Supersymmetric solutions in four-dimensional off-shell curvature-squared supergravity

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(Received 15 December 2012; published 18 March 2013)

Off-shell formulations of supergravities allow one to add closed-form higher-derivative super-invariants that are separately supersymmetric to the usual lower-derivative actions. In this paper we study four-dimensional off-shell $\mathcal{N} = 1$ supergravity where additional super-invariants associated with the square of the Weyl tensor and the square of the Ricci scalar are included. We obtain a variety of solutions where the metric describes domain walls, Lifshitz geometries, and also solutions of a kind known as gyratons. We find that in some cases the solutions can be supersymmetric for appropriate choices of the parameters. In some solutions the auxiliary fields may be imaginary. One may reinterpret these as real solutions in an analytically continued theory. Since the supersymmetry transformation rules now require the gravitino to be complex, the analytically continued theory has a "fake supersymmetry" rather than a genuine supersymmetry. Nevertheless, the concept of pseudosupersymmetric solutions is a useful one, since the Killing spinor equations provide first-order equations for the bosonic fields.

DOI: 10.1103/PhysRevD.87.065014 PACS numbers: 04.65.+e, 04.20.Jb

I. INTRODUCTION

The study of supersymmetric solutions in theories of supergravity has proved to be extremely fruitful over the years. Much of the focus has been on those supergravities that are directly related to string theory or M theory, and mostly at the level of the leading-order theories, such as $D = 11$ supergravity or the type IIA and IIB supergravities in $D = 10$. It is known that in string theory or M theory these supergravities will receive higher-order corrections, including, in particular, terms in the effective actions involving higher powers of the curvature tensor. In fact, these corrections are expected to continue to arbitrarily high order in powers of the curvature. In general, it is inevitable that once any higher-order terms are included in the ten- or eleven-dimensional action, the process of supersymmetrizing them will be an endless one, requiring corrections to the action and to the transformation rules at all orders.

In dimensions $D \leq 6$, things can be rather different, because in these cases there exist off-shell formulations of certain supergravity theories. In such cases, the possibility arises of being able to add a finite number of higher-order terms to an existing supersymmetric action, in the form of complete and self-contained super-invariants, such that the resulting theory is fully supersymmetric in its own right, and with no modifications to the original transformation rules. Such theories can provide interesting insights into the effects of higher-order curvature terms on the solutions of the theories, while retaining the advantages of a theory that is self-contained and allowing the possibility of obtaining corrected solutions in closed form.

Four-dimensional $\mathcal{N} = 1$ off-shell supergravities [1,2], without higher-order curvature terms, were recently used to obtain theories with rigid supersymmetry by taking a limit in which gravity decoupled [3]. Further work leading on from this, including the conditions for the existence of supersymmetric solutions in the $\mathcal{N} = 1$ off-shell supergravity, were studied in detail in Ref. [4], and certain explicit solutions were presented. In Refs. [5,6], solutions were investigated in the more general context of four-dimensional off-shell $\mathcal{N} = 1$ supergravity with an additional Weyl-squared invariant. It had been shown in Ref. [7] that Einstein-Weyl gravity has a critical point for a specific choice of the coefficient of the Weyl-squared term, where the massive graviton disappears and is replaced by a spin-2 mode with a logarithmic falloff. It is a four-dimensional generalization of chiral gravity in three dimensions [8]. The critical behavior of the off-shell supersymmetrization of Einstein-Weyl gravity was studied in Ref. [9].

The focus in Refs. [5,6] was on solutions having the form of Lifshitz or gyrating Schrödinger geometries. Amongst the solutions obtained there were supersymmetric examples. There are in fact two independent off-shell super-invariants involving quadratic-curvature terms that can be added in the $\mathcal{N} = 1$ off-shell theory. In addition to the Weyl-squared invariant already mentioned, there is one other quadratic-curvature invariant, which involves only the square of the Ricci scalar. In the present paper we shall look for supersymmetric solutions, but within the wider class of $\mathcal{N} = 1$ off-shell supergravities involving both of these quadratic-curvature invariants. The higher-order curvature invariants play an essential role in the form of the solutions.

The paper is organized as follows. In Sec. II, we review the four-dimensional $\mathcal{N} = 1$ off-shell supergravity, including all the super-invariants involving powers of curvature ranging from zero to four. For our purposes, it suffices to present only the bosonic Lagrangian and the supersymmetry transformation rule for the gravitino. Owing to the global symmetry of certain of the super-invariants and the way the scalar auxiliary fields S and P couple with the auxiliary vector field A_μ, it is convenient to refer to the off-shell supergravity as the U(1) theory. Motivated by some of the solutions we obtain, it is also natural to consider an analytically continued theory in which the original auxiliary pseudoscalar and vector fields P and A_μ become imaginary. The resulting bosonic Lagrangian remains real, but the supersymmetry transformation rules require the gravitino to be complex, implying that the theory, which we refer to as the O(1, 1) theory, is actually a "fake supergravity." In Sec. III, we find domain-wall solutions supported by the auxiliary scalars and/or the vector, and we study their supersymmetry. In the process, we obtain all the supersymmetric anti-de Sitter (AdS) vacua. Both supersymmetric singular domain walls and wormholes can arise in these higher-order off-shell supergravities.

In Sec. IV we consider Lifshitz solutions and list all possible homogeneous Lifshitz vacua utilizing scalar and/or vector auxiliary fields. We find that Killing spinors can arise for suitable choices of parameters in the Lifshitz solutions, but only in the case of the analytically continued O(1, 1) theory. These solutions are therefore pseudosupersymmetric in the O(1, 1) fake supergravity. In Sec. V, we obtain pseudosupersymmetric asymptotically Lifshitz solutions in the O(1, 1) theory. In Sec. VI, we consider homogeneous gyrating Schrödinger vacua in both the U(1) and the O(1, 1) theories, and we tabulate the general solutions. We then look for parameter choices giving rise to Killing spinors, and find that these arise only in the U(1) theory, implying the existence of supersymmetric gyrating solutions. We extend the discussion in Sec. VII, to consider a more general class of gyrating pp-wave solutions, amongst which we find a large class of supersymmetric solutions. The paper ends with conclusions in Sec. VIII.

II. $\mathcal{N} = 1$, $D = 4$ OFF-SHELL SUPERGRAVITY

The field content of off-shell $\mathcal{N} = 1$, $D = 4$ supergravity comprises the metric e^a_μ, a massive vector A_μ, and a complex scalar $\mathcal{M} = S + \mathrm{i}P$, totalling 12 off-shell degrees of freedom, matching with that of the off-shell gravitino ψ_μ. The general formalism for constructing a supersymmetric action for any chiral superfield was presented in Ref. [10]. For appropriate choices of superfields, one obtains the actions of the supersymmetrizations of the cosmological term, the Einstein-Hilbert term, and higher-order curvature terms.

The supersymmetrization of the Einstein-Hilbert term was obtained in Refs. [1,2]. In this theory, the complex scalar and the massive vector are both auxiliary, with purely algebraic equations of motion. These fields can be integrated out, giving rise to standard on-shell $\mathcal{N} = 1$, $D = 4$ supergravity. One defining property of off-shell supergravity is that the supersymmetry transformation rules close without needing to make use of the equations of motion. This implies that one may construct new theories by adding additional super-invariants, which can in general involve higher-derivatives, without any modification to the supersymmetry transformation rules. It turns out that there are two quadratic-curvature super-invariants: one is the Weyl-squared super-invariant and the other is the R^2 super-invariant [11]. In the former case, the scalars S and P remain auxiliary whilst the vector A_μ acquires a kinetic term. In the latter case, the scalars acquire derivative terms as well. (Note that we shall continue to refer to $S + \mathrm{i}P$ and A_μ as auxiliary fields, even though they start to propagate after the higher-order super-invariants are included.) For our purposes, we shall present only the bosonic Lagrangian and the supersymmetry transformation rule for the gravitino.

A. The U(1) supergravity theory

Together with the supersymmetrization of the cosmological term [9], the bosonic action is given by

$$I = \int d^4x\sqrt{-g}\Big(\sigma\mathcal{L}_0 + \lambda\mathcal{L}_S + \frac{1}{2}\alpha\mathcal{L}_C + \frac{1}{2}\beta\mathcal{L}_{R^2}\Big), \tag{2.1}$$

where σ, λ, α, and β are constants, and

$$\begin{aligned}
\mathcal{L}_0 &= R - \frac{2}{3}(\mathcal{M}\bar{\mathcal{M}} - A^2), \qquad \mathcal{M} = S + \mathrm{i}P,\\
\mathcal{L}_S &= \mathcal{M} + \bar{\mathcal{M}}, \qquad \mathcal{L}_C = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} - \tfrac{2}{3}F^2,\\
\mathcal{L}_{R^2} &= R^2 + \frac{4}{3}\Big(A^2 + \frac{1}{2}\mathcal{M}\bar{\mathcal{M}}\Big)R + 4(\nabla_\mu A^\mu)^2\\
&\quad - 4\partial_\mu\mathcal{M}\partial^\mu\bar{\mathcal{M}} - \frac{4}{3}\,\mathrm{i}A^\mu(\bar{\mathcal{M}}\partial_\mu\mathcal{M} - \mathcal{M}\partial_\mu\bar{\mathcal{M}})\\
&\quad + \frac{4}{9}(\mathcal{M}^2\bar{\mathcal{M}}^2 + \mathcal{M}\bar{\mathcal{M}}A^2 + A^4)\\
&= R^2 + \frac{4}{3}\Big(A^2 + \frac{1}{2}\mathcal{M}\bar{\mathcal{M}}\Big)R + 4(\nabla_\mu A^\mu)^2\\
&\quad - 4|D_\mu\mathcal{M}|^2 + \frac{4}{9}(|\mathcal{M}|^2 + A_\mu A^\mu)^2,
\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}
&F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad A^2 = A^\mu A_\mu,\\
&F^2 = F^{\mu\nu}F_{\mu\nu}, \qquad D_\mu\mathcal{M} = \partial_\mu\mathcal{M} - \frac{1}{3}\,\mathrm{i}A_\mu\mathcal{M},\\
&D_\mu\bar{\mathcal{M}} = \partial_\mu\bar{\mathcal{M}} + \frac{1}{3}\,\mathrm{i}A_\mu\bar{\mathcal{M}}.
\end{aligned} \tag{2.3}$$

The $\mathcal{L}_C$ and $\mathcal{L}_{R^2}$ super-invariants were given in Ref. [11]. The constant σ in general can be set to 1 by appropriate scalings. However, it is convenient here to allow it to remain arbitrary, to emphasize that the terms $\mathcal{L}_0$, $\mathcal{L}_S$, $\mathcal{L}_C$, and $\mathcal{L}_{R^2}$ are independent super-invariants, and each can be turned on or off independently. The supersymmetry transformation rule for the gravitino is universal, and given by

$$\delta\psi_\mu = -D_\mu\epsilon - \frac{i}{6}(2A_\mu - \Gamma_{\mu\nu}A^\nu)\Gamma_5\epsilon - \frac{1}{6}\Gamma_\mu(S + i\Gamma_5 P)\epsilon. \quad (2.4)$$

The equation of motion for the complex scalar $\mathcal{M}$ is given by

$$-\frac{2}{3}\sigma\mathcal{M} + \lambda + \frac{1}{2}\beta\Big(\frac{2}{3}\mathcal{M}R + 4\Box\mathcal{M} - \frac{4}{3}i(2A^\mu\partial_\mu\mathcal{M} + \mathcal{M}\nabla_\mu A^\mu) + \frac{4}{9}\mathcal{M}(2\mathcal{M}\bar{\mathcal{M}} + A_\mu A^\mu)\Big) = 0, \quad (2.5)$$

and the equation of motion for the vector A_μ is given by

$$0 = \frac{2}{3}\alpha\nabla_\mu F^{\mu\nu} + \frac{2}{3}\sigma A^\nu + \beta\Big(\frac{2}{3}RA^\nu - 2\nabla^\nu(\nabla_\mu A^\mu) - \frac{1}{3}i(\bar{\mathcal{M}}\nabla^\nu\mathcal{M} - \mathcal{M}\nabla^\nu\bar{\mathcal{M}}) + \frac{2}{9}\mathcal{M}\bar{\mathcal{M}}A^\nu + \frac{4}{9}A^2A^\nu\Big). \quad (2.6)$$

The Einstein equation of motion is

$$\sigma E^0_{\mu\nu} + \lambda E^S_{\mu\nu} + \alpha E^C_{\mu\nu} + \beta E^{R^2}_{\mu\nu} = 0, \quad (2.7)$$

where

$$E^0_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \frac{1}{3}g_{\mu\nu}\mathcal{M}\bar{\mathcal{M}}g_{\mu\nu} + \frac{2}{3}\Big(A_\mu A_\nu - \frac{1}{2}A^2 g_{\mu\nu}\Big), \qquad E^S_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}(\mathcal{M} + \bar{\mathcal{M}}),$$

$$E^C_{\mu\nu} = -(2\nabla^\rho\nabla^\sigma + R^{\rho\sigma})C_{\mu\rho\sigma\nu} - \frac{2}{3}\Big(F^2_{\mu\nu} - \frac{1}{4}F^2 g_{\mu\nu}\Big),$$

$$\begin{aligned}E^{R^2}_{\mu\nu} = {}& 2RR_{\mu\nu} - 2\nabla_\mu\nabla_\nu R + 2\Box R g_{\mu\nu} + \frac{4}{3}A_\mu A_\nu R + \frac{4}{3}(R_{\mu\nu} - \nabla_\mu\nabla_\nu \\ &+ g_{\mu\nu}\Box)\Big(A^2 + \frac{1}{2}\mathcal{M}\bar{\mathcal{M}}\Big) + 4g_{\mu\nu}\nabla_\rho(A^\rho\nabla_\sigma A^\sigma) - 8A_{(\mu}\nabla_{\nu)}\nabla_\rho A^\rho - 4D_{(\mu}\mathcal{M}D_{\nu)}\bar{\mathcal{M}} + \frac{8}{9}A_\mu A_\nu(\mathcal{M}\bar{\mathcal{M}} + A^2) \\ &- \frac{1}{2}g_{\mu\nu}\Big[R^2 + \frac{4}{3}\Big(A^2 + \frac{1}{2}\mathcal{M}\bar{\mathcal{M}}\Big)R + 4(\nabla_\rho A^\rho)^2 - 4D_\rho\mathcal{M}D^\rho\bar{\mathcal{M}} + \frac{4}{9}(\mathcal{M}\bar{\mathcal{M}} + A^2)^2\Big].\end{aligned} \quad (2.8)$$

Note that the derivatives of the complex scalar M that appear in the action, $D_\mu\mathcal{M}$ and $D_\mu\bar{\mathcal{M}}$, are defined in Eq. (2.3). Thus the complex scalar can be viewed as being "charged" under the U(1) vector. Furthermore, if we set $\lambda = 0$ the complex scalar has a U(1) global symmetry, $\mathcal{M} \to e^{i\theta}\mathcal{M}$. For this reason, we shall refer to this action as the U(1) theory.

B. The O(1, 1) "fake supergravity" theory

If we perform the field redefinitions

$$A_\mu = i\tilde{A}_\mu, \qquad P = -i\tilde{P}, \quad (2.9)$$

where $\tilde{A}_\mu$ and $\tilde{P}$ are taken to be real, the bosonic Lagrangian remains real. We now have

$$\mathcal{L}_0 = R - \frac{2}{3}(\mathcal{M}\check{\mathcal{M}} + \tilde{A}^2), \quad \mathcal{M} = S + \tilde{P}, \quad \check{M} = S - \tilde{P}, \quad \mathcal{L}_S = \mathcal{M} + \check{\mathcal{M}}, \quad \mathcal{L}_C = C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{2}{3}F^2,$$

$$\begin{aligned}\mathcal{L}_{R^2} &= R^2 + \frac{4}{3}\Big(\frac{1}{2}\mathcal{M}\check{\mathcal{M}} - \tilde{A}^2\Big)R - 4(\nabla_\mu\tilde{A}^\mu)^2 - 4\partial_\mu\mathcal{M}\partial^\mu\check{\mathcal{M}} + \frac{4}{3}\tilde{A}^\mu(\check{\mathcal{M}}\partial_\mu\mathcal{M} - \mathcal{M}\partial_\mu\check{\mathcal{M}}) + \frac{4}{9}(\mathcal{M}^2\check{\mathcal{M}}^2 - \mathcal{M}\check{\mathcal{M}}\tilde{A}^2 + \tilde{A}^4) \\ &= R^2 + \frac{4}{3}\Big(\frac{1}{2}\mathcal{M}\check{\mathcal{M}} - \tilde{A}^2\Big)R - 4(\nabla_\mu\tilde{A}^\mu)^2 - 4D_\mu\mathcal{M}D_\mu\check{\mathcal{M}} + \frac{4}{9}(\mathcal{M}\check{\mathcal{M}} - \tilde{A}^2)^2,\end{aligned} \quad (2.10)$$

where

$$D_\mu\mathcal{M} = \partial_\mu\mathcal{M} + \frac{1}{3}\tilde{A}_\mu\mathcal{M}, \qquad D_\mu\check{\mathcal{M}} = \partial_\mu\check{\mathcal{M}} - \frac{1}{3}\tilde{A}_\mu\check{\mathcal{M}}, \quad (2.11)$$

and σ, λ, α, and β are constants. Thus we see that the scalars are gauged in the original Weyl sense. The Lagrangian, if we set $\lambda = 0$, is invariant under an O(1, 1) global symmetry that acts as a boost on the scalars S and $\tilde{P}$. We shall refer to this theory as the O(1, 1) theory.

The analytic continuation of the A_μ and P fields can be thought of as a choice of a different "real section" of the complexification of the original theory. The process of complexifying a supergravity theory was discussed in detail in Ref. [12]. If one first writes the theory in terms of purely holomorphic functions of the original real variables (in particular, in the fermionic sector, in terms of Majorana spinors with all conjugations being performed using the Majorana rather than the Dirac conjugate), then almost trivially the theory remains supersymmetric if all the real fields are now allowed to become complex. Of course, the action will now be complex also, and the numbers of bosonic and fermionic degrees of freedom will be doubled. The question then arises as to whether there exist alternative possibilities for choosing real sections, by imposing appropriate conjugation conditions on all the fields, such that one again obtains a real action and a consistent set of supersymmetry transformation rules for a genuine supergravity theory. Finding a consistent choice of conjugation conditions on the bosonic fields that results again in a real bosonic action is a necessary part of this procedure. If one can at the same time also impose a set of conjugation conditions on the fermionic fields such that their action is real and the supersymmetry transformations are consistent with the conjugation properties, then one has arrived at a genuine supergravity. If, on the other hand, it is not possible to impose such conjugation conditions on the fermions, then one has instead arrived at a ''fake supergravity,'' meaning in particular that the fermions are necessarily complex rather than being purely real or purely imaginary.

In the present case of interest, it turns out that having imposed our conjugation conditions $A^*_\mu = -A_\mu$ and $P^* = -P$ on the original, but now complexified, A_μ and P fields, it is not possible to find a consistent choice of conjugation section of the complexified fermion fields that halves their degrees of freedom again. They must necessarily remain complex, and so the O(1, 1) theory is therefore a ''fake supergravity.'' It is still of interest, however, since it provides us with a real bosonic theory that admits real bosonic ''pseudosupersymmetric'' solutions that obey first-order equations following from the requirement of the existence of complex pseudo-Killing spinors.

The pseudosupersymmetry transformation rule for the off-shell gravitino is now given by

$$\delta\psi_\mu = -D_\mu\epsilon + \frac{1}{6}(2\tilde{A}_\mu - \Gamma_{\mu\nu}\tilde{A}^\nu)\Gamma_5\epsilon - \frac{1}{6}\Gamma_\mu(S + \Gamma_5\tilde{P})\epsilon, \tag{2.12}$$

and the scalar equations of motion (2.5) become

$$\begin{aligned} -\frac{2}{3}\sigma S + \lambda + \frac{1}{2}\beta\bigg[\frac{2}{3}SR + 4\Box S + \frac{4}{3}(2\tilde{A}^\mu\partial_\mu\tilde{P} + \tilde{P}\nabla^\mu\tilde{A}_\mu) \\ + \frac{4}{9}S(2(S^2 - \tilde{P}^2) - \tilde{A}^2)\bigg) = 0, \\ -\frac{2}{3}\sigma\tilde{P} + \frac{1}{2}\beta\bigg[\frac{2}{3}\tilde{P}R + 4\Box\tilde{P} + \frac{4}{3}(2\tilde{A}^\mu\partial_\mu S + S\nabla^\mu\tilde{A}_\mu) \\ + \frac{4}{9}\tilde{P}(2(S^2 - \tilde{P}^2) - \tilde{A}^2)\bigg) = 0. \end{aligned} \tag{2.13}$$

Since many of the solutions that we shall obtain arise (with minor differences as noted) in both the U(1) supergravity theory and in the O(1, 1) fake supergravity theory, we shall sometimes use the generic term ''supersymmetric'' for both cases. It should always be understood that in the case of the O(1, 1) theory the solutions are actually pseudosupersymmetric rather than truly supersymmetric.

C. AdS$_4$ vacua

There may exist several AdS$_4$ vacua in which $A_\mu = 0$ and $\mathcal{M}$ is a constant. For the U(1) theory, the scalar equations of motion imply

$$\left(\lambda - \frac{2}{3}\sigma\mathcal{M}\right) + \frac{4}{3}\beta\mathcal{M}\left(\Lambda + \frac{1}{3}\mathcal{M}\bar{\mathcal{M}}\right) = 0, \tag{2.14}$$

and so the constant $\mathcal{M}$ must be real, i.e., $P = 0$. The equation is a cubic polynomial in S, and so it has at least one real solution, with the possibility of three real solutions. As we shall see later, the supersymmetric AdS$_4$ vacuum has $\Lambda = -\frac{1}{3}S^2$.

The AdS$_4$ solution with $P = 0 = A_\mu$ is also a solution in the O(1, 1) theory. In that theory, however, there also exists a vacuum solution in which $\tilde{P}$ is nonvanishing, provided that $\lambda = 0$. A supersymmetric AdS$_4$ can also arise in this case, which we shall discuss in the next section.

III. SUPERSYMMETRIC DOMAIN WALLS (MEMBRANES)

In this section, we construct supersymmetric domain-wall solutions. The ansatz is given by

$$\begin{aligned} ds^2 = dr^2 + a(r)^2 dx^\mu dx_\mu, \qquad A = \phi(r)dr, \\ S = S(r), \qquad P = P(r). \end{aligned} \tag{3.1}$$

Note that if A_μ were a massless gauge field, it would be pure gauge. However, since A_μ is a massive field, the ansatz is nontrivial. A natural choice for the vielbein is $e^{\bar{r}} = dr$, $e^{\bar{\mu}} = a dx^\mu$. The only nonvanishing components of the corresponding spin connection are then given by $\omega^{\bar{\mu}}{}_{\bar{r}} = (a'/a)e^{\bar{\mu}}$, where a prime denotes a derivative with respect to r. For the U(1) theory, the Killing spinor equations become

$$\begin{aligned} \partial_r\epsilon + \frac{\mathrm{i}}{3}\phi\epsilon + \frac{1}{6}\Gamma_r(S + \mathrm{i}\Gamma_5 P)\epsilon = 0, \\ \partial_\mu\epsilon + \left(\frac{1}{2}a' - \frac{\mathrm{i}}{6}a\phi\right)\Gamma_{\bar{\mu}\bar{r}}\epsilon + \frac{1}{6}a\Gamma_{\bar{\mu}}(S + \mathrm{i}\Gamma_5 P)\epsilon = 0. \end{aligned} \tag{3.2}$$

A. Domain wall with a scalar potential

Let us first consider $\phi = 0 = P$, which applies for both the U(1) and O(1, 1) theories. The solution is supersymmetric provided that

$$S = \frac{3a'}{a}. \tag{3.3}$$

The corresponding Killing spinor is subject to the projection

$$(\Gamma_{\bar{r}} + 1)\epsilon = 0. \tag{3.4}$$

We find that all the equations of motion then reduce to

$$\lambda a^2 - 2\sigma a a' + 6\beta(a a''' - a' a'') = 0. \tag{3.5}$$

If $\sigma = 0 = \lambda$, then the equations of motion simply reduce to

$$a a''' - a' a'' = 0, \tag{3.6}$$

for which the general solution is given by

$$\begin{aligned} a &= a_1 \cosh kr + a_2 \sinh kr \quad \text{or} \\ a &= \tilde{a}_1 \cos kr + \tilde{a}_2 \sin kr. \end{aligned} \tag{3.7}$$

The second choice gives a solution with a naked power-law singularity and we shall not consider it further. For the first choice, we find that not only do AdS_4 vacua with an arbitrary cosmological constant arise, but AdS_4 wormholes can arise also.

If both $\sigma \neq 0$ and $\lambda \neq 0$, then the vacuum solution is AdS_4 with $a = \exp(\lambda/(2\sigma))$. If $\lambda = 0$ but $\sigma \neq 0$, the vacuum solution is Minkowski spacetime with a being constant. If $\sigma = 0$ and $\lambda \neq 0$, neither AdS_4 nor Minkowski spacetime is a solution.

B. Domain wall with $A_\mu \neq 0$

We now consider the case with nonvanishing ϕ and P. The existence of a Killing spinor implies

$$\begin{aligned} &\text{U(1) theory:}\quad \frac{1}{2}\Big(a' - \frac{\mathrm{i}}{3} a\phi\Big)^2 - \frac{1}{18}(S^2 + P^2) = 0, \\ &\text{O(1, 1) theory:}\quad \frac{1}{2}\Big(a' + \frac{1}{3} a\tilde{\phi}\Big)^2 - \frac{1}{18}(S^2 - \tilde{P}^2) = 0, \end{aligned} \tag{3.8}$$

where in the O(1, 1) theory the ansatz for the vector field becomes $\tilde{A} = \tilde{\phi} dr$. We see that for the U(1) theory, the solution cannot be real if ϕ and P are nonvanishing. This reality problem is resolved in the O(1, 1) theory. Substituting the supersymmetry condition into the bosonic equations of motion, we find that if we set $S = \tilde{P}$, the equations are reduced to

$$\sigma a a' - 3\beta(5 a' a'' + a a''') = 0. \tag{3.9}$$

Note that as mentioned in Sec. II, turning on $\tilde{P}$ means we must have $\lambda = 0$. The function $S = \tilde{P}$ is determined by

$$\sigma a S - 3\beta(5 a' S' + a S'') = 0. \tag{3.10}$$

Note that there is no back reaction of the scalars on the metric, and hence the domain wall is supported by the vector field alone.

It is clear from Eq. (3.9) that Minkowski spacetime is a vacuum solution. It also admits an AdS solution with $a = e^{kr}$, where

$$k^2 = \frac{\sigma}{18\beta}. \tag{3.11}$$

IV. LIFSHITZ SOLUTIONS AND THEIR (PSEUDO)SUPERSYMMETRY

In this section we study Lifshitz solutions following from the ansatz

$$\begin{aligned} ds^2 &= \ell^2\Big(\frac{dr^2}{r^2} - r^{2z} dt^2 + r^2(dx^2 + dy^2)\Big), \\ A &= q r^z dt + p\frac{dr}{r}, \end{aligned} \tag{4.1}$$

in the U(1) theory, where p, q and the scalars S and P are constants. Note that since A_μ is massive, with no gauge symmetry, the p term is nontrivial even though it is exact. In the O(1, 1) theory the ansatz for the vector field becomes

$$\tilde{A} = \tilde{q} r^z dt + \tilde{p}\frac{dr}{r}, \tag{4.2}$$

where A is now written as $A = \mathrm{i}\tilde{A}$ with $\tilde{A}$, and hence $\tilde{q}$ and $\tilde{p}$, being real.

Lifshitz solutions were proposed in Ref. [13] as gravity duals for nonrelativistic field theories. (See also Ref. [14].) Although Lifshitz solutions can be embedded in string theories and supergravities [15–22], supersymmetric Lifshitz solutions are rare. Lifshitz solutions arise naturally in higher-derivative gravities. It was shown in Ref. [23] that not only the homogeneous Lifshitz vacua, but also asymptotically Lifshitz black holes can arise in Einstein-Weyl gravity.

A. List of solutions

1. Solutions with $A_\mu = 0$

There are two classes of solutions with $A_\mu = 0$. The first is when $P = 0$, for which

$$\begin{aligned} \lambda &= \frac{2S}{9\ell^2}(2\alpha z(z-4) + 3\beta(3(z^2+2z+3) - \ell^2 S^2)), \\ \sigma &= \frac{1}{3\ell^2}(2\alpha z(z-4) + \beta(6(z^2+2z+3) - \ell^2 S^2)), \\ S^2 &= \frac{3(z^2+2z+3)}{2\ell^2}, \quad \text{or} \\ S^2 &= \frac{\alpha z(z-4) + 3\beta(z^2+2z+3)}{2\beta\ell^2}. \end{aligned} \tag{4.3}$$

The second class is when $P \neq 0$. This implies that we must have $\lambda = 0$. There are then two solutions:

$$\sigma = \frac{\alpha z(z-4)}{3\ell^2}, \qquad \tilde{P}^2 - S^2 = \frac{3(z^2+2z+3)}{\ell^2},$$
$$\beta = -\frac{\alpha z(z-4)}{9(z^2+2z+3)}; \quad \text{or} \quad \sigma = 0,$$
$$S^2 + P^2 = \frac{3(z^2+2z+3)}{2\ell^2}, \qquad \beta = -\frac{4\alpha z(z-4)}{9(z^2+2z+3)}. \tag{4.4}$$

Note that the first solution arises only for the O(1, 1) theory. The second solution, which is presented for the U(1) theory, can also be a solution in the O(1, 1) theory provided that $S^2 + P^2$—which then becomes $S^2 - \tilde{P}^2$—is non-negative. For $z(z-4) = 0$, we must have $\beta = 0$. This implies that $P = 0$, and hence the solution reduces to a special case of Eq. (4.3).

Next, we shall consider solutions with nonvanishing A_μ. We find that the reality of the solution typically tends to select the O(1, 1) rather than the U(1) theory.

2. $A_\mu \neq 0$ and $\alpha \neq 0$

For nonvanishing α, we find that the equations of motion imply that either $p = 0$ or $q = 0$. We then find solutions as follows. First, we can take $p = 0 = P$, with q nonvanishing. We find

$$\tilde{q} = z - 1, \qquad S = \frac{z+2}{\ell},$$
$$\sigma\ell^2 = \beta(z+2)^2 - 2\alpha z, \qquad \lambda = \frac{2\sigma(z+2)}{3\ell}; \tag{4.5}$$

$$\tilde{q} = z - 1, \qquad \sigma = \beta S^2,$$
$$\lambda = \frac{2\beta S(\ell^2 S^2 + 2(z+2)^2)}{9\ell^2}, \tag{4.6}$$
$$3\alpha z + 2\beta((z+2)^2 - \ell^2 S^2) = 0;$$

$$\tilde{q}^2 = (3(z^2+2z+3) - 2\ell^2 S^2), \qquad \sigma = \beta S^2,$$
$$\lambda = \frac{2}{3}\beta S^3, \qquad z\alpha = 0. \tag{4.7}$$

Note that of the above three solutions, the first two are for the O(1, 1) theory, with the ansatz for the vector now taking the form Eq. (4.2). The third solution, with $\alpha = 0$, which is presented for the O(1, 1) theory, could also be real in the U(1) theory if the right-hand side of the expression for $\tilde{q}^2$ were negative.

If instead $q = 0 = P$, we find that there is a solution in the O(1, 1) theory, given by

$$\tilde{p} = 9, \qquad z = 4, \qquad \sigma = \frac{108\beta}{\ell^2}, \qquad \lambda = 0, \qquad S = 0. \tag{4.8}$$

Now we consider the case with nonvanishing P. For this, we find that the equations of motion always require that $\lambda = 0$. For $p = 0$, we find two solutions in the O(1, 1) theory:

$$\tilde{q} = z - 1, \qquad S^2 - \tilde{P}^2 = \frac{(z+2)^2}{\ell^2},$$
$$\sigma = 0, \qquad 2\alpha z = \beta(z+2)^2; \tag{4.9}$$

$$\tilde{q} = z - 1, \qquad S^2 - \tilde{P}^2 = -\frac{2(z+2)^2}{\ell^2},$$
$$\sigma\ell^2 = -2\beta(z+2)^2, \qquad \alpha z = 2\beta(z+2)^2. \tag{4.10}$$

For $\tilde{q} = 0$, we find a solution in the O(1, 1) theory, given by

$$z = 4, \qquad \sigma = \frac{\beta(\tilde{p}^2 + 6\tilde{p} + 81)}{2\ell^2},$$
$$\lambda = \frac{2\beta\tilde{p}(\tilde{p}-9)^2}{\tilde{P}\ell^4}, \qquad S = \frac{(\tilde{p}+9)^2}{12\tilde{p}}\tilde{P}, \tag{4.11}$$
$$\tilde{P} = \pm\frac{6\sqrt{2}\tilde{p}(\tilde{p}-9)}{\sqrt{(\tilde{p}+3)(\tilde{p}+27)(\tilde{p}^2+6\tilde{p}+81)}}.$$

3. $A_\mu \neq 0$ and $\alpha = 0$

In this case, we find solutions in the O(1, 1) theory with both $\tilde{p}$ and $\tilde{q}$ nonvanishing, given by

$$\sigma = \frac{\beta\tilde{p}(z+2)(S \pm \tilde{P})}{\ell^2\tilde{P}}, \qquad \lambda = \frac{2\beta\tilde{p}(z+2)(S^2 - \tilde{P}^2)}{3\ell^2\tilde{P}},$$
$$\tilde{q}^2 - \tilde{p}^2 = 3(z^2+2z+3) - \frac{\tilde{p}(z+2)(2S \pm 3\tilde{P})}{\tilde{P}},$$
$$S^2 - \tilde{P}^2 - \frac{\tilde{p}(z+2)(S \pm 3\tilde{P})}{\ell^2\tilde{P}} = 0. \tag{4.12}$$

A special case arises if $S = \tilde{P}$, implying $\lambda = \sigma = 0$ and $\tilde{q}^2 = 3(z^2+2z+3)$.

B. (Pseudo)supersymmetry analysis

Having obtained a variety of Lifshitz solutions in quadratic-curvature supergravity, we now examine their (pseudo)supersymmetry. Since they arise mostly in the O(1, 1) theory, we shall present the analysis within this framework. For simplicity, and without loss of generality, we shall set $\ell = 1$. A natural choice for the vielbein is given by

$$e^{\hat{0}} = r^z dt, \quad e^{\hat{x}} = r dx, \quad e^{\hat{y}} = r dy, \quad e^{\hat{r}} = \frac{dr}{r}, \tag{4.13}$$

where we use hats to denote tangent-space indices. The nonvanishing components of the corresponding torsion-free spin connection are then given by

$$\omega^{\hat{0}}{}_{\hat{r}} = z e^{\hat{0}}, \qquad \omega^{\hat{x}}{}_{\hat{r}} = e^{\hat{x}}, \qquad \omega^{\hat{y}}{}_{\hat{r}} = e^{\hat{y}}. \tag{4.14}$$

The Killing spinor equations are

$$\partial_t\epsilon+\frac{1}{2}zr^z\Gamma_{0\hat{r}}-\frac{1}{6}r^z(2\tilde{q}-\tilde{p}\Gamma_{0\hat{r}})\Gamma_5\epsilon+\frac{1}{6}r^z\Gamma_{\hat{0}}(S+\tilde{P}\Gamma_5)=0,$$
$$\partial_i\epsilon+\frac{1}{2}r\Gamma_{\hat{i}\hat{r}}\epsilon+\frac{1}{6}r(\tilde{p}\Gamma_{\hat{i}\hat{r}}-\tilde{q}\Gamma_{\hat{i}\hat{0}})\Gamma_5\epsilon+\frac{1}{6}r\Gamma_{\hat{i}}(S+\tilde{P}\Gamma_5)\epsilon=0,$$
$$\partial_r\epsilon-\frac{1}{6}r^{-1}(2\tilde{p}+\tilde{q}\Gamma_{\hat{r}\hat{0}})\Gamma_5\epsilon+\frac{1}{6}r^{-1}(S+\tilde{P}\Gamma_5)\epsilon=0, \quad (4.15)$$

where $i = x, y$.

To establish the supersymmetry of a solution, one need only demonstrate the existence of a Killing spinor, without necessarily solving for it explicitly. This can be done by examining the integrability conditions. We find that

$$0=[\partial_x,\partial_y]\epsilon=\Gamma_{xy}U_{xy}\epsilon,\quad 0=[\partial_r,\partial_i]\epsilon=\Gamma_{ri}U_{ri}\epsilon,$$
$$0=[\partial_t,\partial_i]\epsilon=r^{z+1}\Gamma_{0i}U_{ti}\epsilon,\quad 0=[\partial_r,\partial_t]\epsilon=r^{z-1}\Gamma_{r0}U_{rt}\epsilon, \quad (4.16)$$

where

$$U_{xy}=\frac{1}{18}r^2(9+\tilde{p}^2-\tilde{q}^2-S^2+\tilde{P}^2)+\frac{1}{9}r^2(3\tilde{p}+(\tilde{q}\Gamma_0-\tilde{p}\Gamma_r)(S+\tilde{P}\Gamma_5))\Gamma_5,$$
$$U_{ri}=\frac{1}{18}(9-\tilde{q}^2-S^2+\tilde{P}^2)+\frac{1}{6}\tilde{p}\Gamma_5-\frac{1}{18}\tilde{p}\,\tilde{q}\,\Gamma_{0r}+\frac{1}{9}(\tilde{q}\Gamma_0+\tilde{p}\Gamma_r)(S+\tilde{P}\Gamma_5)\Gamma_5,$$
$$U_{ti}=\frac{1}{18}(9z+\tilde{p}^2-S^2+\tilde{P}^2)-\frac{1}{9}(\tilde{q}\Gamma_0+\tilde{p}\Gamma_r)(S+\tilde{P}\Gamma_5)\Gamma_5+\frac{1}{6}((z+1)\tilde{p}+z\tilde{q}\Gamma_{0r})\Gamma_5-\frac{1}{18}\tilde{p}\,\tilde{q}\,\Gamma_{0r},$$
$$U_{rt}=\frac{1}{18}(9z^2-S^2+\tilde{P}^2)-\frac{1}{9}(\tilde{q}\Gamma_0-\tilde{p}\Gamma_r)(S+\tilde{P}\Gamma_5)\Gamma_5-\frac{z}{6}(\tilde{p}+2\tilde{q}\Gamma_{0r})\Gamma_5. \quad (4.17)$$

It is now straightforward to verify whether the Lifshitz solutions we have obtained are (pseudo)supersymmetric or not. For the $A_\mu = 0$ solutions, we find from the integrability conditions that the only supersymmetric solution is the maximally supersymmetric AdS_4 vacuum. In what follows, we shall enumerate the supersymmetric solutions with nonvanishing A_μ.

Let us first consider $\tilde{p} = 0$ and $\tilde{P} = 0$. We find that the (pseudo)supersymmetric solutions in general satisfy

$$\tilde{q} = z - 1, \qquad S = z + 2. \quad (4.18)$$

The Killing spinor satisfies the projections

$$\Gamma_0\Gamma_5\epsilon - \epsilon = 0, \qquad \Gamma_r\epsilon + \epsilon = 0, \quad (4.19)$$

and so in general the solution preserves $\frac{1}{4}$ of the (pseudo) supersymmetry. It is clear that such a Lifshitz solution does exist, given by Eq. (4.5). There are two cases where a supersymmetry enhancement occurs. We find that when $z = -2$ or $z = 0$, the fraction of preserved supersymmetry is doubled to $\frac{1}{2}$, with now only the single projection given by

$$z=-2:\ \Gamma^0\Gamma_5\epsilon - \Gamma_r\epsilon = 0, \qquad z = 0:\ \Gamma^0\Gamma_5\epsilon + \epsilon = 0. \quad (4.20)$$

Interestingly, there is a maximally (pseudo)supersymmetric solution that is not AdS_4. It is given by Eq. (4.7) with $z = 0$ and $\tilde{q} = -3$, and hence $S = 0$. Thus we have $\sigma = 0$, and so the theory itself is constructed from only quadratic super-invariants. The four Killing spinors can be solved explicitly, and are given by

$$\epsilon = \Big(1-\frac{1}{2}r(x\Gamma_x+y\Gamma_y)(\Gamma_r+\Gamma_0\Gamma_5)\Big)\eta,$$
$$\eta = \frac{e^t}{\sqrt{r}}\eta_1^+ + e^{-t}\sqrt{r}\eta_1^- + e^t\sqrt{r}\eta_2^+ + \frac{e^{-t}}{\sqrt{r}}\eta_2^-, \quad (4.21)$$

where $\eta_i^\pm$ are four constant spinors satisfying

$$(\Gamma_5 \pm 1)\eta_i^\pm = 0, \qquad (\Gamma_{01}-1)\eta_1^\pm = 0, \qquad (\Gamma_{01}+1)\eta_2^\pm = 0. \quad (4.22)$$

Finally, we find that the solution (4.9) also preserves $\frac{1}{4}$ of the (pseudo)supersymmetry. The Killing spinors are subject to the constraints

$$(S+\tilde{P}\Gamma_5+(z+2)\Gamma_r)\epsilon = 0, \qquad (\Gamma_0\Gamma_5+\Gamma_r)\epsilon = 0. \quad (4.23)$$

It is clear that this projection reduces to Eq. (4.19) when $\tilde{P} = 0$. However, we nevertheless treat these as two separate classes of solutions since turning on $\tilde{P}$ will force $\lambda = 0$ in the bosonic equations of motion.

Thus we have obtained all the (pseudo)supersymmetric Lifshitz solutions in the off-shell $\mathcal{N} = 1$ supergravities that use both the quadratic super-invariants. The (pseudo) supersymmetric Lifshitz solutions in Einstein-Weyl supergravity was obtained in Ref. [5].

V. (PSEUDO)SUPERSYMMETRIC T^2-SYMMETRIC SOLUTIONS

In this section, we construct pseudosupersymmetric T^2-symmetric solutions in the O(1, 1) theory. The ansatz is given by

$$ds^2=\frac{dr^2}{f^2}-a^2dt^2+r^2(dx^2+dy^2), \quad \tilde{A}=\phi dt. \quad (5.1)$$

This ansatz encompasses all the Lifshitz solutions we obtained in the previous section that have $\tilde{p} = 0$. We shall not include a term $\tilde{\psi}(r)dr$ in the ansatz for the vector field $\tilde{A}_\mu$ here, because in this section we shall concentrate only on the pseudosupersymmetric T^2-symmetric solutions. As we have seen in the previous section, there is no (pseudo)supersymmetric Lifshitz solution that has nonvanishing $\tilde{p}$.

A. (Pseudo)supersymmetry conditions and equations of motion

As in Ref. [5], the vielbein and the corresponding spin connection are given by

$$e^{\hat{r}} = f^{-1}dr, \qquad e^{\hat{0}} = adt, \qquad e^{\hat{x}} = rdx, \qquad e^{\hat{y}} = rdy,$$
$$\omega^{\hat{0}}{}_{\hat{r}} = \frac{a'f}{a}e^{\hat{0}}, \qquad \omega^{\hat{i}}{}_{\hat{r}} = \frac{f}{r}e^{\hat{i}}, \qquad (i = x, y), \tag{5.2}$$

where a prime denotes a derivative with respect to r. The Killing spinor equations are given by

$$\left(\partial_t + \frac{1}{2}a'f\Gamma_{\hat{0}\hat{r}} - \frac{1}{3}\phi\Gamma_5 + \frac{1}{6}a\Gamma_{\hat{0}}(S + \tilde{P}\Gamma_5)\right)\epsilon = 0,$$
$$\left(\partial_r + \frac{\phi}{6af}\Gamma_{\hat{0}\hat{r}}\Gamma_5 + \frac{1}{6f}\Gamma_r(S + \tilde{P}\Gamma_5)\right)\epsilon = 0, \tag{5.3}$$
$$\left(\partial_i + \frac{1}{2}f\Gamma_{\hat{i}\hat{r}} + \frac{r\phi}{6a}\Gamma_{\hat{0}\hat{i}}\Gamma_5 + \frac{1}{6}r\Gamma_i(S + \tilde{P}\Gamma_5)\right)\epsilon = 0.$$

Following a similar strategy to the one we used for obtaining supersymmetric Lifshitz solutions, we find that for $\tilde{P} = 0$, the existence of a Killing spinor implies

$$\tilde{P} = 0\colon\ \phi = \frac{(ra' - a)f}{r}, \qquad \frac{a'}{a} = \frac{S}{f} - \frac{2}{r}. \tag{5.4}$$

The scalar equation then gives

$$3\lambda - 2\sigma S + 2\beta f(2SS' + 3f'S' + 3fS'') = 0, \tag{5.5}$$

and the vector equation of motion gives

$$0 = r\sigma(rS - 3f) - \beta r(rS - 3f)(2fS' + S^2)$$
$$- \alpha f(rf(3f'' - rS'') - r(rf' + 2rS - 3f)S'$$
$$+ (S - 3f')(5f - 2rS - rf')). \tag{5.6}$$

The Einstein equations of motion are then all satisfied. The Killing spinor is given by $\epsilon = \sqrt{r}\epsilon_0$, and it satisfies the projection (4.19).

For $\tilde{P} \neq 0$, the existence of a Killing spinor implies that

$$\tilde{P} \neq 0\colon\ \phi = \frac{(ra' - a)f}{r}, \qquad S^2 - \tilde{P}^2 = \left(\frac{a'}{a} + \frac{2}{r}\right)^2 f^2. \tag{5.7}$$

The Killing spinor is again given by $\epsilon = \sqrt{r}\epsilon_0$, but now satisfying the projections

$$\left[S + \tilde{P}\Gamma_5 + \left(\frac{a'}{a} + \frac{2}{r}\right)f\Gamma_{\hat{r}}\right]\epsilon = 0, \qquad (\Gamma_0\Gamma_5 + \Gamma_{\hat{r}})\epsilon = 0. \tag{5.8}$$

For nonvanishing $\tilde{P}$, we find, after imposing the supersymmetry conditions, that we must have $\lambda = 0$ and furthermore that $\tilde{P}$ is a constant multiple of S. This may be parametrized as

$$\tilde{P}(r) = \sin\theta S(r), \tag{5.9}$$

where θ is a constant. The scalar and vector equations now become

$$0 = \sigma S - \beta(3f^2S'' + 3f'S' + 2\cos\theta SS'),$$
$$0 = r\sigma(rS\cos\theta - 3f) - \beta r\cos\theta(rS\cos\theta - 3f)$$
$$\times (2fS' + S^2\cos^2\theta) - \alpha f[rf(3f'' - rS''\cos^2\theta)$$
$$- r(rf' + 2rS\cos\theta - 3f)S'\cos\theta$$
$$+ (3f' - S\cos\theta)(rf' + 2rS\cos\theta - 5f)]. \tag{5.10}$$

Note that when $\theta = 0$ we have $\tilde{P} = 0$, but the equations are reduced to the previous $\tilde{P} = 0$ case only for $\lambda = 0$.

B. Some exact solutions

First, we consider the case where $\tilde{P} = 0$. Setting $\lambda = 2$ and $\sigma = 1$, we obtain the solution

$$f = r - r_0, \qquad a = \frac{(r - r_0)^3}{r^2}, \qquad \phi = \frac{3(r - r_0)^3 r_0}{r^3}, \tag{5.11}$$

provided that $\beta = 1/9$. This is also a solution of conformal supergravity with $\sigma = \lambda = \beta = 0$ [5].

Now we consider instead the case where $\tilde{P} \neq 0$. In this case, $\tilde{P}$ is given by Eq. (5.9). One particularly simple situation is when $\sin\theta = 1$ and hence $\tilde{P} = S$. The general solution for the metric functions is then given by

$$a^2 = \frac{1}{r^4}, \qquad f^2 = c_0 + c_1 r^6 + \frac{\sigma r^2}{4\alpha}. \tag{5.12}$$

It appears unlikely that the equations (5.10) are solvable exactly in general, and we have not found any further exact solutions.

VI. GYRATING SCHRÖDINGER GEOMETRIES

In this section, we consider another class of homogeneous metrics, namely the gyrating Schrödinger geometries [6]. The general ansatz is

$$ds^2 = \ell^2\left[\frac{dr^2 - 2dudv + dx^2}{r^2} - \frac{2c_2 dudx}{r^{z+1}} - \frac{c_1 du^2}{r^{2z}}\right],$$
$$A = q\frac{dt}{r^z} + p\frac{dr}{r}, \tag{6.1}$$

with the scalars S and P being constant. In the case of the O(1, 1) theory, the ansatz for the vector A will become $A = \mathrm{i}\tilde{A}$, with

$$\tilde{A} = \tilde{q}\frac{dt}{r^z} + \tilde{p}\frac{dr}{r}. \tag{6.2}$$

The solution is of Schrödinger type if $c_2 = 0$, and the term c_1 adds a further deformation to the Schrödinger metric. The metric is AdS_4 if $z = 1$. There are two other Einstein metrics, given by

$$z = -\frac{1}{2}\colon\ c_2 = 0, \qquad z = -2\colon\ c_1 + \frac{1}{2}c_2^2 = 0. \tag{6.3}$$

The first solution above is the Kaigorodov metric [24]. When $c_2 = 0$, the $z = 2$ solution has Schrödinger symmetry and was proposed as a gravity dual for the Schrödinger system [25,26]. The solutions of Refs. [25,26] make use of a massive vector, which is absent in typical supergravities. However, a massive vector arises naturally in higher-order $\mathcal{N} = 1$, $D = 4$ off-shell supergravity. AdS gyratons were studied in Ref. [27]. Supersymmetric (gyrating) Schrödinger solutions in Einstein-Weyl supergravity were constructed in Refs. [6,28]. Note that the metric of the gyrating Schrödinger solution (6.1) is homogeneous, as is the Schrödinger metric.

We shall now present more general solutions that are not themselves Einstein metrics. As in the case of Lifshitz solutions, we shall present the bosonic solutions first, and then study their supersymmetry.

A. $A_\mu = 0$

In this subsection, we list solutions where the massive vector A_μ vanishes. It can be easily verified that if $P \neq 0$, the scalar equations require that $\lambda = 0$. Thus we shall consider first the case with $P = 0$. For the Schrödinger solutions (i.e., with $c_2 = 0$), we then have

$$\begin{aligned} &S = \frac{3}{\ell}\colon\ \sigma = \frac{9\beta + 2\alpha z(1-2z)}{\ell^2}, \qquad \lambda = \frac{18\beta + 4\alpha z(1-2z)}{\ell^3}, \\ &\sigma = \beta S^2\colon\ \lambda = \frac{2\beta S(18+\ell^2 S^2)}{9\ell^2}, \\ &2\beta(\ell^2 S^2 - 9) + 3\alpha z(2z-1) = 0. \end{aligned} \tag{6.4}$$

For gyrating solutions, namely where $c_2 \neq 0$, we find

$$\begin{aligned} &S = \frac{3}{\ell}\colon\ \sigma = \frac{9\beta - \alpha z(z+1)}{\ell^2}, \qquad \lambda = \frac{18\beta - 2\alpha z(z+1)}{\ell^3}, \\ &\alpha(2c_1 + c_2^2)z(1+2z) = 0; \end{aligned} \tag{6.5}$$

$$\begin{aligned} &\sigma = \beta S^2\colon\ c_1 + \frac{1}{2}c_2^2 = 0, \qquad \lambda = \frac{2\beta S(18+\ell^2 S^2)}{9\ell^2}, \\ &3\alpha z(z+1) + 4\beta(\ell^2 S^2 - 9) = 0. \end{aligned} \tag{6.6}$$

Since P and A_μ are both vanishing here, it follows that these solutions arise in both the U(1) and the O(1, 1) theories.

Now consider the case with $P \neq 0$, for which we must have $\lambda = 0$. We find two Schrödinger solutions:

$$S^2 + P^2 = \frac{9}{\ell^2}, \quad \beta = \frac{2}{9}\alpha z(2z-1), \quad \sigma = 0; \tag{6.7}$$

$$\tilde{P}^2 - S^2 = \frac{18}{\ell^2}, \qquad \beta = \frac{1}{18}\alpha z(2z-1), \qquad \sigma = \frac{\alpha z(1-2z)}{\ell^2}. \tag{6.8}$$

In addition, there are two types of gyrating solution:

$$\begin{aligned} &S^2 + P^2 = \frac{9}{\ell^2}\colon\ (2c_1 + c_2^2)z(1+2z) = 0, \\ &\beta = \frac{1}{9}\alpha z(z+1), \qquad \sigma = 0; \end{aligned} \tag{6.9}$$

$$\begin{aligned} &\tilde{P}^2 - S^2 = \frac{18}{\ell^2}\colon\ (2c_1 + c_2^2)z(1+2z) = 0, \\ &\beta = \frac{1}{36}\alpha z(z+1), \qquad \sigma = -\frac{\alpha z(z+1)}{2\ell^2}. \end{aligned} \tag{6.10}$$

The solutions (6.7) and (6.9) are presented in the U(1) theory, but they could also arise in the O(1, 1) theory, with $P = -\mathrm{i}\tilde{P}$, provided that $\tilde{P}^2$ is sufficiently small that $S^2 - \tilde{P}^2$ remains non-negative. The solutions (6.8) and (6.10) can only arise in the O(1, 1) theory.

B. $A_\mu \neq 0$

When A_μ is turned on, as in the ansatz (6.1), we find that the equations of motion imply the constraints

$$(z+1)\alpha p q = 0. \tag{6.11}$$

Solutions then arise as follows.

1. Case 1: $\alpha \neq 0$

In this case, and if $p = 0 = P$, we find

$$\begin{aligned} &S = \frac{3}{\ell}\colon\ q = \frac{3}{2}\sqrt{2c_1 + c_2^2}(z-1), \\ &\sigma = \frac{1}{2}\ell\lambda = \frac{9\beta - \alpha z(z+1)}{\ell^2}; \end{aligned} \tag{6.12}$$

$$\begin{aligned} &\sigma = \beta S^2\colon\ \lambda = \frac{2\beta S(18+\ell^2 S^2)}{9\ell^2}, \\ &q = \frac{3}{2}\sqrt{2c_1 + c_2^2}(z-1), \\ &3\alpha z(z+1) + 4\beta(\ell S^2 - 9) = 0. \end{aligned} \tag{6.13}$$

For $p = 0$, but $P \neq 0$, we must have $\lambda = 0$. The solutions are

$$S^2+P^2=\frac{9}{\ell^2}: q=3/2\sqrt{2c_1+c_2^2}(z-1),\quad \beta=\frac{1}{9}\alpha z(z+1),\quad \sigma=0; \tag{6.14}$$

$$\tilde{P}^2-S^2=\frac{18}{\ell^2}: q=3/2\sqrt{2c_1+c_2^2}(z-1),\quad \beta=\frac{1}{36}\alpha z(z+1),\quad \sigma=-\frac{\alpha z(z+1)}{2\ell^2}. \tag{6.15}$$

The first solution, written for the U(1) theory, can arise also for the O(1, 1) theory with $P=-i\tilde{P}$ and $A_\mu=i\tilde{A}_\mu$, provided that $S^2-\tilde{P}^2$ is still non-negative. The second solution arises only in the O(1, 1) theory.

If instead $q=0$ and $p\neq 0$, we have

$$\begin{aligned}\lambda&=\frac{2\beta p(P^2+S^2)}{\ell^2 p},\\ \sigma&=\frac{\beta(6pS+\ell^2P(P^2+S^2))}{3\ell^2P},\\ &(-18+p^2)P+3pS+\ell^2P(P^2+S^2)=0,\\ &12\ell^2pS(p^2+S^2)-P(p^2+9)(p^2+36)=0,\\ c_1&=-\frac{c_2^2(z-1)(7z+4)}{4(2z+1)(2z-1)},\quad z=0,\pm1,-2,\end{aligned} \tag{6.16}$$

for the U(1) theory. In the O(1, 1) theory, we have

$$\begin{aligned}\lambda&=-\frac{\beta\tilde{p}(\tilde{p}-3)(\tilde{p}-6)}{\tilde{P}},\\ \sigma&=\frac{1}{2}\beta(\tilde{p}^2+3\tilde{p}+15),\quad z=0,\pm1,-2,\\ S&=-\frac{(\tilde{p}^2+9\tilde{p}+18)\tilde{P}}{2\tilde{p}},\\ \tilde{P}^2&=\frac{18\tilde{p}^2(\tilde{p}-3)(\tilde{p}-6)}{(\tilde{p}^2+3\tilde{p}+18)(\tilde{p}^2+15\tilde{p}+18)},\end{aligned} \tag{6.17}$$

where now the ansatz for the vector field in Eq. (6.1) is written in terms of the tilded field $\tilde{A}_\mu$, as in Eq. (6.2). In the cases $z=-1$ and $z=-2$ there is a further constraint, namely

$$c_1=-\frac{c_2^2(z-1)(7z+4)}{4(2z+1)(2z-1)}. \tag{6.18}$$

2. Case 2: $\alpha=0$

In this case, we find that pq can be nonzero, and the solution in the U(1) theory is given by

$$\begin{aligned}\lambda&=\frac{2\beta p(P^2+S^2)}{\ell^2P},\\ \sigma&=\frac{\beta(6pS+\ell^2P(P^2+S^2))}{3\ell^2P},\\ &18P-p^2P-3pS-\ell^2P(P^2+S^2)=0,\\ &2(p^4-63p^2+81)P+9p(p^2-9)S\\ &\quad+3\ell^2(P^2+S^2)(9P+pS\\ &\quad-\ell^2P(P^2+S^2))=0.\end{aligned} \tag{6.19}$$

In the O(1, 1) theory, we have

$$\begin{aligned}\lambda&=-\frac{\beta\tilde{p}(\tilde{p}-3)(\tilde{p}-6)}{\tilde{P}},\\ \sigma&=\frac{1}{2}\beta(\tilde{p}^2+3\tilde{p}+15),\\ S&=-\frac{(\tilde{p}^2+9\tilde{p}+18)\tilde{P}}{2\tilde{p}},\\ \tilde{P}^2&=\frac{18\tilde{p}^2(\tilde{p}-3)(\tilde{p}-6)}{(\tilde{p}^2+3\tilde{p}+18)(\tilde{p}^2+15\tilde{p}+18)}.\end{aligned} \tag{6.20}$$

It is of interest to note that there is no restriction on the parameters z, c_1, and c_2 in either of these solutions.

C. Supersymmetry analysis

To examine the supersymmetry of the solutions we have obtained in this section, we choose the vielbein

$$e^+=du,\qquad e^-=\frac{dv}{r^2}+\frac{c_2dx}{r^{z+1}}+\frac{c_1du}{2r^{2z}},\qquad e^{\hat{r}}=\frac{dr}{r},\qquad e^{\hat{x}}=\frac{dx}{r}, \tag{6.21}$$

such that the metric is given by $ds^2=-2e^+e^-+e^{\hat{x}}e^{\hat{x}}+e^{\hat{r}}e^{\hat{r}}$. Note that for simplicity we have set $\ell=1$. The corresponding spin connection has nonvanishing components given by

$$\begin{aligned}\omega^{\hat{x}}{}_{\hat{r}}&=-e^{\hat{x}}+\frac{(z-1)c_2}{2r^z}e^+,\\ \omega^{\hat{x}}{}_+&=\frac{(z-1)c_2}{2r^z}e^{\hat{r}},\qquad \omega^{\hat{r}}{}_-=-e^+,\\ \omega^{\hat{r}}{}_+&=-\frac{(z-1)c_2}{2r^z}e^{\hat{x}}-\frac{(z-1)c_1}{r^{2z}}e^+-e^-,\\ \omega^+{}_+&=-e^{\hat{r}},\end{aligned} \tag{6.22}$$

and so the components of the Killing spinor equation are given by

$$0 = \partial_u\epsilon + \frac{(z-1)c_2}{4r^z}\Gamma_{\hat{x}\hat{r}}\epsilon + \frac{1}{2}\Gamma_{\hat{r}+}\epsilon - \frac{(2z-1)c_1}{4r^{2z}}\Gamma_{-\hat{r}}\epsilon + \frac{1}{6}\left(\Gamma_+ + \frac{c_1}{2r^{2z}}\Gamma_-\right)(S + iP\Gamma_5)\epsilon$$
$$+ \frac{i}{6}\left(\frac{q}{r^z}(2+\Gamma_{+-}) - p\left(\Gamma_{+\hat{r}} + \frac{c_1}{2r^{2z}}\Gamma_{-\hat{r}}\right)\right)\Gamma_5\epsilon,$$
$$0 = \partial_v\epsilon - \frac{1}{2r^2}\Gamma_{-\hat{r}}\epsilon - \frac{ip}{6r^2}\Gamma_{-\hat{r}}\Gamma_5\epsilon + \frac{1}{6r^2}\Gamma_-(S + iP\Gamma_5)\epsilon,$$
$$0 = \partial_r\epsilon + \frac{(z-1)c_2}{4r^{z+1}}\Gamma_{-\hat{x}}\epsilon + \frac{1}{2r}\Gamma_{+-}\epsilon - \frac{iq}{6r^{z+1}}\Gamma_{-\hat{r}}\Gamma_5\epsilon + \frac{ip}{3r}\Gamma_5\epsilon + \frac{1}{6r}\Gamma_{\hat{r}}(S + iP\Gamma_5)\epsilon,$$
$$0 = \partial_x\epsilon - \frac{1}{2r}\Gamma_{\hat{x}\hat{r}}\epsilon - \frac{(z+1)c_2}{4r^{z+1}}\Gamma_{-\hat{r}}\epsilon + \frac{1}{6}\left(\frac{1}{r}\Gamma_{\hat{x}} + \frac{c_2}{r^{z+1}}\Gamma_-\right)(S + iP\Gamma_5)\epsilon - \frac{i}{6}\left(\frac{q}{r^{z+1}}\Gamma_{-\hat{x}} + p\left(\frac{1}{r}\Gamma_{\hat{x}\hat{r}} + \frac{c_2}{r^{z+1}}\Gamma_{-\hat{r}}\right)\right)\Gamma_5\epsilon. \tag{6.23}$$

Having obtained the Killing spinor equations, we can study the integrability conditions to determine whether there exists a Killing spinor for a particular background. The Killing spinor equations (6.23) can be expressed as

$$\partial_u\epsilon = U\epsilon, \quad \partial_v\epsilon = V\epsilon, \quad \partial_x\epsilon = X\epsilon, \quad \partial_r\epsilon = R\epsilon. \tag{6.24}$$

This implies, for example,

$$\partial_v\partial_u\epsilon = \partial_v U\epsilon + UV\epsilon, \quad \partial_u\partial_v\epsilon = \partial_u V\epsilon + VU\epsilon, \tag{6.25}$$

and so we have the following derivative-independent equation on ϵ:

$$(\partial_v U - \partial_u V + [U, V])\epsilon = 0. \tag{6.26}$$

There are in total six such equations, from the possible pairs taken from $\{U, V, X, R\}$. Examining these integrability conditions for $A_\mu = 0$, we find that only the Schrödinger solutions (i.e., with $c_2 = 0$) can have Killing spinors. For these solutions, supersymmetry requires that $S^2 + P^2 = 9$, and the Killing spinors satisfy the projections

$$\Gamma_-\epsilon = 0, \qquad \Gamma_{\hat{r}}\epsilon = \frac{1}{3}(S + iP\Gamma_5)\epsilon. \tag{6.27}$$

Thus there is one Killing spinor, and it depends on r only.

An interesting situation arises for these Schrödinger solutions in the special case $P = 0$; i.e., if $S = 3$. It turns out that the integrability conditions are then satisfied if ϵ obeys just the single projection

$$\Gamma_-\epsilon = 0, \tag{6.28}$$

which would suggest that there should be two Killing spinors. However, one finds in this case that the Killing spinor equations (6.23) themselves can only be solved if the second projection condition

$$\Gamma_{\hat{r}}\epsilon = \epsilon \tag{6.29}$$

is also satisfied, and so there is in fact only a single Killing spinor in this special case too. This is an example, not often encountered in practice in supergravity examples, where the second-order integrability conditions obtained by commuting pairs of Killing-spinor derivatives are not sufficient to determine the existence of solutions. In principle, one might have to look at third-order integrability conditions or beyond. (For a discussion of this in the supergravity context, see Ref. [29].) Of course, if one explicitly constructs the most general solution of the Killing-spinor conditions themselves, it is not necessary to examine the higher-order integrability conditions. In practice, as in this example, projection conditions that one learns from the usual second-order integrability conditions—even if they are not providing the complete set of projections—can be helpful when constructing the Killing spinors explicitly.

For $A_\mu \neq 0$, we find that supersymmetry requires $p = 0$. There are two inequivalent solutions. The first is given by

$$q = -\frac{3}{2}c_2(z-1), \qquad c_1 = 0, \qquad S^2 + P^2 = 9. \tag{6.30}$$

In this case, there is only one (constant) Killing spinor, subject to the projection

$$\Gamma_+\epsilon = 0, \qquad \Gamma_{\hat{r}}\epsilon = \frac{1}{3}(S + iP\Gamma_5)\epsilon. \tag{6.31}$$

The second solution is given by

$$q = \frac{1}{2}c_2(z-1), \qquad S^2 + P^2 = 9. \tag{6.32}$$

There is again only one (constant) Killing spinor, subject to the projections

$$\Gamma_-\epsilon = 0, \qquad \Gamma_{\hat{r}}\epsilon = \frac{1}{3}(S + iP\Gamma_5)\epsilon. \tag{6.33}$$

Comparing this with the solutions obtained in the previous subsection, it is straightforward to see that the solutions (6.12) and (6.14) can be made supersymmetric. Both Bogomol'nyi-Prasad-Sommerfield solutions in Einstein-Weyl supergravity ($P = 0$) were obtained in Ref. [6].

VII. MORE GENERAL GYRATING SOLUTIONS

A. A general class of solutions

In general, we can consider the following most general gyraton metrics:

$$ds^2 = \ell^2\Big[\frac{dr^2 - 2dudv + dx^2}{r^2} - 2h(r,u,x)dudx - H(r,u,x)du^2\Big],$$
$$A = \phi(r,u,x)du + \psi(r,u,x)dr, \qquad S = S(r,u,x),$$
$$P = P(r,u,x). \tag{7.1}$$

These become pp-waves when $h = 0$. Such pp-wave solutions in critical gravity and more general higher-derivative gravities can be found in Refs. [30–32].

The general equations of motion are rather complicated to present. There is no u derivative in any of the equations, and so all ‘‘constants of integration’’ can trivially be taken to be functions of u. For simplicity of notation, the freedom to add such arbitrary u dependence will be understood, but not explicitly indicated. A further simplification can be achieved by considering cases where $P = 0$ and S is a constant, in which case two possible choices arise:

$$S^2 = \frac{9}{\ell^2} \quad \text{or} \quad S^2 = \frac{\sigma}{\beta}. \tag{7.2}$$

The equations become completely solvable if we then make the further assumption that the functions are all independent of x, leading to the ansatz

$$ds^2 = \ell^2\Big[\frac{dr^2 - 2dudv + dx^2}{r^2} - 2h(r,u)dudx - H(r,u)du^2\Big],$$
$$A = \phi(r)dt, \quad S = \text{const.}, \quad P = 0. \tag{7.3}$$

We then find that the functions ϕ, h, and H are given by

$$\phi = \frac{q_1}{r^z} + q_2 r^{z+1},$$
$$h = h_1 r^{-z-1} + h_2 r^z + h_3 r^{-2} + h_4 r,$$
$$H = H_1 r^{-z-1} + H_2 r^z + H_3 r^{-2} + H_4 r - \frac{1}{2}h_4^2 r^4 - h_1 h_4 r^{2-z} - \frac{1}{2}h_1^2 r^{-2z} - h_2 h_4 r^{3+z} - \frac{1}{2}h_2^2 r^{2z+2} + \frac{2q_1^2}{9(z-1)^2 r^{2z}} + \frac{2q_2^2 r^{2z+2}}{9(z+2)^2}. \tag{7.4}$$

For the constants, there are two possibilities:

$$\ell^2 S^2 = 9, \qquad \sigma = \frac{1}{2}\lambda\ell = \frac{9\beta - \alpha z(z+1)}{\ell^2}, \tag{7.5}$$

or

$$\sigma = \beta S^2, \qquad \lambda = \frac{2\beta S(18 + \ell^2 S^2)}{9\ell^2},$$
$$3\alpha z(z+1) + 4\beta(\ell^2 S^2 - 9) = 0. \tag{7.6}$$

There exist two critical values of z, namely $z = 1$ or $z = -2$, for which the solution degenerates. The functions ϕ, h, and H are now given by

$$\phi = \frac{q_1}{r} + q_2 r^2, \qquad h = \frac{(h_1 + h_2 r^3)\log r}{r^2} + \frac{h_3 + h_4 r^3}{r^2},$$
$$H = \frac{(H_1 + H_2 r^3)\log r}{r^2} + \frac{H_3 + H_4 r^3}{r^2} + \frac{2(6q_1^2 + q_2^2 r^6)}{81 r^2} + \frac{2q_1^2 \log r(4 + 3\log r)}{27 r^2} - \frac{2h_1^2 + 4h_1 h_2 r^3 + 3h_4^2 r^6}{6r^2} - \frac{(2h_1^2 - 4h_1 h_2 r^3 + 3h_2 h_4 r^6)\log r}{3r^2} - \frac{(h_1 + h_2 r^3)^2 (\log r)^2}{2r^2}. \tag{7.7}$$

Logarithmic behavior can also arise when $z = -1/2$, for which we have

$$\phi = \sqrt{r}(q_1 + q_2 \log r),$$
$$h = \frac{(h_1 + h_2 \log r)}{\sqrt{r}} + \frac{h_3}{r^2} + h_4 r,$$
$$H = \frac{(H_1 + H_2 \log r)}{\sqrt{r}} + \frac{H_3}{r^2} + H_4 r + \frac{16}{243} q_1 q_2 r(3\log r - 5) + \frac{8}{729} q_2^2 r(3\log r - 2)^2 - \frac{1}{18} r(-30 h_1 h_2 - 16 h_2^2 + 18 h_1 h_4 r^{3/2} + 9h_4^2 r^3 + 18 h_2(h_1 + h_4 r^{3/2})\log r + 9h_2^2 (\log r)^2). \tag{7.8}$$

Note that for these solutions, the parameters h_3 and H_3 are trivial.

B. Supersymmetry analysis

We shall choose the vielbein basis

$$e^+ = du, \qquad e^- = \frac{dv}{r^2} + \frac{1}{2}Hdu + hdx,$$
$$e^{\hat{r}} = \frac{dr}{r}, \qquad e^{\hat{x}} = \frac{dx}{r}. \tag{7.9}$$

The nonvanishing components of the spin connection are then given by

$$\begin{aligned}
\omega^{\hat{x}}{}_{\hat{r}} &= -e^{\hat{x}} - \left(rh + \frac{1}{2}r^2h'\right)e^+,\\
\omega^{\hat{x}}{}_{+} &= -\left(rh + \frac{1}{2}r^2h'\right)e^{\hat{r}},\\
\omega^{\hat{r}}_{+} &= \left(rh + \frac{1}{2}r^2h'\right)e^{\hat{x}} + \left(H + \frac{1}{2}rH'\right)e^+ - e^-,\\
\omega^{\hat{r}}{}_{-} &= -e^+, \qquad \omega^{+}{}_{+} = -e^{\hat{r}}.
\end{aligned} \tag{7.10}$$

The Killing spinor equations are

$$\begin{aligned}
0 &= \partial_u\epsilon - \frac{1}{4}(2rh + r^2h')\Gamma_{\hat{x}\hat{r}}\epsilon\\
&\quad + \frac{1}{2}\Gamma_{\hat{r}+}\epsilon - \frac{1}{4}(H + rH')\Gamma_{\hat{r}-}\epsilon + \frac{\mathrm{i}}{6}\phi(\Gamma_{+-} + 2)\Gamma_5\epsilon\\
&\quad + \frac{1}{6}\left(\Gamma_+ + \frac{1}{2}H\Gamma_-\right)(S + \mathrm{i}P\Gamma_5)\epsilon,\\
0 &= \partial_v\epsilon + \frac{1}{2r^2}\Gamma_{\hat{r}-}\epsilon + \frac{1}{6r^2}\Gamma_-(S + \mathrm{i}P\Gamma_5)\epsilon,\\
0 &= \partial_r\epsilon + \frac{1}{4}(2h + rh')\Gamma_{\hat{x}-}\epsilon + \frac{1}{2r}\Gamma_{+-}\epsilon\\
&\quad + \frac{\mathrm{i}}{6r}\phi\Gamma_{\hat{r}-}\Gamma_5\epsilon + \frac{1}{6r}\Gamma_{\hat{r}}(S + \mathrm{i}P\Gamma_5)\epsilon,\\
0 &= \partial_x\epsilon - \frac{1}{2r}\Gamma_{\hat{x}\hat{r}}\epsilon - \frac{1}{4}rh'\Gamma_{\hat{r}-}\epsilon + \frac{\mathrm{i}}{6r}\phi\Gamma_{\hat{x}-}\Gamma_5\epsilon\\
&\quad + \frac{1}{6}\left(\frac{1}{r}\Gamma_{\hat{x}} + h\Gamma_-\right)(S + \mathrm{i}P\Gamma_5)\epsilon.
\end{aligned} \tag{7.11}$$

For the solutions with $P = 0$ that we considered earlier, it is clear that if we turn off h and ϕ, they then reduce to a special class of AdS pp-waves and hence preserve $\frac{1}{4}$ of the supersymmetry, provided that $S = 3/\ell$. In fact, it was shown in Ref. [33] that the most general pp-waves with r, u, and x dependence and with A_μ turned off all preserve $\frac{1}{4}$ of the supersymmetry. The Killing spinor satisfies the projections

$$\Gamma_{\hat{r}}\epsilon = \epsilon, \qquad \Gamma_-\epsilon = 0. \tag{7.12}$$

For nonvanishing ϕ and h, Killing spinors with the same projections (7.12) also exist, provided that

$$\phi = -\frac{1}{2}(2rh + r^2h'). \tag{7.13}$$

Thus the bosonic solution (7.4) becomes supersymmetric provided that the condition

$$\begin{aligned}
&\frac{3}{2}h_4r^2 + r^{-z}\left(q_1 - \frac{1}{2}(z-1)h_1\right)\\
&\quad + r^{z+1}\left(q_2 + \frac{1}{2}(z+2)h_2\right) = 0
\end{aligned} \tag{7.14}$$

holds for all r. For generic z, we must therefore have

$$h_4 = 0, \qquad q_1 = \frac{1}{2}(z-1)h_1, \qquad q_2 = -\frac{1}{2}(z+2)h_2. \tag{7.15}$$

For the critical solution (7.7), we find that supersymmetry implies

$$q_1 = -\frac{1}{2}h_1, \qquad h_2 = 0, \qquad q_2 = -\frac{3}{2}h_4. \tag{7.16}$$

For the $z = -1/2$ solution (7.18), supersymmetry implies

$$h_4 = 0, \qquad q_1 = -\frac{1}{4}(3h_1 + 2h_2), \qquad q_2 = -\frac{3}{4}h_2. \tag{7.17}$$

Finally, we find that there exists another type of Killing spinor, satisfying

$$\Gamma_{\hat{r}}\epsilon = \epsilon, \qquad \Gamma_+\epsilon = 0. \tag{7.18}$$

It requires that

$$\phi = -\frac{3}{2}r(2h + rh'), \qquad 2H + rH' = 0. \tag{7.19}$$

Applying this condition to the three solutions, we find that $H_1 = H_2 = H_4 = 0$, and that

$$\begin{aligned}
\text{Generic } z:&\quad q_1 = \frac{3}{2}(z-1)h_1, \qquad q_2 = -\frac{3}{2}(z+2)h_2, \qquad h_4 = 0;\\
z = 1, -2:&\quad q_1 = -\frac{3}{2}h_1, \qquad q_2 = -\frac{9}{2}h_4, \qquad h_2 = 0;\\
z = -\frac{1}{2}:&\quad q_1 = -\frac{3}{4}(3h_1 + 2h_2), \qquad q_2 = -\frac{9}{4}h_2, \qquad h_4 = 0.
\end{aligned} \tag{7.20}$$

VIII. CONCLUSIONS

In this paper, we have considered four-dimensional $\mathcal{N} = 1$ off-shell supergravity including all four super-invariants up to and including quadratic order in curvature. These comprise a "cosmological term," the Einstein-Hilbert term, and two quadratic-curvature terms: one formed using the square of the Weyl tensor, and the other formed using the square of the Ricci scalar. In addition to the graviton and the gravitino, the fields of the off-shell multiplet include a complex scalar $S + \mathrm{i}P$ and a vector A_μ. In the Einstein plus cosmological supergravity, the complex scalar and the vector are auxiliary and possess no physical degrees of freedom. The supersymmetric solution space is then rather limited. Examples of such solutions were given in Ref. [4].

However, when the curvature-squared super-invariants are included, the auxiliary fields can develop dynamics, and in particular the vector becomes a massive Proca field. For lack of a more satisfactory name, one may continue to call these fields auxiliary, even though they may now propagate. (The supersymmetry algebra still closes off-shell, however.) In Einstein-Weyl supergravity, Lifshitz solutions and also a new type of supersymmetric gyrating Schrödinger solution were obtained in Refs. [5,6]. In this paper, we included both of the curvature-squared super-invariants, namely the one based on the square of the Weyl tensor and the one based on the square of the Ricci scalar. We found large classes of domain-wall solutions, as well as Lifshitz and gyrating Schrödinger vacua. Amongst these solutions, we found subsets that were supersymmetric or pseudosupersymmetric. We also obtained (pseudo)supersymmetric solutions that were asymptotic to the Lifshitz and gyrating Schrödinger vacua. It is worth pointing out that these supersymmetric solutions depend upon nontrivial contributions from the auxiliary fields. Thus the mechanism for supersymmetry in our solutions is rather different from that in an on-shell theory, where typically supersymmetry is associated with a balance between mass and conserved charges carried by form fields. In fact, the massive vector that is essential for the supersymmetry has no conserved charge.

The wealth of supersymmetric vacua of the AdS, Lifshitz, and gyrating Schrödinger types leads to many new avenues for investigation in off-shell higher-derivative supergravities. They may provide a rich source of gravity backgrounds for studying the correspondences of both AdS/CFT and AdS/CMT physics. In particular, the existence of supersymmetric Schrödinger and gyrating Schrödinger vacua provides a supersymmetric framework for studying nonrelativistic field theories. The inclusion of higher-derivative terms generically requires that one specify additional boundary data in order to make the variational problem well defined. In the context of holography, the additional data corresponds to a new operator in the dual CFT, in addition to the stress-energy tensor which is dual to the bulk metric.

For generic higher-derivative terms added to the action, the dual operator has complex dimension and/or negative norm, reflecting the fact that the addition of higher-derivative terms violates unitarity. However, if the effects from these higher-derivative terms are treated perturbatively, the corrections to the leading-order solution satisfy second-order inhomogeneous differential equations, rather than higher-order differential equations. Since the equations are of second order, only the field itself need be specified on the boundary in order that the variational problem be well defined. Within such a perturbative framework, corrections to thermodynamic quantities and transport coefficients from higher-derivative interactions have been studied extensively in the context of the AdS/CFT correspondence [34,35].

ACKNOWLEDGMENTS

We are grateful to Ergin Sezgin for useful conversations, and to the KITPC, Beijing, for hospitality during the course of this work. Y. P. thanks Qing-Guo Huang at the ITP, Beijing, for hospitality. The research of H. L. is supported in part by NSFC Grants No. 11175269 and No. 11235003. C. N. P. is supported in part by DOE Grant No. DE-FG03-95ER40917.

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