

Holographic thermalization with radial flow

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Recently, a lot of effort has been put into describing the thermalization of the quark-gluon plasma using the gauge/gravity duality. In this context, we here present a full numerical solution of the early far-from-equilibrium formation of the plasma, which is expanding radially in the transverse plane and is boost invariant along the collision axis. This can model the early stage of a head-on relativistic heavy ion collision. The resulting momentum distribution quickly reaches local equilibrium, after which it can be evolved using ordinary hydrodynamics. We comment on general implications for these hydrodynamic simulations, both for central and noncentral collisions, and including fluctuations in the initial state.

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I. INTRODUCTION

Describing the very early stage of a relativistic heavy ion collision has remained challenging, in particular when the state is far from equilibrium. In this stage, QCD is partially strongly coupled and impossible to solve in general. However, a lot of progress has been made by employing a dual gravitational description (see Ref. [1] and its references), the two seminal examples being fast thermalization [2,3] and very small viscosity in the consequent hydrodynamic regime [4].

These dual descriptions have been studied for both near-equilibrium [4–6] and (numerically) far-from-equilibrium physics [3,7–12]. Yet these studies have always assumed homogeneity in the transverse plane, which in particular makes it impossible to study radial flow. Recently, Bantilan *et al.* [13] have performed a simulation having both longitudinal and radial expansion. However, in their setup, it was not possible to have boost invariance in the longitudinal direction, and more importantly, the evolution was always in the hydrodynamic regime. In this paper, we will present a numerical simulation having radial flow in the transverse plane, boost invariance in the longitudinal direction, and being far from equilibrium initially.

Essentially, we follow the numerical scheme worked out in Ref. [8], but we assume boost invariance in the longitudinal direction and allow for nontrivial radial dynamics in the transverse plane. This keeps the gravitational problem $2 + 1$ dimensional, which means it can be solved using pseudospectral evolution. As initial conditions, we present two simple models: the first starts with a blob of energy with a diameter of approximately 14 fm in vacuum, whereas the second has a blob of about 1 fm in a bath of half the peak energy density. These initial states can model the overall thermalization of a central collision and the evolution of an initial fluctuation in such a collision. (Fluctuations recently became a topic of much interest;

see, for instance, Ref. [14].) For the bulk metric, we started with vacuum anti-de Sitter (AdS) but adapted the near-boundary coefficients for the energy density and the pressures according to the Glauber model.

The results of our simulations are both intuitive and encouraging. Firstly, we find that our geometry thermalizes very quickly, confirming previous studies. Secondly, our radial velocity profile at the end of our evolution is similar to typical initial conditions used for a hydrodynamical evolution. So these radial velocities serve as a confirmation that current hydrodynamic simulations do not have to be modified dramatically, but they also provide an improvement for hydrodynamic evolutions.

II. HOLOGRAPHIC MODEL

As our coordinates in the field theory, it is natural to use proper time τ and rapidity y , defined by $t = \tau \cosh y$ and $x_{\parallel} = \tau \sinh y$, and angular coordinates ρ and θ in the transverse plane. The assumptions of boost invariance and rotational symmetry then imply that all functions are independent of y and θ . In these coordinates, the flat metric of the field theory reads

$$ds_B^2 = -d\tau^2 + d\rho^2 + \rho^2 d\theta^2 + \tau^2 dy^2. \quad (1)$$

Given these symmetries and using generalized Eddington-Finkelstein coordinates, we can write the dual bulk metric as

$$ds^2 = -Ad\tau^2 + \Sigma^2(e^{-B-C}dy^2 + e^B d\rho^2 + e^C d\theta^2) + 2drd\tau + 2Fd\rho d\tau, \quad (2)$$

where A , B , C , Σ and F are all functions of τ , ρ and the bulk radial coordinate r . Solving Einstein's equations (with cosmological constant $\Lambda = -6$) order by order in r , demanding that $ds^2|_{r=\infty} = r^2 ds_B^2$, gives us the near-boundary expansion of the bulk metric:

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$$\begin{aligned}
A &= r^2 + \frac{a_4(\tau, \rho)}{r^2} + O(r^{-3}), \\
B &= -\frac{2}{3} \log(\tau \rho) + \frac{3r\tau(1-2r\tau) - 2}{9r^3\tau^3} + \frac{b_4(\tau, \rho)}{r^4} + O(r^{-5}), \\
C &= -\frac{2}{3} \log(\tau/\rho^2) + \frac{3r\tau(1-2r\tau) - 2}{9r^3\tau^3} + \frac{c_4(\tau, \rho)}{r^4} + O(r^{-5}), \\
\Sigma &= \rho^{1/3} \frac{3r\tau(9r\tau(3r\tau(3r\tau+1) - 1) + 5) - 10}{243r^3\tau^{11/3}} + O(r^{-4}), \\
F &= \frac{f_4(\tau, \rho)}{r^2} + O(r^{-3}), \tag{3}
\end{aligned}$$

where in these expressions we have fixed a residual gauge freedom $r \rightarrow r + \xi(\tau, \rho)$ by demanding $\partial_r A|_{r=\infty} = 2r$. The normalizable modes of the metric, a_4 , b_4 , c_4 and f_4 , depend on the full bulk geometry and cannot be determined from a near-boundary expansion. Using the gauge/gravity duality, we can now determine the stress tensor of the dual field theory [15], which has five independent nonzero components:

$$\begin{aligned}
\varepsilon &\equiv -T_\tau^\tau = -\frac{27}{8\pi^2} a_4, & s &\equiv T_\rho^\rho = \frac{9}{2\pi^2} f_4, \\
p_\rho &\equiv T_\rho^\rho = \frac{9}{2\pi^2} \left(-\frac{1}{6\tau^4} - \frac{1}{4} a_4 + b_4 \right), \\
p_\theta &\equiv T_\theta^\theta = \frac{9}{2\pi^2} \left(-\frac{1}{6\tau^4} - \frac{1}{4} a_4 + c_4 \right), \\
p_y &\equiv T_y^y = \varepsilon - p_\rho - p_\theta, \tag{4}
\end{aligned}$$

all functions of τ and ρ , where we set the number of colors $N_c = 3$. The conservation of the stress tensor implies that

$$\begin{aligned}
\partial_\tau a_4 &= -\frac{12\tau^4(\rho(\tau\partial_\rho f_4 + a_4 + b_4 + c_4) + \tau f_4) - 4\rho}{9\rho\tau^5}, \\
\partial_\tau f_4 &= -\frac{1}{4} \partial_\rho a_4 + \partial_\rho b_4 + \frac{b_4 - c_4}{\rho} - \frac{f_4}{\tau}. \tag{5}
\end{aligned}$$

Our model basically contains two scales: the initial energy density and the characteristic scale in the radial direction. We can, however, make use of the scale invariance of the field theory to rescale our coordinates such that at $\tau = 0.6$ fm the energy density at the origin equals $\varepsilon_0 = 187$ GeV/fm³ [16]. We choose this combination to reproduce the final multiplicities of central heavy-ion collisions at LHC [18]. For the radial profile, we then consider two types of initial conditions, specified at some small time $\tau_{\text{in}} \approx 0.12$ fm [19]. The first is a model for a head-on collision, where the shape of the energy density is provided by the Glauber model, having an approximate radius of 6.5 fm. The second energy density profile models one fluctuation in the initial state of such a collision. We take a Gaussian of width 0.5 fm for this profile (see Fig. 1). For both initial conditions, we assume that initially there is no radial momentum, such that $f_4(\tau_{\text{in}}, \rho) = 0$.

Importantly, we must also specify the metric functions $B(r, \tau_{\text{in}}, \rho)$ and $C(r, \tau_{\text{in}}, \rho)$ on a full time slice of the bulk

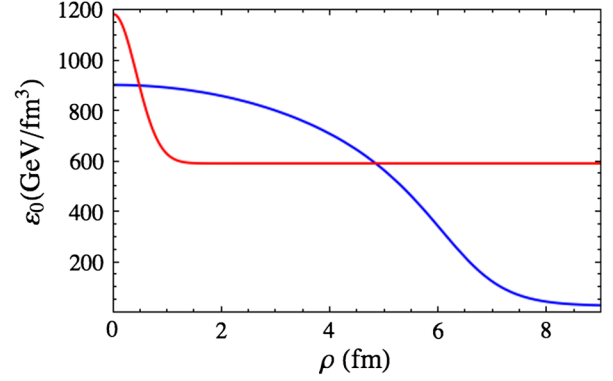


FIG. 1 (color online). The initial energy density profiles at $\tau_{\text{in}} = 0.12$ fm as a function of the distance to the origin. The wide blue curve models a central heavy-ion collision; the narrow red curve models a fluctuation in such a collision.

AdS geometry. These two functions, together with a_4 , f_4 and the Einstein equations, specify the complete metric and its time derivative on a time slice [8]. In principle, these functions should follow from a model describing the very first weakly coupled stage after the collision, such as the Glauber model or the Color Glass Condensate. However, these models themselves contain significant uncertainties and, more importantly, it is not clear how to map them to this gravitational setting. Therefore, we made a simple choice, where B and C are the same functions as in vacuum AdS, but with modified b_4 and c_4 , such that the longitudinal pressure p_y vanishes initially.

Having specified the initial and boundary conditions, we can solve Einstein's equations numerically [20], using essentially the same scheme as in Ref. [8]. One small difference is in the required boundary conditions in the ρ direction, which in this case means smoothness at the origin and at infinity. As in Ref. [8], we added a small (3%) regulator energy density and checked that our results do not depend on this regulator.

III. RESULTS

After determining the stress tensor, one can extract the radial velocity, defined by the boost after which there is no momentum flow. Figure 2 shows this velocity times the energy density, which gives a good measure of the momentum flow. The radial velocity, together with the stress tensor in the local rest frame, can be used to compute the stress tensor according to hydrodynamics. Although initially there will not be local equilibrium, at late times a hydrodynamic expansion is expected to be valid. It is therefore interesting to compare the actual pressures with the pressures which follow from a hydrodynamic expansion [21,22].

In Fig. 3, we plot the difference of p_ρ and the corresponding first-order hydrodynamic prediction of our model of a nucleus. The stress tensor is excellently described by

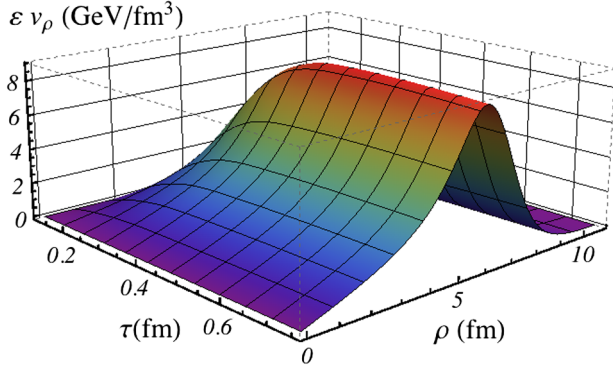


FIG. 2 (color online). The radial velocity times the energy density as a function of proper time τ and distance to the origin ρ for our model of a nucleus. Note that at late times, the increasing velocity is almost exactly compensated by the decreasing energy density (which is due to the longitudinal expansion). The slope at the origin at the end of our simulation equals 0.66 GeV/fm⁴.

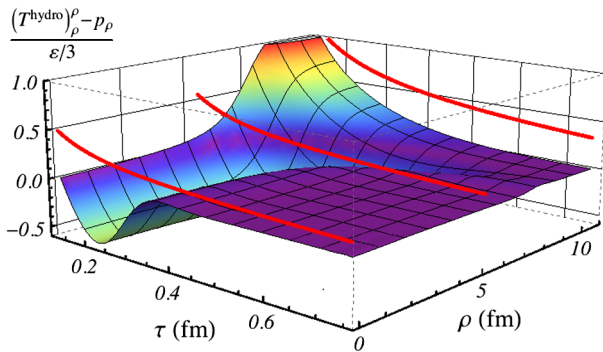
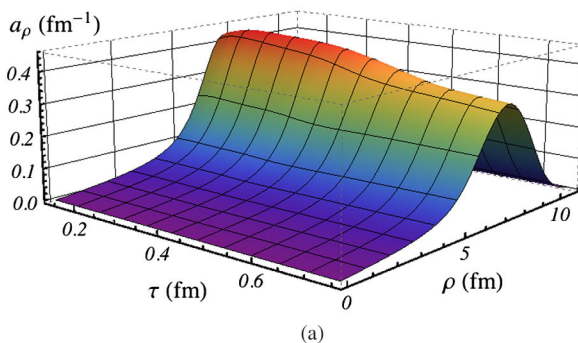


FIG. 3 (color online). The difference between the full non-equilibrium p_ρ and the pressure given by first-order hydrodynamics. Although hydrodynamics applies very quickly, the viscous contribution is still large (shown by red lines). The relatively high values for $\rho > 7$ fm are a consequence of the very small energy density. For the model of a fluctuation, the graph is similar, with equally quick thermalization.



hydrodynamics as soon as $\tau = 0.35$ fm. At the border of our nucleus this is slightly subtler, since the stress tensor is rather small there, and it becomes comparable to our regulator energy density. We therefore cannot say too much about this, but the agreement with hydrodynamics there is also encouraging. We note that in previous studies [10,11], somewhat larger thermalization times (with respect to the local temperature) were found, so we expect more exotic initial conditions in our bulk AdS to give somewhat later thermalization.

In Fig. 4(b), we plot the radial acceleration of our model of a fluctuation. We notice the acceleration already decreases considerably during our simulation, in contrast with the model for the nucleus. Also, the acceleration increases rapidly near the origin, whereas for the nucleus it is rather narrowly peaked near the boundary of the nucleus. This means that fluctuations are expected to spread out rather quickly. Perhaps surprisingly, the stress tensor for the fluctuation is also governed by hydrodynamics within 0.35 fm.

IV. DISCUSSION

The main motivation for this study is to provide a description of the far-from-equilibrium stage of heavy-ion collisions, including nontrivial dynamics in the transverse plane. While we kept rotational symmetry in the transverse plane, we believe our study can be used more generally. One reason for this is an old result in asymptotically flat space [23], recently studied in asymptotically AdS [11], that during black hole formation gravity can be well approximated by linearizing around the final state. We therefore believe that an initial energy profile with many fluctuations could be well approximated by superposing the result of our fluctuation presented above.

Also, it should be possible to use our results for non-central collisions. This can be seen by comparing with

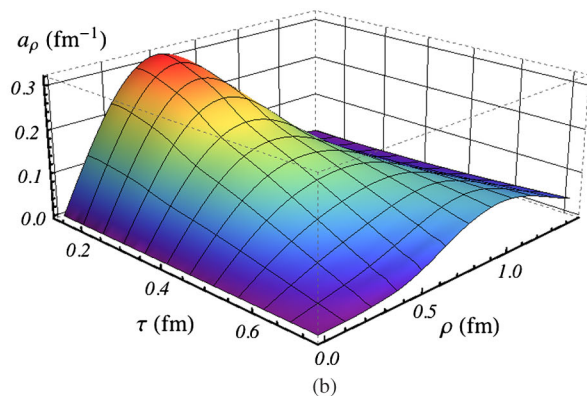


FIG. 4 (color online). (a) The radial acceleration of our nucleus model. The acceleration decreases after some time, which is mainly a consequence of the decrease in radial pressure, due to the isotropization. Thereafter, the acceleration is quite steady and mainly localized near the boundary of the nucleus. (b) The radial acceleration of our fluctuation model. Since the bump of energy is much smaller, one can clearly see the spreading out and the decrease in acceleration. As will also be clear from Fig. 5, this model reaches a lower radial speed than the model for the nucleus.

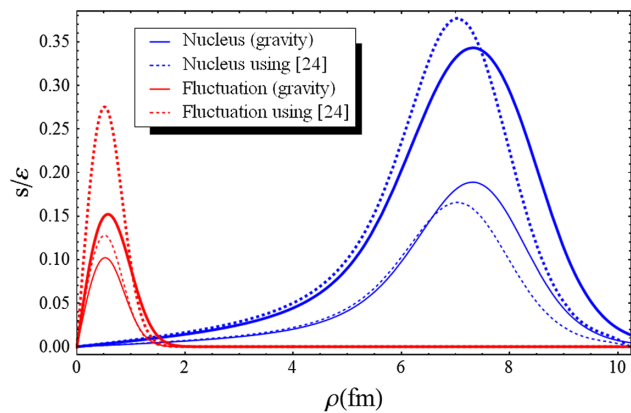


FIG. 5 (color online). Here we plot the momentum flow s divided by the energy density, at time $\tau = 0.75$ fm (thick lines) and $\tau = 0.4$ fm (thin lines), as a function of ρ . The plots compare our gravitational results with Eq. (6), which was found in Ref. [24]. The two results are remarkably similar, especially at earlier times, and considering the dynamics takes place at very different scales. We suggest Eq. (6) as an initial condition for nonsymmetric hydrodynamic simulations, where a simulation should start at about $\tau = 0.4$ fm if fluctuations are present.

Ref. [24]. There, they assume that the anisotropy is independent of ρ , that the transverse pressures are equal and that the velocity is approximately linear in time. Without using any hydrodynamics, they used the conservation of the stress tensor to arrive at the following local formula for the transverse momentum of the stress tensor:

$$\vec{s}/\varepsilon \approx -\frac{\vec{\nabla}_{\perp} \varepsilon_0}{2\varepsilon_0} (\tau - \tau_{\text{in}}), \quad (6)$$

where ε_0 is the initial energy density. This formula (see Fig. 5) works remarkably well at early times, and also later on, for the nucleus model. At later times, the transverse velocities of fluctuations are smaller, which is due to the decreasing acceleration [displayed in Fig. 4(b)]. This result therefore increases confidence in the result of Ref. [24],

which can be used in less symmetric situations. When including fluctuations, however, one should use hydrodynamics as soon as $\tau = 0.4$ fm to get more accurate results.

Apart from the well-known fact that the gauge/gravity duality in our setting describes a supersymmetric theory at infinite coupling and a large number of colors, there is another reason why one should take care in applying our results directly to experimental settings. Our initial bulk metric should, in principle, follow from the way the experimental state was created. It is not clear how to do this, and we therefore adopted a simple model, where the metric is basically close to vacuum. In future work, we plan to study the dependence of the outcomes on this initial state, but preliminary findings suggest that the results do not depend strongly on the initial bulk metric.

Even though our model is much simpler than a real heavy-ion collision in QCD, we carried out a far-from-equilibrium calculation at strong coupling with non-trivial dynamics in the transverse plane. These results can therefore be very useful as initial conditions for the hydrodynamic modeling of heavy-ion collisions. In particular, the radial velocity at this initial time was basically unknown, and it is hence usually taken to be zero. Our radial velocities would be a more natural start for a hydrodynamic simulation. Although the final effect on experimental observables would be rather moderate at the moment, it will become increasingly important as the experimental data improve.

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