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From configuration to dynamics: Emergence of Lorentz signature in classical field theory

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The Lorentzian metric structure used in any field theory allows one to implement the relativistic notion of causality and to define a notion of time dimension. This article investigates the possibility that at the microscopic level the metric is Riemannian, i.e., locally Euclidean, and that the Lorentzian structure, that we usually consider as fundamental, is in fact an effective property that emerges in some regions of a four-dimensional space with a positive definite metric. In such a model, there is no dynamics nor signature flip across some hypersurface; instead, all the fields develop a Lorentzian dynamics in these regions because they propagate in an effective metric. It is shown that one can construct a decent classical field theory for scalars, vectors, and (Dirac) spinors in flat spacetime. It is then shown that gravity can be included but that the theory for the effective Lorentzian metric is not general relativity but of the covariant Galileon type. The constraints arising from stability, the equivalence principle, and the constancy of fundamental constants are detailed and a phenomenological picture of the emergence of the Lorentzian metric is also given. The construction, while restricted to classical fields in this article, offers a new view on the notion of time.

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I. INTRODUCTION

When constructing a physical theory, there is a large freedom in the choice of the mathematical structures. The developments of theoretical physics taught us that some of these structures are well suited to describe some classes of phenomena (e.g., the use of a vector field for electromagnetism, of spinors for some class of particles, the use of some symmetries, etc.). However, these choices can only be validated by the mathematical consistency of the theory and the agreement between the consequences of these structures and experiments. It may even be that different structures are possible to reproduce what we know about physics and one may choose one over the other on the basis of less well-defined criteria such as simplicity and economy.

At each step, some properties such as the topology of space [1], the number of spatial dimensions, or the numerical values of the free parameters that are the fundamental constants [2], may remain *a priori* free in a given framework, or imposed in another framework (e.g., the number of space dimensions is fixed in string theory [3]).

Among all these structures, and in the framework of metric theories of gravitation, the signature of the metric is in principle arbitrary. Indeed, it seems that on the scales that have been probed so far there is the need for only one time dimension and three spatial dimensions. In special and general relativity, time and space are geometrically different because the geometry of spacetime is locally Minkowskian, i.e., it enjoys a Lorentzian metric with signature $(-,+,+,+)$, i.e., the line element is $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}\ell^2 \equiv \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$. While the existence of two time directions may lead to confusion [4], it is not clear if there are theoretical obstacles to have more than one time direction, as even suggested by some framework [5] (see the argument for such possibilities made by Ref. [6] and detailed further in Ref. [7]). Several models for the birth of the universe [8] are based on a change of signature via an instanton in which a Riemannian and a Lorentzian manifold are joined across a hypersurface which may be thought of as the origin of time. While there is no time in the Euclidean region, where the signature is $(+,+,+,+)$, it flips to $(-,+,+,+)$. Eddington even suggested [9] that it can flip across some surface to $(-,-,+,+)$. Signature flip also arises in brane-world scenarios [10] (see Ref. [7] for a review of these possibilities) or in loop quantum cosmology [11]. These discussions however let the problem of the origin of the time direction open [12].

In Newtonian theory, time is a fundamental concept. It is assumed to flow and is described by a real variable. It can be measured by good clocks and any observers shall, irrespective of their motion, agree on the time elapsed between two events [13]. The laws of dynamics describe the change of configurations of a system with time. In relativity, first the notions of space and time are set on the same footing and, second, the notion of time is no more unique. One has to distinguish between a coordinate time, with no physical meaning, and the proper time that can be

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measured by an observer. Quantum mechanics offers another insight on time: there, while there may be operators or observables corresponding to spatial positions, time is not an observable, and thus not an operator [14]. As detailed in Ref. [7], by an argument going back to Pauli, commutation relations like $[\hat{x}^\mu, \hat{P}_\nu] = i\delta^\mu_\nu$ are incompatible with the spectrum of $\hat{P}^\mu$ lying in the future light cone and the notion of time is intimately related to the complex (Hilbert space) structure of quantum mechanics [7].

The question of whether time does actually ''exist'' has been widely debated in the context of classical physics [15], relativity [16], and quantum mechanics [14]. The debate on the nature of time has shifted with quantum gravity where the recovery of a classical notion of time is considered as a problem. In that case, the Schrödinger equation becomes the Wheeler-DeWitt equation, of the form $\hat{H}|\Psi\rangle = 0$, so that the allowed states are those for which the Hamiltonian vanishes. Thus, it determines in which states the universe can be but does not give any evolution through time. We refer to Refs. [17–19] for general discussions on the nature of time. This has led to numerous works on the emergence of time in different versions of quantum gravity [20–25] (and indeed the reverse opinion has been argued [26]). Also, the thermodynamical aspects of gravity, the existence of dualities between gauge theories and gravity theories [27], and holography [28] have led to the idea that the metric itself may have to be thought of as the result of a coarse graining of underlying more fundamental degrees of freedom [29].

The local Minkowski structure is an efficient way to implement the notion of causality in realistic theories and is today accepted as a central ingredient of the construction of the relativistic theory of fields. When gravity is included, the equivalence principle implies (this is not a theoretical requirement, but just an experimental fact, required at a given accuracy) that all the fields are universally coupled to the same Lorentzian metric. From the previous discussion, we may wonder whether the signature of this metric is only a convenient way to implement causality or whether it is just a property of an effective description of a microscopic theory in which there is no such notion.

This article proposes the view according to which the fundamental physical theory is intrinsically purely Euclidean so that its field equations determine a static four-dimensional field configuration. The Lorentzian dynamics that we can observe in our universe has then to be thought of as an emergent property, that is as an illusion holding in a small patch of a Euclidean mathematical space. This is thus an attempt to go further than early proposals [30–32] and see to which extent this can be an open possibility. We emphasize that it is different from the models discussed above involving a signature change across a boundary or obtained by rotating to an Euclidean space. We consider it important to take the freedom to see how far one can go in such a direction. As we shall later discuss, if possible, such a setting may shed a new light on several theoretical issues from the nature of singularities to quantum gravity.

Our attitude is however more modest and we want to start by constructing a decent classical field theory under this hypothesis. Section II explains the basics of our mechanism and then describes the construction of the scalar, vector, and spinor sectors in flat spacetime. We show that the whole standard model of particle physics can be constructed from a Euclidean theory, at the classical level. Section III addresses the more difficult question of gravity. While general relativity is not recovered in general, it shows that an extended K-essence theory of gravity called covariant Galileon can be obtained. We then show in Sec. IV that the dynamics of scalar and vector in curved spacetime can also be obtained. We then discuss the experimental and theoretical constraint on our construction in Sec. V and also propose a way to understand phenomenologically the emergence of the effective Lorentzian dynamics. It is however to be remembered that there is no dynamics at the fundamental level and that this illusion is restricted to a domain of a large Euclidean space.

II. FIELD THEORY IN FLAT SPACE

This section introduces the mechanism in the simple case of a flat space (Sec. II A). It shows how scalars (Sec. II B), vectors (Secs. II C and II D), and spinors (Sec. II E) defined in Euclidean space can have an apparent Lorentzian dynamics. We finish by pointing out the properties and limits of this mechanism in Sec. II G, many of them being discussed in a more realistic version in the following sections.

A. Clock field

In order to understand the basics of our model, let us consider a four-dimensional Riemannian manifold $\mathcal{M}$ with a positive definite Euclidean metric $g^{\mathrm{E}}_{\mu\nu} = \delta_{\mu\nu}$ in a Cartesian coordinate system. As a consequence, the theory we shall consider on this manifold does not have a natural concept of time. In order to make such a notion emerge locally, we introduce a scalar field ϕ and assume that its derivative has a nonvanishing vacuum expectation value (vev) in a region $\mathcal{M}_0$ of the Riemannian space (see Fig. 1). To be more precise, we assume that $\partial_\mu\phi = \mathrm{const} \neq 0$ in $\mathcal{M}_0$. It follows that we can set

$$\partial_\mu\phi = M^2 n_\mu \quad \text{in } \mathcal{M}_0 \tag{2.1}$$

with n_μ a unit constant vector ($\delta^{\mu\nu}n_\mu n_\nu = 1$). We have introduced a mass scale M so that n_μ is dimensionless. By construction, its norm $X_{\mathrm{E}} \equiv \delta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = M^4$ is constant and satisfies

$$X_{\mathrm{E}} > 0 \quad \text{in } \mathcal{M}_0. \tag{2.2}$$

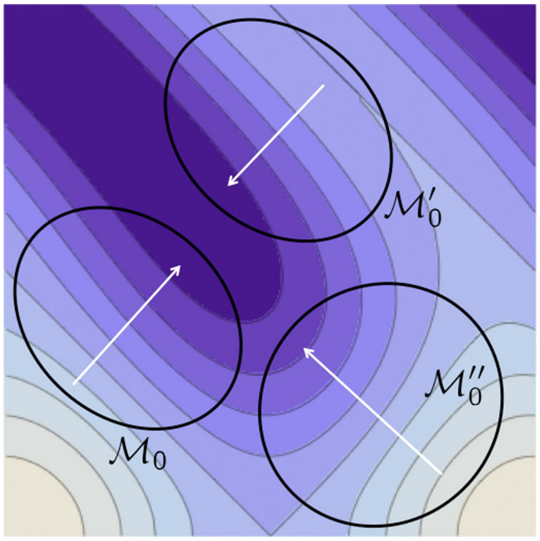

FIG. 1 (color online). Example of a spatial configuration of the clock field. Locally, one can define regions such as $\mathcal{M}_0$, $\mathcal{M}_0'$, and $\mathcal{M}_0''$, in each of which a time direction emerges. Indeed this direction does not preexist at the microscopic level and can be different from patches to patches.

Now, under this assumption, one of the coordinates can be chosen as

$$\mathrm{d}t = n_\mu \mathrm{d}x^\mu. \tag{2.3}$$

This accounts for choosing

$$t \equiv \frac{\phi}{M^2} \tag{2.4}$$

up to a constant that can be set to zero without loss of generality. The metric of the four-dimensional Riemannian space (with Euclidean geometry) can be rewritten as

$$\begin{aligned} \mathrm{d}s_\mathrm{E}^2 &= \delta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = (n_\mu \mathrm{d}x^\mu)^2 + (\delta_{\mu\nu} - n_\mu n_\nu)\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= \mathrm{d}t^2 + \delta_{ij}\mathrm{d}x^i \mathrm{d}x^j, \end{aligned} \tag{2.5}$$

by introducing a set of three independent coordinates x^i ($i = 1, \ldots, 3$) on the hypersurfaces Σ_t normal to n^μ. Note that the geometry on Σ_t would not be Euclidean if n_μ were not constant. As we shall now discuss, the scalar field ϕ will be related to what we usually call "time," so that we shall call such a scalar field a *clock field*.

B. Scalar field

The Euclidean configuration of a scalar field χ can be obtained by combining the usual action for a scalar field, with a kinetic term and a potential,

$$-\int \mathrm{d}^4x\left[\frac{1}{2}\delta^{\mu\nu}\partial_\mu\chi\partial_\nu\chi + V(\chi)\right],$$

with a coupling to the clock field ϕ as

$$\int \mathrm{d}^4x(\delta^{\mu\nu}\partial_\mu\phi\partial_\nu\chi)^2.$$

Let us consider the action obtained by the following combination:

$$\begin{aligned} S_\chi = \int \mathrm{d}^4x\bigg[&-\frac{1}{2}\delta^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - V(\chi) \\ &+ \frac{1}{M^4}(\delta^{\mu\nu}\partial_\mu\phi\partial_\nu\chi)^2\bigg]. \end{aligned} \tag{2.6}$$

It is straightforward to conclude that since $\delta^{\mu\nu}\partial_\mu\chi\partial_\nu\chi = (\partial_t\chi)^2 + \delta^{ij}\partial_i\chi\partial_j\chi$ and, when restricted to $\mathcal{M}_0$, $(\delta^{\mu\nu}\partial_\mu\phi\partial_\nu\chi)^2 = M^4(\partial_t\chi)^2$, the action (2.6) reduces to

$$S_\chi = \int \mathrm{d}t\mathrm{d}^3x\left[\frac{1}{2}(\partial_t\chi)^2 - \frac{1}{2}\delta^{ij}\partial_i\chi\partial_j\chi - V\right] \tag{2.7}$$

in $\mathcal{M}_0$. This can indeed be rewritten as

$$S_\chi = \int \mathrm{d}t\mathrm{d}^3x\left[-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - V\right]. \tag{2.8}$$

The action (2.6) thus describes, when restricted to $\mathcal{M}_0$, the dynamics of a scalar field propagating in a four-dimensional Minkowski spacetime with metric $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$. The apparent Lorentzian dynamics, with a preferred time direction, is thus the result of the coupling to the scalar clock field.

C. Vector field

Usually, the dynamics of a vector field A_μ is dictated by the action $F_{\mu\nu}F_\mathrm{E}^{\mu\nu}$, where $F_{\mu\nu}$ is the Faraday tensor defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and where the subscript E indicates that the indices are raised with the Euclidean metric $\delta^{\mu\nu}$.

The standard action of the vector field can be extended to include a coupling to the clock field of the form $F_\mathrm{E}^{\mu\rho}F_{E\rho}^{\nu}\partial_\mu\phi\partial_\nu\phi$ so that the action for the vector field we consider is

$$S_A = \frac{1}{4}\int \mathrm{d}^4x\left[-F_{\mu\nu}F_\mathrm{E}^{\mu\nu} + \frac{4}{M^4}F_\mathrm{E}^{\mu\rho}F_{\mathrm{E}\rho}^{\nu}\partial_\mu\phi\partial_\nu\phi\right]. \tag{2.9}$$

Since $F_{\mu\nu}F_\mathrm{E}^{\mu\nu} = 2\delta^{ij}F_{0i}F_{0j} + \delta^{ik}\delta^{jl}F_{ij}F_{kl}$ and since $F_\mathrm{E}^{\mu\rho}F_{\mathrm{E}\rho}^{\nu}\partial_\mu\phi\partial_\nu\phi = M^4\delta^{ij}F_{0i}F_{0j}$ in $\mathcal{M}_0$, it is easily concluded that this action can be rewritten as

$$S_A = \frac{1}{4}\int \mathrm{d}t\mathrm{d}^3x[2\delta^{ij}F_{0i}F_{0j} - \delta^{ik}\delta^{jl}F_{ij}F_{kl}], \tag{2.10}$$

or more simply as

$$S_A = -\frac{1}{4}\int \mathrm{d}t\mathrm{d}^3x\eta^{\mu\alpha}\eta^{\nu\beta}F_{\mu\alpha}F_{\nu\beta}. \tag{2.11}$$

Because of the coupling of the Faraday tensor to the clock field in the Euclidean theory, the vector field propagates effectively in a Minkowski metric and we recover the standard Maxwell action for a vector field. The generalization to a non-Abelian group is straightforward.

D. Charged scalar field

The construction of Sec. II B can easily be generalized to a complex scalar field charged under a $U(1)$. Considering a complex scalar field ω, we add to the standard kinetic term $\delta^{\mu\nu}(D_\mu\omega)^*(D_\nu\omega)$ a coupling to the clock field of the form $\delta^{\mu\nu}|\partial_\mu\phi D_\nu\omega|^2$, where $D_\mu \equiv \partial_\mu - iqA_\mu$. The Euclidean action is then chosen to be

$$S_\omega = \int \mathrm{d}^4x\Big[-\frac{1}{2}\delta^{\mu\nu}(D_\mu\omega)^*(D_\nu\omega) - U(|\omega|^2) + \frac{1}{M^4}\delta^{\mu\nu}|\partial_\mu\phi D_\nu\omega|^2\Big]. \tag{2.12}$$

Following the same arguments as for the real scalar field χ, this action takes the form

$$S_\omega = \int \mathrm{d}t\mathrm{d}^3x\Big[-\frac{1}{2}\eta^{\mu\nu}(D_\mu\omega)^*(D_\nu\omega) - U\Big]. \tag{2.13}$$

Again, the coupling to the clock field implies that the Euclidean dynamics leads to an effectively Minkowskian dynamics for ω.

E. Spinor fields

The next step is to include fermions in such a way that the standard Dirac dynamics emerges from a Euclidean action. Let us start by comparing the standard Dirac algebra in Minkowski spacetime (Sec. II E 1) and that in Euclidean space (Sec. II E 2) before we propose a choice of Euclidean action for the fermions (Sec. II E 3).

1. Dirac matrices in Minkowski spacetime

In a Minkowski spacetime with signature $(-+++)$, Dirac matrices are 4×4 matrices satisfying the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}. \tag{2.14}$$

For concreteness, throughout this section we shall adopt the following form of the Dirac matrices in Minkowski spacetime:

$$\gamma^0 = \boldsymbol{\sigma}_0\otimes\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & \boldsymbol{\sigma}_0 \\ \boldsymbol{\sigma}_0 & 0\end{pmatrix}, \quad \gamma^i = i\boldsymbol{\sigma}_i\otimes\boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & \boldsymbol{\sigma}_i \\ -\boldsymbol{\sigma}_i & 0\end{pmatrix}, \tag{2.15}$$

where $\boldsymbol{\sigma}_0$ is the 2×2 unit matrix and $\boldsymbol{\sigma}_i$ $(i = 1, 2, 3)$ are Pauli matrices,

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0\end{pmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}. \tag{2.16}$$

While γ^0 is Hermitian, γ^i are anti-Hermitian. One then defines γ^5 by

$$\gamma^5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3 = \boldsymbol{\sigma}_0\otimes\boldsymbol{\sigma}_3 = \begin{pmatrix} \boldsymbol{\sigma}_0 & 0 \\ 0 & -\boldsymbol{\sigma}_0\end{pmatrix}, \tag{2.17}$$

which satisfies

$$(\gamma^5)^2 = \mathbf{1}, \qquad \{\gamma^5, \gamma^\mu\} = 0 \quad (\mu = 0, \ldots, 3). \tag{2.18}$$

The matrices

$$S^{\mu\nu} \equiv \frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{2.19}$$

satisfy the algebra of Lorentz generators

$$[S^{\mu\nu}, S^{\rho\sigma}] = i(\eta^{\nu\rho}S^{\mu\sigma} - \eta^{\mu\rho}S^{\nu\sigma} - \eta^{\nu\sigma}S^{\mu\rho} + \eta^{\mu\sigma}S^{\nu\rho}). \tag{2.20}$$

Hence, the Lorentz transformation for a Dirac field ψ is

$$\psi \to \Lambda_{\frac{1}{2}}\psi, \qquad \Lambda_{\frac{1}{2}} = \exp\Big[-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\Big], \tag{2.21}$$

where $\omega_{\mu\nu}$ are real numbers. Concretely,

$$S^{0i} = -\frac{i}{2}\begin{pmatrix} \boldsymbol{\sigma}^i & 0 \\ 0 & -\boldsymbol{\sigma}^i\end{pmatrix}, \qquad S^{ij} = \frac{1}{2}\sum_{k=1}^{3}\epsilon^{ijk}\begin{pmatrix} \boldsymbol{\sigma}^k & 0 \\ 0 & \boldsymbol{\sigma}^k\end{pmatrix}. \tag{2.22}$$

While S^{ij} are Hermitian, S^{0i} are anti-Hermitian. As a consequence, $\Lambda_{\frac{1}{2}}$ is not unitary in general. In particular this means that

$$\psi^\dagger \to \psi^\dagger\Lambda_{\frac{1}{2}}^\dagger \neq \psi^\dagger\Lambda_{\frac{1}{2}}^{-1} \tag{2.23}$$

and that $\psi^\dagger\psi$ is not a scalar under Lorentz transformation. However, it is easy to check that

$$\bar\psi \to \bar\psi\Lambda_{\frac{1}{2}}^{-1}, \qquad \bar\psi \equiv \psi^\dagger\gamma^0 \tag{2.24}$$

so that $\bar\psi\psi$ is a scalar under Lorentz transformations. This is the reason why the Dirac action in Minkowski spacetime is usually constructed as

$$S_\psi^{\mathrm{M}} = \int \mathrm{d}^4x\bar\psi(\gamma^\mu\partial_\mu - m)\psi. \tag{2.25}$$

2. γ matrices in Euclidean space

In a four-dimensional Euclidean space with metric $\delta_{\mu\nu}$, one can also define matrices γ_{E}^μ according to

$$\gamma_{\mathrm{E}}^0 \equiv i\gamma^5, \qquad \gamma_{\mathrm{E}}^i \equiv \gamma^i \tag{2.26}$$

so that they obey the anticommutation relation

$$\{\gamma_{\mathrm{E}}^\mu, \gamma_{\mathrm{E}}^\nu\} = -2\delta^{\mu\nu}. \tag{2.27}$$

Then, we can define

$$\gamma_{\mathrm{E}}^5 \equiv \gamma_{\mathrm{E}}^0\gamma_{\mathrm{E}}^1\gamma_{\mathrm{E}}^2\gamma_{\mathrm{E}}^3 = \gamma^0, \tag{2.28}$$

which satisfies

$$(\gamma_{\mathrm{E}}^5)^2 = \mathbf{1}, \qquad \{\gamma_{\mathrm{E}}^5, \gamma_{\mathrm{E}}^\mu\} = 0 \quad (\mu = 0, \ldots, 3). \tag{2.29}$$

It follows that the matrices

$$S_{\mathrm{E}}^{\mu\nu} \equiv \frac{i}{4}[\gamma_{\mathrm{E}}^\mu, \gamma_{\mathrm{E}}^\nu] \tag{2.30}$$

satisfy the algebra of $SO(4)$ rotation generators

$$[S_{\rm E}^{\mu\nu}, S_{\rm E}^{\rho\sigma}] = i(\delta^{\nu\rho} S_{\rm E}^{\mu\sigma} - \delta^{\mu\rho} S_{\rm E}^{\nu\sigma} - \delta^{\nu\sigma} S_{\rm E}^{\mu\rho} + \delta^{\mu\sigma} S_{\rm E}^{\nu\rho}). \tag{2.31}$$

Hence, the $SO(4)$ rotation for the Dirac field ψ is

$$\psi \to \Lambda_{{\rm E},\frac{1}{2}}\psi, \qquad \Lambda_{{\rm E},\frac{1}{2}} = \exp\left[-\frac{i}{2}\omega^{\rm E}_{\mu\nu} S_{\rm E}^{\mu\nu}\right], \tag{2.32}$$

where $\omega^{\rm E}_{\mu\nu}$ are real numbers. Since all $S_{\rm E}^{\mu\nu}$ are Hermitian, $\Lambda_{{\rm E},\frac{1}{2}}$ is unitary. In particular, this implies that

$$\bar\psi \to \bar\psi \Lambda^{-1}_{{\rm E},\frac{1}{2}}, \qquad \psi^\dagger \to \psi^\dagger \Lambda^{-1}_{{\rm E},\frac{1}{2}}, \tag{2.33}$$

and that both $\bar\psi\psi$ and $\bar\psi\gamma^5_{\rm E}\psi (= \psi^\dagger\psi)$ are scalars under a $SO(4)$ transformation (see e.g., Refs. [33,34]). Note also that $\bar\psi = \psi^\dagger\gamma^0$ can be written as

$$\bar\psi = \psi^\dagger \gamma^5_{\rm E}. \tag{2.34}$$

3. *Euclidean action and emergence of the Lorentzian Dirac action*

As in the previous sections, we will need to couple the spinor field ψ to the clock field ϕ in order for the spinor to have an apparent Lorentzian dynamics. Starting from the Euclidean Dirac action in flat space with the metric $\delta_{\mu\nu}$,

$$\int dx^4 \bar\psi\left(\frac{i}{2}\gamma^\mu_{\rm E}\overleftrightarrow{\partial}_\mu - m\right)\psi,$$

and assuming that the clock field ϕ has derivative couplings to the Euclidean Dirac field ψ of the form

$$\int dx^4 \delta^{\mu\nu}(i\bar\psi\gamma^5_{\rm E}\overleftrightarrow{\partial}_\mu\psi)\partial_\nu\phi,$$
$$\int dx^4 \delta^{\mu\nu}(i\bar\psi\gamma^\rho_{\rm E}\overleftrightarrow{\partial}_\mu\psi)\partial_\rho\phi\partial_\nu\phi,$$

we can consider a Euclidean action for the Dirac spinor of the form

$$S_\psi = \int dx^4\left\{\bar\psi\left(\frac{i}{2}\gamma^\mu_{\rm E}\overleftrightarrow{\partial}_\mu - m\right)\psi + \frac{1}{2M^2}\delta^{\mu\nu}[(i\bar\psi\gamma^5_{\rm E}\overleftrightarrow{\partial}_\mu\psi) - (i\bar\psi\gamma^\rho_{\rm E}\overleftrightarrow{\partial}_\mu\psi)\partial_\rho\phi]\partial_\nu\phi\right\}. \tag{2.35}$$

As in the previous sections, the action S_ψ reduces to

$$S_\psi = \int dx^4 \bar\psi\left[\frac{i}{2}\gamma^0\overleftrightarrow{\partial}_0 + \frac{i}{2}\gamma^i\overleftrightarrow{\partial}_i - m\right]\psi. \tag{2.36}$$

The coupling to the clock field implies that ψ effectively propagates in an effective Lorentzian metric and we recover the standard Minkowskian Dirac action (2.25) with the usual algebra (2.14) for the γ matrices.

F. Massive point particle

The dynamics of massive object is usually derived from an action defined from the length of their worldline. In order to recover a proper dynamics, we start from the Euclidean action for a point particle

$$\frac{1}{2}\int\left(\mathcal{N}^{-1}\delta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - \mathcal{N}m^2\right)d\tau$$

to which we add the coupling to the clock field of the form

$$\int \mathcal{N}^{-1}\partial_\mu\phi\partial_\nu\phi\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}d\tau.$$

The Euclidean action for a point particle is thus given by

$$S_{\rm pp} = \frac{1}{2}\int\left[\mathcal{N}^{-1}\left(\delta_{\mu\nu} - \frac{2}{M^4}\partial_\mu\phi\partial_\nu\phi\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - \mathcal{N}m^2\right]d\tau. \tag{2.37}$$

The equation of motion is thus simply given by the geodesic equation for the effective metric $g^{(m)}_{\mu\nu} = g^{\rm E}_{\mu\nu} - \frac{2}{M^4}\partial_\mu\phi\partial_\nu\phi$. It is obvious that in $\mathcal{M}_0$ this effective metric reduces to the Minkowski metric $\eta_{\mu\nu}$.

G. Discussion

This section has provided the general construction of a mechanism that allows for scalar, vector, and spinor fields to actually propagate in an effective Lorentzian metric even though the underlying theory is purely Euclidean and written in terms of the Euclidean metric $\delta_{\mu\nu}$. This general construction assumes the existence of a scalar field ϕ, called *clock field*, that couples to all fields (scalar, vector, and spinor fields). In particular, this implies that we can construct the whole standard model of particle physics.

Let us now discuss some properties and limitations of such a construction.

(1) It requires that the clock field satisfies $\partial_\mu\phi = \text{const} \neq 0$ in a region $\mathcal{M}_0$ of the Euclidean space. It follows that the effective Lorentzian description is local and holds in $\mathcal{M}_0$. The properties of this model when $\partial_\mu\phi$ is not constant will be discussed in Sec. V C below. As we shall see in the next section the clock field should enjoy a shift symmetry in order for the system to exhibit the time translation symmetry after the emergence of time. In $\mathcal{M}_0$, both the shift symmetry and the translational symmetry along the direction of $\partial_\mu\phi$ are spontaneously broken, but a combination of them remains unbroken and is responsible for the existence of a conserved quantity that reduces in $\mathcal{M}_0$ to the usual notion of energy.

(2) It is limited to classical field theory in flat space. The extension to curved space is discussed in Secs. III and IV below and the quantum aspects are left for future investigations.

(3) The origin of the effective Lorentzian dynamics in $\mathcal{M}_0$ can be intuitively understood for scalars and vectors. For scalars, the action (2.6) is equivalent to the coupling to the effective metric

$$\hat{g}^{\mu\nu} = \delta^{\mu\nu} - \frac{2}{M^4}\delta^{\mu\alpha}\delta^{\nu\beta}\partial_\alpha\phi\partial_\beta\phi. \quad (2.38)$$

For vectors, one could have simply used a coupling to $\hat{g}^{\mu\nu}$ and a Lagrangian of the form $\hat{g}^{\mu\alpha}\hat{g}^{\nu\beta}F_{\mu\nu}F_{\alpha\beta}$ since the extra term quartic in $\partial_\mu\phi$ compared to the action (2.9) is of the form $4M^{-8}\delta^{\mu\lambda}\delta^{\alpha\lambda'}\delta^{\nu\sigma}\delta^{\beta\sigma'}\partial_\lambda\phi\partial_{\lambda'}\phi\partial_\sigma\phi\partial_{\sigma'}\phi F_{\mu\nu}F_{\alpha\beta}$ and does not contribute (note that it reduces to $4F_{00}^2 = 0$ in $\mathcal{M}_0$). Hence, the apparent Lorentzian dynamics for scalars and vectors boils down to the fact that $\hat{g}^{\mu\nu}|_{\mathcal{M}_0} = \eta^{\mu\nu}$. Massive point particles also propagate in this metric.

(4) This interpretation cannot be extended to spinors mostly because of the γ matrices, at least straightforwardly.

(5) It is however important to realize that despite this, when restricted to $\mathcal{M}_0$ all fields propagate in the same effective Minkowski metric so that the equivalence principle is safe in first approximation.

(6) The couplings to the clock field have been tuned in order to recover the exact Minkowski actions. For instance, the action (2.6) for a scalar field could have been chosen as

$$S_\chi = \int d^4x\left[-\frac{\kappa_\chi}{2}\delta^{\mu\nu}\partial_\mu\chi\partial_\nu\chi - V(\chi) + \frac{\alpha_\chi}{2M^4}(\delta^{\mu\nu}\partial_\mu\phi\partial_\nu\chi)^2\right]. \quad (2.39)$$

In such a case, a Lorentzian signature is recovered only if $\alpha_\chi > \kappa_\chi > 0$. In the case where these constants are not tuned, different fields can have different light cones. This will be discussed in Sec. V.

(7) In the bosonic sector, since the theory is invariant under the Euclidean parity ($x^\mu \to -x^\mu$) as well as the field parity ($\phi \to -\phi$), both P and T invariances in the emergent Lorentzian theory are ensured. Without the field parity invariance, the T invariance would be spontaneously broken by a nonvanishing vacuum expectation value (vev) of the derivative of the clock field. This explains the reason why we have included only quadratic terms in $\partial_\mu\phi$ in the actions for scalars and vectors.

(8) In the fermionic sector, let us first remark that one could have constructed 16 independent Euclidean γ matrices, explicitly given by

$$\mathbf{1},\ \gamma_E^5,\ \gamma_E^\mu,\ \gamma_E^5\gamma_E^\mu,\ S_E^{\mu\nu}. \quad (2.40)$$

From the Dirac spinor ψ, we can thus construct bilinear combinations that transform as scalars under $SO(4)$ rotations. Among them, Hermitian bilinears that do not include more than one derivative acting on spinors are the following ten possibilities:

$$\begin{array}{ll}
\bar{\psi}\psi, & \bar{\psi}\gamma_E^5\psi, \\
i\bar{\psi}\gamma_E^\mu\overleftrightarrow{\partial}_\mu\psi, & \bar{\psi}\gamma_E^5\gamma_E^\mu\overleftrightarrow{\partial}_\mu\psi, \\
(\bar{\psi}\gamma_E^\mu\psi)\partial_\mu\phi, & (i\bar{\psi}\gamma_E^5\gamma_E^\mu\psi)\partial_\mu\phi, \\
\delta^{\mu\nu}(i\bar{\psi}\overleftrightarrow{\partial}_\mu\psi)\partial_\nu\phi, & \delta^{\mu\nu}(i\bar{\psi}\gamma_E^5\overleftrightarrow{\partial}_\mu\psi)\partial_\nu\phi, \\
\delta^{\mu\nu}(i\bar{\psi}\gamma_E^\rho\overleftrightarrow{\partial}_\mu\psi)\partial_\rho\phi\partial_\nu\phi, & \delta^{\mu\nu}(\bar{\psi}\gamma_E^5\gamma_E^\rho\overleftrightarrow{\partial}_\mu\psi)\partial_\rho\phi\partial_\nu\phi.
\end{array}$$

The first two of the left column correspond to the standard mass and kinetic terms while six among the eight others describe possible couplings to the clock field. Among these six couplings, we have only used the two which were sufficient as an existence proof of our mechanism for Dirac spinors, namely $\delta^{\mu\nu}(i\bar{\psi}\gamma_E^5\overleftrightarrow{\partial}_\mu\psi)\partial_\nu\phi$ and $\delta^{\mu\nu}(i\bar{\psi}\gamma_E^\rho\overleftrightarrow{\partial}_\mu\psi)\partial_\rho\phi\partial_\nu\phi$. It has to be remarked that the second term is not CPT invariant after the clock field has a vev. Hence, unless the coefficient of this term is exactly the value shown in (2.35), the CPT invariance is violated. We also need to emphasize that we have been able to construct the Dirac spinor but that we also need to construct Majorana and Weyl spinors. This is an open problem at the moment.

(9) The mass scale M is related to $\partial_\mu\phi$ and is arbitrary. It is important to realize that it does not appear in the final expressions of the effective Lorentzian actions.

(10) X_E may not be constant if $g^E_{\mu\nu} \neq \delta_{\mu\nu}$ (curved space) and/or if $\partial_\mu\phi$ is not strictly constant in $\mathcal{M}_0$. This will be discussed in Secs. III and V

(11) The configuration of the clock field is not arbitrary but should be determined by solving the equation of motion. Since the action for the clock field enjoys a shift symmetry, its equation of motion takes the form of a current conservation. This will be addressed in Sec. III, where we will show that $\partial_\mu\phi = \text{const} \neq 0$ can be a solution, e.g., with $g^E_{\mu\nu} = \delta_{\mu\nu}$.

III. GRAVITATION AND CURVED SPACE

So far, our description has been restricted to the classical dynamics of standard fields in flat spacetime. The first natural generalization we must consider is the way to include gravity, i.e., a theory that will mimic or be close to general relativity.

For this purpose, we now consider a general four-dimensional Riemannian[1] manifold $\mathcal{M}$ with a positive definite metric $g^{\rm E}_{\mu\nu}$. Again, the theory we shall consider on this manifold does not have a microscopic concept of time. As previously, we introduce a clock field ϕ and assume it enjoys a shift symmetry that, as we have already seen, is necessary for the system to exhibit the time translation symmetry after the emergence of time.

A. Generic couplings to the clock field

In order to minimize the number of physical degrees of freedom, we demand that the equation of motion for ϕ is a second-order differential equation. Hence, the action for ϕ is restricted to the Riemannian version of the Horndeski theory [35] with shift symmetry. Equivalently, it is given by the shift-symmetric generalized Galileon [36] as

$$S_g = \int {\rm d}x^4 \sqrt{g_{\rm E}}(L_2 + L_3 + L_4 + L_5), \tag{3.1}$$

where the Lagrangians are explicitly given by

$$\begin{aligned} L_2 &= \mathcal{K}(X_{\rm E}),\\ L_3 &= -G_3(X_{\rm E})\nabla^2_{\rm E}\phi,\\ L_4 &= G_4(X_{\rm E})R_{\rm E} - 2G_4'(X_{\rm E})[(\nabla^2_{\rm E}\phi)^2 - (\nabla^{\rm E}_\mu\nabla^{\rm E}_\nu\phi)^2],\\ L_5 &= -g_5 G^{\mu\nu}_{\rm E}\partial_\mu\phi\partial_\nu\phi + \tilde{G}_5(X_{\rm E})G^{\mu\nu}_{\rm E}\nabla^{\rm E}_\mu\nabla^{\rm E}_\nu\phi\\ &\quad + \frac{1}{3}\tilde{G}_5'(X_{\rm E})[(\nabla^2_{\rm E}\phi)^3 - 3(\nabla^2_{\rm E}\phi)(\nabla^{\rm E}_\mu\nabla^{\rm E}_\nu\phi)^2\\ &\quad + 2(\nabla^{\rm E}_\mu\nabla^{\rm E}_\nu\phi)^3]. \end{aligned} \tag{3.2}$$

Here, $\nabla^{\rm E}_\mu$, $R_{\rm E}$, and $G^{\mu\nu}_{\rm E}$ are the covariant derivative associated with the Riemannian metric $g^{\rm E}_{\mu\nu}$, its Ricci scalar, and Einstein tensor. The coefficient g_5 is a constant and $\mathcal{K}(X_{\rm E})$, $G_{3,4}(X_{\rm E})$, and $\tilde{G}_5(X_{\rm E})$ are arbitrary functions of $X_{\rm E}$ and a prime refers to a derivative with respect to $X_{\rm E}$ that is defined as

$$X_{\rm E} \equiv g^{\mu\nu}_{\rm E}\partial_\mu\phi\partial_\nu\phi. \tag{3.3}$$

We use the following short-hand notations:

$$\begin{aligned} \nabla^2_{\rm E}\phi &\equiv g^{\mu\nu}_{\rm E}\nabla^{\rm E}_\mu\nabla^{\rm E}_\nu\phi,\\ (\nabla^{\rm E}_\mu\nabla^{\rm E}_\nu\phi)^2 &\equiv g^{\nu\rho}_{\rm E}g^{\sigma\mu}_{\rm E}(\nabla^{\rm E}_\mu\nabla^{\rm E}_\nu\phi)(\nabla^{\rm E}_\rho\nabla^{\rm E}_\sigma\phi),\\ (\nabla^{\rm E}_\mu\nabla^{\rm E}_\nu\phi)^3 &\equiv g^{\nu\rho}_{\rm E}g^{\sigma\alpha}_{\rm E}g^{\beta\mu}_{\rm E}(\nabla^{\rm E}_\mu\nabla^{\rm E}_\nu\phi)(\nabla^{\rm E}_\rho\nabla^{\rm E}_\sigma\phi)(\nabla^{\rm E}_\alpha\nabla^{\rm E}_\beta\phi), \end{aligned} \tag{3.4}$$

where $g^{\mu\nu}_{\rm E}$ is the inverse of $g^{\rm E}_{\mu\nu}$.

[1]We use the term *Riemannian* for a curved spacetime with a positive definite metric and *Lorentzian* for a curved spacetime with a Lorentz signature. We keep the terms *Euclidean* and *Minkowskian* for the analog in flat space. However, for simplicity, we use the same subscript E for the Riemannian and Euclidean cases.

For the effective equations, i.e., once the Lorentzian structure and the notion of time have emerged, we would like to ensure that the system is invariant not only under time translation but also under *CPT*.[2]

For this reason, we require that besides the shift symmetry ($\phi \to \phi + {\rm const}$) the theory also enjoys a Z_2 symmetry ($\phi \to -\phi$) for the clock field action. With these symmetries, the action reduces to

$$\begin{aligned} S_g = \int {\rm d}x^4\sqrt{g_{\rm E}}\{&G_4(X_{\rm E})R_{\rm E} - g_5 G^{\mu\nu}_{\rm E}\partial_\mu\phi\partial_\nu\phi + \mathcal{K}(X_{\rm E})\\ &- 2G_4'(X_{\rm E})[(\nabla^2_{\rm E}\phi)^2 - (\nabla^{\rm E}_\mu\nabla^{\rm E}_\nu\phi)^2]\}, \end{aligned} \tag{3.5}$$

since only L_2, L_4, and the first term of L_5 can contribute.

It is easy to show that the constant g_5 in the action (3.5) can be absorbed into the redefinition of $G_4(X_E)$ up to a boundary term. Hence, by setting $g_5 = 0$, hereafter we consider the Riemannian gravity action of the form

$$\begin{aligned} S_g = \int {\rm d}x^4\sqrt{g_{\rm E}}\{&G_4(X_{\rm E})R_{\rm E} + \mathcal{K}(X_{\rm E})\\ &- 2G_4'(X_{\rm E})[(\nabla^2_{\rm E}\phi)^2 - (\nabla^{\rm E}_\mu\nabla^{\rm E}_\nu\phi)^2]\}. \end{aligned} \tag{3.6}$$

B. Action for the gravitational sector

Following the logic developed in Sec. II, we restrict our analysis to a region $\mathcal{M}_0$ in which $X_{\rm E} > 0$ so that we can define a preferred direction, that we shall call t, defined as in Eq. (2.4),

$$t \equiv \frac{\phi}{M^2}, \tag{3.7}$$

that is chosen as one of coordinates of the four-dimensional Riemannian manifold. We refer to such a coordinate choice (3.7) as *unitary gauge*.

1. Decomposition of the Riemannian metric

One can then introduce a set of three other independent coordinates x^i ($i = 1, 2, 3$) so that the Riemannian metric is decomposed as

$$g^{\rm E}_{\mu\nu}{\rm d}x^\mu{\rm d}x^\nu = N^2_{\rm E}{\rm d}t^2 + \gamma_{ij}({\rm d}x^i + N^i{\rm d}t)({\rm d}x^j + N^j{\rm d}t), \tag{3.8}$$

where the lapse $N_{\rm E}$ is given, thanks to Eq. (3.7), by

$$N_{\rm E} \equiv \frac{1}{\sqrt{g^{tt}_{\rm E}}} = \frac{M^2}{\sqrt{X_{\rm E}}}. \tag{3.9}$$

The 3-metric γ_{ij} is given by

[2]As we have seen in the previous section, this requirement is not obviously fulfilled for spinors without fine-tuning. An additional mechanism is needed to naturally ensure the *CPT* invariance for spinors. In the present article we shall thus focus on the bosonic sector.

$$\gamma_{ij} \equiv g^{\rm E}_{ij}, \tag{3.10}$$

and γ^{ij} is its inverse. To finish, the shift vector N^i is given by

$$N^i \equiv \gamma^{ij} g^{\rm E}_{tj}. \tag{3.11}$$

One can then easily check that the inverse Riemannian metric is given by

$$g^{tt}_{\rm E} = \frac{1}{N^2_{\rm E}}, \qquad g^{ti}_{\rm E} = g^{it}_{\rm E} = -\frac{N^i}{N^2_{\rm E}}, \qquad g^{ij}_{\rm E} = \gamma^{ij} + \frac{N^i N^j}{N^2_{\rm E}}. \tag{3.12}$$

2. Riemannian geometrical quantities

With the decomposition (3.8), it is straightforward to show that the Einstein-Hilbert term reduces to

$$\sqrt{g_{\rm E}} R_{\rm E} = N_{\rm E}\sqrt{\gamma}(-K^{ij}_{\rm E} K^{\rm E}_{ij} + K^2_{\rm E} + R^{(3)}) - 2\partial_i(\sqrt{\gamma}\gamma^{ij}\partial_j N_{\rm E}) - 2\partial_t(\sqrt{\gamma} K_{\rm E}) + 2\partial_i(\sqrt{\gamma} N^i K_{\rm E}), \tag{3.13}$$

in terms of the extrinsic curvature of the constant-t hypersurface, $K^{\rm E}_{ij}$, defined by

$$K^{\rm E}_{ij} \equiv \frac{1}{2N_{\rm E}}(\partial_t \gamma_{ij} - D_i N_j - D_j N_i), \tag{3.14}$$

where D_i is the spatial covariant derivative compatible with γ_{ij}, and $R^{(3)}$ is its Ricci scalar. We have used the notations $K^{ij}_{\rm E} \equiv \gamma^{ik}\gamma^{jl} K^{\rm E}_{kl}$, $K_{\rm E} \equiv \gamma^{ij} K^{\rm E}_{ij}$, and $N_i \equiv \gamma_{ij} N^j$.

3. Riemannian action in $\mathcal{M}_0$

With the use of the quantities introduced above, the Riemannian action (3.6) takes the form

$$S_g = \int {\rm d}t{\rm d}x^3 N_{\rm E}\sqrt{\gamma}\{-G_4(K^{ij}_{\rm E} K^{\rm E}_{ij} - K^2_{\rm E}) + G_4 R^{(3)} + L_\phi\}, \tag{3.15}$$

where the Lagrangian L_ϕ is given by

$$L_\phi = -2(\partial^{\rm E}_\perp \partial^{\rm E}_\perp + D^2) G_4 - 2G_4'[(\nabla^2_{\rm E}\phi)^2 - (\nabla^\mu_{\rm E}\nabla^\nu_{\rm E}\phi)(\nabla^{\rm E}_\mu \nabla^{\rm E}_\nu \phi)] + \mathcal{K}(X_{\rm E}), \tag{3.16}$$

in which the three-dimensional Laplacian is defined as usual as $D^2 \equiv \gamma^{ij} D_i D_j$. The perpendicular derivative $\partial^{\rm E}_\perp$ is defined in terms of the unit vector normal to the constant ϕ hypersurfaces, $n^{\rm E}_\mu = \partial_\mu\phi/\sqrt{X_E}$, as

$$\partial^{\rm E}_\perp \equiv n^\mu_{\rm E}\partial_\mu \equiv \frac{1}{N_{\rm E}}(\partial_t - N^i\partial_i), \tag{3.17}$$

with $n^\mu_{\rm E} = g^{\mu\nu}_{\rm E} n^{\rm E}_\mu$.

In order to further simplify L_ϕ, note that $\nabla^{\rm E}_\mu \nabla^{\rm E}_\nu \phi = -M^2 \Gamma^t_{E\mu\nu}$ in terms of the Christoffel symbols for the metric $g^{\rm E}_{\mu\nu}$, $\Gamma^\rho_{{\rm E}\mu\nu}$. Its components are explicitly given by

$$\begin{aligned}\phi^{\rm E}_{;ij} &\equiv \nabla^{\rm E}_i \nabla^{\rm E}_j \phi = \sqrt{X_{\rm E}} K^{\rm E}_{ij},\\ \phi^{\rm E}_{;\perp i} &\equiv \phi^{\rm E}_{;i\perp} \equiv n^\mu_{\rm E} \nabla^{\rm E}_\mu \nabla^{\rm E}_i \phi = \frac{1}{2}\sqrt{X_{\rm E}}\partial_i \ln X_{\rm E},\\ \phi^{\rm E}_{;\perp\perp} &\equiv n^\mu_{\rm E} n^\nu_{\rm E} \nabla^{\rm E}_\mu \nabla^{\rm E}_\nu \phi = \frac{1}{2}\sqrt{X_{\rm E}}\partial^{\rm E}_\perp \ln X_{\rm E}.\end{aligned} \tag{3.18}$$

It implies that the term $(\nabla^2_{\rm E}\phi)^2 - (\nabla^\mu_{\rm E}\nabla^\nu_{\rm E}\phi)(\nabla^{\rm E}_\mu\nabla^{\rm E}_\nu\phi)$ appearing in Eq. (3.16) takes the form

$$(\gamma^{ij}\gamma^{kl} - \gamma^{ik}\gamma^{jl})\phi^{\rm E}_{;ij}\phi^{\rm E}_{;kl} + 2\gamma^{ij}(\phi^{\rm E}_{;\perp\perp}\phi^{\rm E}_{;ij} - \phi^{\rm E}_{;\perp i}\phi^{\rm E}_{;\perp j})$$

and thus reduces to

$$-X_{\rm E}(K^{ij}_{\rm E} K^{\rm E}_{ij} - K^2_{\rm E}) + K_{\rm E}\partial^{\rm E}_\perp X_{\rm E} - \frac{1}{2} X_{\rm E}(D_i \ln X_{\rm E})^2.$$

Inserting this into Eq. (3.16), it follows that L_ϕ takes the form

$$L_\phi = 2G_4' X_{\rm E}(K^{ij}_{\rm E} K^{\rm E}_{ij} - K^2_{\rm E}) + \mathcal{K}(X_{\rm E}) + \frac{\Delta_\phi}{N_{\rm E}\sqrt{\gamma}}, \tag{3.19}$$

where the last term is given by

$$\Delta_\phi = -2\partial_t(\sqrt{\gamma}\partial^{\rm E}_\perp G_4) + 2\partial_i(\sqrt{\gamma} N^i \partial^{\rm E}_\perp G_4) - 2\partial_i(\sqrt{\gamma}\gamma^{ij} N_{\rm E}\partial_j G_4) \tag{3.20}$$

and is a total derivative. We finally obtain the expression of the Riemannian action

$$S_g = \int {\rm d}t{\rm d}x^3 N_{\rm E}\sqrt{\gamma}\{(2G_4' X_{\rm E} - G_4)(K^{ij}_{\rm E} K^{\rm E}_{ij} - K^2_{\rm E}) + G_4 R^{(3)} + \mathcal{K}(X_{\rm E})\}, \tag{3.21}$$

where it is understood that $X_{\rm E}$ defined in Eq. (3.3) is given by

$$X_{\rm E} = \frac{M^4}{N^2_{\rm E}}$$

and that the time coordinate is fixed according to the unitary gauge (3.7).

4. Lorentzian metric

We now introduce a Lorentzian metric $g_{\mu\nu}$ and decompose it as

$$g_{\mu\nu}{\rm d}x^\mu {\rm d}x^\nu = -N^2{\rm d}t^2 + \gamma_{ij}({\rm d}x^i + N^i{\rm d}t)(dx^j + N^j{\rm d}t), \tag{3.22}$$

where the lapse N is defined by

$$N{\rm d}N = -N_{\rm E}{\rm d}N_{\rm E} \tag{3.23}$$

so that the Riemannian and Lorentzian lapses are related to each other as

$$N = \sqrt{N^2_c - N^2_{\rm E}}, \tag{3.24}$$

N_c being an arbitrary positive constant. As above, we can define the extrinsic curvature of this metric as

$$K_{ij} \equiv \frac{1}{2N}(\partial_t \gamma_{ij} - D_i N_j - D_j N_i) \tag{3.25}$$

and $K^{ij} \equiv \gamma^{ik}\gamma^{jl}K_{kl}$, $K \equiv \gamma^{ij}K_{ij}$. It is related to the Riemannian extrinsic curvature by

$$K_{ij} = \frac{N_\mathrm{E}}{N}K^\mathrm{E}_{ij}, \qquad K^{ij} = \frac{N_\mathrm{E}}{N}K^{ij}_\mathrm{E}, \qquad K = \frac{N_\mathrm{E}}{N}K_\mathrm{E}. \tag{3.26}$$

The Ricci scalar of $g_{\mu\nu}$ can be expressed in terms of the extrinsic curvature by the well-known formula

$$\sqrt{-g}R = N\sqrt{\gamma}[K^{ij}K_{ij} - K^2 + R^{(3)}] - \Delta \tag{3.27}$$

with $\Delta = 2\partial_i(\sqrt{\gamma}\gamma^{ij}\partial_j N) - 2\partial_t(\sqrt{\gamma}K) + 2\partial_i(\sqrt{\gamma}N^i K)$ and $g = \det g_{\mu\nu}$.

5. Lorentzian action in unitary gauge

In the unitary gauge we have been using so far, the action (3.21) is now rewritten as

$$S_g = \int \mathrm{d}t\mathrm{d}x^3 N\sqrt{\gamma}\{[f(X) - 2Xf'(X)](K^{ij}K_{ij} - K^2) + f(X)R^{(3)} + P(X)\}, \tag{3.28}$$

where the functions f and P are defined by

$$f(X) \equiv \frac{N_\mathrm{E}}{N}G_4(X_E), \quad f'(X) \equiv \frac{\mathrm{d}f(X)}{\mathrm{d}X}, \quad P(X) \equiv \frac{N_\mathrm{E}}{N}\mathcal{K}(X_\mathrm{E}) \tag{3.29}$$

in terms of

$$X \equiv \frac{M^4}{N^2}. \tag{3.30}$$

To show the equivalence between Eqs. (3.21) and (3.28), we have noted that Eq. (3.24) implies that

$$\frac{1}{X} + \frac{1}{X_\mathrm{E}} = \frac{N_c^2}{M^4}, \qquad \frac{\mathrm{d}X}{\mathrm{d}X_\mathrm{E}} = -\frac{X^2}{X_\mathrm{E}^2}. \tag{3.31}$$

Now, using the property (3.27), the action (3.28) can be further simplified to

$$S_g = \int \mathrm{d}t\mathrm{d}x^3 N\sqrt{\gamma}\Big\{f(X)R - 2Xf'(X)(K^{ij}K_{ij} - K^2) + f'(X)\Big[\frac{(D_i X)^2}{X} + 2K\partial_\perp X\Big] + P(X)\Big\}, \tag{3.32}$$

where the perpendicular derivative $\partial_\perp$ is defined similarly as Eq. (3.17) in terms of the normal vector to the constant ϕ hypersurfaces, $n_\mu = \partial_\mu\phi/\sqrt{X}$, as

$$\partial_\perp = n^\mu\partial_\mu = \frac{1}{N}(\partial_t - N^i\partial_i), \tag{3.33}$$

with $n^\mu = g^{\mu\nu}n_\mu$.

6. Covariant expression

In the previous section the action has been derived assuming that the time coordinate was fixed according to the unitary gauge (3.7).

The action (3.32) can be rewritten in a covariant way by noting that $\nabla_\mu\nabla_\nu\phi = -M^2\Gamma^t_{\mu\nu}$, where ∇_μ is the covariant derivative compatible with the Lorentzian metric $g_{\mu\nu}$ and $\Gamma^\rho_{\mu\nu}$ are its Christoffel symbols for $g_{\mu\nu}$. Concretely, its components are given by

$$\begin{aligned} \phi_{;ij} &\equiv \nabla_i\nabla_j\phi = -\sqrt{X}K_{ij}, \\ \phi_{;\perp i} &\equiv \phi_{;i\perp} \equiv n^\mu\nabla_\mu\nabla_i\phi = \frac{1}{2}\sqrt{X}\partial_i \ln X, \\ \phi_{;\perp\perp} &\equiv n^\mu n^\nu\nabla_\mu\nabla_\nu\phi = \frac{1}{2}\sqrt{X}\partial_\perp \ln X. \end{aligned} \tag{3.34}$$

Hence, the term $(\nabla^2\phi)^2 - (\nabla^\mu\nabla^\nu\phi)(\nabla_\mu\nabla_\nu\phi)$ can be expressed as

$$(\gamma^{ij}\gamma^{kl} - \gamma^{ik}\gamma^{jl})\phi_{;ij}\phi_{;kl} - 2\gamma^{ij}(\phi_{;\perp\perp}\phi_{;ij} - \phi_{;\perp i}\phi_{;\perp j}),$$

which reduces to

$$-X(K^{ij}K_{ij} - K^2) + K\partial_\perp X + \frac{1}{2}\frac{(D_i X)^2}{X}.$$

Finally, the Lorentzian action takes the form

$$S_g = \int \mathrm{d}x^4\sqrt{-g}\{f(X)R + 2f'(X)[(\nabla^2\phi)^2 - (\nabla^\mu\nabla^\nu\phi)(\nabla_\mu\nabla_\nu\phi)] + P(X)\}. \tag{3.35}$$

Whilst this form of the action was derived assuming the unitary gauge (3.7), it can become manifestly covariant by promoting X to a scalar defined by

$$X = -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi. \tag{3.36}$$

It is thus well defined without the unitary gauge condition. Actually, the covariant action (3.35) is a special case of the covariant Galileon considered in Ref. [36] coupled to the Lorentzian metric $g_{\mu\nu}$. In particular, the equations of motion are second order (see Ref. [37] for comparison).

C. Correspondence

The derived Lorentzian theory (3.35) and the parent Riemannian theory (3.6) are related to each other by the following relations:

$$\begin{aligned} g_{\mu\nu} &= g^\mathrm{E}_{\mu\nu} - \frac{\partial_\mu\phi\partial_\nu\phi}{X_c}, \\ g^{\mu\nu} &= g^{\mu\nu}_\mathrm{E} + \frac{g^{\mu\rho}_\mathrm{E} g^{\nu\sigma}_\mathrm{E}\partial_\rho\phi\partial_\sigma\phi}{X_c - X_\mathrm{E}}, \qquad \frac{1}{X} = \frac{1}{X_c} - \frac{1}{X_\mathrm{E}}, \\ \frac{f(X)}{\sqrt{X}} &= \frac{G_4(X_\mathrm{E})}{\sqrt{X_\mathrm{E}}}, \qquad \frac{P(X)}{\sqrt{X}} = \frac{\mathcal{K}(X_\mathrm{E})}{\sqrt{X_\mathrm{E}}}, \end{aligned} \tag{3.37}$$

where X_c is an arbitrary positive constant given by

$$X_c = \frac{M^4}{N_c^2}. \tag{3.38}$$

These relations are well defined even without the unitary gauge condition as far as X_E/X_c is large enough. It is straightforward to express various quantities defined in the Lorentzian theory in terms of those in the Riemannian theory.

D. Stability analysis

We now analyze the stability of a general flat Friedmann-Lemaître (FL) background using the Lorentzian action (3.35) with the Lorentzian Arnowitt-Deser-Misner decomposition (3.22).

1. Cosmological background

We consider a flat FL background spacetime for which the metric in cosmic time reduces to

$$N = 1, \qquad N_i = 0, \qquad \gamma_{ij} = a(t)^2\delta_{ij}, \tag{3.39}$$

where a is the scale factor, and for which the clock field $\phi = \phi_0(t)$.

The action (3.22) being invariant under a constant shift of the clock field ϕ, there is a conserved current associated with the shift symmetry so that the equation of motion for ϕ takes the form

$$\dot{J}_\phi + 3HJ_\phi = 0, \tag{3.40}$$

where $H = \dot{a}/a$ is the Hubble function and

$$J_\phi \equiv [P'_0 + 6H^2(2X_0 f''_0 + f'_0)]\dot{\phi}_0. \tag{3.41}$$

We are using notations according to which

$$X_0 = \dot{\phi}_0^2, \qquad P_0^{(n)} = P^{(n)}(X_0), \qquad f_0^{(n)} = f^{(n)}(X_0), \tag{3.42}$$

where n stands for the order of the derivation. Thus, Eq. (3.40) implies that J_ϕ decays as $J_\phi \propto 1/a^3$.

By using the correspondence (3.37), J_ϕ can be expressed in the language of the Riemannian theory as

$$\begin{aligned} J_\phi\dot{\phi}_0 = \{&[4G''_4X_E^2 + 4G'_4X_E - G_4]r^{3/2} \\ &+ [2G'_4X_E - G_4]r^{1/2}\}3H^2 \\ &+ \frac{1}{2}\left[(\mathcal{K} - 2\mathcal{K}'X_E)r^{1/2} + \frac{\mathcal{K}}{r^{1/2}}\right], \end{aligned} \tag{3.43}$$

where a prime in the right-hand side represents derivative with respect to X_E, and where the ratio r is defined by

$$r \equiv \frac{X_E}{X} = \frac{X_E}{X_c} - 1. \tag{3.44}$$

From Eq. (3.24), we have that $N_c^2 > (N_E^2, N^2)$ which implies that $r > 0$.

The equation of motion for the metric reduces, as usual, to the Friedmann equation that takes the form

$$3M_{\rm eff}^2H^2 = 2J_\phi\dot{\phi}_0 - P_0, \tag{3.45}$$

where the effective mass scale is defined by

$$M_{\rm eff}^2 \equiv 2(f_0 - 2X_0f'_0). \tag{3.46}$$

To understand the qualitative behavior of the system, let us suppose that $H^2/M^2 \ll 1$ and Taylor expand $P'(X)$ and $f'(X)$ around a local minimum of $P(X)$ (which we denote as $X \equiv qM^4$) as

$$P'(X) = p_2\delta + \mathcal{O}(\delta^2), \qquad f'(X) = \frac{f_1 + f_2\delta}{M^2} + \mathcal{O}(\delta^2), \tag{3.47}$$

where q, p_2, and $f_{1,2}$ are dimensionless constants of order unity, and $\delta \equiv \frac{X}{M^4} - q$ is a small quantity. Accordingly,

$$J_\phi = \left[p_2\delta + 6\frac{H^2}{M^2}(2f_2q + f_1) + \mathcal{O}\left(\frac{H^2}{M^2}\delta, \delta^2\right)\right]\dot{\phi}_0. \tag{3.48}$$

As already stated above, J_ϕ behaves as $\propto 1/a^3 \to 0$ ($a \to \infty$). Hence, apart from the trivial behavior with $\dot{\phi}_0 \to 0$, the system has a nontrivial attractor with $\delta + \mathcal{O}(H^2/M^2) \propto 1/a^3 \to 0$. This implies that $\dot{\phi}_0 \to \sqrt{q}M^2[1 + \mathcal{O}(H^2/M^2)]$ and that $M_{\rm eff}^2$ and P_0 approach constant values up to $\mathcal{O}(H^2/M^2)$ corrections. Therefore, Eq. (3.45) is no more than the standard Friedmann equation for a universe containing a pressureless fluid (from $J_\phi \propto 1/a^3$) and a cosmological constant (from $P_0 \to$ const). This behavior is similar to the one obtained in ghost condensate models [38,39]. To be consistent with the cosmic expansion as understood today, we need to have

$$P_0 < 0. \tag{3.49}$$

More precisely, we even need P_0 to be tuned so that

$$P_0 \sim -3\Omega_{\Lambda 0}M_{\rm eff}^2H_0^2 \sim -2.1M_{\rm eff}^2H_0^2, \tag{3.50}$$

where $\Omega_{\Lambda 0} \sim 0.7$ is the standard density parameter for the cosmological constant. Note also that since J_ϕ contributes to the dark matter component, it has to be bounded so that we shall require

$$\frac{2}{3}\frac{J_{\phi_0}}{M_{\rm eff}^2}\sqrt{q}\frac{M^2}{H_0^2} \le \Omega_{m0} \sim 0.3, \tag{3.51}$$

today. Note that the term can even be negative at the expense of introducing more dark matter. These two last bounds are the only indicative form that can be derived from cosmology, but a full cosmological analysis will be presented elsewhere.

2. Tensor perturbations

We now consider tensor (T) perturbations around the FL background so that the metric is given by

$$N = 1, \qquad N_i = 0, \qquad \gamma_{ij} = a(t)^2[e^h]_{ij}, \tag{3.52}$$

where h_{ij} is transverse and traceless (i.e., $\partial_i h^i_k = 0 = \delta^{ij} h_{ij}$). We still have that $\phi = \phi_0(t)$.

In Fourier space, the quadratic action for each polarization of the tensor mode is given by

$$\delta S^{(2)}_{\mathrm{T},\boldsymbol{k}} = \frac{1}{8}\int \mathrm{d}t a^3\left[M^2_{\mathrm{eff}}\dot{h}^2_{\boldsymbol{k}} - 2f_0\frac{\boldsymbol{k}^2}{a^2}h^2_{\boldsymbol{k}}\right]. \qquad (3.53)$$

Note that this result can be easily inferred from the expression (3.28). Hence, the stability of the tensor sector requires that

$$M^2_{\mathrm{eff}} > 0, \qquad f_0 > 0. \qquad (3.54)$$

By using the correspondence (3.37), M^2_{eff} and f_0 are expressed in the language of the Riemannian theory as

$$M^2_{\mathrm{eff}} = 2(2G_4' X_{\mathrm{E}} - G_4)\sqrt{r}, \qquad f_0 = \frac{1}{\sqrt{r}}G_4, \qquad (3.55)$$

where a prime in the right-hand side of these expressions represents derivative with respect to X_{E} and r is defined in Eq. (3.44). Thus, the stability of the tensor sector gives the constraint

$$2G_4' X_E > G_4 > 0. \qquad (3.56)$$

3. Scalar perturbations

For scalar perturbations around the FL background, the metric in the unitary gauge is given by

$$N = 1 + \alpha, \qquad N_i = \partial_i\beta, \qquad \gamma_{ij} = a(t)^2 \mathrm{e}^{2\zeta}\delta_{ij}, \qquad (3.57)$$

and, by definition, $\phi = \phi_0(t)$.

It is then straightforward to calculate the quadratic perturbed action since the time derivatives of α and β do not appear in the action. Thus, the equations of motion for α and β become constraint equations. After solving for those constraint equations with respect to α and β, one gets that the perturbed action for ζ, in Fourier space, is

$$\delta S^{(2)}_{\mathrm{S},\boldsymbol{k}} = \frac{1}{2}\int \mathrm{d}t a^3\left[\mathcal{A}\dot{\zeta}^2_{\boldsymbol{k}} - \mathcal{B}\frac{\boldsymbol{k}^2}{a^2}\zeta^2_{\boldsymbol{k}}\right], \qquad (3.58)$$

where $\mathcal{A}$ and $\mathcal{B}$ are given by

$$\mathcal{A} = \frac{M^2_{\mathrm{eff}}}{H^2\mathcal{G}^2}(6 + M^2_{\mathrm{eff}}\mathcal{F}), \qquad \mathcal{B} = \frac{1}{a}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{aM^4_{\mathrm{eff}}}{H\mathcal{G}^2}\right) + 4f_0, \qquad (3.59)$$

with $\mathcal{F}$ and $\mathcal{G}$ given by

$$\begin{aligned}\mathcal{F} &= P_0'' X_0^2 + \frac{1}{2}J_\phi\dot{\phi}_0 + 3H^2[4f_0''' X_0^3 + 14 f_0'' X_0^2 \\ &\quad + 6f_0' X_0 - f_0], \\ \mathcal{G} &= 4f_0'' X_0^2 + 4f_0' X_0 - f_0. \end{aligned} \qquad (3.60)$$

The quadratic action (3.58) agrees with a special case of the action derived in Ref. [40]. The stability of scalar perturbations requires that

$$\mathcal{A} > 0, \qquad \mathcal{B} > 0. \qquad (3.61)$$

By using the correspondence (3.37), this sets two other constraints on the Riemannian theory since $\mathcal{F}$ and $\mathcal{G}$ can be expressed in the language of the Riemannian theory as

$$\begin{aligned}\mathcal{F} &= -\left\{\left[4G_4''' X_{\mathrm{E}}^3 + 18G_4'' X_{\mathrm{E}}^2 + 9G_4' X_{\mathrm{E}} - \frac{3}{2}G_4\right]r^{5/2} \right.\\ &\quad \left. + \left[2G_4'' X_{\mathrm{E}}^2 + 2G_4' X_{\mathrm{E}} - \frac{1}{2}G_4\right]r^{3/2}\right\}3H^2 \\ &\quad + \left(\mathcal{K}'' X_{\mathrm{E}}^2 + \mathcal{K}' X_{\mathrm{E}} - \frac{1}{4}\mathcal{K}\right)r^{3/2} \\ &\quad - \frac{1}{4}(\mathcal{K} - 2\mathcal{K}' X_{\mathrm{E}})r^{1/2}, \\ \mathcal{G} &= [4G_4'' X_{\mathrm{E}}^2 + 4G_4' X_{\mathrm{E}} - G_4]r^{3/2}, \end{aligned} \qquad (3.62)$$

where a prime in the right-hand side of these expressions represents derivative with respect to X_{E} and r is defined in Eq. (3.44).

E. Summary

Starting from the Riemannian action (3.6) for a positive definite metric $g^{\mathrm{E}}_{\mu\nu}$, we have been able to derive an action for a Lorentzian metric $g_{\mu\nu}$. The key ingredient is the coupling of the Einstein tensor of $g^{\mathrm{E}}_{\mu\nu}$ to the clock field. In $\mathcal{M}_0$, the dynamics of $g_{\mu\nu}$ is dictated by the covariant action (3.35), which is a special case of the covariant Galileon considered in Ref. [36]. (See Ref. [41] for the original Galileon theory.)

The theory has two free functions, $\mathcal{K}$ and G_4, and we have shown that the stability of the FL spacetime with respect of both scalar and tensor perturbations at linear level sets four constraints on these quantities. An extra constraint appears from the requirement that the constant term entering the Friedmann equation reproduces a positive cosmological constant.

IV. BOSONIC MATTER FIELDS IN CURVED SPACE

Given the formulation of gravity described in the previous section, we shall now extend the constructions presented in Sec. II to describe the proper dynamics of the matter fields in curved spacetime.

A. Scalar field

Following Sec. II C we assume that the clock field ϕ has a derivative coupling to a real scalar field χ of the form

$$\int \mathrm{d}x^4\sqrt{g_{\mathrm{E}}}(g^{\mu\nu}_{\mathrm{E}}\partial_\mu\phi\partial_\nu\chi)^2. \qquad (4.1)$$

Adding this to the Riemannian kinetic term and a potential term $\tilde{V}$ for χ, the general action for χ is of the form

$$S_\chi = \int \mathrm{d}x^4 \sqrt{g_\mathrm{E}} \bigg[-\frac{\kappa_\chi}{2} g_\mathrm{E}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - \tilde{V}(\chi) + \frac{\alpha_\chi}{2M^4} (g_\mathrm{E}^{\mu\nu} \partial_\mu \phi \partial_\nu \chi)^2 \bigg]. \tag{4.2}$$

It involves two dimensionless constants, κ_χ and α_χ. There is a freedom to rescale χ and thus we can set $\kappa_\chi = \pm 1$ if needed.

With the decomposition (3.8), the Riemannian kinetic and the derivative coupling terms are correspondingly rewritten as

$$\begin{aligned} g_\mathrm{E}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi &= (\partial_\perp^\mathrm{E} \chi)^2 + \gamma^{ij} \partial_i \chi \partial_j \chi, \\ (g_\mathrm{E}^{\mu\nu} \partial_\mu \phi \partial_\nu \chi)^2 &= \frac{M^4}{N_\mathrm{E}^2} (\partial_\perp^\mathrm{E} \chi)^2, \end{aligned} \tag{4.3}$$

where $\partial_\perp^\mathrm{E}$ is defined in Eq. (3.17). Therefore, the action of the scalar field χ reduces to

$$S_\chi = \int \mathrm{d}t\mathrm{d}x^3 N_\mathrm{E} \sqrt{\gamma} \bigg[\frac{1}{2} \bigg(\frac{\alpha_\chi}{N_\mathrm{E}^2} - \kappa_\chi \bigg) (\partial_\perp^\mathrm{E} \chi)^2 - \tilde{V}(\chi) - \frac{\kappa_\chi}{2} \gamma^{ij} \partial_i \chi \partial_j \chi \bigg]. \tag{4.4}$$

If

$$\frac{\alpha_\chi}{N_\mathrm{E}^2} > \kappa_\chi > 0, \tag{4.5}$$

then S_χ describes a scalar field propagating in a Lorentzian spacetime. To see this explicitly let us define a Lorentzian effective metric $g^\chi_{\mu\nu}$ by

$$g^\chi_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = -N_\chi^2 \mathrm{d}t^2 + \Omega_\chi^2 \gamma_{ij} (\mathrm{d}x^i + N^i \mathrm{d}t)(\mathrm{d}x^j + N^j \mathrm{d}t), \tag{4.6}$$

where

$$N_\chi = N_\mathrm{E} \bigg[\frac{\kappa_\chi^3}{\frac{\alpha_\chi}{N_\mathrm{E}^2} - \kappa_\chi} \bigg]^{1/4}, \qquad \Omega_\chi = \bigg[\kappa_\chi \bigg(\frac{\alpha_\chi}{N_\mathrm{E}^2} - \kappa_\chi \bigg) \bigg]^{1/4}. \tag{4.7}$$

The scalar field action S_χ is rewritten as

$$S_\chi = -\int \mathrm{d}x^4 \sqrt{-g^\chi} \bigg[\frac{1}{2} g_\chi^{\mu\nu} \partial_\mu \chi \partial_\nu \chi + V(\chi, X) \bigg], \tag{4.8}$$

where g^χ and $g_\chi^{\mu\nu}$ are the determinant and the inverse of $g^\chi_{\mu\nu}$, and

$$\begin{aligned} V(\chi, X) &= \tilde{V}(\chi) \bigg[\kappa_\chi^3 \bigg(\frac{\alpha_\chi}{N_\mathrm{E}^2} - \kappa_\chi \bigg) \bigg]^{-1/2} \\ &= \tilde{V}(\chi) \bigg[\kappa_\chi^3 \bigg(\frac{\alpha_\chi X_\mathrm{E}}{M^4} - \kappa_\chi \bigg) \bigg]^{-1/2}. \end{aligned} \tag{4.9}$$

Note that α_χ and κ_χ may depend on X_E and that X_E is related to X via the correspondence (3.37).

B. Vector field

Let us now consider the case of a gauge field A^μ. Similarly as in Sec. II C, we can add a coupling to the clock field ϕ of the form

$$\int \mathrm{d}x^4 \sqrt{g_\mathrm{E}} F_\mathrm{E}^{\mu\rho} F_{\mathrm{E}\rho}^{\nu} \partial_\mu \phi \partial_\nu \phi \tag{4.10}$$

to the standard (Riemannian) Maxwell action. Again, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Faraday tensor of A_μ and we use the notations $F^\mu_{\mathrm{E}\nu} \equiv g_\mathrm{E}^{\mu\rho} F_{\rho\nu}$ and $F_\mathrm{E}^{\mu\nu} \equiv g_\mathrm{E}^{\nu\rho} F^\mu_{\mathrm{E}\rho}$. This leads to the general gauge-invariant action for the vector field,

$$S_A = \frac{1}{4} \int dx^4 \sqrt{g_\mathrm{E}} \bigg[-\kappa_A F_\mathrm{E}^{\mu\nu} F_{\mu\nu} + 2\frac{\alpha_A}{M^4} F_\mathrm{E}^{\mu\rho} F^\nu_{\mathrm{E}\rho} \partial_\mu \phi \partial_\nu \phi \bigg], \tag{4.11}$$

where κ_A and α_A are two dimensionless constants.

With the decomposition (3.8) for the Riemannian metric, the Riemannian kinetic term and the nonminimal coupling term can be respectively written as

$$\begin{aligned} F_\mathrm{E}^{\mu\nu} F_{\mu\nu} &= 2\gamma^{ij} \tilde{F}_{\perp i} \tilde{F}_{\perp j} + \gamma^{ik} \gamma^{jl} F_{ij} F_{kl}, \\ F_\mathrm{E}^{\mu\rho} F^\nu_{\mathrm{E}\rho} \partial_\mu \phi \partial_\nu \phi &= \frac{M^4}{N_\mathrm{E}^2} \gamma^{ij} \tilde{F}_{\perp i} \tilde{F}_{\perp j}, \end{aligned} \tag{4.12}$$

where

$$\tilde{F}_{\perp i} \equiv \frac{1}{N_\mathrm{E}} (F_{ti} - N^j F_{ji}). \tag{4.13}$$

Therefore, the gauge-invariant action takes the form

$$S_A = \frac{1}{4} \int \mathrm{d}t\mathrm{d}x^3 N_\mathrm{E} \sqrt{\gamma} \bigg[2 \bigg(\frac{\alpha_A}{N_\mathrm{E}^2} - \kappa_A \bigg) \gamma^{ij} \tilde{F}_{\perp i} \tilde{F}_{\perp j} - \kappa_A \gamma^{ik} \gamma^{jl} F_{ij} F_{kl} \bigg]. \tag{4.14}$$

If

$$\frac{\alpha_A}{N_\mathrm{E}^2} > \kappa_A > 0, \tag{4.15}$$

then S_A describes a $U(1)$ gauge field propagating in a Lorentzian spacetime. To see this explicitly, let us define a Lorentzian effective metric $g^A_{\mu\nu}$ by

$$g^A_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = -N_A^2 dt^2 + \Omega_A^2 \gamma_{ij} (\mathrm{d}x^i + N^i \mathrm{d}t)(\mathrm{d}x^j + N^j \mathrm{d}t), \tag{4.16}$$

where Ω_A is an arbitrary positive function and

$$N_A = N_\mathrm{E} \Omega_A \bigg[\frac{\kappa_A}{\frac{\alpha_A}{N_\mathrm{E}^2} - \kappa_A} \bigg]^{1/2}. \tag{4.17}$$

The vector field action S_A takes the form of the usual Maxwell action

$$S_A = -\int \mathrm{d}x^4 \sqrt{-g^A}\frac{1}{4e^2} g_A^{\mu\rho} g_A^{\nu\sigma} F_{\mu\nu}F_{\rho\sigma}, \tag{4.18}$$

where g^A and $g_A^{\mu\nu}$ are the determinant and the inverse of $g^A_{\mu\nu}$, and the effective coupling constant e^2 is given by

$$e^2 = \left[\kappa_A\left(\frac{\alpha_A}{N_\mathrm{E}^2} - \kappa_A\right)\right]^{-1/2}. \tag{4.19}$$

Note that α_A and κ_A may depend on X_E and that X_E is related to X via the correspondence (3.37).

C. Generalization to a complex scalar field

The generalization to a complex scalar field charged under the $U(1)$ is straightforward and follows the construction presented in Sec. II D. Consider the action for a complex scalar ψ,

$$S_\psi = \int \mathrm{d}x^4\sqrt{g_\mathrm{E}}\left\{-\frac{\kappa_\psi}{2} g_\mathrm{E}^{\mu\nu}(\partial_\mu + iqA_\mu)\psi^*(\partial_\nu - iqA_\nu)\psi + \frac{\alpha_\psi}{2M^4}|g_\mathrm{E}^{\mu\nu}\partial_\mu\phi(\partial_\nu - iqA_\nu)\psi|^2 - \tilde{U}(|\psi|^2)\right\}, \tag{4.20}$$

where q, κ_ψ, and α_ψ are dimensionless constants and $\tilde{U}(|\psi|^2)$ is a function of $|\psi|^2$.

Supposing that

$$\frac{\alpha_\psi}{N_\mathrm{E}^2} > \kappa_\psi > 0, \tag{4.21}$$

it is easy to show that

$$S_\psi = -\int \mathrm{d}x^4\sqrt{-g^\psi}\left[\frac{1}{2} g_\psi^{\mu\nu}(\partial_\mu + iqA_\mu)\psi^*(\partial_\nu - iqA_\nu)\psi + U(|\psi|^2, X)\right], \tag{4.22}$$

where we have introduced a Lorentzian metric $g^\psi_{\mu\nu}$ by

$$g^\psi_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -N_\psi^2\mathrm{d}t^2 + \Omega_\psi^2\gamma_{ij}(\mathrm{d}x^i + N^i\mathrm{d}t)(\mathrm{d}x^j + N^j\mathrm{d}t), \tag{4.23}$$

$$N_\psi = N_\mathrm{E}\left[\frac{\kappa_\psi^3}{\frac{\alpha_\psi}{N_\mathrm{E}^2} - \kappa_\psi}\right]^{1/4}, \qquad \Omega_\psi = \left[\kappa_\psi\left(\frac{\alpha_\psi}{N_\mathrm{E}^2} - \kappa_\psi\right)\right]^{1/4}, \tag{4.24}$$

g^ψ and $g_\psi^{\mu\nu}$ are the determinant and the inverse of $g^\psi_{\mu\nu}$, and

$$U(|\psi|^2, X) = \tilde{U}(|\psi|^2)\left[\kappa_\psi^3\left(\frac{\alpha_\psi}{N_\mathrm{E}^2} - \kappa_\psi\right)\right]^{-1/2} = \tilde{U}(|\psi|^2)\left[\kappa_\psi^3\left(\frac{\alpha_\psi X_\mathrm{E}}{M^4} - \kappa_\psi\right)\right]^{-1/2}. \tag{4.25}$$

Note that α_ψ and κ_ψ may depend on X_E and that X_E is related to X via the correspondence (3.37).

Generalization to a non-Abelian group is trivial.

D. Massive point particle

For a massive point particle, we assume that the action is given by

$$S_\mathrm{pp} = \frac{1}{2}\int\left[\mathcal{N}^{-1}\left(\bar{\kappa}_\mathrm{pp} g^\mathrm{E}_{\mu\nu} - \frac{\bar{\alpha}_\mathrm{pp}}{M^4}\partial_\mu\phi\partial_\nu\phi\right)\times\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - \mathcal{N}m^2\right]d\tau, \tag{4.26}$$

where $\mathcal{N}$ is a function of τ. Then, a test particle propagates in the effective metric $g^\mathrm{pp}_{\mu\nu} = \bar{\kappa}_\mathrm{pp} g^\mathrm{E}_{\mu\nu} - \frac{\bar{\alpha}_\mathrm{pp}}{M^4}\partial_\mu\phi\partial_\nu\phi$ so that its equation of motion is simply a geodesic equation for this metric

$$u^\mu\nabla^\mathrm{pp}_\mu u^\nu = 0, \tag{4.27}$$

with $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$, where λ is an affine parameter defined by $d\lambda = \mathcal{N}d\tau$. Using the decomposition (3.8) of the Riemannian metric, we obtain that

$$g^\mathrm{pp}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -(\bar{\alpha}_\mathrm{pp} - \bar{\kappa}_\mathrm{pp}N_\mathrm{E}^2)\mathrm{d}t^2 + \bar{\kappa}_\mathrm{pp}\gamma_{ij}(\mathrm{d}x^i + N^i\mathrm{d}t)(\mathrm{d}x^j + N^j\mathrm{d}t). \tag{4.28}$$

This effective metric has Lorentzian signature if

$$\frac{\bar{\alpha}_\mathrm{pp}}{N_\mathrm{E}^2} > \bar{\kappa}_\mathrm{pp} > 0. \tag{4.29}$$

V. PHENOMENOLOGY

A. Summary

We have proposed a Riemannian field theory for gravity, vector, and scalar fields that, with the expense of the introduction of a scalar field ϕ called a clock field, leads to an effective Lorentzian dynamics.

This construction involves a set of free parameters:

(i) For the gravitational sector, we have two free functions of X_E, $\mathcal{K}$ and G_4 in terms of which the two free functions of the Lorentzian theory $f(X)$ and $P(X)$ are defined; see the correspondence between the two sets given in Eq. (3.37).

(ii) For the matter sector, we have derived the actions for scalar and vector fields. Each action depends on two parameters (κ, α) that are allowed to be functions of X_E, or equivalently X, in general but may as well be assumed constant.

(iii) Besides, there is an environmental parameter which characterizes the clock field configuration on the patch $\mathcal{M}_0$, N_E [see Eqs. (3.3) and (3.9)] and the associated integration constant N_c [see Eq. (3.24)]. They combine in the parameter r [see Eq. (3.44)].

With these parameters the actions for gravity, scalar, and vector fields are respectively given by Eqs. (3.35), (4.8), and (4.18).

We have already shown that these parameters are subject to a series of constraints.

(i) For the gravitational sector, we have two sets of constraints. The first one arises from the stability analysis and is given by Eqs. (3.56) and (3.61). The second is related to the dynamics of the homogeneous model. Interestingly, the model induces two components which respectively behave as dark matter and dark energy. This sets the two constraints (3.50) and (3.51) in order for the cosmology to be consistent with the standard cosmology [42], at least at the background level.

(ii) For the matter sector, the constants α have to satisfy [see Eqs. (4.5) and (4.15)]

$$\alpha_\chi > N_{\rm E}^2\kappa_\chi, \qquad \alpha_A > N_{\rm E}^2\kappa_A. \tag{5.1}$$

From the effective Lagrangians (4.8) and (4.18), we see that scalars and vectors propagate in two different effective metrics In order for the weak equivalence principle to hold, we have to impose that these two metrics coincide. This can be obtained by imposing that $c_A^2 = c_\chi^2$, with $c_A^2 = \Omega_A^{-2}N_A^2/N_{\rm E}^2$ and $c_\chi^2 = \Omega_\chi^{-2}N_\chi^2/N_{\rm E}^2$. This sets the following constraints:

$$\frac{\kappa_A}{\alpha_A} = \frac{\kappa_\chi}{\alpha_\chi}. \tag{5.2}$$

In the simplest situation in which the coefficients (κ, α) are assumed to be constant, we can always set $\kappa = 1$ in both sectors so that we are left with the constraint $\alpha_A = \alpha_\chi$ for the two coupling constants. This is similar to what we performed in Sec. II in which the couplings to the clock field were chosen *a priori* so that effectively all fields propagate in the same effective Minkowski metric. Interestingly, in this class of models, one requires a tuning on the parameters of the Lagrangians, but once it is done, it is satisfied whatever the configuration of the clock field, that is whatever $N_{\rm E}$ or $X_{\rm E}$. In this sense the tuning is not worse than the one usually does by assuming that all the fields propagate in the same metric. This conclusion holds even if κ's and α's are functions of $X_{\rm E}$ as long as their ratios agree between different sectors. Again, once this condition is satisfied, it holds whatever the field configuration.

The Lorentzian effective metric (4.28) for a point particle coincides with that for the vector if $\bar\alpha_{\rm pp}/\bar\kappa_{\rm pp} - N_{\rm E}^2 = N_A^2/\Omega_A^2$, that is if

$$\frac{\bar\alpha_{\rm pp}}{\bar\kappa_{\rm pp}N_{\rm E}^2} - 1 = \left[\frac{\alpha_A}{\kappa_A N_{\rm E}^2} - 1\right]^{-1}. \tag{5.3}$$

This may look as a functional fine-tuning depending on the local value of $X_{\rm E}$. Actually, this arises from the fact that $\bar\alpha_{\rm pp}$ and $\bar\kappa_{\rm pp}$ have been introduced with reference to $g^{\rm E}_{\mu\nu}$ while $\alpha_{A,\chi}$ and $\kappa_{A,\chi}$ have been introduced with reference to $g_{\rm E}^{\mu\nu}$. Shifting to the inverse metric and redefining these coefficients leads to a constraint similar to Eq. (5.2).

Only the condition (5.1) for the emergence of the Lorentzian signature is environmentally dependent so that there are regions in the configuration space where the dynamics is effectively Lorentzian while other regions remain Riemannian. This could drive us toward a multiverse description in the configuration space but we do not have any anthropic reasons associated.

Note that without such a tuning so that all fields propagate in the same effective Lorentzian metric, one can choose one metric as reference so that the equations of motion of other fields will exhibit an explicit coupling to the clock field.

B. Other constraints

The fundamental parameters entering our effective Lorentzian actions are environmentally determined. This means that if $X_{\rm E}$ is not strictly constant on $\mathcal{M}_0$, fundamental constants may be spacetime dependent, which can induce a violation of the equivalence principle [2].

(i) From Eq. (4.19), it can be deduced that the coupling constant of any gauge field will be environment dependent. The first implication is that the coupling constants of the three nongravitational interactions have to be spacetime varying. There exist strong constraints on such a possibility [2].

(ii) The action for the gravitational sector implies that the Newton constant is also expected to be spacetime dependent.

(iii) Furthermore, even if $c_A = c_\chi$ so that scalars and vectors propagate on the same light cone, we have to compare the propagation speeds of gravity waves and photons. The first is given by

$$c_\gamma^2 = c_A^2 = \frac{N_A^2}{\Omega_A^2 N_{\rm E}^2} = \left[\frac{\alpha_A X_{\rm E}}{\kappa_A M^4} - 1\right]^{-1}. \tag{5.4}$$

The propagation speed of the gravity waves can be obtained from the action (3.53) rewritten as

$$\delta S^{(2)}_{{\rm T},\boldsymbol{k}} = \frac{1}{8}\int {\rm d}t a^3 N_{\rm E}\left[M_{\rm eff}^2\frac{N_{\rm E}}{N}\left(\frac{\dot h_{\boldsymbol{k}}}{N_{\rm E}}\right)^2 - 2f_0\frac{N}{N_{\rm E}}\frac{\boldsymbol{k}^2}{a^2}h_{\boldsymbol{k}}^2\right],$$

from which we read off

$$c_{\rm GW}^2 = \frac{2f_0}{M_{\rm eff}^2}\frac{N^2}{N_{\rm E}^2} = \frac{2f_0}{M_{\rm eff}^2}r. \tag{5.5}$$

It can be rewritten in terms of the function $G_4(X_E)$ entering the Euclidean gravitational action (3.6) as

$$c_{\rm GW}^2 = \left[\frac{2G_4' X_{\rm E}}{G_4} - 1\right]^{-1}. \tag{5.6}$$

As long as both light cones are nondegenerate, there is no *a priori* intrinsic problem even if these two propagation speeds are different [43,44] and similar features indeed appear in many bimetric theories such as TeVeS [45] or many other extensions of general relativity [43]. This difference can be tested by future experiments by comparing e.g., the arrival time of gravity waves and light emitted during the explosion of supernovae; see e.g., Ref. [46]. Models in which $c_{\rm GW}^2 < c_A^2$ are very constrained by the observations of cosmic rays [47] because particles propagating faster than the gravity waves emit gravi-Cerenkov radiation. They lead to the constraint [48]

$$\frac{c_\gamma - c_{\rm GW}}{c_\gamma} < 2\times 10^{-15}. \tag{5.7}$$

C. Emergence of Lorentz symmetry on intermediate scales

Let us first consider our mechanism in flat space. As discussed in Sec. II, M is not the most important mass scale of the problem. More important is the scale characterizing the variation of $X_{\rm E}/M^4$. In order to illustrate this and to capture the way the Lorentz dynamics appear on relevant scales, let us assume that the clock field configuration is given by

$$\phi(x^\mu) = M[m_x x + \cos(m_y y) + \beta\cos(M_y y)]$$

with $m_y \ll M_y$ two mass scales that characterize respectively large- and small-scale variations of ϕ and β a dimensionless number. We neglect the two other dimensions for simplicity. With such a form it is obvious that $\partial_\mu\phi$ is not constant.

Now, assume that we are smoothing the dynamics at a scale $R \sim m_x^{-1}$ with, e.g., a top-hat window function. (See Fig. 2.) On the scale R, the clock field is given by

$$\phi = M\left[m_x x + 2\frac{J_1(m_y R)}{m_y R}\cos(m_y y) + 2\beta\frac{J_1(M_y R)}{M_y R}\cos(M_y y)\right],$$

where J_1 is the Bessel function of order 1. It follows that $\partial_\mu\phi$ is given by

$$\frac{\partial_\mu\phi}{M^2} = \begin{pmatrix} m_x/M \\ -2\frac{J_1(m_y R)}{MR}\sin(m_y y) - 2\beta\frac{J_1(M_y R)}{MR}\sin(M_y y)\end{pmatrix}.$$

This can be considered as constant only if $2J_1(m_y R) \ll m_x R$ and $2\beta J_1(m_y R) \ll m_x R$. By choosing $R \sim m_x^{-1}$, the first condition reduces to

$$\frac{m_y}{m_x} \ll 1,$$

since $J_1(x) \sim x/2$ at small x, while the second gives

$$2\sqrt{\frac{2}{\pi}}\beta\cos\left(\frac{M_y}{m_x} + \frac{\pi}{4}\right)\sqrt{\frac{m_x}{M_y}} \ll 1.$$

This condition is clearly fulfilled if $m_y \ll m_x \ll \beta^{-2} M_y$. For example, if we assume that M_y is of the order of the Planck mass, $M_y \sim M_{\rm p} \sim 10^{19}$ GeV, and that the large-scale variations appear on Hubble scales, $m_y \sim H_0 \sim 10^{-41}$ GeV, then we end up with the conclusion that $\partial_\mu\phi$ can be considered as constant at the level 10^{-n} on scales

$$10^{n-41}\ {\rm GeV} < m_x < 10^{21-2n}\left(\frac{\beta}{0.1}\right)^{-2}\ {\rm GeV}. \tag{5.8}$$

For $n = 9$ and $\beta = \mathcal{O}(0.1)$, this means that we can work with scales

$$10^{-19}\ {\rm m} < m_x^{-1} < 10^{16}\ {\rm m}. \tag{5.9}$$

In such a range of scales, we expect no deviations larger than 10^{-9} to the standard field theory. On cosmological scales, we can probably relax the bound to deviations of the order if 10^{-1}–10^{-2} so that our model may be compatible

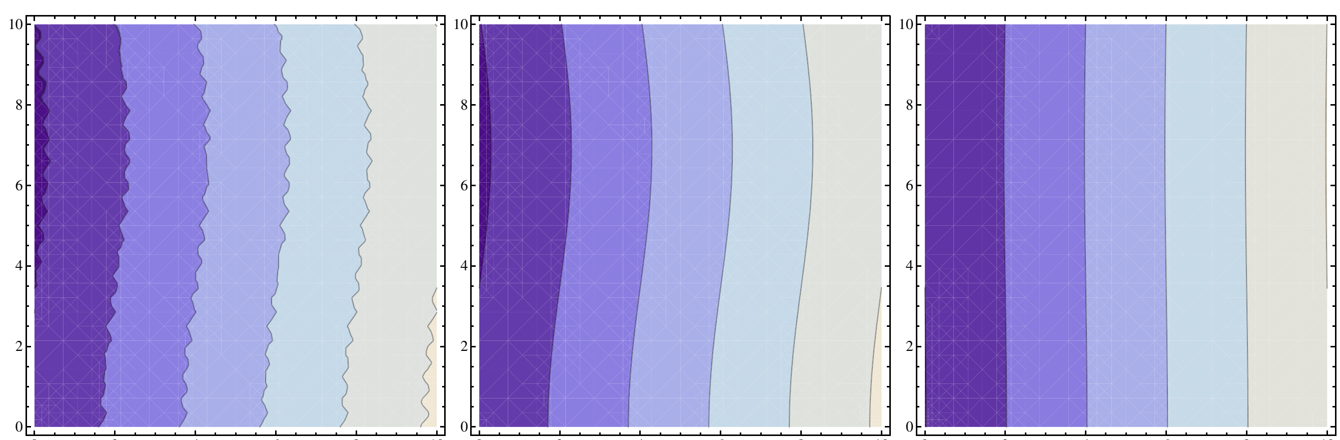

FIG. 2 (color online). Example of a field configuration with fluctuations on scales larger than M_y^{-1} and shorter than m_y^{-1} (left). When smoothed on scales m_x^{-1} (middle) and $10m_x^{-1}$ (right), the distribution of the clock field is such that $\partial_\mu\phi$ can be considered as constant on scales smaller than m_x^{-1}.

with standard cosmology on scales of the order of the observable universe.

VI. SUMMARY AND DISCUSSION

We have followed the idea that the apparent Lorentzian dynamics of the usual field theories is an emergent property and that the underlying field theory is in fact strictly Riemannian. This requires the introduction of the clock field, a scalar field playing the role of the physical time. We emphasize that the microscopic theory is Euclidean, and that time evolution is just an effective and emergent property, which holds on some energy scales, and in some regions of the Euclidean space. We have thus to think of time and dynamics as illusions in our local patch $\mathcal{M}_0$. This has to be distinguished from the mathematical trick of a Wick rotation used to effectively study genuine Lorentzian theories in a Euclidean space.

We have been able to perform such a construction in flat spacetime for scalar, vector, and spinor, hence allowing for the construction of the standard model of particle physics. Our construction is however restricted to classical field theory and the spinor sector suffers from the severe fine-tuning to ensure the *CPT* invariance. (See e.g., Ref. [49] and references therein for recent constraints on *CPT* violation.)

We have then generalized our construction to curved spacetimes. This generalizes an early attempt [31] that ended up with a Nordström theory of gravity. Our construction leads to an extended K-essence model for gravity called covariant Galileon, which can be close enough to general relativity to be experimentally acceptable. We have then generalized the scalar and vector sectors to curved spacetimes. This requires the introduction of four arbitrary functions. Again, so far we have not generalized the construction of spinors to curved spacetime. We have expressed the effective Lorentzian action that can emerge in a patch $\mathcal{M}_0$ of the Euclidean space. It allowed us to list a set of constraints arising from the stability of the cosmological solution, and the requirement for the different test fields to propagate in the same metric, in order for the weak equivalence principle to hold. The effective fundamental constants, such as the three nongravitational coupling constants and the gravitational constant, are spacetime dependent and the difference of the propagation speeds of gravity and electromagnetic waves can also set constraints on our model. We have proposed a heuristic description on the way the Lorentz symmetry can emerge on a band of energy scales.

From a theoretical point of view, our construction gives a new insight into the need for Lorentzian metric as a fundamental entity. As we have shown, this is not a mandatory requirement and a decent field theory, at least at the classical level, can be constructed from a Riemannian metric. Such a formalism may be fruitful in the debate on the emergence of time and, speculating, for the development of quantum gravity.

It also opens up a series of questions and possibilities that will be addressed in a companion article. We can list (1) the construction of Majorana and Weyl spinors, (2) the development of a quantum theory and study of particle creation [50], (3) the possibility from the classical viewpoint that singularities in our local Lorentzian region may be related to singularities in the clock field (e.g., similar to topological defects) and not in the metric of the Euclidean theory (see Ref. [51] for a similar idea in a totally different setup), (4) the possibility that a de Sitter spacetime may be an "illusion" in an anti-de Sitter Riemannian space. It then follows that a Euclidean AdS/CFT correspondence at the microscopic level would reveal itself as a dS/CFT correspondence in our effective Lorentzian universe.

All these are indeed, for now, bold speculations but they illustrate that this framework may be fruitful for extending our current field theories, including general relativity.

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