Causal structure of spacetime and geometric algebra for quantum gravity

I. Dukovski*

Aksius Research Institute, P.O. Box 600008, Newton, Massachusetts 02460-0001, USA

(Received 25 March 2011; revised manuscript received 22 October 2012; published 18 March 2013)

We construct a background independent model of spacetime from a minimal set of postulates of causality. The topology and geometry of spacetime can be derived from the model, instead of being postulated. We define a measure of causality and relate it to the noncommutative geometric algebra of spacetime coordinates. The geometric algebra formalism leads to the definition of a Euclidean action. The Euclidean action is positive definite and allows the path integral formulation of quantum gravity to be treated as statistical physics of a causal stochastic Markov field.

DOI: 10.1103/PhysRevD.87.064022 PACS numbers: 04.60.Gw, 02.10.Ox, 04.20.Gz, 04.60.Nc

I. INTRODUCTION

Quantization of the gravitational field has proven to be one of the most difficult problems in contemporary physics. One of the reasons for this is that it is not a straightforward task to relate the quantized gravitational field to a local stochastic Markov field [1] and treat it with the analytical and computational methods of statistical physics. This situation is evident in the problem of conformal divergence in the theory of Euclidean quantum gravity (EQG) [2]. There, one starts with the path integral formulation of the theory of quantum gravity, and transitions to a corresponding Markov field theory by generalizing the Wick rotated action for flat spacetime to a Euclidean action for spacetime with curvature. The EQG action, however, is not positive definite and cannot be used to relate the path integral to a partition function of a Markov field. This is seen in the presence of a negative kinetic term, after a conformal transformation $\tilde{g}_{ab} = \Omega^2 g_{ab}$ in the Euclidean action:

$$I_E[\tilde{g}, \phi] = \frac{1}{16\pi G} \int \Big(-6 g^{ab} \Omega_{;a} \Omega_{;b} - \Omega^2 R + 2\Omega^4 \Lambda + \frac{1}{2} \Omega^2 g^{ab} \phi_{;a} \phi_{;b} + \frac{1}{2} \Omega^4 m^2 \phi^2 \Big) \sqrt{g} d^4 x.$$

The problem of nonpositive definiteness of the EQG action has been, in large part, resolved by the causal dynamical triangulations (CDT) approach [3–7]. The numeric-experimental results of this methodology show that, if the causal order in the Lorentzian manifold is preserved and carried over to the Wick rotated Euclidean manifold [8], the partition function and the corresponding path integral are well defined and convergent. In the CDT model this is done by introducing a time coordinate and performing a Wick rotation on it. The Lorentzian manifold background [9] in the CDT model is inherited in its entirety from the spacetime model of general relativity. This is the difference between the CDT model and another discrete model, the causal sets model of spacetime [10,11].

The causal sets model manages to eliminate a large portion of the mathematical background from the theory by building the model and its mathematical structure starting with the causal structure of spacetime [12]. In this paper we will attempt to merge these two models of spacetime. We will attempt to derive the spacetime background (including its topology, dimensionality, geometry, metric etc.) from the causal structure among events in spacetime, with the additional requirement that the causal relation between any two events in the model can be expressed in a quantitative way. It has already been shown that both Lorentz invariance [13] and the metric of spacetime [14] can be deduced from causality alone. Some of these results have led to the formulation of the theory of causal sets [10]. The causal sets theory postulates the discreteness of spacetime, and constructs its model as a partially ordered set, a poset, of causally related events. In contrast to the causal sets theory, here we will assume neither a discrete nor continuous nature of spacetime. The model presented here can be understood either as an exact representation of a discrete spacetime, or as a discrete subset, an approximation, of a continuum.

Posets and order lattices in mathematics are typically used as abstractions, tools for understanding the relations among specific mathematical objects. A geometric lattice, for example, is one that carries the information about hierarchical relationships among concrete geometric objects. The lattices themselves are very seldom given geometric, topological, or even metric properties. Here we will depart from this tradition and show that abstract order lattices in fact can be understood as frameworks for building the topology and geometry of spacetime. The order lattice model in our case is not a mere abstract representation of the causal relationships among the points of the physical spacetime, but it is in fact a model, an approximation of the physical spacetime.

In the following section we will construct the model from a series of postulates. In Sec. III we will give a quantitative meaning to the notion of causality in the model. Section IV is a derivation of the topology and geometry of spacetime from the causal structure of the

*ilija.dukovski@aksius.org

model. In Sec. V we will introduce the positive definite action and we will conclude with Sec. VI.

II. POSTULATES OF THE MODEL

The central entity in our model of spacetime is the notion of *event*. The concept of event is well defined in general relativity. There, one postulates the existence of a continuum (i.e., a set of the same cardinality as the cardinality of the set of real numbers) of points, each one carrying a value of a physical field, and an event is one such point. Here, we will construct our model of spacetime as a set of events as well. The difference between the set of events in general relativity and in our model is in its postulated mathematical properties. Besides its cardinality (i.e., being a continuum), the spacetime continuum of events in general relativity carries a range of assumed mathematical properties. Most importantly, the spacetime continuum in general relativity is assumed to be a topological manifold. These assumptions constitute the mathematical background of the theory and are not necessarily required by the results of physical experiments.

One of the aims of this work is to minimize the mathematical background in the theory. We will not make any assumption about the topology and geometry of spacetime. Instead, we will attempt to derive these properties from a minimal set of assumptions, or postulates. These assumptions, if possible, will be made on the level of the mathematical set theory. The higher-level topological and geometrical properties of the model will be derived from the set-theoretic postulates rather than being postulated themselves. With this we hope to accomplish two major goals. First, we will eliminate as much of the assumed background as possible. This goal is, we believe, in the very heart of general relativity, and it is at least desirable, if not necessary, for a quantum theory of gravitation to eliminate as much of the background as theoretically possible. Second, we will attempt to provide a way of bookkeeping for the mathematical assumptions in our theory. Our model is certainly not completely background free. However, as we will address further on in the text, there are indications that, minimal as it is, the set of postulates could be further reduced in a future theory. With this goal in mind, we will build our model constructively, starting from the minimal notion of a set of events and then building up, while keeping track of all the assumptions built into the model.

The axiomatic structure of our model is very close to the one of the causal sets theory [10–12]. However, we will attempt to formulate the model in the language of the mathematical theory of sets, specifically partially ordered sets and lattices. We will do this in order to utilize the existing powerful mathematical formalism of the mathematical theory of partially ordered sets.

Although seldom treated in the physics literature, the consistency of the claim of the existence of a set of events is not a trivial matter from a mathematical point of view. Before we relate any of the mathematical properties of our model to the physical properties of spacetime, we have to assume a set-theoretic system of axioms, within which the concept of a set of events is defined. The spacetime set of events in our theory will therefore obey the standard axioms of set theory. We will assume the validity of the Zermelo-Fraenkel axioms, although other set-theoretic approaches, such as the von Neumann-Robinson-Bernays-Gödel approach, may be taken as starting points as well.

The consideration of the set-theoretic system of axioms here may seem unnecessary. The issues of consistency of a definition of a set of physical points are typically considered purely mathematical, if not philosophical, and are very rarely discussed in the physics literature. These set-theoretic assumptions are silently present in any physical theory and perhaps from a physicist's point of view practically trivial. It is important, however, to emphasize the consequences of the assumed validity of one particular set-theoretic axiom, the axiom of choice. It is typically included among, but independent from, the rest of the Zermelo-Fraenkel axioms or Neumann–Robinson–Bernays–Gödel axioms. The axiom of choice states that given a collection of sets, one can choose a single element from each and every set in the collection. The first impression about the axiom of choice may be that it is trivial too, and therefore irrelevant. That, however, is not true and in fact it will play an important role in our order model of spacetime. Its importance is partially due to the fact that the very possibility to well order a set is in fact equivalent to the possibility of choosing an element in it. What is meant by well ordering here is that one can order the elements of a set in a way that each of its subsets has a minimal element. The possibility of choosing such minimal elements after the appropriate well ordering gives an indication why the axiom of choice is equivalent to the possibility of well ordering a set. The assumption of the validity of the axiom of choice is equivalent to the assumption that any set can be well ordered, i.e., a minimal element can be defined in the set, and each of its subsets.

A. Postulate of partially ordered set

Let us start by postulating the general existence and causality of the model [10]:

P1: Physical spacetime is modeled by a partially ordered set of spacetime events.

Spacetime, therefore, is modeled by the partially ordered set (poset) $(P, \geq)$, where P is the set of events and $\geq$ is the relation of causality, defined as a partial order that is reflexive

$$\forall\ a \in P, \quad a \geq a,$$

antisymmetric

$$\text{if } a \geq b \quad \text{and} \quad b \geq a, \quad \text{then } a = b,$$

and transitive [15]

$$\text{if } a \geq b \quad \text{and} \quad b \geq c, \quad \text{then } a \geq c.$$

We will also write $a > b$, if $a \geq b$ and $a \neq b$.

Not all pairs of elements in P are causally related and comparable, and we will call P linear or one-dimensional in the special case when all pairs indeed are comparable. An arbitrary poset can be constructed as an intersection of a number of linear posets; the poset's dimension is defined as the smallest such number.

The poset's dimension as defined above is not a geometrical concept. No geometry, not even topology, has been defined on the set P. The poset's dimensionality however mirrors closely the analogous geometric concept. An element of the poset $(P, \geq)$ can be given a "coordinate" by its position in the linear orders whose intersection gives the partial ordering $\geq$ in P.

An intuitive insight in the correspondence of the two concepts of dimensionality, the geometric and the ordinal (i.e., relating to the partial order), can be gained by looking at the graphical representation of a poset, the Hasse diagram. The elements (events) of the set P are represented in a Hasse diagram by dots. If two elements a and b are related by the partial order, e.g., $b \geq a$, and there are no other elements between them, i.e.,

$$\text{if } b > a \quad \text{and} \quad b > c \geq a, \quad \text{then } c = a,$$

then we say that b covers a and write $b \succ a$ [16]. This covering relation in a Hasse diagram is graphically represented by a line between two (covering) neighboring elements, with b being higher up on the page than a.

Figure 1 illustrates the Hasse diagram of a two-dimensional poset. The partial order is given by the set of ordered pairs (a, b) if and only if $b \geq a$. The partial order in the poset in Fig. 1 is specified by the pairs

$$(R, \geq) = \{(1, 2), (1, 3)\}.$$

It can be obtained from the intersection of the linear posets

$$(L_1, \geq_1) = \{(1, 2), (1, 3), (2, 3)\}$$

and

$$(L_2, \geq_2) = \{(1, 2), (1, 3), (3, 2)\}.$$

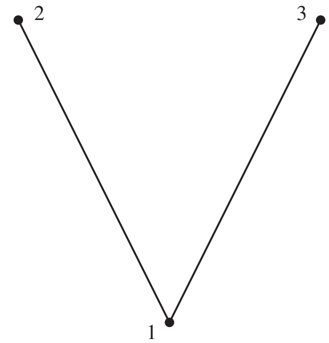

FIG. 1. Two-dimensional poset.

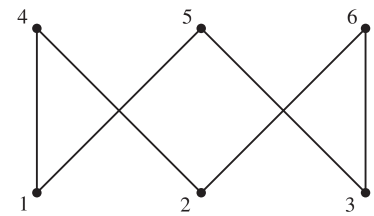

FIG. 2. Three-dimensional poset.

This is the minimal number of linear orders whose intersection gives the poset R. By the definition of poset's dimension above, the poset R is two dimensional. The analogy to the geometric concept of dimension can be seen in the fact that the Hasse diagram of the poset R can be drawn flat on a sheet of paper, without any intersecting covering lines.

That is not the case with the three-dimensional poset Q in Fig. 2. This poset is given by

$$(Q, \geq) = \{(1, 4), (1, 5), (2, 4), (2, 6), (3, 5), (3, 6)\}.$$

It can be obtained from the intersection of the three linear posets

$$(M_1, \geq_1) = \{(1, 2), (1, 4), (1, 3), (1, 5), (1, 6), (2, 4), (2, 3), (2, 5), (2, 6), (4, 3), (4, 5), (4, 6), (3, 5), (3, 6), (5, 6)\},$$

$$(M_2, \geq_2) = \{(2, 3), (2, 6), (2, 1), (2, 4), (2, 5), (3, 6), (3, 1), (3, 4), (3, 5), (6, 1), (6, 4), (6, 5), (1, 4), (1, 5), (4, 5)\},$$

and

$$(M_3, \geq_3) = \{(3, 1), (3, 5), (3, 2), (3, 4), (3, 6), (1, 5), (1, 2), (1, 4), (1, 6), (5, 2), (5, 4), (5, 6), (2, 4), (2, 6), (4, 6)\}.$$

The choice of the linear posets $M1$, $M2$ and $M3$ is not unique. One could construct a different collection of linear posets with the same property. However, this is the minimal number of linear posets whose intersection gives the poset Q. Hence its dimensionality.

B. Postulate of measure of causality

In the second postulate we require that causality be quantifiable within the model. The causal order in a Lorentzian manifold is given by its metric and the corresponding light cones, and is fully determined by two quantities, typically the square path length and the time coordinate. The existence of a quantitative measure of "distance" and order between two events a and b in an arbitrary poset, without defined geometry and metric, however, is not trivial. The length (cardinality) of a maximal linear chain between two events, for example, is not well defined in a poset where two events can be connected by infinite, as well as finite, chains. Another reason for the failure of the chain length alone as a measure of causality, even in finite posets, is the fact that nonrelated events, $a \parallel b$, are not connected by causal chains. In principle one can assign zero value in that case; however that would make it impossible for the measure to distinguish between the cases $a = b$ and $a \parallel b$. In order to model spacetime quantitatively, we must be able to define a real and at least double-valued order preserving function on P.

With the second postulate we will put a constraint on the set of posets as possible candidates for our model of spacetime, to the ones whose order structure allows the definition of an order preserving measure of causal distance.

P2: The partial order in P allows the definition of an order preserving function $\hat{d}(\cdot,\cdot)$: $P \times P \to \mathbb{R}^2$ *as a measure of causality.*

The function $\hat{d}$ is order preserving in the sense that a partial order $\geq_d$ in $\mathbb{R}^2$, the set of ordered real pairs, exists such that $\hat{d}(a,a) = (0,0)$, and if $a \geq b$ then $\hat{d}(a,b) \geq_d (0,0)$ and $(0,0) \geq_d \hat{d}(b,a)$.

The metric of general relativity carries the information if two points are casually related. It is not, however, order preserving. The value of the relativistic distance between two causally related spacetime points a and b is the same regardless of whether $a \geq b$ or $b \geq a$. In order to fully quantify the information of the causal ordering in the spacetime of general relativity, we need to specify a timelike coordinate. This is, in fact, one of the major reasons why in some treatments of spacetime quantization, the time coordinate is singled out and treated separately from the spacelike coordinates. Specifically, in the methodology of causal dynamical triangulations [3], causality plays the crucial role for the regularization of the otherwise nonpositive definite path integral over spacetime geometries. In the absence of an order preserving function of distance, the time coordinate is singled out and serves to preserve the causal order in the dynamically triangulated model.

The singled out, Wick rotated time coordinate is part of the background in the causal dynamical triangulation models of spacetime. One of our main goals in this paper is to eliminate as much of the spacetime background as possible and treat time and space coordinates on the same footing. We will attempt to define the measure of causality without using a separate time coordinate. In much of the rest of this paper we will attempt to define a background independent, order preserving analogue of the spacetime metric. It is not clear, however, if such a function exists at all. In this section we postulate its existence. This postulate puts a constraint on the properties of our spacetime model. The mathematical structure of the spacetime poset must be such that it will allow the definition of an order preserving function on it. In this sense, the physics of the model, the possibility of an experimental measurement of the causal relations among events, is what dictates its mathematical structure. The possibility of measuring causality is the model's building principle, since we will construct it by eliminating all the possible posets that do not allow a definition of an order preserving distance.

As mentioned in the introduction, we will show that the constraint imposed by the requirement of the existence of causal distance constrains the set of possible posets to the ones that share the topological and geometrical properties of spacetime of general relativity. In other words, the causality itself, armed with the requirement that it is a physically measurable quantity, leads to the emergence of the topology and geometry of spacetime. This reduces significantly the background assumptions in the model of spacetime. In general relativity, one starts with a model with presumed manifold topology and presumed geometry given by the postulated spacetime metric. Here we presume only the causality of events and the possibility of measurement of causes and effects. The topology and geometry of spacetime are emerging from its causality. Before we show that, however, we will have to construct a poset model of spacetime with an order preserving function of causal distance, one that will serve the roles of the metric and the time coordinate put together in general relativity.

The main reason that the standard spacetime metric of general relativity is not order preserving is that it is a quadratic form. Taking the square of the timelike coordinate annihilates the information about the causal order between events. It is natural then to try to define the causal distance as a linear form, a square root of the quadratic metric. This is indeed possible; however it requires change in the algebra of its values. Instead of expressing the values of the causal distance in ordinary (commutative) real numbers, taking the square root of the metric leads naturally to the emergence of geometric algebras (also known as Clifford algebra). This is similar to Dirac's procedure of taking the square root of the Lorentzian, which leads to the emergence of the geometric algebra in the Dirac's equation.

Here we will take a different route towards the formulation of the causal distance. We will not start from the spacetime model of general relativity and then take the square root of its metric. Instead, we will first show that a class of posets exists, such that it allows the definition of a multivalued order preserving causal distance function. The geometric algebra in fact will emerge from the construction of the distance function.

To show this, we will use the powerful mathematical formalism of a specific class of posets, the order lattices. There are indications that the two above postulates are the minimal requirements for the spacetime background. However, the task of constructing the class of posets that allows the definition of a causal distance can be made significantly easier if we include four more assumptions about the mathematical structure of the poset. These are the existence of a supremum, local finiteness, the property of semidistributivity and uniform poset dimensionality. We will see that although all four postulates may seem trivial at first, they simplify the mathematical structure of the poset and make the task of model construction easier.

We will leave the task of understanding the structure of a model emerging from the constraints imposed only by the postulates P1 and P2 for a future study.

C. Postulate of supremum (infimum)

The *upper bound* of a subset S of the poset P is defined as a poset element $a \in P$, such that for all $s \in S$, $a \geq s$. Similarly, the lower bound is $b \in P$, such that for all $s \in S$, $s \geq b$. The *supremum* (*infimum*) of a poset's subset is defined as the subset's least upper (greatest lower) bound. A supremum (infimum) of a subset does not necessarily exist. An element $a \in P$ is a supremum of a subset S if and only if [16]

$$(\forall\ b \in P)[((\forall\ s \in S) b \geq s) \Longleftrightarrow b \geq a],$$

and similarly it is an infimum if and only if

$$(\forall\ b \in P)[((\forall\ s \in S) s \geq b) \Longleftrightarrow a \geq b].$$

If it exists, the supremum (infimum) of a subset is unique. In the terminology of the order theory literature, the supremum and infimum are called "join" and "meet" and denoted as $a \vee b$ and $a \wedge b$ respectively.

A very special class of posets, *order lattices*, is defined by insisting that in a lattice, any two elements have a supremum and infimum [16]. Let L be an order lattice; then

$$(\forall\ a, b \in L)(a \vee b \in L \text{ and } a \wedge b \in L).$$

The term lattice here is used strictly for a poset with an existing supremum and infimum, rather than a crystallographic lattice, and the two should not be confused. A *sublattice* of a given lattice is defined as a subset that is a lattice too.

We will postulate the existence of a supremum and an infimum in our poset model P:

P3: Every pair of events in P has a supremum and an infimum.

Postulating the existence of a supremum and infimum in a model of spacetime may seem trivial at first. The existence of a supremum (least upper bound) in a past oriented light cone in general relativity follows naturally from its existence in the set of real numbers. The existence of the supremum in the set of real numbers, however, is not trivial at all, and must be postulated independently. In fact, it is the postulate of supremum that distinguishes the set of real numbers from the set of rational numbers. In the set of rational numbers, the subset of rationals smaller than $\sqrt{2}$, for example, is bound from above; however, the least bound is $\sqrt{2}$ which is not a rational number. One important consequence of the postulate of supremum for the set of real numbers is reflected in the difference in the possibility of defining open and closed intervals in the sets of rationals and reals. Consequently, the topology of the rationals is significantly different than the one of the reals.

The spacetime manifold of general relativity naturally inherits this essential role of the supremum from the set of real numbers. Taking a single point out of the spacetime manifold, for example, changes its topology. The presence of such a singularity in the spacetime manifold, on the other hand, is equivalent to the statement that its past oriented light cone does not have a supremum. Therefore, one should be very careful when considering singularities in the light of supremum and infimum existence. Imposing the requirement that any two points have a supremum and infimum may restrict the possibility of the existence of singularities in the model.

Imposing the postulate of supremum and infimum on our model has global topological consequences too. In a certain class of posets called "crowns," resembling crystallographic lattices with periodic boundary conditions, the lowest upper bound may not be unique, as illustrated in Fig. 3. This analogy of crowns to lattices with periodic boundary conditions points to the fact that we will not be able to treat periodic boundary conditions in our lattice model. Instead, we will have to restrict the applicability of our model to spacetime with asymptotically flat boundary conditions.

The constraints imposed by postulate P3 are therefore very restrictive regarding the topology of the model, and constitute a part of the background in the model. The reason for restricting the model to a lattice is that it allows us to use the powerful mathematical apparatus of mathematical lattice theory. We will show that the manifold topology of spacetime emerges from the model without postulating it as a part of the background. For this we will need some results from the theory of order lattices.

This, on the other hand, does not mean that we cannot use the model in the case of periodic or more general

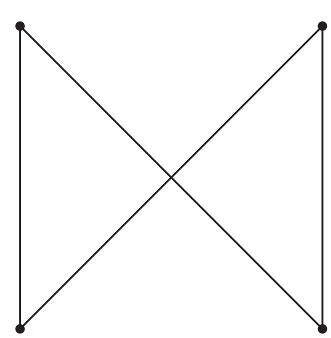

FIG. 3. Poset without supremum or infimum.

nonflat boundary conditions. With the next postulate we will assume that all intervals in the model are finite. We could, in principle, reformulate postulate P3 to be valid only to finite poset intervals that are not crowns. We will avoid these complications for the time being and restrict the model of spacetime to an order lattice. The price that we have to pay is that we have to consider asymptotically flat boundary conditions and exclude the presence of singularities from our model.

D. Postulate of local finiteness

Next we will consider the assignment of measure to subsets in our model of spacetime. In other words, we need to define the concepts of distance, area and volume. In general relativity the manifold topology of spacetime relates its measures of distance and volume to the ones defined on the set of real numbers and its tuples. The measure of an interval $[a, b]$ of reals is simply given by $\mu = b - a$. Although perhaps trivial, this definition carries a range of assumptions about the set of real numbers. One of the crucial assumptions is that the supremum of an interval is also its maximum. This is not trivial and is certainly not true in the set of rational numbers, for example.

Similarly, we would like to define a measure of volume for a poset interval

$$[a, b] = \{\forall\ c \in P | c \geq a \text{ and } b \geq c\}.$$

Let us, however, consider the measure of linear distance first. One possible way of assigning a measure of distance in a poset is to consider its linear subposets, chains, and come up with a mapping from the set of all linear chains to the set of real numbers. This is straightforward in the case of finite chains. The measure of a distance between the end points of a chain can be defined as the total number of points in it. The definition of measure, in that case, is given by simple counting.

The definition of a distance between the end points of an infinite chain, however, is not a simple task. Let us, for example, observe an infinite linear chain L, with a and b being its end points. Let us assume that we have assigned a measure of distance $\mu(L)$ to the chain. Now let us try to construct a chain between a (its minimal point) and a point c at half the length, $\mu(L)/2$, of the chain L. In other words, we will try to cut the chain L in half and take the half that contains a. In the case of a chain (interval) of real numbers this procedure is trivial. The result is the interval $[a, a + \mu(L)/2]$. In our case, however, we do not have a well defined procedure for finding the midpoint between a and b. One would be tempted to establish a 1-to-1 relation among the points in the chain starting with the point a and going upwards, and the chain ending with b and going downwards. The midpoint would be the last of the chosen points. In the case of an infinite chain, however, one would never be able to find the last point. In fact one can establish a 1-to-1 correspondence between the chain $[a, c]$ and a proper subchain of the chain $[c, b]$, by taking, for example, every other point in it. Clearly, this obtained measure of the two resulting chains would be ill defined, and we would not be able to compare the lengths of the chains $[a, c]$ and $[c, b]$. Similarly, we could put in 1-to-1 correspondence the entire chain L and any of its infinite proper subchains. The possibility of putting in 1-to-1 correspondence a set with its proper subset, in fact, can be taken as a definition of infinity.

One possible way out of this situation is to postulate the existence of a midpoint between a and b. After all, the interval $[a, c]$, where c is a point between a and b, is clearly bound from above, and it is natural to assume that a lowest upper bound exists in a linear chain. The existence of a supremum and its inclusion in the real numbers interval $[a, c]$ as its maximal is in fact postulated; otherwise it would be impossible to define the above measure μ on the set of reals. In our model, we have already postulated the existence of a supremum; therefore the set of upper bounds in a linear chain $[a, c]$ must have a least element, the supremum. This however does not guarantee the existence of c, i.e., the maximum element in the interval.

One way to solve the above problems is to postulate the existence of a maximal element in all bound intervals. This would be equivalent to postulating the existence of a well ordering in the poset. This means that we can postulate the existence of a partial order, one in which the element c is the maximum. The well ordering principle states that every set can be well ordered. This statement is not intuitively trivial, certainly not for the set of real numbers, and it is taken as an axiom. We have mentioned already that the well ordering principle, in turn, is equivalent to the axiom of choice. We have therefore assumed the existence of maximal elements in bound posets, indirectly, by assuming the validity of the axiom of choice. Our problems with defining a measure on the interval, however, do not stop here. Although postulating the axiom of choice guarantees the existence of midpoints in any interval in the poset P, it does not say anything about the possibility of defining a measure given the points a, b and a point between them. Quite to the contrary, given the axiom of choice, in certain circumstances, one can create nonmeasurable sets as subsets of the interval $[a, b]$. Namely, it has been shown that in the continuum, in the set of real numbers, one can formulate a noncountable family of subsets of a continuous interval, choose an element from each, and put them together in a set. The newly formed set sometimes is identical (more precisely, it is congruent) to the original set. This is certainly not consistent with our intuition about sets and measures and it is often considered a paradox emerging from the axiom of choice.

The most striking of these paradoxes is the Banach-Tarski paradox [17]. Banach and Tarski have shown that one can take a solid three-dimensional ball, cut it into congruent pieces, then put the pieces together and form two separate balls, each congruent to the original ball. This striking paradox is sometimes illustrated as taking an orange, slicing it into pieces, and putting the pieces together into two new oranges, each identical to the original orange. It is important to emphasize here that the slices never resemble solid orange pieces. The process of slicing and reconstructing the ball resembles a play with needles rather than slicing.

The typical, and strongly nonmathematical, answer to the objection that real-life oranges cannot be multiplied, is the fact that they are not continuous or infinite objects. An orange is namely made of a discrete, and more importantly, finite set of atoms. The axiom of choice is never the cause of a paradox in finite sets. This explanation certainly has little value when considering the spacetime of general relativity. We will however use a similar way to escape from the problems related to the measure of spacetime.

The problems we are facing here regarding the definition of a measure in our model are related, although not directly, to similar problems that have led the proponents of the causal sets theory to assume that the physical spacetime is discrete. The starting point of the causal sets theory is the fact that the topology, geometry and even the differential structure of spacetime can be derived from its causal structure. The metric however is given only up to a conformal factor, which is a reflection of some of the problems we are facing here. The causal structure determines the angles, but not the measure of distance and volume in spacetime. Causal sets theory concludes that spacetime must be discrete [10], since in a discrete set the problem of measure can be solved trivially by reducing it to counting the elements of subsets. This however does not solve the problem in the case of a discrete but infinite bound chain, as we argued above.

We will avoid the above problems related to the definition of a measure in our model by postulating the finiteness of intervals in P:

P4: Every interval $[a, b]$ *in* P *is finite*.

This postulate defines the model as locally finite [15]. We do not, however, assume that the total number of events in the model is finite too. In fact, our model could be understood as one extending to infinity in both the spacelike and timelike directions. Only local intervals, between two concrete events a and b, consist of a finite number of events. Also, postulate P4 implies that our model is discrete. We do not, however, postulate the discreteness of the physical spacetime. The model can equally well be understood as an approximation of a continuous spacetime.

E. Postulate of semidistributivity

One consequence of postulate P4 is the existence of a covering relation $b \succ a$ among the events. Intuitively, the covers of an element are its "closest" neighbors from above. Covers can be present only in discrete posets and they have no analogs in the continuum with the standard partial order of greater or equal $\geq$.

The existence of covers in a discrete poset has profound consequence for the questions related to the definition of a measure, questions that we addressed in the previous section. The covering relation provides a minimal and universal (constant) unit of causal distance. The length of a linear chain can be defined as the number of events in the chain. Similarly, two- and three-dimensional intervals can be assigned surface and volume, respectively, as the total number of events in them. The intuitive picture here is that spacetime is divided into elementary "chunks," with each event being in the center of such a chunk. One can easily think of an analogy to a crystalline lattice of atoms. This picture however is somewhat misleading from the constructive approach we have adopted here. At this point, our model of spacetime is simply a collection of abstract elements with a relation of partial order in it, and several constraints imposed on the partial order by postulates P2 through P4. On the other hand, the possibility of defining volume as the number of atoms in the lattice has at its core the assumption that the model has a topology analogous to that of a crystalline lattice.

Figures 4 and 5 illustrate this point. Only a small portion of the lattices is shown in the figures, and they are assumed to spread to infinity in a uniform way. By uniformity here we mean that each interval of 25 elements in the lattice is identical to the one shown in Figs. 4 and 5 respectively. Both lattices are two dimensional. The difference between the two however is that the two-dimensional interval in

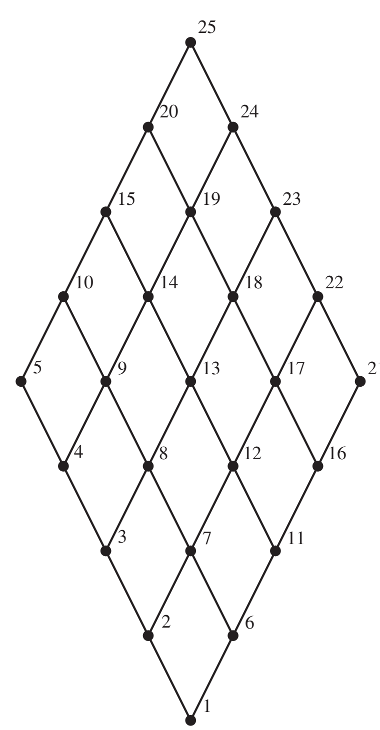

FIG. 4. Uniformly two-dimensional lattice.

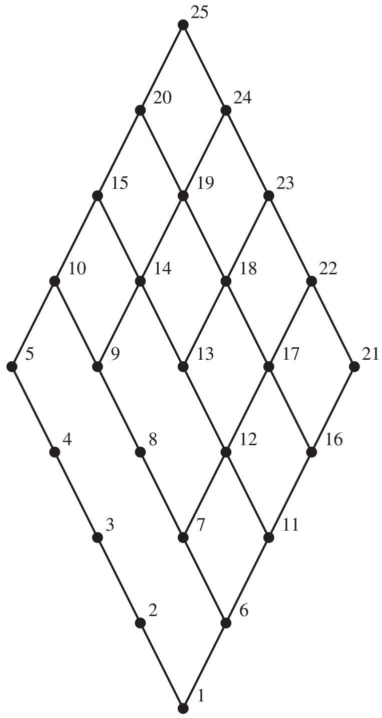

FIG. 5. Two-dimensional lattice with one-dimensional intervals.

Fig. 4 has *uniform dimensionality*, meaning that all intervals, other than the covering pairs, are two dimensional too. It is intuitively clear that this interval can be understood as a discrete approximation of a continuous plane polyhedron. The area of the polyhedron can be naturally expressed as a multiple of an elementary building block with a given unit of area.

That is not the case with the lattice in Fig. 5. Here the interval [7,9] is not two dimensional. Also, although we can assign a measure of area to the interval [13,18], that cannot be done consistently for the interval [7,18], for example. Furthermore, assigning area to the two-dimensional interval [1,13], for example, is meaningless.

If understood as discrete approximations of continuous geometric figures in the plane, the lattices in Figs. 4 and 5 have very different topologies. The one in Fig. 4 can be understood as a discrete approximation of a manifold, i.e., a well defined surface, while the one in Fig. 5 cannot.

The question that arises from the above considerations is whether the manifold topology is essential for the possibility of consistent definition of a measure on a discrete set of events. Does one need a background surface, on which the events are presumed to lie, when constructing a poset as a model of spacetime? This would be a procedure similar to the one adopted by the causal sets methodology of "sprinkling" [10]. There, one assumes the existence of a constraint, a background surface, upon which one sprinkles the discrete set of events. Then a network of causal relations is formed according to the ordering among the coordinates of the events on the original surface. We will show in the next section that the answer to the above question is negative. We do not need to assume the existence of a background that is topologically and geometrically as complex as a manifold. Instead we will attempt in the next section to derive these properties from the set of postulates of order.

With the above considerations in mind, it follows that we need to have two additional postulates that will rule out the existence of the lattices of the type shown in Fig. 5. First, we will forbid the existence of intervals with events that are covered by a single event and cover a single event, i.e., the interval [7,9] in Fig. 5. Another way of saying this is that the fifth postulate forbids creating an intermediate event between two covering ones:

$$a \succ b \succ c \quad \text{and} \quad \nexists d \neq a, c \quad d \succ b \quad \text{or} \quad b \succ d.$$

We will postulate the absence of such cases:

P5: A linearly ordered subset of P, in which only the meet and the join cover, or are covered by, events in the rest of P, is equivalent to (can be reduced to) a covering pair.

Postulate $P5$ imposes a very strict constraint on the poset, by forbidding, among others, the Hasse diagrams: M_3, shown in Fig. 6, L_3 in Fig. 7 and the diagrams L_4 and L_4^∂ in Fig. 8. These lattices play an important role in lattice theory, since their absence as sublattices indicates that the lattice has the special property of being *semidistributive*. Namely, it has been shown that it is sufficient and necessary for a lattice not to contain the lattice S_7^∂, shown in Fig. 9, or any of the lattices M_3, L_3, L_4 and L_4^∂, to satisfy the condition [19]

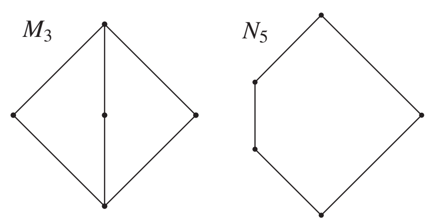

FIG. 6. Lattices M_3 and N_5.

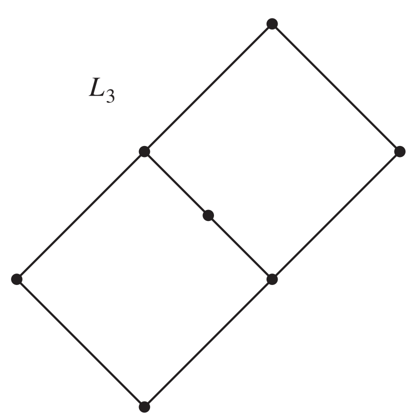

FIG. 7. The nonsemidistributive lattice L_3.

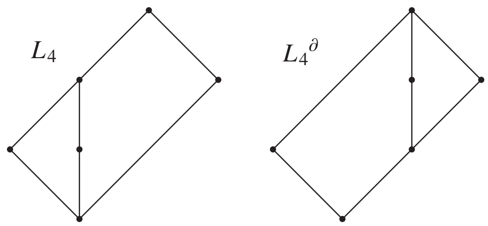

FIG. 8. The nonsemidistributive lattice L_4 and its dual.

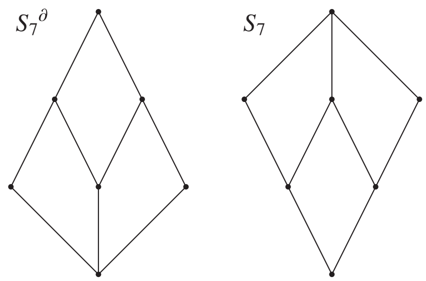

FIG. 9. Locally distributive lattice S_7 and its dual.

$$a \wedge b = a \wedge c \Rightarrow a \wedge b = a \wedge (b \vee c).$$

A lattice that satisfies the above condition is called "meet-semidistributive" [20]. On the other hand, if it does not contain as a sublattice S_7, also shown in Fig. 9, or any of the lattices M_3, L_3, L_4 and L_4^∂, the lattice satisfies the condition

$$a \vee b = a \vee c \Rightarrow a \vee b = a \vee (b \wedge c),$$

and is called "join-semidistributive."

It is important to note that postulate 5 forbids the existence of the Hasse diagrams M_3, L_3, L_4 and L_4^∂ only locally. This means the diagrams made only of covering relations are forbidden. The postulate does not restrict the existence of a sublattice of one of the above types. In that sense the postulate P5 is less restrictive than a requirement of (global) semidistributivity. Postulate P5 provides the model with the necessary condition towards local semidistributivity—hence our choice for the name of this postulate. We will show further in the text that local semidistributivity is one of the key ingredients for the introduction of topology in our model.

F. Postulate of uniform dimensionality

The last postulate addresses the impossibility of assigning measure to intervals of the type [1,12] shown in Fig. 5. The problem here is that although the interval is two dimensional, it is a concatenation of the one-dimensional interval [1,6] and the two-dimensional interval [6,12]. The interval [1,12] therefore is neither uniformly one dimensional nor uniformly two dimensional, and it is not clear how we could assign a measure of either length or area to it.

The notion of uniform dimensionality that we introduce here is illustrated in Fig. 4. The only intervals that are not two dimensional are either covering pairs, or they are on the lattice "boundary." Here we have, however, assumed that the lattice does not have a boundary and it continues to infinity, i.e., none of its elements has a single cover or is covered by a single element. The lattice shown in Fig. 5, although two dimensional, does not have uniform dimensionality, which makes it impossible to assign a measure to some of its intervals. In order to be able to assign measure to each and every interval in the lattice in a consistent way, we will restrict our model to a lattice with uniform dimensionality.

P6: Every interval in the lattice model, with maximal chain length $d + 1$, has the dimensionality d of the model itself.

Here by maximal chain of an interval $[a, b]$ we mean the largest linearly ordered subset of the interval that contains a and b.

III. DUALITY AND CAUSAL DISTANCE

Postulate P6 completes the set of postulates of our lattice model of spacetime. Next, we will attempt to find an explicit form for the causal distance, whose existence is postulated by P2. Also, in conjunction with defining the causal distance, we will construct the topology and geometry of spacetime from the set of constraints imposed on our model by its postulates.

The function of causal distance will serve a dual role. First, in the same way as in the case of the standard metric in general relativity, it will provide a quantification of the distance between two events. The causal distance will provide the information if two events are closer or further apart in the way analogous to the standard Euclidean distance. The (pseudo)metric in general relativity is not positive definite and carries the information about the timelike or spacelike nature of the distance between two events. We will require the same from the causal distance. Second, an aspect of the causal distance, that will not have an analog in the metric of general relativity, is that we will require it to carry the information about the order. We will require that the causal distance is an order preserving function, in the way we specified it when we postulated its existence.

A. Duality

One of the most important consequences of the requirement for causal distance to be order preserving is that it must make the distinction between the interval $[a, b]$ and its reversal $[b, a]$. This is certainly not the case with the metric in general relativity. There, the metric being a quadratic form, the information about the causal order of the event is lost and one needs to specify a global time coordinate in order to keep track of the ordering between them. One of the main consequences of this requirement is that the causal distance cannot be the same for $\hat{d}(a, b)$ and $\hat{d}(b, a)$. This brings us to the concept of duality.

The duality in a partial order is defined as reversal of the order. Given a pair of events (a, b), defined as $b \geq a$, its dual is denoted as $(a, b)^\partial$, and defined as $a \geq b$. Clearly $(a, b)^\partial = (b, a)$. The *principle of duality* for posets states that if a statement is true for a class of posets, then its dual, i.e., the statement with the order relation reversed, is true in the class of duals of the original posets. In the special case when a statement is true for all posets, then its dual is true for all posets too. It is important to note that the principle of duality is valid for a global reversal of the order. That means if a statement is globally true, i.e., true for a Hasse diagram, by the principle of duality its dual will be true if we turn the Hasse diagram upside down. A statement about the uniqueness of the supremum, for example, by duality is valid if and only if its dual statement about the uniqueness of the infimum is valid too.

The mathematical principle of duality clearly mirrors the time reversal symmetry in general relativity. This is another property of spacetime emerging naturally from its causal structure. We will make use of it in our definition of causal distance, keeping in mind the global nature of the principle of duality. The principle applies only to the general properties of the spacetime model, ones that do not depend on the specifics of a given realization (configuration) of the model.

B. Causal distance and Clifford algebras

The existence of a covering relation and the principle of duality will be the two key ingredients in the construction of the causal distance. The causal distance is a mapping from the set of ordered pairs of events to pairs of real numbers. The order in the pair of events and in the pair of real numbers must be preserved by the mapping. This is the key difference between our causal distance and the standard metric of general relativity. We will attempt to define the causal distance in a way that the function will carry the information of order in a pair of events. In general relativity it is the sign of the time coordinate, in addition to the metric, that carries the information of causal order. The metric itself is a quadratic function of the coordinates and it does not carry the information about their sign. This is why we will construct the causal distance as a linear form. Our procedure of defining the causal metric is somewhat analogous to taking a square root of the metric. This procedure of taking the "square root" is such that a new algebra, the Clifford spacetime algebra, will emerge from it. It is similar to the procedure of going from a Laplacian to the Dirac's operator, in which case a specific Clifford algebra, the Dirac's algebra, emerges.

Let us start by considering the covering pairs. These are the minimal sets with defined partial order on them. In other words, they are the elementary units of partial order consisting of a single ordered pair. The only distinction between two such pairs is the partial order; (a, b) is not the same covering pair as (b, a). Given the fact that the fifth postulate specifies there are no other points, or events, between a and b, all covering pairs are equivalent in the sense that taken alone they carry the same amount of information about the partial order.

This fact must be reflected by the "amount" of causal distance to be assigned to each and every covering pair. The causal distance between each and every covering pair must be the same tuple of real numbers. The only distinction must be made when taking the dual of a covering pair. The reversal of the causal order must be reflected in the function of causal distance.

Let us assign a unit of causal distance to a covering pair and its dual. For all covering pairs in P, we have

$$b \succ a \Rightarrow \hat{d}(a, b) = \hat{D}$$

and

$$a \succ b \Rightarrow \hat{d}(a, b) = \hat{D}^\partial.$$

At this point we do not know anything about the mathematical structure of $\hat{D}$ and its dual $\hat{D}^\partial$. They are tuples of real numbers, assigned as constants to each and every covering pair. In order to expand the assignment of causal distance to pairs that are not covers, we will have to define addition and multiplication for the tuples $\hat{D}$ and $\hat{D}^\partial$. At this point, however, we only require that the constant tuples $\hat{D}$ and $\hat{D}^\partial$ are not necessarily identical. Furthermore, by duality we have

$$(\hat{D}^\partial)^\partial = \hat{D}$$

or more generally

$$\hat{d}(a, b) = \hat{d}^\partial(b, a).$$

In order to find the explicit form of $\hat{d}(a, b)$ and its dual, let us observe their symmetric and antisymmetric linear combinations

$$\hat{r} = \hat{d} + \hat{d}^\partial$$

and

$$\hat{t} = \hat{d} - \hat{d}^\partial.$$

The reason for using the letters r and t for the symmetric and antisymmetric parts of $\hat{d}$ is that they behave in a way analogous to the space and time coordinates upon reversal of causality. The reversal of causality in general relativity is given by the change of sign of the time coordinate. In the case of our lattice model it is given by the duality operation. The function $\hat{r}$ does not change upon reversal of causality:

$$\hat{r}^\partial = (\hat{d} + \hat{d}^\partial)^\partial = \hat{d}^\partial + (\hat{d}^\partial)^\partial = \hat{d}^\partial + \hat{d} = \hat{r}.$$

Here we used the fact that reversal of causality, i.e., taking the dual, is an involution. The function $\hat{t}$ is antisymmetric upon duality:

$$\hat{t}^{\partial} = (\hat{d} - \hat{d}^{\partial})^{\partial} = \hat{d}^{\partial} - (\hat{d}^{\partial})^{\partial} = -(\hat{d} - \hat{d}^{\partial}) = -\hat{t}.$$

These are the key properties of the functions $\hat{r}$ and $\hat{t}$ that relate them to the standard relativistic space and time coordinates. We will take this analogy further by requiring that the square of the causal distance corresponds to the standard relativistic metric

$$s^2 = r^2 - t^2.$$

It is important to notice the absence of "hats" in the standard metric. Here r and t are single-valued real coordinates. The reversal of causality does not affect the standard relativistic metric. We will require the same from the square of the causal distance

$$\hat{d}^2 = (\hat{d}^{\partial})^2 = s^2.$$

Taking the squares of $\hat{d}$ and $\hat{d}^{\partial}$ we get

$$\hat{d}^2 = (\hat{r} + \hat{t})^2 = \hat{r}^2 + \hat{r}\,\hat{t} + \hat{t}\,\hat{r} + \hat{t}^2.$$

The same procedure for $\hat{d}^{\partial}$ gives

$$(\hat{d}^{\partial})^2 = (\hat{r}^{\partial} + \hat{t}^{\partial})^2 = (\hat{r} - \hat{t})^2 = \hat{r}^2 - \hat{r}\,\hat{t} - \hat{t}\,\hat{r} + \hat{t}^2.$$

The requirement $\hat{d}^2 = (\hat{d}^{\partial})^2 = s^2$ can be fulfilled if

$$\hat{r}\,\hat{t} + \hat{t}\,\hat{r} = 0, \qquad \hat{r}^2 = r^2$$

and

$$\hat{t}^2 = -t^2.$$

These are the defining relations of the Clifford algebra $Cl_{1,1}(\mathbb{R})$ [21]. The Clifford algebras are typically defined as geometric algebras. This means they "live" in a vector space with a defined scalar and a defined antisymmetric product. Typically the Clifford product is defined as the sum of the scalar and the antisymmetric product. Here we obtained the Clifford algebra by identifying the symmetric part of the causal distance as a measure of space and the antisymmetric as a measure of time.

Next, we will define the causal distance beyond a single covering pair. Let us start with the antisymmetric time coordinate. Let us impose

$$\hat{d}(a, a) = \hat{d}^{\partial}(a, a) = 0.$$

This allows us to define

$$\hat{t}(a, a) = 0.$$

Let us assign a unit time distance from a to a covering b:

$$\hat{t}(b, a) = \hat{D}_t.$$

By duality, from $\hat{t}^{\partial} = -\hat{t}$, we also have

$$\hat{t}(a, b) = -\hat{D}_t.$$

On the other hand, from

$$\hat{t}^2 = -t^2,$$

by convention, we can set

$$\hat{D}_t^2 = -1.$$

This means $\hat{t}$ can be expressed as

$$\hat{t} = t\hat{e}_t,$$

where $\hat{e}_t$ is the basis vector of the Clifford algebra,

$$\hat{e}_t^2 = -1,$$

and t is a single-valued real function $t(a, b)$. From $\hat{t}(a, a) = 0$ it follows that

$$t(a, a) = 0.$$

By imposing $\hat{D}_t^2 = -1$ we made a choice for the unit of t for a covering pair $b \succ a$:

$$t(a, b) = 1.$$

The assignment of a unit distance is universal among all covering pairs; it is the same for all covering pairs in P. The function $t(a, b)$ can be expressed as a difference of an order preserving (isotone) single-valued function. By "order preserving or isotone function" we mean a mapping τ: $P \longrightarrow \mathbb{R}$ such that

$$\text{if } b \geq a \quad \text{then } \tau(b) \geq \tau(a).$$

We have

$$t(a, b) = \tau(b) - \tau(a).$$

In the case of a covering pair, we have

$$\tau(b) = \tau(a) + 1.$$

The existence of an isotone function in P, one that satisfies the above expression, defines P as a *graded* poset [22]. The intuitive picture of a graded poset is one made by piling layers of antichains. By "antichain" here we mean a subset of P in which no two events are causally related. In Fig. 4, for example, the subset $A = 4, 8, 12, 16$ is an antichain, while the subset $C = 1, 2, 3, 8$ is a chain.

The property of a lattice being graded can be equivalently expressed by the Jordan-Dedekind chain condition [20], which states that all maximal chains in an interval have the same length. Here we need to point out that we could have, in principle, adopted the Jordan-Dedekind condition as a postulate of the model and then derived the existence of a causal distance from it. The choice of postulating the causal distance as a fundamental property is due to the fact that it is more physical rather than the purely mathematical statement of the Jordan-Dedekind condition.

By being graded, our lattice model is a discrete approximation of a globally hyperbolic spacetime. Globally hyperbolic spacetime consists of layers of spacelike sets, ordered along a family of timelike curves. The difference between our graded poset and a continuous globally hyperbolic spacetime is that the distance between two spacelike layers in general relativity is infinitesimally small. A single spacelike layer in general relativity does not have a predecessor in the sense of causality. Here, we imposed the discreteness of the model by postulating its finiteness and therefore assured the existence of a predecessor in the structure of antichains layered along chains of events.

Another consequence of P being graded is that the Hasse diagram N_5, shown in Fig. 6, is forbidden in P. In order to ensure that our model is a graded lattice and that the Hasse diagram N_5 does not exist in it, it is sufficient to assume the validity of the condition

$$a \succ a \wedge b \Rightarrow a \vee b \succ b,$$

or its dual, which defines the lattice as *upper* or *lower semimodular*, respectively [20]. This is one of the key properties that will lead us to the topology and geometry of our model.

Figure 9 illustrates the two smallest lattices that satisfy the condition for either upper or lower semimodularity, but not both. An important property of the lattices S_7 and S_7^∂ is that they can intuitively be understood as "built" from "squares," i.e., lattices 2^2 shown in Fig. 10. We can formalize the notion of a square order lattice by defining it as *Boolean*. The two lattices in Fig. 10 are Boolean, meaning that each element a, except the bottom 0 and the top 1, has a complement a', such that

$$a \vee a' = 1$$

and

$$a \wedge a' = 0.$$

The intuitive notion of built from squares can be further formalized by the definition of *local distributivity*. A lattice is locally upper (lower) distributive [20] if every interval $[x, x_+]$ ($[x_-, x]$) is a Boolean lattice, where x_+ (x_-) is the join (meet) of all covers of (covered by) x. Upper local distributivity is a conjunction of meet semidistributivity and upper semimodularity [23]. Lower local distributivity is a conjunction of join semidistributivity and lower semimodularity. These two properties, semidistributivity and semimodularity, on the other hand, are properties of our model, but only locally. This is reflected in the fact that the Hasse diagrams M_3 and N_5 are excluded from our model. Semidistributivity excludes the lattice M_3 and semimodularity excludes the lattice N_5. We should note that the sublattices M_3 and N_5 are excluded only locally, with their order relation restricted to covers only. Global exclusion of M_3 and N_5 as sublattices would define the model as a *distributive* lattice, equivalent to the conjunction of upper and lower local distributivity. In our model, instead, local intervals satisfy upper and/or lower local distributivity, but not necessarily both at the same time. We will define it as *locally Boolean*; a lattice with the property that each interval with a maximal chain length of $d + 1$, where d is the lattice dimension, is a Boolean lattice. Thus, our model of spacetime locally resembles a Boolean lattice.

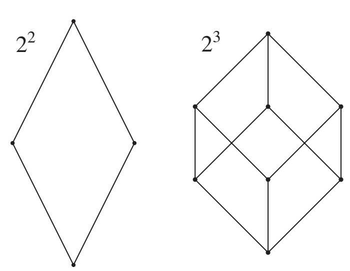

FIG. 10. Two- and three-dimensional Boolean lattice.

This is one of the key results in this work. We managed to construct a poset model (postulate P1), intuitively resembling a crystalline lattice structure built from squares. In order to do that, however, we did not have to assume any geometry, or even topology, for our model. Instead, the structure emerges from the property of local distributivity, or equivalently, the conjunction of the two constraints: semidistributivity and semimodularity. The "crystallinelike" structure, therefore, was obtained by imposing these two rules, or constraints, on the causal (partial order) structure in the model. These rules, on the other hand, were consequence of the postulates of the model. First, from the requirement of the existence of a causal measure (postulate P2) and the discreteness of the model (postulate P4), our poset model emerged as a graded one. We expanded this property by defining our model as semimodular. Second, we imposed the absence of trivial covers in postulate P5. From this, the property of semidistributivity followed. Now our insistence on postulate P3 and the model as an ordered lattice becomes more clear. Namely, we would not be able to come up with any of these results if we did not have at hand the powerful tools of the formalism of (order) lattice theory. Hence the need for postulate P3. Finally, postulate P6 ensures that the model is built from squares in a uniform way. It is important to emphasize here that the postulates were expressed in purely lattice theoretic terminology, without having defined any topology or geometry in the model. In that sense the topology and geometry emerge from the order structure of the model.

Having established the local structure of the spacetime poset, we can now go back to the causal distance and finish the definition of its (by duality) symmetric part, the spatial distance $\hat{r}$. We can define $\hat{r}(a, b)$ locally in each n-dimensional Boolean interval:

$$r(0, x) = t(0, x) \Longleftrightarrow t(0, x) \leq n/2$$
$$r(x, 1) = t(x, 1) \Longleftrightarrow t(x, 1) \leq n/2$$
$$r(0, 1) = 0;$$
$$r(x, x') = 2r(0, x) = 2r(0, x').$$

Defined this way, the time coordinate plays a crucial role, while the space coordinate is expressed through time. This is, however, only possible within a building block, an elementary "hypercube," formed by the local Boolean interval sublattice. In the following sections we will generalize the causal distance to non-Boolean intervals, and we will see that each Boolean interval can be understood as a locally flat spacetime.

C. Matrix representation

The Clifford algebra $\mathrm{Cl}_{1,1}(\mathbb{R})$ can be defined as a vector space $V(\mathbb{R})$ with a defined symmetric scalar product

$$\hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{x},$$

and an antisymmetric "wedge" product

$$\hat{x} \bigwedge \hat{y} = -\hat{y} \bigwedge \hat{x}.$$

The Clifford product is defined as

$$\hat{x}\,\hat{y} = \hat{x} \cdot \hat{y} + \hat{x} \bigwedge \hat{y}.$$

Here we used the $\bigwedge$ symbol for the antisymmetric product to distinguish it from the $\wedge$ symbol for join, typically used in lattice theory.

Given an orthonormal (in the sense of the scalar product) basis $\hat{e}_t$ and $\hat{e}_r$ in $V(\mathbb{R})$, we have

$$\hat{e}_t \hat{e}_r = -\hat{e}_r \hat{e}_t.$$

Given the Clifford product in $V(\mathbb{R})$, the Clifford algebra $\mathrm{Cl}_{1,1}(\mathbb{R})$ is defined by

$$\hat{e}_t^2 = -1$$

and

$$\hat{e}_r^2 = 1.$$

The standard matrix representation of $\mathrm{Cl}_{1,1}(\mathbb{R})$ is

$$\hat{e}_r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{e}_t = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The role of the Clifford product is taken by the standard matrix product.

With the above definitions we can represent the causal distance by matrices and their product. The causal distance of a local pair of events is given by

$$\hat{d}(a,b) = \hat{r}(a,b) + \hat{t}(a,b) = r(a,b)\hat{e}_r + t(a,b)\hat{e}_t.$$

The matrix representation of $\hat{d}$ is

$$\hat{d}(a,b) = \begin{pmatrix} 0 & r-t \\ r+t & 0 \end{pmatrix}.$$

It is important to note that the scalar product in $V(\mathbb{R})$ is related to the Clifford product by

$$2\hat{x} \cdot \hat{y} = \hat{x}\,\hat{y} + \hat{y}\,\hat{x}.$$

For a couple of vectors in $V(\mathbb{R})$ of $\mathrm{Cl}_{1,1}(\mathbb{R})$

$$\hat{x} = x_r \hat{e}_r + x_t \hat{e}_t$$

and

$$\hat{y} = y_r \hat{e}_r + y_t \hat{e}_t,$$

the scalar product is

$$\hat{x}\,\hat{y} = x_r y_r - x_t y_t,$$

where the product on the right-hand side is the standard commutative product of real numbers. We see that $V(\mathbb{R})$ of $\mathrm{Cl}_{1,1}(\mathbb{R})$ is the hyperbolic plane or the two-dimensional Minkowski space, $\mathbb{R}^{1,1}$.

The standard four-dimensional Minkowski space, $\mathbb{R}^{3,1}$, can be obtained by adding two extra spacelike dimensions. The addition of an extra dimension to a vector space certainly has consequences on its Clifford algebra. The Clifford algebra of $\mathbb{R}^{1,1}$ is $\mathrm{Cl}_{1,1}(\mathbb{R})$, while the Clifford algebra of $\mathbb{R}^{3,1}$ is $\mathrm{Cl}_{3,1}(\mathbb{R})$, given by the basis

$$\hat{e}_i^2 = 1, \qquad i = 1,2,3,$$

and

$$\hat{e}_4^2 = -1.$$

Finally, we should note the following important expression for the direct product of Clifford algebras:

$$\mathrm{Cl}_{p+1,q+1}(\mathbb{R}) \simeq \mathrm{Cl}_{p,q}(\mathbb{R}) \otimes \mathrm{Cl}_{1,1}(\mathbb{R}).$$

The matrix representation of the pseudoscalar (bivector) of the $\mathrm{Cl}_{1,1}(\mathbb{R})$ algebra is

$$\hat{e}_{rt} = \hat{e}_r \hat{e}_t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and plays a crucial role in the construction of the matrix representation of $\mathrm{Cl}_{p+1,q+1}(\mathbb{R})$. Given a basis $\hat{e}_i$, $i = 1, \ldots, p+q$ of $\mathrm{Cl}_{p,q}(\mathbb{R})$, we can construct the matrix representation of $\mathrm{Cl}_{p+1,q+1}(\mathbb{R})$ by directly multiplying it with the bivector of $\mathrm{Cl}_{1,1}(\mathbb{R})$ and including the basis of $\mathrm{Cl}_{1,1}(\mathbb{R})$. The basis of $\mathrm{Cl}_{p+1,q+1}(\mathbb{R})$ is therefore given by

$$\hat{e}_i = \begin{pmatrix} \hat{e}_i & 0 \\ 0 & -\hat{e}_i \end{pmatrix}, \qquad i = 1, \ldots, p+q,$$

$$\hat{e}_+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and

$$\hat{e}_- = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The representation of the direct product of the type $\mathrm{Cl}_{p,q}(\mathbb{R}) \otimes \mathrm{Cl}_{1,1}(\mathbb{R})$ will play an important role later on, when we will include a physical field, other than the gravitational, in our model.

IV. MANIFOLD TOPOLOGY AND SPACETIME GEOMETRY FROM LOCALLY BOOLEAN LATTICE

The definition of the causal distance can be generalized globally only in the special case of a distributive, locally Boolean lattice of uniform dimensionality. By *distributive lattice* we mean an order lattice L such that

$$(\forall\ a, b, c \in L)(a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)).$$

The lattices in Figs. 4 and 10, for example, are distributive, while the lattices in Figs. 6 and 9 are not.

In the case when our model, in addition to the structure imposed by the postulates, is also a distributive lattice, it is a model of a flat Minkowski spacetime. Then t and r take the role of time and space coordinates and $\hat{d}$ is related to the Minkowski metric η as $\mathrm{Tr}(\hat{d}^2)/2 = \eta_{\mu\nu} dx^\mu dx^\nu$. In order to generalize the definition of the causal distance to a model of curved spacetime, we will have to introduce topology and geometry, with a corresponding metric, in the lattice model.

To illustrate the difference between a distributive and a nondistributive locally Boolean lattice, let us consider Figs. 4, 11, and 12. The two-dimensional locally Boolean lattice in Fig. 4 is distributive and it is intuitively clear how it relates to the geometry of a flat spacetime. The way the Hasse diagram of the lattice was drawn suggests that it suffices to identify each of the square Boolean sublattices with a geometric square of a flat space.

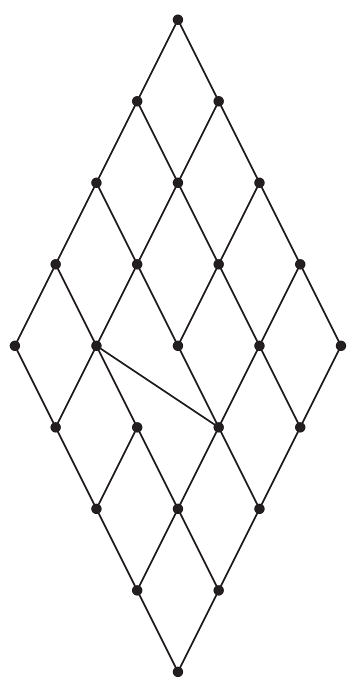

FIG. 11. Hasse diagram of a nondistributive lattice.

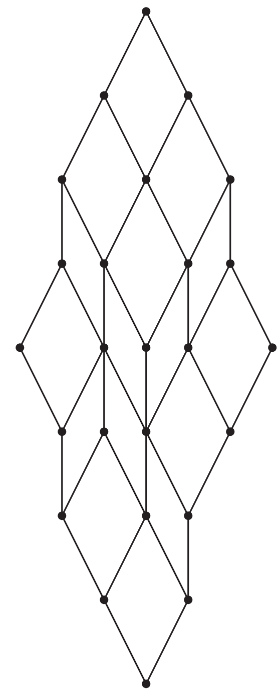

FIG. 12. Alternative Hasse diagram of the lattice in Fig. 11.

The lattice layout in a Hasse diagram is certainly not unique; we can easily come up with a different image of the same order lattice. This is illustrated in Figs. 11 and 12. The two lattices are identical. They are locally Boolean and two dimensional too, but they are not distributive. The only difference between Fig. 11 and 12 is the way we drew the two, equally valid, Hasse diagrams. Intuitively, however, it is clear from the diagram in Fig. 12 that if we identify each of the Boolean sublattices with a geometric square, the nondistributivity of this lattice translates as curvature in the geometric space.

The lattice structures in Figs. 11 and 12 are analogous to the discrete spacetime structures in the dynamically triangulated models. In the dynamically triangulated models, the building blocks, or elementary volume units, are generalized n-dimensional tetrahedrons. In the case of a two-dimensional spacetime, the dynamically triangulated models are collections of triangles. In the language of algebraic topology, a unit of volume (a triangle in 2D and a tetrahedron in a general case) is called a "simplex" and the collection of simplexes is called a "simplicial complex." The lattice in Fig. 11 is practically identical to a 2D CDT model, except for the missing horizontal spacelike connections between two neighboring, but causally unrelated events. Adding such connections would produce a triangulated simplicial complex, identical to a 2D CDT model.

In our model, in the case of a two-dimensional lattice, the unit "building block" is the square lattice in Fig. 10. The three-dimensional unit is the cube in Fig. 10. In general, the building block of an n-dimensional spacetime lattice model is an n-dimensional hypercube Boolean lattice. We will use these building blocks to construct the topology of our model in a way similar to the one used for the dynamical triangulations. One way of doing this is to triangulate the Boolean sublattices, in analogy to a triangulation of a cube. This, however, would introduce connections between events that are not causal in nature. In other words, we would have two types of connected pairs of events, timelike and spacelike. This is true for the dynamically triangulated models, but it is not in the spirit of our model, where we included only causal relations between events, and we will avoid it.

The question that we must ask here is whether we can define the topology of our model without the usage of a simplicial complex. The answer is confirmative, and a powerful mathematical theory of algebraic topology based on a generalization of the notion of a simplicial complex has been developed. The generalized simplex structure is called a "cell" in algebraic topology, and the corresponding complex is called a "cell complex." A cell complex is defined inductively by defining points as 0-cells, pairs of connected points as 1-cells, and inductively, n-cells are constructed by "gluing" n-1-cells along their borders, where the notion of border has a specific topological meaning.

Given the notion of a cell complex, it is fairly straightforward to define the topology of our model by defining events as 0-cells, covering pairs as 1-cells and, inductively, n-dimensional Boolean intervals as n-cells forming a cell complex [24]. Having defined it as a cell complex, we can now investigate the topological properties of our model that follow from the causal (order) structure imposed by the postulates.

The most important property of the cell complex structure of our model is that it is a topological *manifold*. This result is illustrated in Figs. 11 and 12, where it is intuitively clear that a two-dimensional locally Boolean lattice can be understood as a discrete approximation of a continuous two-dimensional surface.

Here we need to mention that we could have obtained our model with a reverse procedure of embedding our lattice in an existing continuous surface (with defined causality flow, i.e., a time coordinate, on it), rather than construct it as an order poset and imposing the constraints on its order structure. Regardless of the curvature of the surface (except in the presence of singularities), we could have found a discretization fine enough to build the surface out of forward oriented squares. This in fact is the procedure used in the causal dynamical triangulations, except that in those cases one uses triangles, not squares, to form the discrete mesh. A similar mesh is used in the Regge calculus [25]. Our model is similar to that of the Regge calculus, except that the only connections in our model are causal, without any spacelike relations explicitly defined.

Another procedure, as mentioned above, of defining a discrete approximation of spacetime is the sprinkling used in the causal sets theory. There, one starts with the continuous manifold of spacetime and sprinkles points at random on it. Then, neighboring points, in the sense of causality inherited from the spacetime manifold, are connected, thus forming a random mesh with well defined causal order, a *causet*. The causet of causal sets theory in many ways is analogous to our model. The difference is that causal sets theory postulates the discreteness of spacetime. The procedure of sprinkling is necessary in causal sets theory, since there is no constructive procedure to create a causet that would be a discrete approximation of a continuous manifold. Furthermore, the causal sets theory considers the continuous manifold an approximation of the actual physical discrete spacetime, since the assumed discreteness is essential for obtaining some of the theory's key results. In our theory we do not assume the discreteness of spacetime. Our model may be understood as a discrete approximation of a continuum.

A. Discrete manifold topology

Let us go back to the topological properties of our model, specifically its manifold structure [18]. The manifold structure of a simplicial complex, and more generally a cell complex, is a discrete analog of the manifold topology in the continuous case. A discrete manifold, in fact, can be understood, and constructed, as a discrete approximation of a continuous manifold. Such discrete cell structures are extensively used in computer aided engineering designs, where a smooth surface, for example, is represented by finite elements, typically triangles, but sometimes also by a more general collection of connected polygons. What they have in common is that they are connected in a way that allows the approximation to be embedded in a continuous surface.

The intuitive notion of a discrete manifold above has a precise meaning, independent of any continuous approximation, in the theory of discrete cell complexes. In order to define a discrete manifold, we will need to define some of the essential concepts in algebraic (discrete) topology. Having defined inductively a cell complex above, we will now define its boundary. The precise algebraic topological definition of a *boundary* of an n-cell complex is the collection of n-1-cells not shared by any pair of n-cells in the complex. An n-cell *element* (n-ball) is a cell complex topologically equivalent to an n-simplex. A cell sphere is a cell complex whose boundary vanishes, i.e., the sphere is its own boundary. In other words, an n-sphere is an n-complex equivalent to the boundary of an $n+1$-simplex. Clearly, a ball and a sphere are homeomorphic to (can be continuously deformed into) a two-dimensional ball and sphere, respectively. Finally, we will define the *neighborhood* of a point (0-cell) as the cell complex constructed from all n-cells to which the point belongs.

With the definitions above we are ready to give a precise meaning to the notion of a discrete manifold. In our model of spacetime, the cells are identified with the Boolean sublattices, so a neighborhood of an event (0-cell) is formed by all the Boolean lattices to which the event belongs. A *cell manifold* in combinatorial topology is defined as a cell complex such that each point (0-cell) has a neighborhood that is a ball or a sphere.

Similarly to the continuous case, the discrete manifold structure is given by the existence of local discrete analogs of topological balls as neighborhoods of each point. In the continuous case, each local region of a manifold can be approximated with a Euclidean space by mapping a surrounding ball at each point of the manifold to such Euclidean space. Similarly, a cell manifold is characterized by the fact that each of its points is surrounded by points that form an n-cell ball or an n-1-cell sphere.

Now we are ready to investigate the topology of our model of spacetime. The very fact that our model is built from n-cubes glued to one another provides an intuitive picture that each point in it will be surrounded by a complex topologically equivalent to a ball. We have seen that the n-cube building blocks emerge from the fact that our model is a locally Boolean lattice. More precisely each interval of length equal to the lattice global dimensionality is a Boolean lattice. In a uniformly two-dimensional lattice this means that if an event x is the supremum (or dually infimum) of more than one Boolean sublattice, pairs of them share at most one covering pair.

Next, we need to consider all of the Boolean sublattices (squares) in which x is neither the supremum nor the infimum. In two dimensions there can be only two such sublattices, formed by the two covers on the "sides" of the Boolean intervals. More precisely, in the two-dimensional case, the Boolean sublattices to which x belongs form a polygon built from squares, i.e., a cell 2-ball (homeomorphic to a Euclidean 2-ball or polygon). This implies that, in the case of uniform local two-dimensionality, the model is a manifold.

In the case of dimensionality higher than 2, the above conjecture, although perhaps intuitively clear, is not trivial to prove and we will assume its validity at this point.

B. Geometry and clifford-valued metric

The geometry of the model [26,27] is defined by identifying the interior of each n-cell with a bound region of an n-dimensional Minkowski spacetime. This assignment is analogous to the geometry of spacetime in Regge calculus [25]. It is important to note that the causal structure of our model defines the geometry of spacetime up to a volume factor. In other words, we need to assign a unit of volume to each n-cell. As we mentioned before, since the model is a discrete one, the assignment of a volume of a region in the model is reduced to counting the number of n-cells that make up the region.

While each n-cell is considered a region of flat spacetime, the curvature is concentrated on n-2-cells. This can be easily illustrated in a two-dimensional spacetime. Each Boolean sublattice is representing a geometrical square in the physical spacetime. An event x and its two covers are considered to form a 90° angle. If an event belongs to four Boolean lattices (i.e., four squares), the curvature of spacetime at that event is zero. In that case the event covers and is covered by two other events. The event and its neighborhood, the interval $[x_-, x_+]$, can be shown flattened on a sheet of paper, as is the case with any of the events in Fig. 4, for example. If the event however is covered by more or less than two events, as is the case with some of the events in Fig. 9, the curvature of the model at that point is not zero. This is illustrated in Fig. 12. It is intuitively clear that if the square Boolean lattices in Fig. 12 were geometric squares, it would not be possible to lay out the model flat on a sheet of paper.

In addition to the curvature, most of the concepts of differential geometry can be generalized to the discrete case. The concept of tangent space, for example, generalizes naturally by assigning a distributive, locally Boolean, uniformly n-dimensional lattice, and a corresponding flat spacetime to an n-cell. It is straightforward to define coordinates and geometric algebra on it by $\hat{e}_0 = \hat{e}_t$ and the direct product $\hat{e}_i = \hat{e}_r \otimes \hat{\sigma}_i$, where $\hat{\sigma}_i$ is the basis in $\mathrm{Cl}_{n-1,0}(\mathbb{R})$. The matrix representation of a basis of the flat n-dimensional Minkovski spacetime is

$$\hat{e}_i = \begin{pmatrix} 0 & \hat{e}_i \\ -\hat{e}_i & 0 \end{pmatrix}, \qquad i = 1, \ldots, n-1,$$

$$\hat{e}_n = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where $\hat{\sigma}_i$, $i = 1, \ldots, n-1$, is the basis of the Clifford algebra $\mathrm{Cl}_{n-1,0}(\mathbb{R})$. For a four-dimensional spacetime, this is the three-dimensional Euclidean space, and the basis $\hat{\sigma}_i$ is typically represented by the Pauli matrices. The inclusion of the $\hat{\sigma}_i$ basis, however, is not necessary and we will omit it in the rest of the paper.

The relation between the metric and the causal distance in a flat Minkovski spacetime can now be generalized as

$$\hat{d}^2 = \hat{g}_{\mu,\nu} d\hat{x}^\mu d\hat{x}^\nu,$$

where $\hat{x}_\mu$ are Clifford-valued local coordinates, and $\hat{g}$ is the Clifford-valued metric with

$$\hat{g}_{0,i} = g_{0,i}\hat{e}_{rt}$$

and

$$g_{0,i} = -g_{i,0}$$

for $i \neq 0$ and

$$\det \hat{g} > 0.$$

The Clifford-valued metric, therefore, has scalar symmetric and pseudoscalar antisymmetric components and, more importantly, it is Euclidean. It is important to note that operation of multiplication in the determinant of the metric is the noncommutative Clifford product.

The Clifford-valued metric of the flat spacetime is generalized naturally by the procedure analogous to the one in the Regge calculus. For each pair of n-cells, a coordinate system can be introduced, such that the metric is constant in both cells, except at the n-2-cells on the boundary between the two n-cells. Each of the n-2-cells is associated with an excess angle, given by the excess of n-cells that share it, which in turn is the measure of the curvature on it.

It is important to remind ourselves that we obtained the geometric picture above constructively, by identifying each Boolean sublattice as an n-dimensional geometric hypercube. It is the lattice order properties of the model, i.e., the fact that the model is a locally Boolean lattice, that allow this identification to result in a discrete version of spacetime with all of its topological and geometric properties. This is a reversal of the procedure in Regge calculus, where the continuous spacetime, with all of its topological and geometric properties, is shown to allow its discretization. The discrete model in Regge calculus is an approximation of the continuous spacetime. Here, in our model, the continuum can be considered an approximation of our model. We do not, however, postulate the discreteness of the physical spacetime itself. Our model can also be understood as a discrete approximation of a continuous spacetime, which in turn is recovered in the limit of the volume of each n-cell being infinitesimally small.

Our model is discrete; therefore its cardinality is the same as the one of a subset of the set of natural numbers. In other words, it is countable. The spacetime continuum model of general relativity has the cardinality of the set of real numbers. It is uncountably infinite. It is important to note that when going from the discrete to a continuous model, and vice versa, we have made a silent assumption about a transition of the cardinality of the model. The question of the cardinality of the physical spacetime is certainly not a trivial one; the proponents of the causal sets theory, for example, are decidedly on the side of a discrete spacetime. In that sense, we should emphasize again that although we have postulated our model as discrete, we do not postulate the discreteness of the physical spacetime. A future modification of our model may be defined as a continuous one, i.e., as a poset generalization of the continuum. We will leave this for a future study.

V. CLIFFORD-VALUED ACTION

Having defined the topology and geometry of our model of spacetime, we can now go on and add physical fields in it [28–34]. The inclusion of physical fields is crucial for the solution of the sign problem. It is exactly the presence of physical field terms that prevents the formulation of a positive definite Lagrangian of the gravitational field.

Let us include a physical real scalar field ϕ in the model [35], in a way consistent with the causal order of spacetime. We will do this by defining it as an ordered pair of an event and a one-dimensional vector field $(x, \phi(x))$. This can also be expressed as an ordered pair of the tuple of coordinates of the event, $\hat{x} = (\hat{r}, \hat{t})$ and the field $\hat{\phi}(\hat{x})$. Here, we will have to specify the meaning of the hat on the field $\hat{\phi}$. In other words, our model is quantified by the values of a Clifford algebra, so we will have to specify how the field fits in it.

Let us remind ourselves that the reason for using the Clifford algebra is that the concept of causality and its quantitative expression, the causal distance, are expressed naturally in it. Our physical field, being a function with a Clifford-valued argument, must also obtain values in the Clifford algebra. One choice is to insist on the fields being Clifford scalars, i.e., represented by the identity matrix $\hat{1}$ multiplied by a number ϕ. This however will make the field fully insensitive to the inversion of causality, something we would like to avoid. We have seen above that reversal of causality is reflected in our model by the involution of duality

$$(\hat{d}^{\partial})^{\partial} = \hat{d}.$$

This means that the reversal of causality will be represented by one of the matrix involutions—matrix transposition, for example. In general, the causal reversal can be expressed as an involution in the Clifford algebra, independent of the matrix representation. Going back to the physical field, if we assign to it only scalar values, it will not be affected by an involution. This means we will not be able to capture properly any effect of causality reversal on it.

In order to include the field in a nontrivial way into the causal structure of our model we will assign it values of the pseudoscalar, the bivector of the Clifford algebra $\mathrm{Cl}_{1,1}(\mathbb{R})$. Another aspect of this choice is that the Clifford algebra of spacetime and the field is given by the direct product $\mathrm{Cl}_{n-1,1}(\mathbb{R}) \otimes \mathrm{Cl}_{1,0}(\mathbb{R})$. The standard way of defining this direct product is to directly multiply the basis bivector (pseudoscalar) of $\mathrm{Cl}_{1,1}(\mathbb{R})$ with the basis vector of $\mathrm{Cl}_{1,0}(\mathbb{R})$ [21]. The field ϕ then becomes Clifford pseudoscalar $\hat{\phi} = \phi \hat{e}_{rt}$, with the standard representation

$$\hat{\phi} = \begin{pmatrix} \phi & 0 \\ 0 & -\phi \end{pmatrix}.$$

This way, the field obtains a geometrical interpretation. It can be understood as a degree of freedom in addition to the ones of spacetime. This is certainly desirable when considering fields beyond a simple scalar one.

Having defined the scalar field as a Clifford-valued bivector, we are now ready to define the Clifford-valued action too. Having taken the limit of infinitesimally small elementary volume assigned to the Boolean lattice, our model can be approximated by a continuous manifold. The continuous approximation carries over the Clifford algebra, now defined at each of its points. We can therefore define Clifford derivatives

$$\hat{\partial}_0 = \hat{e}_t \partial_0$$

and

$$\hat{\partial}_i = \hat{e}_r \partial_i,$$

where ∂ is the ordinary partial derivative. These expressions are analogs to the ones for a vector derivative $\vec{e}_x \partial/\partial x$. The Clifford derivatives carry the algebraic structure of the Clifford product, so we have

$$\hat{\partial}_t \hat{\partial}_r = -\hat{\partial}_r \hat{\partial}_t.$$

Also, the field $\hat{\phi}$ anticommutes with the derivatives

$$\hat{\partial}_\mu \hat{\phi} \hat{\partial}_\nu \hat{\phi} = -\hat{\partial}_\mu \phi \hat{\partial}_\nu \phi.$$

Given that the metric has either scalar or bivector values, we have

$$g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = -\hat{g}^{\mu\nu} \hat{\partial}_\mu \hat{\phi} \hat{\partial}_\nu \hat{\phi}.$$

With this, we are ready to define the Clifford-valued action. Rescaling Ω by $k = \sqrt{4\pi G/3}$ and having

$$\hat{\Omega}_{;a} = \hat{\partial}_a \Omega,$$

the kinetic and mass terms in the Einstein-Hilbert action, with scalar field ϕ, become

$$I[\tilde{g}, \phi] = \frac{1}{16\pi G} \int \bigg(-6 \hat{g}^{ab} \hat{\Omega}_{;a} \hat{\Omega}_{;b} - \frac{1}{2} \Omega^2 \hat{g}^{ab} \hat{\phi}_{;a} \hat{\phi}_{;b} + \frac{1}{2} \Omega^4 m^2 \hat{\phi}^2 \bigg) \sqrt{\hat{g}} d^4 x.$$

We should note the absence of trace in the Clifford-valued action. The reason is that the action is Clifford scalar. This means that the expression can be understood either in matrix representation, in which case the trace would add a trivial multiplicative factor, or as an abstract Clifford-valued expression resulting in a real number, i.e., a Clifford scalar. In other words, the expression for action is representation free, and can be used for direct calculation of the partition function.

In the absence of a gravitational field, the case of flat spacetime, the above action is equivalent to the Klein-Gordon one. Varying $\hat{\phi}$ and having

$$\hat{\partial}(\delta \hat{\phi}\, \hat{\partial}\, \hat{\phi}) = (\hat{\partial} \delta \hat{\phi}) \hat{\partial}\, \hat{\phi} - \delta \hat{\phi} \hat{\partial}^2 \hat{\phi},$$

in the case of flat spacetime, leads to the Euler-Lagrange equation

$$\hat{\eta}^{\mu\nu} \hat{\partial}_\mu \hat{\partial}_\nu \hat{\phi} - m^2 \hat{\phi} = 0, \quad \det(\hat{\eta}) > 0,$$

from which we get the Klein-Gordon equation

$$\eta^{\mu\nu} \partial_\mu \partial_\nu \phi - m^2 \phi = 0, \quad \det(\eta) < 0,$$

by multiplying with $\hat{e}_{rt}$ and calculating the Clifford products.

Perhaps the most important result in this letter is the fact the two kinetic terms in the action I have the same sign. Given that $\det(\hat{g}) > 0$, the action is Euclidean. This opens up the possibility to treat $I[\hat{g}, \hat{\phi}]$ as a positive definite and Euclidean action. The mass term, however, has the sign opposite of the kinetic terms. We will therefore introduce a negative mass parameter

$$m_n^2 = -m^2$$

and the action can be defined as positive definite

$$I_n = -I[\hat{g}, \hat{\phi}].$$

The partition function of the model can be obtained from the corresponding path integral by defining an imaginary-valued temperature $\beta = i/\hbar$,

$$Z = \int D[g] D[\phi] e^{-\beta I_n[\hat{g}, \hat{\phi}]}.$$

VI. CONCLUSION

We constructed a model of spacetime as a set of causally related events, with as little of an assumed background as possible. Its background independence is evident in the fact that we postulated only the order structure, not its topology or geometry. The topology and the geometry of the model are emerging from its causal structure [36–38].

We build the model from six postulates. The first two of the postulates are motivated by the physics of spacetime. The first postulate states that the causal relationship among events in spacetime is expressed as a relation of partial order in the model. The second postulate is given as a general statement that the causality can be physically measured, and therefore can be expressed in a quantitative way in our model. The difference between the quantitative expression of causality in our case and general relativity is that we do not require the presence of a time coordinate, a part of the background, in addition to the background independent metric.

The last four postulates fix the mathematical structure of the model, one that allows the definition of a background independent measure of causality, the causal distance. One of the novelties of this work is the use of Clifford algebra to express the causal order in a quantitative way. The causal distance is a Clifford-valued function on the poset model. One significant consequence of this is that the physical fields included in the model cannot be expressed as commutative real numbers, but must be introduced in a way consistent with the spacetime Clifford algebra.

Another significant consequence of the introduction of the Clifford algebra is that the kinetic terms in the action have the same sign, which in turn leads to a positive definite action, perhaps the most important result of this work. The positive action will allow us to create a computation model, similar to the causal dynamical triangulations model, but without the need to single out the time coordinate. The time "foliation," imposed as a constraint in the causal dynamical triangulations, is mirrored in our model by the fact that it is a graded poset.

The discrete model, as formulated here, can be understood either as exact, or as an approximation of the continuous physical spacetime. If the former case is true, a physical scale must exist, the Planck length being the obvious candidate, at which Lorentz invariance in spacetime fails. Regardless of whether this is correct or not, we hope that future computational studies of the discrete model presented here may help us understand the elementary excitations of the quantized gravitational field, and perhaps provide clues if, and how, the model is related to other approaches to quantum gravity.

---

[1] E. Nelson, J. Funct. Anal. **12**, 97 (1973).
[2] *Euclidean Quantum Gravity*, edited by G. W. Gibbons and S. W. Hawking (World Scientific Publishing, Singapore, 1993).
[3] J. Ambjørn, J. Jurkiewicz, and R. Loll, Phys. Rev. Lett. **85**, 924 (2000).
[4] J. Ambjorn, J. Jurkiewicz, and R. Loll, Nucl. Phys. **B610**, 347 (2001).
[5] J. Ambjorn, J. Jurkiewicz, and R. Loll, Phys. Rev. D **64**, 044011 (2001).
[6] J. Ambjorn, J. Jurkiewicz, and R. Loll, Phys. Rev. Lett. **93**, 131301 (2004).
[7] J. Ambjorn, J. Jurkiewicz, and R. Loll, Phys. Rev. D **72**, 064014 (2005).
[8] J. Ambjorn, A. Dasgupta, J. Jurkiewicz, and R. Loll, Nucl. Phys. **106–107**, 977 (2002).
[9] L. Smolin, arXiv:hep-th/0507235v1.
[10] L. Bombelli, J. Lee, D. Meyer, and R. D. Sorkin, Phys. Rev. Lett. **59**, 521 (1987).
[11] D. P. Rideout and R. D. Sorkin, Phys. Rev. D **61**, 024002 (1999).
[12] S. Major, D. Rideout, and S. Surya, J. Math. Phys. (N.Y.) **48**, 032501 (2007).
[13] E. C. Zeeman, J. Math. Phys. (N.Y.) **5**, 490 (1964).
[14] S. W. Hawking, A. R. King, and P. J. McCarthy, J. Math. Phys. (N.Y.) **17**, 174 (1976).
[15] R. P. Stanley, *Enumerative Combinatorics* (Cambridge University Press, Cambridge, England, 2011), Vol. 1.
[16] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order* (Cambridge University Press, Cambridge, England, 2010).
[17] T. J. Jech, *The Axiom of Choice* (Dover Publications, Inc., New York, 1973).
[18] L. Bombelli, A. Corichi, and O. Winkler, Classical Quantum Gravity **26**, 245012 (2009).
[19] B. A. Davey, W. Poguntke, and I. Rival, Algebra Universalis **5**, 72 (1975).
[20] M. Stern, *Semimodular Lattices: Theory and Applications* (Cambridge University Press, Cambridge, England, 1999).
[21] P. Lounesto, *Clifford Algebras and Spinors* (Cambridge University Press, Cambridge, England, 2001).
[22] G. Birkhoff, *Lattice Theory* (American Mathematical Society, Providence, 1995).
[23] R. P. Dilworth and P. Crawley, Trans. Am. Math. Soc. **96**, 1 (1960).
[24] A. Hatcher, *Algebraic Topology* (Cambridge University Press, Cambridge, England, 2002).
[25] T. Regge, Nuovo Cimento **19**, 558 (1961).
[26] J. Ambjorn, A. Gorlich, J. Jurkiewicz, and R. Loll, Phys. Lett. B **690**, 420 (2010).
[27] L. Bombelli and D. A. Meyer, Phys. Lett. **141A**, 226 (1989).
[28] M. R. Atkin and S. Zohren, Phys. Lett. B **712**, 445 (2012).
[29] J. Ambjorn, K. N. Anagnostopoulos, R. Loll, and I. Pushkina, Nucl. Phys. **B807**, 251 (2009).
[30] J. Ambjorn, A. T. Gorlich, J. Jurkiewicz, and H.-G. Zhang, Nucl. Phys. **B863**, 421 (2012).
[31] D. Benedetti and R. Loll, Gen. Relativ. Gravit. **39**, 863 (2007).
[32] I. Khavkine, R. Loll, and P. Reska, Classical Quantum Gravity **27**, 185025 (2010).
[33] R. D. Sorkin, J. Phys. Conf. Ser. **306**, 012017 (2011).
[34] S. Johnston, Phys. Rev. Lett. **103**, 180401 (2009).
[35] R. Sverdlov and L. Bombelli, Classical Quantum Gravity **26**, 075011 (2009).
[36] J. Ambjorn, R. Loll, and I. Pushkina, Ann. Phys. (Berlin) **19**, 186 (2010).
[37] R. Loll, Classical Quantum Gravity **25**, 114006 (2008).
[38] D. Rideout and P. Wallden, J. Phys. Conf. Ser. **174**, 012017 (2009).