2| \rightarrow 0$ :

$$\lim_{x_2 \rightarrow x_1} \langle 0 | \mathcal{T}[\phi_e(x_1)\phi_e(x_2)] | 0 \rangle = \langle 0 | \phi_e^2(x_1) | 0 \rangle \propto \frac{1}{\epsilon^2}. \quad (4)$$

In what follows, I will show that the addition of certain nonrenormalizable terms to the action density (25) sufficiently damps the field fluctuations of the resulting nonrenormalizable theory as to make its Green's functions less singular or even finite.

**III. TOY THEORIES IN EUCLIDIAN SPACE**

Toy theories without derivatives are easy to analyze because their functional integrals are infinite products of ordinary integrals. In the toy theory with  $L_e = m^2 \phi^2$  and no derivative terms, the 2-point function  $G_e(x, x)$  is a ratio of products of integrals, all but one of which cancel:

$$\begin{aligned}\langle 0|\phi^2(x)|0\rangle &= \frac{\int \phi^2(x) \exp\{-\int m^2 \phi^2(x') d^4x'\} D\phi}{\int \exp\{-\int m^2 \phi^2(x') d^4x'\} D\phi} \\ &= \frac{\int \phi^2(x) \exp\{-m^2 \phi^2(x) d^4x\} d\phi(x)}{\int \exp\{-m^2 \phi^2(x) d^4x\} d\phi(x)}.\end{aligned}\quad (5)$$

Setting  $d^4x = \epsilon^4$  and  $y = m\phi(x)\epsilon^2$ , we find

$$\langle 0|\phi^2(x)|0\rangle = \lim_{\epsilon \rightarrow 0} \frac{1}{m^2 \epsilon^4} \frac{\int y^2 e^{-y^2} dy}{\int e^{-y^2} dy} = \frac{1}{2m^2} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^4}.\quad (6)$$

The 2-point function of this toy theory without derivatives diverges quartically. This makes perfect sense because if we remove the  $p^2$  from the denominator of the 2-point function (26), then it too diverges quartically. The derivatives of the soluble theory (25) tether the field  $\phi(x)$  to its values at neighboring points and so reduce the divergence of the mean value of its square  $\langle 0|\phi^2(x)|0\rangle$  from quartic to quadratic.

We now add a quartic interaction and consider the toy theory with action density  $L_e = m^2 \phi^2(x) + \lambda \phi^4(x)$ . The 2-point function  $G_e(x, x)$  is again a ratio of products of integrals, all but one of which cancel:

$$\begin{aligned}\langle 0|\phi^2(x)|0\rangle &= \frac{\int \phi^2(x) \exp\{-\int m^2 \phi^2(x') + \lambda \phi^4(x') d^4x'\} D\phi}{\int \exp\{-\int m^2 \phi^2(x') + \lambda \phi^4(x') d^4x'\} D\phi} \\ &= \frac{\int \phi^2(x) \exp\{-[m^2 \phi^2(x) + \lambda \phi^4(x)] d^4x\} d\phi(x)}{\int \exp\{-[m^2 \phi^2(x) + \lambda \phi^4(x)] d^4x\} d\phi(x)}.\end{aligned}\quad (7)$$

Setting  $d^4x = \epsilon^4$  and  $y = \lambda^{1/4} \epsilon \phi(x)$ , we have

$$\begin{aligned}\langle 0|\phi^2(x)|0\rangle &= \lim_{\epsilon \rightarrow 0} \frac{1}{\sqrt{\lambda} \epsilon^2} \frac{\int y^2 \exp[-(\epsilon^2 m^2 y^2 / \sqrt{\lambda} + y^4)] dy}{\int \exp[-(\epsilon^2 m^2 y^2 / \sqrt{\lambda} + y^4)] dy} \\ &= \frac{\Gamma(3/4)}{\Gamma(1/4) \sqrt{\lambda}} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2}\end{aligned}\quad (8)$$

apart from finite terms. The quartic term  $\lambda \phi^4(x)$  in the action density has provided enough damping to reduce the divergence of  $G_e(x, x)$  from quartic to quadratic.

The action density of our third toy model is  $L_e = m^2 \phi^2(x) + \lambda \mu^{4-2n} \phi^{2n}(x)$  in which  $n \geq 2$ , and  $\mu$  is a mass parameter. Boettcher and Bender have studied an  $n \rightarrow \infty$  limit of this model [6]. Now after canceling identical integrals in the ratio of path integrals and setting  $d^4x = \epsilon^4$  and  $y = \lambda^{1/2n} \mu^{2/n-1} \epsilon^{2/n} \phi(x)$ , we get

$$\begin{aligned}\langle 0|\phi^2(x)|0\rangle &= \lim_{\epsilon \rightarrow 0} \frac{\int \phi^2(x) \exp\{-[m^2 \phi^2(x) + \lambda \mu^{4-2n} \phi^{2n}(x)] d^4x\} d\phi(x)}{\int \exp\{-[m^2 \phi^2(x) + \lambda \mu^{4-2n} \phi^{2n}(x)] d^4x\} d\phi(x)} \\ &= \lim_{\epsilon \rightarrow 0} \frac{\mu^{2-4/n}}{\lambda^{1/n} \epsilon^{4/n}} \frac{\int y^2 \exp\{-[\epsilon^{4-4/n} m^2 \mu^{2-4/n} \lambda^{-1/n} y^2 + y^{2n}]\} dy}{\int \exp\{-[\epsilon^{4-4/n} m^2 \mu^{2-4/n} \lambda^{-1/n} y^2 + y^{2n}]\} dy} \\ &= \frac{\mu^{2-4/n}}{\lambda^{1/n}} \frac{\Gamma(3/2n)}{\Gamma(1/2n)} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{4/n}}\end{aligned}\quad (9)$$

apart from terms that are finite. As  $n$  rises, the singularity in the Green's function  $G_e(x, x)$  softens. For  $n = 4$ , the divergence is linear; for  $n = 8$ , it is a square root.

The action density of our fourth and final nonderivative toy theory is

$$L_e = m^2 M^2 \left( \frac{1}{1 - \phi^2/M^2} - 1 \right) \equiv m^2 M^2 \sum_{\ell=1}^{\infty} \frac{\phi^{2\ell}}{M^{2\ell}}.\quad (10)$$

It is infinite for  $\phi^2 \geq M^2$ . This singularity effectively limits the path integral to fields in the range  $-M < \phi(x) < M$  for all space-time points  $x$ . Setting  $d^4x = \epsilon^4$  and  $y = \phi(x)/M$ , we find, after a cancellation in which only the integration over  $\phi(x)$  survives, that even the  $2n$ -point function

$$\begin{aligned}
\langle 0 | \phi^{2n}(x) | 0 \rangle &= \frac{\int \phi^{2n}(x) \exp \left\{ - \left[ m^2 M^2 \left( \frac{1}{1 - \phi^2(x)/M^2} - 1 \right) \right] d^4 x \right\} d\phi(x)}{\int \exp \left\{ - \left[ m^2 M^2 \left( \frac{1}{1 - \phi^2(x)/M^2} - 1 \right) \right] d^4 x \right\} d\phi(x)} = \lim_{\epsilon \rightarrow 0} \frac{\int_{-M}^M \phi^{2n} \exp \left[ - \epsilon^4 \left( \frac{m^2 M^2}{1 - \phi^2/M^2} \right) \right] d\phi}{\int_{-M}^M \exp \left[ - \epsilon^4 \left( \frac{m^2 M^2}{1 - \phi^2/M^2} \right) \right] d\phi} \\
&= M^{2n} \lim_{\epsilon \rightarrow 0} \frac{\int_{-1}^1 y^{2n} \exp \left[ - \epsilon^4 \left( \frac{m^2 M^2}{1 - y^2} \right) \right] dy}{\int_{-1}^1 \exp \left[ - \epsilon^4 \left( \frac{m^2 M^2}{1 - y^2} \right) \right] dy} = M^{2n} \frac{\int_{-1}^1 y^{2n} dy}{\int_{-1}^1 dy} = \frac{M^{2n}}{2n + 1}
\end{aligned} \tag{11}$$

is finite.

#### IV. LATTICE MODELS OF THEORIES WITH DERIVATIVES IN EUCLIDIAN SPACE

We now add derivative terms to our toy models. The first toy model becomes the soluble theory with 2-point function (26). The action density of the second toy model with derivatives is

$$L_4 = \frac{1}{2} [\dot{\phi}^2 + (\nabla \phi)^2 + m^2 \phi^2] + \lambda \phi^4. \tag{12}$$

We can put it on a lattice of spacing  $a$  if we take the action  $S$  to be a sum over all vertices  $v$  of the vertex action

$$\begin{aligned}
S_{4,v} &= \frac{a^4}{4} \sum_{j=1}^4 \left( \frac{\phi(v) - \phi(v + \hat{j})}{a} \right)^2 + \frac{a^4 m^2}{2} \phi^2(v) + a^4 \lambda \phi^4(v) \\
&= \frac{1}{4} \sum_{j=1}^4 (\varphi(v) - \varphi(v + \hat{j}))^2 + \frac{1}{2} a^2 m^2 \varphi^2(v) + \lambda \varphi^4(v)
\end{aligned} \tag{13}$$

in which each vertex is labeled by four integers  $v = (n_1, n_2, n_3, n_4)$ , the field  $\varphi = a\phi$  is dimensionless, and  $\hat{j}_k = \delta_{j,k}$ . Apart from the mass term, the lattice spacing  $a$  has disappeared from the action, but it reappears in the Green's functions:

$$\begin{aligned}
\langle 0 | \mathcal{T}[\phi_e(v_1) \dots \phi_e(v_n)] | 0 \rangle &= \frac{\int \phi(v_1) \dots \phi(v_n) \exp \left[ - \int L_4(\phi) d^4 x \right] D\phi}{\int \exp \left[ - \int L_4(\phi) d^4 x \right] D\phi} \\
&= \lim_{a \rightarrow 0} \frac{1}{a^n} \frac{\int \varphi(v_1) \dots \varphi(v_n) \exp \left( - \sum_v S_{4,v} \right) \prod_v d\varphi(v)}{\int \exp \left( - \sum_v S_{4,v} \right) \prod_v d\varphi(v)}.
\end{aligned} \tag{14}$$

For instance, the 2-point function is

$$\begin{aligned}
\langle 0 | \mathcal{T}[\phi_e(v_1) \phi_e(v_2)] | 0 \rangle &= \lim_{a \rightarrow 0} \frac{1}{a^2} \frac{\int \varphi(v_1) \varphi(v_2) \exp \left( - \sum_v S_{4,v} \right) \prod_v d\varphi(v)}{\int \exp \left( - \sum_v S_{4,v} \right) \prod_v d\varphi(v)}.
\end{aligned} \tag{15}$$

As  $v_2 \rightarrow v_1$ , this ratio still diverges quadratically, like its toy twin (31), so the quartic and derivative terms do not conspire to further reduce this divergence in  $G_e(v_1, v_1)$ .

The third toy model with derivatives has action density

$$L_{2n} = \frac{1}{2} [\dot{\phi}^2 + (\nabla \phi)^2 + m^2 \phi^2] + \lambda \mu^{4-2n} \phi^{2n}. \tag{16}$$

Boettcher and Bender have studied an  $n \rightarrow \infty$  limit of this model [6]. Its lattice action  $S$  is a sum over all vertices  $v$  of

$$\begin{aligned}
S_{2n,v} &= \frac{a^4}{4} \sum_{j=1}^4 \left( \frac{\phi(v) - \phi(v + \hat{j})}{a} \right)^2 + \frac{a^4 m^2}{2} \phi^2(v) \\
&\quad + a^4 \lambda \mu^{4-2n} \phi^{2n}(v) \\
&= \frac{a^{2-4/n} \lambda^{-1/n} \mu^{2-4/n}}{4} \sum_{j=1}^4 (\varphi(v) - \varphi(v + \hat{j}))^2 \\
&\quad + \frac{1}{2} a^{4-4/n} \lambda^{-1/n} m^2 \mu^{2-4/n} \varphi^2(v) + \varphi^{2n}(v)
\end{aligned} \tag{17}$$

in which the field  $\varphi(v) = \lambda^{1/2n} \mu^{2/n-1} a^{2/n} \phi(v)$  is dimensionless. The 2-point function

$$\begin{aligned}
\langle 0 | \mathcal{T}[\phi_e(v_1) \phi_e(v_2)] | 0 \rangle &= \lim_{a \rightarrow 0} \frac{\mu^{2-4/n} \int \varphi(v_1) \varphi(v_2) \exp \left( - \sum_v S_{2n,v} \right) \prod_v d\varphi(v)}{a^{1/n} \lambda^{1/n} \mu^{4/n} \int \exp \left( - \sum_v S_{2n,v} \right) \prod_v d\varphi(v)}
\end{aligned} \tag{18}$$

for  $n > 2$  and  $v_1 = v_2$  is less singular than  $1/a^2$ .

The fourth toy model with derivatives is

$$\begin{aligned}
L_e &= \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 M^2 \left( \frac{1}{1 - \phi^2/M^2} - 1 \right) \\
&= \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} m^2 M^2 \sum_{n=1}^{\infty} \frac{\phi^{2n}}{M^{2n}}.
\end{aligned} \tag{19}$$

Its lattice action is a sum over all vertices  $v$  of the vertex action

$$\begin{aligned}
S_{M,v} &= \frac{a^4}{4} \sum_{j=1}^4 \left( \frac{\phi(v) - \phi(v + \hat{j})}{a} \right)^2 \\
&\quad + \frac{a^4 m^2 M^2}{2} \left( \frac{1}{1 - \phi^2(v)/M^2} - 1 \right) \\
&= \frac{a^2 M^2}{4} \sum_{j=1}^4 (\varphi(v) - \varphi(v + \hat{j}))^2 \\
&\quad + \frac{a^4 m^2 M^2}{2} \left( \frac{1}{1 - \varphi^2(v)} - 1 \right)
\end{aligned} \tag{20}$$

in which the field  $\varphi = \phi/M$  is dimensionless. The essential singularity in the functional integrals effectively restricts the field  $\varphi(v)$  to the interval  $-1 < \varphi(v) < 1$ . The  $2n$ -point function

$$\begin{aligned} & \langle 0 | \mathcal{T}[\phi_e(v_1) \dots \phi_e(v_{2n})] | 0 \rangle \\ &= M^{2n} \lim_{a \rightarrow 0} \frac{\int_{-1}^1 \varphi(v_1) \dots \varphi(v_{2n}) \exp(-\sum_v S_{M,v}) \prod_v d\varphi(v)}{\int_{-1}^1 \exp(-\sum_v S_{M,v}) \prod_v d\varphi(v)} \end{aligned} \quad (21)$$

is finite for all  $n$ , even when all the points coincide,  $v_j = v_0$ ,

$$\begin{aligned} & \langle 0 | \phi_e^{2n}(v_0) | 0 \rangle \\ &= M^{2n} \lim_{a \rightarrow 0} \frac{\int_{-1}^1 \varphi^{2n}(v_0) \exp(-\sum_v S_{M,v}) \prod_v d\varphi(v)}{\int_{-1}^1 \exp(-\sum_v S_{M,v}) \prod_v d\varphi(v)} \\ &= \frac{M^{2n}}{2n+1}. \end{aligned} \quad (22)$$

## V. GREEN'S FUNCTIONS IN MINKOWSKI SPACE

We have seen that the addition of certain nonrenormalizable terms to the Euclidian action density of a theory of scalar fields can make the Euclidian Green's functions of the resulting nonrenormalizable theory less singular or even finite. To extend these results to Green's functions in Minkowski space and avoid extra notation, I will continue to focus on theories of a single scalar field. The mean value in the ground state of a time-ordered product of fields is a ratio of path integrals [7]

$$\begin{aligned} G(x_1, \dots, x_n) &\equiv \langle 0 | \mathcal{T}[\phi(x_1) \dots \phi(x_n)] | 0 \rangle \\ &= \frac{\int \phi(x_1) \dots \phi(x_n) \exp[i \int L(\phi) d^4x] D\phi}{\int \exp[i \int L(\phi) d^4x] D\phi} \end{aligned} \quad (23)$$

in which  $L$  is the action density and the time dependence of the field  $\phi(x)$  is  $\phi(t, \vec{x}) = e^{iH} \phi(t, \vec{x}) e^{-iH}$  where  $H$  is the Hamiltonian. The symbol  $D\phi$  means that we should integrate over all real functions  $\phi(x)$  of space-time and also should include in both the numerator and the denominator the factors  $\langle 0 | \phi(\infty, \vec{x}) \rangle$  and  $\langle \phi(-\infty, \vec{x}) | 0 \rangle$ , which lead to the  $i\epsilon$  terms in propagators. Green's functions play a central role in quantum field theory; they occur, for example, in the Lehmann-Symanzik-Zimmermann reduction formula [8] for the scattering of  $n$  incoming particles of momenta  $p_1 \dots p_n \equiv \{p\}$  into  $n'$  outgoing particles of momenta  $p'_1 \dots p'_n \equiv \{p'\}$

$$\begin{aligned} \langle p' | p \rangle &= \prod_{\ell=1}^n \prod_{\ell'=1}^{n'} \int d^4x_\ell d^4x'_{\ell'} e^{ip_\ell x_\ell - ip'_{\ell'} x'_{\ell'}} (-\partial_\ell^2 + m^2) \\ &\quad \times (-\partial_{\ell'}^2 + m^2) \langle 0 | \mathcal{T}[\phi(x_1) \dots \phi(x_{n+n'})] | 0 \rangle. \end{aligned} \quad (24)$$

If the action density is the quadratic form

$$\begin{aligned} L &= -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \\ &= \frac{1}{2} (\dot{\phi})^2 - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} m^2 \phi^2, \end{aligned} \quad (25)$$

then we can compute all the Green's functions. The 2-point function, for instance, is

$$\begin{aligned} G(x_1, x_2) &= \langle 0 | \mathcal{T}[\phi(x_1) \phi(x_2)] | 0 \rangle \\ &= \frac{\int \phi(x_1) \phi(x_2) \exp[i \int L(\phi) d^4x] D\phi}{\int \exp[i \int L(\phi) d^4x] D\phi} \\ &= \Delta(x_1 - x_2) = \int \frac{e^{ip(x_1 - x_2)}}{p^2 + m^2 - i\epsilon} \frac{d^4p}{(2\pi)^4}. \end{aligned} \quad (26)$$

It diverges quadratically as  $\epsilon \equiv |x_1 - x_2| \rightarrow 0$ :

$$\lim_{x_2 \rightarrow x_1} \langle 0 | \mathcal{T}[\phi(x_1) \phi(x_2)] | 0 \rangle = \langle 0 | \phi^2(x_1) | 0 \rangle \propto \frac{1}{\epsilon^2}. \quad (27)$$

In what follows, I will show that the addition of certain nonrenormalizable terms to the action density (25) makes its Green's functions (23) less singular or even finite.

## VI. TOY THEORIES IN MINKOWSKI SPACE

In the toy theory with  $L = -m^2 \phi^2$  and no derivative terms, the 2-point function  $G(x, x)$  is a ratio of products of integrals, all but one of which cancel:

$$\begin{aligned} \langle 0 | \phi^2(x) | 0 \rangle &= \frac{\int \phi^2(x) \exp\{-i \int m^2 \phi^2(x') d^4x'\} D\phi}{\int \exp\{-i \int m^2 \phi^2(x') d^4x'\} D\phi} \\ &= \frac{\int \phi^2(x) \exp\{-im^2 \phi^2(x) d^4x\} d\phi(x)}{\int \exp\{-im^2 \phi^2(x) d^4x\} d\phi(x)}. \end{aligned} \quad (28)$$

Setting  $d^4x = \epsilon^4$  and  $y = m\phi(x)\epsilon^2$ , we find

$$\begin{aligned} \langle 0 | \phi^2(x) | 0 \rangle &= \lim_{\epsilon \rightarrow 0} \frac{1}{m^2 \epsilon^4} \frac{\int_0^\infty y^2 e^{-iy^2} dy}{\int_0^\infty e^{-iy^2} dy} \\ &= \frac{2\sqrt{2}}{(1-i)\sqrt{\pi} m^2} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^4} \int_0^\infty y^2 e^{-iy^2} dy \end{aligned} \quad (29)$$

in which the final integral does not converge. The 2-point function of this toy theory without derivatives diverges a bit worse than quartically.

We now add a quartic interaction and consider the toy theory with action density  $L = -m^2 \phi^2(x) - \lambda \phi^4(x)$ . The 2-point function  $G(x, x)$  is again a ratio of products of integrals, all but one of which cancel:

$$\begin{aligned} \langle 0 | \phi^2(x) | 0 \rangle &= \frac{\int \phi^2(x) \exp\{-i \int m^2 \phi^2(x') + \lambda \phi^4(x') d^4x'\} D\phi}{\int \exp\{-i \int m^2 \phi^2(x') + \lambda \phi^4(x') d^4x'\} D\phi} \\ &= \frac{\int \phi^2(x) \exp\{-i[m^2 \phi^2(x) + \lambda \phi^4(x)] d^4x\} d\phi(x)}{\int \exp\{-i[m^2 \phi^2(x) + \lambda \phi^4(x)] d^4x\} d\phi(x)}. \end{aligned} \quad (30)$$

Setting  $d^4x = \epsilon^4$  and  $y = \lambda^{1/4}\epsilon\phi(x)$ , we have

$$\begin{aligned} \langle 0|\phi^2(x)|0\rangle &= \lim_{\epsilon \rightarrow 0} \frac{1}{\sqrt{\lambda}\epsilon^2} \frac{\int y^2 \exp[-i(\epsilon^2 m^2 y^2 / \sqrt{\lambda} + y^4)] dy}{\int \exp[-i(\epsilon^2 m^2 y^2 / \sqrt{\lambda} + y^4)] dy} \\ &= \frac{\Gamma(3/4)}{(-1)^{1/4}\Gamma(1/4)\sqrt{\lambda}} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^2} \end{aligned} \quad (31)$$

or  $0.238994(1-i)/\sqrt{\lambda}\epsilon^2$  apart from finite terms. The quartic term  $-\lambda\phi^4(x)$  in the action density has reduced

$$\begin{aligned} \langle 0|\phi^2(x)|0\rangle &= \lim_{\epsilon \rightarrow 0} \frac{\int \phi^2(x) \exp\{-i[m^2\phi^2(x) + \lambda\mu^{4-2n}\phi^{2n}(x)]d^4x\} d\phi(x)}{\int \exp\{-i[m^2\phi^2(x) + \lambda\mu^{4-2n}\phi^{2n}(x)]d^4x\} d\phi(x)} \\ &= \lim_{\epsilon \rightarrow 0} \frac{\mu^{2-4/n}}{\lambda^{1/n}\epsilon^{4/n}} \frac{\int y^2 \exp\{-i[\epsilon^{4-4/n}m^2\mu^{2-4/n}\lambda^{-1/n}y^2 + y^{2n}]\} dy}{\int \exp\{-i[\epsilon^{4-4/n}m^2\mu^{2-4/n}\lambda^{-1/n}y^2 + y^{2n}]\} dy} \\ &= e^{-i\pi/2n} \frac{\mu^{2-4/n}}{3\lambda^{1/n}} \frac{\Gamma(1+3/2n)}{\Gamma(1+1/2n)} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{4/n}} \end{aligned} \quad (32)$$

apart from terms that are finite. As  $n$  rises, the singularity in the Green's function  $G(x, x)$  softens. For  $n = 4$ , the divergence is linear; for  $n = 8$ , it is a square root.

The action density of our fourth and final nonderivative toy theory is

$$L = -m^2 M^2 \left( \frac{1}{1 - \phi^2/M^2} - 1 \right) \equiv -m^2 M^2 \sum_{\ell=1}^{\infty} \frac{\phi^{2\ell}}{M^{2\ell}}. \quad (33)$$

$$\begin{aligned} \langle 0|\phi^{2n}(x)|0\rangle &= \frac{\int \phi^{2n}(x) \exp\left\{-i \int \left[ m^2 M^2 \left( \frac{1}{1 - \phi^2(x)/M^2} - 1 \right) \right] d^4x' \right\} D\phi}{\int \exp\left\{-i \int \left[ m^2 M^2 \left( \frac{1}{1 - \phi^2(x)/M^2} - 1 \right) \right] d^4x' \right\} D\phi} \\ &= \frac{\int \phi^{2n}(x) \exp\left\{-i \left[ m^2 M^2 \left( \frac{1}{1 - \phi^2(x)/M^2} - 1 \right) \right] d^4x \right\} d\phi(x)}{\int \exp\left\{-i \left[ m^2 M^2 \left( \frac{1}{1 - \phi^2(x)/M^2} - 1 \right) \right] d^4x \right\} d\phi(x)} \\ &= \lim_{\epsilon \rightarrow 0} \frac{\int_{-M}^M \phi^{2n} \exp\left[-i\epsilon^4 \left( \frac{m^2 M^2}{1 - \phi^2/M^2} \right) \right] d\phi}{\int_{-M}^M \exp\left[-i\epsilon^4 \left( \frac{m^2 M^2}{1 - \phi^2/M^2} \right) \right] d\phi} = M^{2n} \lim_{\epsilon \rightarrow 0} \frac{\int_{-1}^1 y^{2n} \exp\left[-i\epsilon^4 \left( \frac{m^2 M^2}{1 - y^2} \right) \right] dy}{\int_{-1}^1 \exp\left[-i\epsilon^4 \left( \frac{m^2 M^2}{1 - y^2} \right) \right] dy} \\ &= M^{2n} \frac{\int_{-1}^1 y^{2n} dy}{\int_{-1}^1 dy} = \frac{M^{2n}}{2n+1} \end{aligned} \quad (34)$$

is finite. By symmetry, one has  $\langle 0|\phi^{2n+1}(x)|0\rangle = 0$ .

## VII. LATTICE MODELS OF THEORIES WITH DERIVATIVES IN MINKOWSKI SPACE

We now add derivative terms to our toy models. The first toy model becomes the soluble theory with 2-point function (26). The action density of the second toy model with derivatives is

$$L_4 = \frac{1}{2} [\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2] - \lambda\phi^4. \quad (35)$$

the divergence of  $G(x, x)$  from a little more than quartic to quadratic.

The action density of our third toy model is  $L = -m^2\phi^2(x) - \lambda\mu^{4-2n}\phi^{2n}(x)$  in which  $n \geq 2$  and  $\mu$  is a mass parameter. Boettcher and Bender have studied an  $n \rightarrow \infty$  limit of this model [6]. Now after canceling identical integrals in the ratio of path integrals and setting  $d^4x = \epsilon^4$  and  $y = \lambda^{1/2n}\mu^{2/n-1}\epsilon^{2/n}\phi(x)$ , we get

It is infinite for  $\phi^2 \geq M^2$ . This singularity effectively limits the path integral to fields in the range  $-M < \phi(x) < M$  for all space-time points  $x$ . Setting  $d^4x = \epsilon^4$  and  $y = \phi(x)/M$ , we find, after a cancellation in which only the integration over  $\phi(x)$  survives, that even the  $2n$ -point function

We can put it on a lattice of spacing  $a$  if we take the action  $S$  to be a sum over all vertices  $v$  of the vertex action

$$\begin{aligned} S_{4,v} &= -\frac{a^4}{4} \sum_{j=0}^3 \eta^{jj} \left( \frac{\phi(v) - \phi(v + \hat{j})}{a} \right)^2 \\ &\quad - \frac{a^4 m^2}{2} \phi^2(v) - a^4 \lambda \phi^4(v) \\ &= -\frac{1}{4} \sum_{j=0}^3 \eta^{jj} (\varphi(v) - \varphi(v + \hat{j}))^2 \\ &\quad - \frac{1}{2} a^2 m^2 \varphi^2(v) - \lambda \varphi^4(v) \end{aligned} \quad (36)$$

in which each vertex is labeled by four integers  $v = (n_0, n_1, n_2, n_3)$ , the field  $\varphi = a\phi$  is dimensionless,  $\hat{j}_k = \delta_{j,k}$ , and  $\eta$  is the diagonal metric of flat space  $\text{diag}(\eta) = (-1, 1, 1, 1)$ . Apart from the mass term, the lattice spacing  $a$  has disappeared from the action, but it reappears in the  $n$ -point functions

$$\begin{aligned} \langle 0 | \mathcal{T}[\phi(v_1) \dots \phi(v_n)] | 0 \rangle &= \frac{\int \phi(v_1) \dots \phi(v_n) \exp[-i \int L_4(\phi) d^4x] D\phi}{\int \exp[-i \int L_4(\phi) d^4x] D\phi} \\ &= \lim_{a \rightarrow 0} \frac{1}{a^n} \frac{\int \varphi(v_1) \dots \varphi(v_n) \exp(-i \sum_v S_{4,v}) \prod_v d\varphi(v)}{\int \exp(-i \sum_v S_{4,v}) \prod_v d\varphi(v)}. \end{aligned} \quad (37)$$

For instance, the 2-point function is

$$\langle 0 | \mathcal{T}[\phi(v_1)\phi(v_2)] | 0 \rangle = \lim_{a \rightarrow 0} \frac{1}{a^2} \frac{\int \varphi(v_1)\varphi(v_2) \exp(-i \sum_v S_{4,v}) \prod_v d\varphi(v)}{\int \exp(-i \sum_v S_{4,v}) \prod_v d\varphi(v)}. \quad (38)$$

When  $v_2 \rightarrow v_1$ , this ratio still diverges quadratically, like its toy twin (31), so the quartic and derivative terms do not conspire to further reduce this divergence in  $G(v_1, v_1)$ .

The third toy model with derivatives has action density

$$L_{2n} = \frac{1}{2} [\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2] - \lambda\mu^{4-2n}\phi^{2n}. \quad (65)$$

Boettcher and Bender have studied an  $n \rightarrow \infty$  limit of this model [6]. Its lattice action  $S$  is a sum over all vertices  $v$  of

$$\begin{aligned} S_{2n,v} &= -\frac{a^4}{4} \sum_{j=0}^3 \eta^{jj} \left( \frac{\phi(v) - \phi(v + \hat{j})}{a} \right)^2 \\ &\quad - \frac{a^4 m^2}{2} \phi^2(v) - a^4 \lambda \mu^{4-2n} \phi^{2n}(v) \\ &= -\frac{a^{2-4/n} \lambda^{-1/n} \mu^{2-4/n}}{4} \sum_{j=0}^3 \eta^{jj} (\varphi(v) - \varphi(v + \hat{j}))^2 \\ &\quad - \frac{1}{2} a^{4-4/n} \lambda^{-1/n} m^2 \mu^{2-4/n} \varphi^2(v) - \varphi^{2n}(v) \end{aligned} \quad (40)$$

in which the field  $\varphi(v) = \lambda^{1/2n} \mu^{2/n-1} a^{2/n} \phi(v)$  is dimensionless. The 2-point function

$$\langle 0 | \mathcal{T}[\phi(v_1)\phi(v_2)] | 0 \rangle = \lim_{a \rightarrow 0} \frac{\mu^{2-4/n} \int \varphi(v_1)\varphi(v_2) \exp(-i \sum_v S_{2n,v}) \prod_v d\varphi(v)}{\lambda^{1/n} a^{4/n} \int \exp(-i \sum_v S_{2n,v}) \prod_v d\varphi(v)} \quad (41)$$

for  $n > 2$  and  $v_1 = v_2$  is less singular than  $1/a^2$ .

The fourth toy model with derivatives is

$$\begin{aligned} L &= -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - m^2 M^2 \left( \frac{1}{1 - \phi^2/M^2} - 1 \right) \\ &\equiv -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 M^2 \sum_{n=1}^{\infty} \frac{\phi^{2n}}{M^{2n}}. \end{aligned} \quad (42)$$

Its lattice action is a sum over all vertices  $v$  of the vertex action

$$\begin{aligned} S_{M,v} &= -\frac{a^4}{4} \sum_{j=0}^3 \eta^{jj} \left( \frac{\phi(v) - \phi(v + \hat{j})}{a} \right)^2 \\ &\quad - \frac{a^4 m^2 M^2}{2} \left( \frac{1}{1 - \phi^2(v)/M^2} - 1 \right) \\ &= -\frac{a^2 M^2}{4} \sum_{j=0}^3 \eta^{jj} (\varphi(v) - \varphi(v + \hat{j}))^2 \\ &\quad - \frac{a^4 m^2 M^2}{2} \left( \frac{1}{1 - \varphi^2(v)} - 1 \right) \end{aligned} \quad (43)$$

in which the field  $\varphi = \phi/M$  is dimensionless. The essential singularity in the functional integrals effectively restricts the field  $\varphi(v)$  to the interval  $-1 < \varphi(v) < 1$ . The  $2n$ -point function

$$\begin{aligned} \langle 0 | \mathcal{T}[\phi(v_1) \dots \phi(v_{2n})] | 0 \rangle &= M^{2n} \lim_{a \rightarrow 0} \frac{\int_{-1}^1 \varphi(v_1) \dots \varphi(v_{2n}) \exp(-i \sum_v S_{M,v}) \prod_v d\varphi(v)}{\int_{-1}^1 \exp(-i \sum_v S_{M,v}) \prod_v d\varphi(v)} \end{aligned} \quad (44)$$

is finite for all  $n$ , even when all the points coincide,  $v_j = v_0$ ,

$$\begin{aligned} \langle 0 | \phi^{2n}(v_0) | 0 \rangle &= M^{2n} \lim_{a \rightarrow 0} \frac{\int_{-1}^1 \varphi^{2n}(v_0) \exp(-i \sum_v S_{M,v}) \prod_v d\varphi(v)}{\int_{-1}^1 \exp(-i \sum_v S_{M,v}) \prod_v d\varphi(v)} \\ &= \frac{M^{2n}}{2n + 1}. \end{aligned} \quad (45)$$

## VIII. CONCLUSION

The addition of terms like  $\phi^{2n}$  for  $n > 2$  or  $(m^2 M^2/2) [(1 - \phi^2/M^2)^{-1} - 1]$  to the usual action density (25) of a scalar field leads to nonrenormalizable theories whose exact Green's functions in Euclidian and Minkowski space are less singular than those of the free theory. In some cases, they are finite. One may use lattice methods to extract physical information from these less singular, nonrenormalizable theories.



If the results of this paper can be extended to fields of higher spin, then nonrenormalizable theories may have more to teach us, and their lessons may be important because of the nonrenormalizability of general relativity and the absence of experimental evidence for supersymmetry [9].

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- [1] G. 't Hooft and M. Veltman, *Nucl. Phys.* **B44**, 189 (1972).
  - [2] S. Weinberg, *Foundations, The Quantum Theory of Fields*, Vol. I (Cambridge University Press, Cambridge, England, 1995).
  - [3] K. G. Wilson, *Phys. Rev. D* **10**, 2445 (1974); M. Creutz, *Phys. Rev. Lett.* **43**, 553 (1979).
  - [4] R. Benzi, G. Martinelli, and G. Parisi, *Nucl. Phys.* **B135**, 429 (1978); C. M. Bender, F. Cooper, G. S. Guralnik, and D. H. Sharp, *Phys. Rev. D* **19**, 1865 (1979); C. M. Bender, F. Cooper, G. S. Guralnik, R. Roskies, and D. H. Sharp, *Phys. Rev. D* **23**, 2976 (1981).
  - [5] K. Becker, M. Becker, and J. H. Schwarz, *String Theory and M-Theory* (Cambridge University Press, Cambridge, England, 2007).
  - [6] S. Boettcher and C. M. Bender, *J. Math. Phys. (Cambridge, Mass.)* **31**, 2579 (1990).
  - [7] K. Cahill, *Physical Mathematics* (Cambridge University Press, Cambridge, England, 2013), Chap. 16.
  - [8] M. Srednicki, *Quantum Field Theory* (Cambridge University Press, Cambridge, England, 2007), Chap. 5.
  - [9] M. Shifman, *Mod. Phys. Lett. A* **27**, 1230043 (2012).